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Research article

Extended Brauer analysis of some Dynkin and Euclidean diagrams

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Abstract: The analysis of algebraic invariants of algebras induced by appropriated multiset systems called Brauer configurations is a Brauer analysis of the data defining the multisets. Giving a complete description of such algebraic invariants (e.g., giving a closed formula for the dimensions of algebras induced by significant classes of Brauer configurations) is generally a tricky problem. Ringel previously proposed an analysis of this type in the case of Dynkin algebras, for which so-called Dynkin functions were used to study the numerical behavior of invariants associated with such algebras. This paper introduces two additional tools (the entropy and the covering graph of a Brauer configuration) for Brauer analysis, which is applied to Dynkin and Euclidean diagrams to define Dynkin functions associated with Brauer configuration algebras. Properties of graph entropies defined by the corresponding covering graphs are given to establish relationships between the theory of Dynkin functions, the Brauer configuration algebras theory, and the topological content information theory.

Keywords: Brauer configuration algebra (BCA); Dynkin diagram; Dynkin function; Euclidean diagram; graph entropy; integer categorification; path algebra; quiver representation

1. Introduction

This paper establishes relationships between the theory of Dynkin functions, the theory of Brauer configuration algebras (BCAs), and the topological content information theory (graph entropy theory). On the one hand, Ringel and Fahr [1] introduced the notion of *categorification of a set of numbers*, which means considering suitable objects in a category instead of these numbers (e.g., representation of quivers) so that the numbers in question occur as invariants of the objects.

According to Ringel and Fahr [1], the equality of such numbers may be visualized by isomorphisms of objects and functional relations by functorial ties. For instance, they proved that certain formulas for Fibonacci numbers can be interpreted via filtrations of some Fibonacci modules [2]. In this line, Ringel [3] introduced *Dynkin functions* which assign to a given Dynkin diagram $\Delta \in \{\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2\}$ an integer, or, more generally, a real number, and sometimes even a set or a sequence of real numbers (see Figure 2). Thus, a Dynkin function f consists of four sequences $f(\mathbb{A}_n)$, $f(\mathbb{B}_n)$, $f(\mathbb{C}_n)$, $f(\mathbb{D}_n)$ as well as five additional single values $f(\mathbb{E}_6)$, $f(\mathbb{E}_7)$, $f(\mathbb{E}_8)$, $f(\mathbb{F}_4)$, and $f(\mathbb{G}_2)$.

The number of indecomposable modules and the number of tilting modules over the corresponding Dynkin algebras are examples of Dynkin functions. We remind that a module T over an algebra $\mathcal{A}$ is said to be a tilting module if the projective dimension of T denoted $\mathrm{pd}_{\mathcal{A}}\, T$ is less than or equal to 1. Furthermore, $\mathrm{Ext}^1_{\mathcal{A}}(T,T) = 0$, and there exists a short exact sequence $0 \to T'_{\mathcal{A}} \to T''_{\mathcal{A}} \to 0$, where $T'_{\mathcal{A}}$ and $T''_{\mathcal{A}}$ are objects of the category and add T generated by the module T [4].

Ringel [3] also suggested the creation of an online encyclopedia of Dynkin functions (OEDF) with the same goals as the online Encyclopedia of integer sequences (OEIS).

Brauer configuration algebras are examples of the application of combinatorial tools in the theory of representation of algebras. Such algebras arise from multisets and their induced bounded directed graphs called Brauer quivers [5–8].

It is worth noting that the confluence between the theory of representation of algebras and combinatorics realized by the Brauer configuration algebras allows for finding applications of these algebras in several fields of mathematics and science [9–11].

Since graphs are examples of systems or collections of multisets, they naturally induce Brauer configuration algebras. Green and Schroll explored this point of view [6] and defined Brauer graph algebras, a particular case of Brauer configuration algebras. Moreover, Brauer analysis allowed Cañadas et al. [10] to give formulas for the dimension of Brauer configuration algebras induced by particular kinds of graphs used to build quantum entanglements.

Rashevsky [12] and Trucco [13] introduced the concept of graph entropy to measure complexity of networks. The construction of such measures was originally called topological content information of a graph.

It is worth pointing out that most of the Brauer analyses have been focused on using the structure of indecomposable projective modules and the dimension of the Brauer configuration algebras induced by the data underlying a finite collection of multisets without paying attention to the amount of information arising from these data [9–11]. This paper fills the gap by introducing the entropy associated with a Brauer configuration.

We are focused on constructing some Dynkin functions based on algebraic invariants associated with Brauer configuration algebras (induced by Dynkin and Euclidean diagrams) and the topological content information arising from their corresponding covering graphs. Such use of algebraic invariants and graph entropies is said to be an *extended Brauer analysis* of the diagrams.

1.1. Contributions

This paper extends Brauer analysis by introducing the covering graph of a Brauer configuration (see (2.14)) and its entropy (see (2.21)). Some of the main results of this paper are Theorems 7, 8, 10–14, and Corollaries 1–3.

Theorems 7 and 8 give some properties of the covering graph of a Brauer configuration. Theorem 11 gives bounds for the entropy associated with a Brauer configuration algebra with the length grading property, and Theorem 12 compares some degree-based entropies with the entropy of the Brauer configuration induced by Dynkin and Euclidean diagrams.

Theorem 10 and Corollaries 1–3 define Dynkin functions based on algebraic invariants, for instance, admissible ideals that define Brauer configuration algebras, covering graphs, the dimension of Brauer configuration algebras, and degree-based entropies.

Theorem 13 provides results similar to Euclidean diagrams, as those presented for Dynkin diagrams. Theorem 14 gives relationships between dimensions of Brauer configuration algebras and entropies.

Tables 4–7 summarize the results obtained.

The organization of this paper goes as follows: Background, main definitions and notations are given in Section 2; we give definitions of multisets (Section 2.4), Brauer configuration algebras (Section 2.4.1), graph entropies (Section 2.5), and some of their properties. We present the main results in Section 3. Discussions regarding the results obtained are given in Section 4. Concluding remarks are given in Section 5.

2. Preliminaries

This section describes the research development of Dynkin functions, Brauer configuration algebras, and graph entropy. It also recalls basic notations and definitions.

2.1. Background

Ringel and Fahr introduced the notion of categorification of a set of numbers [1]; as a consequence of this work, the sequence A132262=$\{1, 2, 7, 29, 130, \dots\}$ was introduced in the OEIS. They used the theory of representation of the 3-Kronecker quiver to obtain an integer partition for even-index Fibonacci numbers. Moreover, they introduced Fibonacci modules to categorify the sequence of Fibonacci numbers [2].

Dynkin functions associated with Dynkin algebras were introduced by Ringel [3]. He gave the number of indecomposable modules $r(\Delta)$ over the corresponding algebras as an example of a Dynkin function (see Table 1). In [3] Ringel proposes to define new Dynkin functions and to create an OEDF. This paper introduces Dynkin functions associated with Brauer configuration algebras induced by Dynkin and Euclidean diagrams.

Ringel et al. [14] proved that the nonzero numbers $a_s(\Delta)$ (the number of support-tilting modules with support rank s) is a Dynkin function and that the Dynkin diagrams $\mathbb{A}, \mathbb{B}$, and $\mathbb{D}$ yield three triangles having similar properties as the Catalan triangle (A009766 in the OEIS, [15]), Pascal's triangle (A059481 in the OEIS, [16]) and Lucas triangle (A029635 in the OEIS, [17]), respectively.

Table 1. Example of a Dynkin function.

Dynkin diagrams	$r(\Delta)$
$\mathbb{A}_n$	$t_n = \binom{n+1}{2}$
$\mathbb{B}_n$	n^2
$\mathbb{D}_n$	$n(n-1)$
$\mathbb{E}_6$	36
$\mathbb{E}_7$	63
$\mathbb{E}_8$	120
$\mathbb{F}_4$	24
$\mathbb{G}_2$	6

Green and Schroll introduced Brauer configuration algebras and Brauer graph algebras [5,6] to deal with research regarding the classification of algebras of wild representation type. Since then, these algebras and their combinatorial nature have been used by Espinosa et al. [11] to investigate fields in mathematics and different sciences. They were used to give multimedia security in cryptography and to establish relationships between quantum entanglement theory and Brauer configuration algebras [9, 10]. Relationships between Brauer configuration algebras and Dynkin functions of type ADE were given by Bravo in [18].

Rashevsky and Trucco [12, 13] introduced the concept of graph entropy to measure the structural complexity of several graph invariants. It is worth noting that the construction of entropy-based measures was originally called the topological information content of a graph. According to Mowshowitz [19], the probabilistic and deterministic categories are the main approaches to measuring the complexity of graphs. The probabilistic category deals with applying an entropy function to a probability distribution associated with a graph. The deterministic category embraces the encoding, substructure count, and generative approaches. Graph entropies were used recently by Kulkarni et al. [20] to study networks induced by note transitions in Bach's work.

2.2. Path algebras

This section recalls basic notations and definitions regarding the theory of representation of algebras. The authors refer the interested reader to [4, 21] and [22] for more insights on the development of the theory and a detailed explanation of the notions described.

A *graph* is a pair $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$ of sets satisfying $E_{\mathcal{G}} \subseteq V_{\mathcal{G}} \times V_{\mathcal{G}}$. Thus, the elements of $E_{\mathcal{G}}$ are 2-element subsets of $V_{\mathcal{G}}$. It is assumed that $V_{\mathcal{G}} \cap E_{\mathcal{G}} = \emptyset$. The elements of $V_{\mathcal{G}}$ are the *vertices* (or *nodes*, or *points*), and the elements of $E_{\mathcal{G}}$ are its *edges* (or *lines*). The number of vertices of a graph $\mathcal{G}$ is its *order* written as $|V_{\mathcal{G}}| = |\mathcal{G}|$, and its number of edges is denoted $||\mathcal{G}||$ [23].

A vertex v is *incident* with an edge e if $v \in e$. The edge e is an edge at v.

The two vertices incident with an edge are its *endvertices* or *ends*, and an edge joins its ends. An edge $\{x, y\}$ is usually written as a *word* of the form $w(e) = xy$ (or $w(e) = yx$).

A *path* is a nonempty graph $P = (V_P, E_P)$ with $V_P = \{x_0, x_1, \ldots, x_m\}$, and $E_P = \{x_0x_1, x_1x_2, \ldots, x_{m-1}x_m\}$, where the x_i are all distinct. The vertices x_0 and x_m are *linked* by P and are called its *ends*; the vertices $x_1, x_2, \ldots, x_{m-1}$ are the *inner* vertices of P. The sequence of vertices can be used to denote a path, say $P = x_0x_1 \ldots x_m$ and calling P a path from x_0 to x_m [23].

If $P = x_0 \ldots x_{m-1}$ is a path and $m \geq 3$, then the graph $C = P + x_{m-1}x_0$ is called a *cycle*.

A nonempty graph $\mathcal{G}$ is called *connected* if any two of its vertices are linked by a path in $\mathcal{G}$. A maximal connected subgraph of $\mathcal{G}$ is called a *component*.

An acyclic graph (a graph without cycles) is called a *forest*. A connected forest is a *tree*, i.e., a forest is a graph whose components are trees [23].

Let $\mathcal{G}$ be a graph and $A = (a_1, a_2, ..., a_k)$ be an ordered subset of vertices of $\mathcal{G}$. A *hair graph* of $\mathcal{G}$ with respect to A, denoted by $\mathcal{G}[(a_1, a_2, ..., a_k); (n_1, n_2, ..., n_k)]$ is the graph obtained from $\mathcal{G}$ by attaching a path P_{n_i} with n_i (≥ 1) vertices to vertex a_i at the first (or last) vertex of P_{n_i}, for all $i = 1, ..., k$ [24]. Furthermore, the set of all hair graphs obtained from the graph $\mathcal{G}$ are denoted by $Hair(\mathcal{G})$. Figure 1 shows a hair graph $C_4[(v_1, v_2, v_3, v_4); (1, 2, 1, 3)]$ of the 4-point cycle $C_4 = \{v_1, v_2, v_3, v_4\}$.

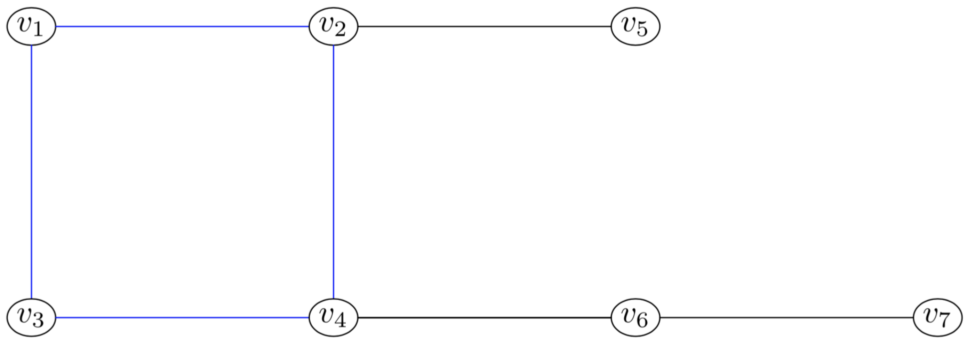

Figure 1. Hair graph $C_4[(v_1, v_2, v_3, v_4); (1, 2, 1, 3)]$.

A *quiver* is a directed graph $Q = (Q_0, Q_1, s, t)$, where Q_0 (Q_1) is the set of vertices (arrows) of the quiver whereas $s, t : Q_1 \to Q_0$ are maps assigning the source $s(\alpha)$ and the target $t(\alpha)$ of an arrow $\alpha \in Q_1$.

If k is an algebraically closed field, then the paths of a quiver Q generate a corresponding path algebra kQ, for which each vertex $a \in Q_0$ has associated a stationary path $e_a = a||a$. The set $\{e_a \mid a \in Q_0\}$ constitutes a system of primitive orthogonal idempotents of the algebra kQ [4].

Ideals in path algebras are generated by a set of relations ρ which are paths with the same ends and length ≥ 1. For $l \geq 1$, we let R^l_Q denote the ideal consisting of paths of length l.

If Q is a finite quiver and R_Q is the arrow ideal of the path algebra kQ, then a two-sided ideal I of kQ is said to be *admissible* if there exists $m \geq 2$ such that $R^m_Q \subseteq I \subseteq R^2_Q$. The pair (Q, I) is said to be a *bound quiver* and the quotient algebra kQ/I is said to be a *bound quiver algebra* [4].

Since the category mod kQ of finitely generated kQ-modules is abelian, the main problem regarding the theory of representation of algebras consists of giving a complete description of the indecomposable kQ-modules (or indecomposable representations of Q) and the irreducible morphisms [4].

Drozd [25] proved that the algebras over an algebraically closed field are either of tame or wild representation type, and not both. We remind that a quiver Q is of finite (tame) representation type if it has finitely many isomorphism clases of indecomposable representations (it is of infinite type and the indecomposable representations occur in families of at least two parameters).

Gabriel [26] proved that any finite dimensional basic algebra is isomorphic to a suitable bounded path algebra and that a finite dimensional algebra $\Lambda = kQ$ is of finite representation type, if and only if, the underlying undirected graph $\overline{Q}$ is a Dynkin diagram of type $\mathbb{A}_n, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, $n \geq 1$ and $m \geq 4$. Dlab and Ringel [27] proved that a connected hereditary Artin algebra is representation-finite, if and only if, the underlying undirected graph $\overline{Q}$ is one of the Dynkin diagrams $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4$, or $\mathbb{G}_2$ (the new diagrams $\mathbb{B}_n, \mathbb{C}_n, \mathbb{F}_4$, and $\mathbb{G}_2$ appear for finite type when the field k is not algebraically closed).

Path algebras $\Lambda = kQ$ defined by a quiver Q whose underlying graph $\overline{Q}$ is a Euclidean diagram of the form $\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_m, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ are of tame representation type. Figure 2 shows Dynkin and Euclidean diagrams.

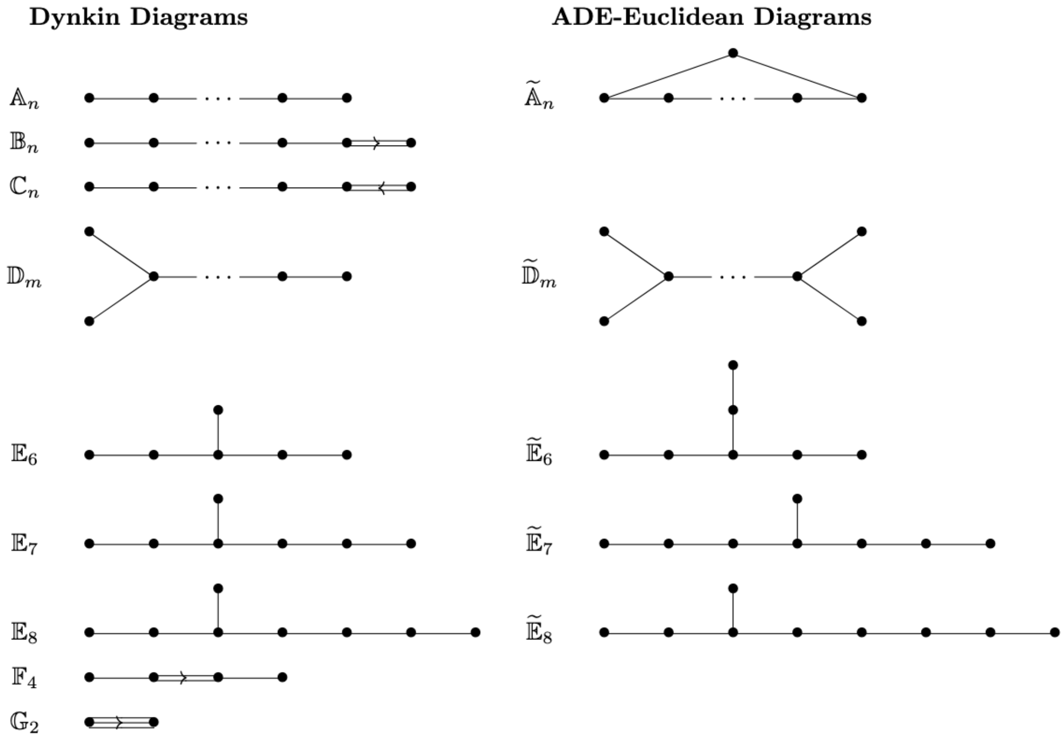

Figure 2. Dynkin and Euclidean diagrams.

2.3. Dynkin functions

As described in the introduction, any Dynkin function f consists of four sequences $f(\mathbb{A}_n)$, $f(\mathbb{B}_n)$, $f(\mathbb{C}_n)$, $f(\mathbb{D}_n)$ and five single values $f(\mathbb{E}_s)$, $s \in \{6, 7, 8\}$, $f(\mathbb{F}_4)$, and $f(\mathbb{G}_2)$ (see [3] and [14]). The counting problems for which f arises deal with invariants of Dynkin algebras.

Given any Dynkin algebra Λ an *exceptional sequence* for Λ is a sequence $(M_1, M_2, \ldots, M_t)$ of indecomposable Λ-modules such that $\mathrm{Hom}(M_i, M_j) = 0 = \mathrm{Ext}^1(M_i, M_j)$ for $i > j$. It is well-known that if $h(\Delta)$ is the Coxeter number (the order of a Coxeter element) of the Dynkin diagram Δ and $W(\Delta)$ is the corresponding *Weyl group* (for a quiver $Q = (Q_0, Q_1, s, t)$ the Weyl group is the group of automorphisms of $E = \mathbb{Q}^n$ generated by a set of refections $\{s_i \mid i \in Q_0\}$ its elements are said to be Coxeter elements), then the number of exceptional sequences of the associated Dynkin algebra Λ is given by the identity $e(\Delta) = \frac{n!h(\Delta)^n}{|W(\Delta)|}$. Table 2 shows the Dynkin functions defined by $e(\Delta)$, $h(\Delta)$, and $|W(\Delta)|$.

Table 2. Dynkin functions defined by the number of exceptional sequences, the Coxeter number, and the size of the corresponding Weyl group [3].

Dynkin diagrams	Dynkin functions		
	$e(\Delta)$	$h(\Delta)$	$\|W(\Delta)\|$
$\mathbb{A}_n$	$(n+1)^{n-1}$	$n+1$	$(n+1)!$
$\mathbb{B}_n$	n^n	$2n$	$2^n n!$
$\mathbb{D}_n$	$2(n-1)^n$	$2(n-1)$	$2^{(n-1)}n!$
$\mathbb{E}_6$	$2^9 3^4$	$(2)^2(3)$	$(2)^7(3)^4(5)$
$\mathbb{E}_7$	$2(3)^{12}$	$(2)(3)^2$	$(2)^{10}(3)^4(5)(7)$
$\mathbb{E}_8$	$2(3)^5(5)^7$	$(2)(3)(5)$	$(2)^{14}(3)^5(5)^2(7)$
$\mathbb{F}_4$	$(2)^4(3)^3$	$(2)^2(3)$	$(2)^7(3)^2$
$\mathbb{G}_2$	$(2)(3)$	$(2)(3)$	$(2)^2(3)$

This paper introduces Dynkin functions defined by Brauer configuration algebras Λ_Δ induced by Dynkin and Euclidean diagrams. The construction of such functions is said to be the *extended Brauer analysis* of the corresponding Brauer configuration.

2.4. *Multisets*

This section reminds basic definitions regarding multisets. Authors referred the interested reader to [7,8,11], and [28] for more details dealing with multisets and their messages. We also introduce basic constructions as multiset systems of type M which can be considered Brauer configurations in the sense of Green and Schroll [5]. Thus, we will see that multiset systems of type M induce suitable Brauer configuration algebras, for which the analysis of their invariants is said to be a Brauer analysis.

A *multiset* is a pair (M, f) where M is a set and $f : M \to \mathbb{N}$ is a map from M to the nonnegative integers set. Roughly speaking, multisets allow repeated elements [8, 28].

According to Andrews [8], a *permutation* of a multiset (M, f) is determined by a word w whose letters are the elements of M with letter occurrences given by the map f. In this paper, we assume that words associated with multisets are given by fixed permutations of the following form:

$$w(M) = m_1^{f(m_1)} m_2^{f(m_2)} \dots m_s^{f(m_s)}. \tag{2.1}$$

Note that there are $(f(m_1) + f(m_2) + \cdots + f(m_s))!$ permutations associated with the multiset (M, f).

Operations between multisets are defined by da Fontoura in [28]. According to him, if (M_1, f_1) and (M_2, f_2) are multisets with $M_1 = \{m_{1,1}, \dots, m_{1,r}\}$ and $M_2 = \{m_{2,1}, \dots, m_{2,s}\}$, then

$$\begin{aligned} &(M_1, f_1) \cup (M_2, f_2) = (M_1 \cup M_2, f_{1,2}) \text{ with } f_{1,2}(x) = \max\{f_1(x), f_2(x)\} \text{ if } x \notin M_j, \text{ then } f_j(x) = 0 \text{ for } j \in \{1, 2\}. \\ &(M_1, g_1) \cap (M_2, g_2) = (M_1 \cap M_2, g_{1,2}) \text{ with } g_{1,2}(x) = \min\{f_1(x), f_2(x)\}. \end{aligned} \tag{2.2}$$

Let $\mathcal{M} = \{(M_1, f_1), (M_2, f_2), \ldots, (M_h, f_h)\}$ be a finite collection of multisets satisfying the following condition (2.3).

$$\text{If } w(M_i) = (m_{i,1})^{f_i(m_{i,1})}(m_{i,2})^{f_i(m_{i,2})}\ldots(m_{i,s_i})^{f_i(m_{i,s_i})}, \text{ then } \sum_{j=1}^{s_i} f_i(m_{i,j}) > 1. \tag{2.3}$$

If $M = \bigcup_{i=1}^{h} M_i$, with $(M_i, f_i) \in \mathcal{M}$, $\bigcap_{x\in I} M_x$ gives the intersection of all multisets containing an element $y \in M$ with $I \subseteq \{1, 2, \ldots, h\}$ a fixed set of indices, and $f_x(y)$ is the frequency of y in M_x, then $\sum_{x\in I} f_x(y)$ is said to be the *valency* of y denoted $val(y)$ [5].

For each $y \in M,$ the set $\mathfrak{I}_y = \{M_x \mid x \in I\}$ is endowed with a linear order $\prec$ in such a way that if $I = \{i_1, i_2, \ldots, i_n\}$ and $i_1 < i_2 < \cdots < i_n$ with the usual order of natural numbers then $M_{i_1} \prec M_{i_2} \prec \cdots \prec M_{i_n}$.
(2.4)

For each $y \in M$, it holds that any fixed $M_x \in \mathfrak{I}_y$, with $f_x(y) \geq 1$ has associated a subchain $\mathfrak{M}_{x,y}$ with the form $\mathfrak{M}_{x,y} = M_x^{(1)} \prec M_x^{(2)} \prec \cdots \prec M_x^{f_x(y)}$, named the *expansion of* M_x induced by y, where M_x^i is associated with a unique copy of M_x. $M_x^{(1)} = M_x$ if $f_x(y) = 1$.

According to Stanley [7] if x and y are points in a poset $(P, \leq)$ then y covers x if $x < y$ and if no element $z \in P$ satisfies $x < z < y$. Thus, y covers x if and only if $x < y$ and $[x, y] = \{x, y\}$.

For any $y \in M$, we assume that if $M_x \prec M_{x'}$ is a covering in $\mathfrak{I}_y$, then $\mathfrak{M}_{x,y} \prec \mathfrak{M}_{x',y}$ is also a covering and $M_x^{(1)} \prec M_x^{(2)} \prec \cdots \prec M_x^{f_x(y)} \prec M_{x'}^{(1)} \prec M_{x'}^{(2)} \prec \cdots \prec M_{x'}^{f_{x'}(y)}$.

Let $\{M_1, M_2, \ldots, M_h\}$ be a collection of nonempty finite sets and $M = \bigcup_{i=1}^{h} M_i$ with $M_i = \{m_{i,j} \mid 1 \leq j \leq s_i\}$ then a collection of multisets $\mathcal{M} = \{(M_1, f_1), \ldots, (M_h, f_h)\}$ is said to be a *multiset system of type M,* if $\sum_{j=1}^{s_i} f_i(m_{i,j}) > 1$ for each $1 \leq i \leq h$. Furthermore, for all $y \in M,$ there exists a poset $(\mathfrak{I}_y, \prec)$ of type (2.4), which is a linearly ordered set or chain with corresponding expansions associated with sets $M_x \in \mathfrak{I}_y$, and there exists a map, $\nu : M \to \mathbb{N}^+ \times \mathbb{N}^+$ such that $\nu(m) = (j, val(m)),$ for each $m \in M$.
(2.5)

Green and Schroll [5] named vertices the elements of M, if they are associated with a so-called Brauer configuration. The positive integer j in a pair $(j, val(m))$ associated with a vertex m is the multiplicity value $\mu(m)$ according to them. It is worth pointing out that $\mu(m)$ does not deal with the multiplicity of m as an element of a multiset. In particular, $m \in M$ is said to be nontruncated (truncated) provided that $\mu(m)val(m) > 1$ ($\mu(m)val(m) = 1$). Thus, the multiplicity function μ allows them to classify the set of vertices M into the set of truncated and nontruncated vertices. In this paper, if $val(m) = 1$, it is assumed that $\mu(m) = 2$, otherwise $\mu(m) = 1$. Therefore, the considered vertices are nontruncated.

Every $y \in M$ in a multiset system of type M defines a *successor sequence* S_y *of type* M. In such a case, if $\mathfrak{I}_y = \{M_{i_1}, M_{i_2}, \ldots, M_{i_t}\}$, then the successor sequence S_y at vertex y is a chain of type (2.4) with the following forms:

$$M_{i_1}^{(1)} \prec \cdots \prec M_{i_1}^{(f_{i_1}(y))} \prec M_{i_2}^{(1)} \prec \cdots \prec M_{i_2}^{(f_{i_2}(y))} \prec \cdots \prec M_{i_t}^{(1)} \prec \cdots \prec M_{i_t}^{(f_{i_t}(y))}. \tag{2.6}$$

Successor sequences of type (2.6) at vertices $y \in M$ give rise to circular orderings by adding a relation of the form $M_{i_t}^{(f_{i_t}(y))} \prec M_{i_1}^{(1)}$ (see (2.8)). In such a case, $M_{i_{s'}}^{(f_{i_{s'}}(y))}$ is the successor of $M_{i_s}^{(f_{i_s}(y))}$ in a covering of the form $M_{i_s}^{(f_{i_s}(y))} \prec M_{i_{s'}}^{(f_{i_{s'}}(y))}$. Furthermore, $M_{i_1}^{(1)}$ is the successor of $M_{i_t}^{(f_{i_t}(y))}$.

Remark 1. *In this paper, we assume that in successor sequences, it holds that* $(M_i^{(r)}, f_i) \prec (M_j^{(s)}, f_j)$ *if* $i < j$ *for any* $r, s \geq 1$, $(M_i^{(r)}, f_i) \prec (M_i^{(s)}, f_i)$ *if* $r < s$. *If no confusion arises we will omit the use of super indices in successor sequences.*

A *compressed version of the successor sequence (2.6)* is obtained if the expansions are not considered. In such a case we use the ordered sequence (2.7) to denote the compressed successor sequence.

$$\mathfrak{M}_{i_1,y} \prec \mathfrak{M}_{i_2,y} \prec \cdots \prec \mathfrak{M}_{i_t,y}. \tag{2.7}$$

Note that a circular order of type (2.6) at a vertex $y \in M$ is equivalent to one of the following form:

$$M_{i_s}^{(j)} \prec \cdots \prec M_{i_1}^{(f(i_1,y))} \prec \cdots \prec M_{i_t}^{(f(i_t,y))} \prec M_{i_1}^{(j)} \prec M_{i_2}^{(j)} \cdots \prec M_{i_s}^{(j)}. \quad 1 \leq s \leq t, \quad 1 \leq j \leq f_{i_s}(y). \tag{2.8}$$

Henceforth, if no confusion arises we will use the symbol $<$ instead of the symbol $\prec$ to denote relations in a multiset system of type M.

Example 1. *As a toy example, let us consider a multiset system* $\mathcal{M}_\mathcal{G}$ *of type* M *induced by a graph* $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G})$ *with* $\{m_1, m_2, m_3, m_4\}$ *as the set* $V_\mathcal{G}$ *of vertices and set of edges* $E_\mathcal{G} = \{M_1, M_2, M_3, M_4\}$ *defining the multisets of* $\mathcal{M}_\mathcal{G}$ *such that*

$$\begin{aligned}
(M_1, f_1) &= \{m_1, m_2 \mid f_1(m_i) = 1, i \in \{1,2\}\},\\
(M_2, f_2) &= \{m_2, m_3 \mid f_2(m_i) = 1, i \in \{2,3\}\},\\
(M_3, f_3) &= \{m_3, m_4 \mid f_3(m_i) = 1, i \in \{3,4\}\},\\
(M_4, f_4) &= \{m_2, m_4 \mid f_4(m_i) = 1, i \in \{2,4\}\},\\
M &= \bigcup_{j=1}^{4} M_i = V_\mathcal{G},\\
M_1 \cap M_2 \cap M_4 = \{m_2\}, &\quad M_2 \cap M_3 = \{m_3\}, \quad M_3 \cap M_4 = \{m_4\}.
\end{aligned} \tag{2.9}$$

In this case, vertices $m_i \in M$ *are nontruncated. Note that,*

$$\begin{aligned}
val(m_1) = 1, \quad val(m_2) = 3, \quad val(m_i) = 2, \text{ for } i \in \{3,4\},\\
\mu(m_1) = 2, \quad \mu(m_i) = 1, \text{ for } i \in \{2,3,4\}.
\end{aligned} \tag{2.10}$$

The successor sequences of type M, S_{m_i} *associated with the set* $M = \{m_1, m_2, m_3, m_4\}$ *are*

$$
\begin{aligned}
&S_{m_1} = M_1^{(1)}, \quad \mathfrak{I}_{m_1} = \{M_1\} \\
&S_{m_2} = M_1^{(1)} < M_2^{(1)} < M_4^{(1)}, \quad \mathfrak{I}_{m_2} = \{M_1 < M_2 < M_4\}, \\
&S_{m_3} = M_2^{(1)} < M_3^{(1)}, \quad \mathfrak{I}_{m_3} = \{M_2 < M_3\}, \\
&S_{m_4} = M_3^{(1)} < M_4^{(1)}, \quad \mathfrak{I}_{m_4} = \{M_3 < M_4\}.
\end{aligned} \tag{2.11}
$$

2.4.1. Brauer configuration algebras induced by multiset systems of type M

This section recalls basic definitions and notation regarding Brauer configurations and their induced Brauer configuration algebras see [5,6,10] and [11] for more details dealing with Brauer graph algebras and Brauer configuration algebras.

According to Green and Schroll [5], a Brauer configuration is a combinatorial data $\Gamma = (\Gamma_0, \Gamma_1, \mu, O)$, where Γ_0 is the (finite) set of vertices of Γ, Γ_1 is a finite collection of finite labeled multisets called *polygons*, whose elements are in Γ_0, μ is function called multiplicity and O is called the orientation of Γ. In such a case, every vertex in Γ_0 is a vertex in at least one polygon, every polygon in Γ_1 has at least two vertices, and every polygon in Γ_1 has at least one nontruncated vertex α such that $val(\alpha)\mu(\alpha) > 1$. The role of the multiplicity function μ was described in Section 2.4 of this paper.

An *orientation* O for Γ is a choice, for each vertex $\alpha \in \Gamma_0$, of a cycling ordering of the polygons in which α occurs as a vertex, including repetitions. In such a case, if α occurs in polygons or multisets of the form $(V_1, f_1), \dots, (V_u, f_u)$ then the polygon (V_i, f_i) occurs $f_i(\alpha)$ times in the list. The cyclic order at vertex α is obtained by linearly ordering the list, say $(V_{i_1}, f_{i_1}) < (V_{i_2}, f_{i_2}) < \cdots < (V_{i_u}, f_{i_u})$ and by adding $(V_{i_u}, f_{i_u}) < (V_{i_1}, f_{i_1})$, $u = val(\alpha)$.

According to Green and Schroll [5], the cyclically ordered list of polygons $(V_1, f_1), \dots, (V_u, f_u)$ or simply $V_1, V_2, \dots, V_u$ is the *successor sequence at* α. In such a case, if $V_1 < V_2 < \cdots < V_u$ is the successor sequence at some nontruncated vertex α with $val(\alpha) = u$, then V_{i+1} is the successor of V_i at α, for $1 \le i \le u$, where $V_{u+1} = V_1$. If α is a vertex in polygon V and $val(\alpha) = 1$ then the successor sequence at α is just V.

The definitions described in this section allow us to conclude that multiset systems of type M are Brauer configurations, named *Brauer configurations of type M* (or simply Brauer configurations, if no confusion arises) whose set of vertices is given by a set $M = \bigcup_{j=1}^{u} M_j$, where $\{(M_1, f_1), \dots (M_u, f_u)\}$ is the set of polygons. The multiplicity function μ is given by the first coordinate of the function ν (see (2.5)). The orientation in this case is given by successor sequences (2.6) and their compressed versions (2.7) of type M.

Since function ν associated with multiset systems of type M includes the multiplicity function μ then we can assume the notation (2.12) for Brauer configurations of type $\mathcal{M}$.

$$
\mathcal{M} = (M, \mathcal{M}_1, \mu, O) \tag{2.12}
$$

The *message* or *Brauer message* $M(\mathcal{M})$ of a Brauer configuration is the concatenation of the fixed words $w(M_i)$ [11], i.e.,

$$
M(\mathcal{M}) = w(M_1)w(M_2)\dots w(M_h). \tag{2.13}
$$

Remark 2. *The Brauer configurations considered in this paper are multiset systems of type M with successor sequences defined as in (2.6), (2.7), and Remark 1.*

Example 2. *As an example, we note that the Brauer message $M(\mathcal{M}_{\mathcal{G}}) = w(M_1)w(M_2)w(M_3)w(M_4)$ induced by the Brauer configuration (2.9) described in Example 1 can be graphically represented by assigning a labeled line of the form given in Figure 3 to each word $w(M_i) = m_{i_1}m_{i_2}$, $i, i_j \in \{1, 2, 3, 4\}$.*

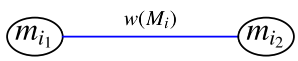

Figure 3. Graphical representation of a word $w(M_i) = m_{i_1}m_{i_2}$.

Figure 4 shows words, $w(M_i)$, $i \in \{1, 2, 3, 4\}$.

Figure 4. Graphical representations of words $w(M_i)$, $i \in \{1, 2, 3, 4\}$.

Word concatenation is obtained by gluing lines with the same ends represented by labeled circles. Figure 5 shows concatenations $w(M_1)w(M_2)$, $w(M_1)w(M_2)w(M_3)$, and the Brauer message $M(\mathcal{M}_{\mathcal{G}}) = \mathcal{G} = C_3[(m_2, m_3, m_4); (2, 1, 1)] = w(M_1)w(M_2)w(M_3)w(M_4)$.

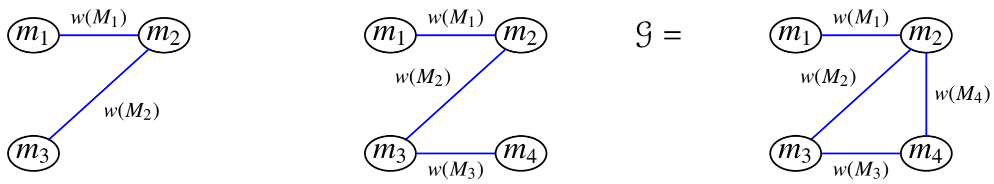

Figure 5. A hair graph $\mathcal{G} = C_3[(m_2, m_3, m_4); (2, 1, 1)]$ of a 3-point cycle $C_3 = \{m_2, m_3, m_4\}$ viewed as a Brauer message $w(M_1)w(M_2)w(M_3)w(M_4)$ (see Figure 1).

The *Brauer quiver* $Q_{\mathcal{M}} = (Q_0, Q_1, s, t)$ (or simply Q, if no confusion arises) induced by a Brauer configuration $\mathcal{M} = (M, \mathcal{M}_1, \mu, O)$ is defined in such a way that there is a bijective correspondence between its set of vertices Q_0 and the collection of polygons $\mathcal{M}_1$. In other words, vertices in a Brauer quiver are given by the multisets in $\mathcal{M}_1$. If (M_j, f_j) is a successor to (M_i, f_i) in the successor sequence at the nontruncated vertex $m \in M$ then there is a unique arrow $a \in Q_1$ such that $s(a) = (M_i, f_i)$ and $t(a) = (M_j, f_j)$. In particular, each covering $(M_i, f_i) < (M_j, f_j)$ in a circular ordering defines an arrow from (M_i, f_i) to (M_j, f_j) in Q_1. The cycles defined by equivalent circular orderings associated with nontruncated vertices $y \in M$ starting and ending at a given vertex $M \in Q_0$ are named *special cycles* by Green and Schroll [5].

Since in this paper, it is assumed that $\mu(v) = 2$ for any vertex $v \in M$ with valency $val(v) = 1$, then circular orderings associated with vertices with valency 1 define an arrow (loop) α^v from the polygon containing it to itself [5]. In such a case $(\alpha^v)^2$ is the unique special cycle associated with $v \in M$.

Example 3. *Figure 6 shows the Brauer quiver induced by graph $\mathcal{G}$ (see Figure 5) and the Brauer configuration (2.9) defined in Example 1.*

$Q_{\mathcal{M}_{\mathcal{G}}} =$

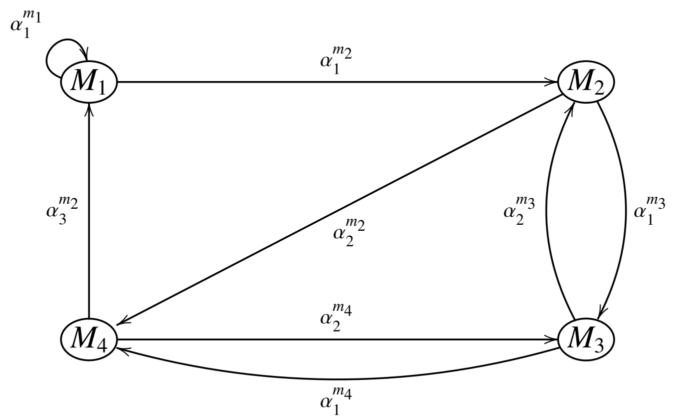

Figure 6. Brauer quiver $Q_{\mathcal{M}_{\mathcal{G}}}$ induced by graph $\mathcal{G}$ (see Figure 5).

The *covering graph* or *nerve* of a Brauer quiver $Q_{\mathcal{M}}$ induced by a fixed Brauer configuration $\mathcal{M}$ (or the covering graph of a Brauer configuration $\mathcal{M}$) is a graph $\mathfrak{c}(Q_{\mathcal{M}}) = (V_{\mathfrak{c}(Q_{\mathcal{M}})}, E_{\mathfrak{c}(Q_{\mathcal{M}})})$ whose set of vertices $V_{\mathfrak{c}(Q_{\mathcal{M}})}$ consists of polygons in $\mathcal{M}_1$, and there exists an edge $M_i - M_j \in E_{\mathfrak{c}(Q_{\mathcal{M}})}$ connecting two polygons $M_i, M_j \in \mathcal{M}_1$ if and only if either $\mathfrak{M}_{i,y} < \mathfrak{M}_{j,y}$ or $\mathfrak{M}_{j,y} < \mathfrak{M}_{i,y}$ is a covering in the compressed successor sequence of type (2.7) at some vertex $y \in M$. Note that $\mathfrak{c}(Q_{\mathcal{M}})$ has no multiple edges or loops. So, we write just one edge to represent all the coverings of a given form. We provide alternative conditions for vertices and edges of a covering graph in (2.14).

$$\begin{aligned} &V_{\mathfrak{c}(Q_{\mathcal{M}})} = \mathcal{M}_1. \\ &\{M_i, M_j\} \in E_{\mathfrak{c}(Q_{\mathcal{M}})}, \text{ if and only if, either } M_i < M_j \text{ or } M_j < M_i \text{ is a covering for some } M_i, M_j \in \mathfrak{I}_y \\ &\text{and some } y \in M. \end{aligned} \tag{2.14}$$

Graph $\mathcal{G}$ (see Figure 5) is the covering graph of the Brauer quiver $Q_{\mathcal{M}_{\mathcal{G}}}$ (see Figure 6). If no confusion arises, we also write $\mathfrak{c}(Q_{\mathcal{G}})$ instead of $\mathfrak{c}(Q_{\mathcal{M}_{\mathcal{G}}})$ to denote the covering graph induced by a graph $\mathcal{G}$.

A Brauer configuration $\mathcal{M} = (M, \mathcal{M}_1, \mu, O)$ is said to be *reduced* if the set M has only nontruncated vertices. It is *connected* if the corresponding Brauer quiver is.

A Brauer configuration algebra $\Lambda_{\mathcal{M}}$ (or simply Λ if no confusion arises) is a bound quiver algebra $\Lambda_{\mathcal{M}} = kQ_{\mathcal{M}}/I_{\mathcal{M}}$ induced by a Brauer quiver $Q_{\mathcal{M}}$ bounded by an admissible ideal $I_{\mathcal{M}} = \langle \rho \rangle$ generated by a set of relations ρ of the following types [5, 6]:

(ρ_1) $C_i^{\mu(i)} - C_j^{\mu(j)}$ if i and j are vertices in the same polygon (M_i, f_i) and C_x is a special cycle x-cycle at the polygon (M_i, f_i).

(ρ_2) $C_i^{\mu(i)}a$ if a is the first arrow of a special cycle C_i associated with a vertex i. In particular, $C_i^3 \in I_{\mathcal{M}}$ if $val(i) = 1$ and $\mu(i) = 2$.

(ρ_3) ab if $a, b \in Q_1$ are arrows of different special cycles and ab is an element of the path algebra induced by $Q_{\mathcal{M}}$.

We let $\mathfrak{I}_{\mathcal{M}}$ denote the minimum set of generators of the admissible ideal $I_{\mathcal{M}}$ named the *fundamental set of relations* of $I_{\mathcal{M}}$.

$\mathfrak{I}_{\mathcal{M}} = \{\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_{\iota_{\mathcal{M}}} \mid \iota_{\mathcal{M}} \geq 1$ and $\mathfrak{r}_i$ is a relation of type (ρ_j), $j \in \{1,2,3\}\}$, such that any other relation $r \in I_{\mathcal{M}}$ can be written as a sum of the form $r = \sum\limits_{h,h' \in \Lambda_{\mathcal{M}}} \sum\limits_{j=1}^{\iota_{\mathcal{M}}} h\mathfrak{r}_j h'$.

It is worth noting that giving $\iota_{\mathcal{M}} = |\mathfrak{I}_{\mathcal{M}}|$ for any Brauer configuration algebra is a hard problem.

We say that a Brauer configuration algebra is indecomposable (as an algebra) and reduced if it is induced by a connected and reduced Brauer configuration.

Example 4. *As an example the Brauer configuration algebra $\Lambda_{\mathcal{M}_{\mathcal{G}}}$ induced by the Brauer configuration $\mathcal{M}$ defined by identities (2.9) and the Brauer quiver $Q_{\mathcal{M}_{\mathcal{G}}}$ shown in Figure 6 is bounded by an admissible ideal $I_{\mathcal{M}_{\mathcal{G}}}$. The following are examples of relations contained in $I_{\mathcal{M}_{\mathcal{G}}}$ (see Examples 1–3):*

1) $(\alpha_1^{m_1})^2 - \alpha_1^{m_2}\alpha_2^{m_2}\alpha_3^{m_2}$, $\quad \alpha_2^{m_2}\alpha_3^{m_2}\alpha_1^{m_2} - \alpha_1^{m_3}\alpha_2^{m_3}$, $\quad \alpha_2^{m_3}\alpha_1^{m_3} - \alpha_1^{m_4}\alpha_2^{m_4}$, $\quad \alpha_3^{m_2}\alpha_1^{m_2}\alpha_2^{m_2} - \alpha_2^{m_4}\alpha_1^{m_4}$.

2) $\alpha_1^{m_2}\alpha_2^{m_2}\alpha_3^{m_2}\alpha_1^{m_2}$, $\quad \alpha_2^{m_3}\alpha_1^{m_3}\alpha_2^{m_3}$, $\quad \alpha_1^{m_4}\alpha_2^{m_4}\alpha_1^{m_4}$, $\quad (\alpha_1^{m_1})^3$.

3) $\alpha_1^{m_2}\alpha_1^{m_3}$, $\quad \alpha_1^{m_1}\alpha_1^{m_2}$, $\quad \alpha_1^{m_3}\alpha_1^{m_4}$, $\quad \alpha_2^{m_2}\alpha_2^{m_4}$, $\quad \alpha_3^{m_2}\alpha_1^{m_1}$, $\quad \alpha_1^{m_4}\alpha_3^{m_2}$, $\quad \alpha_2^{m_4}\alpha_2^{m_3}$, $\quad \alpha_2^{m_3}\alpha_2^{m_2}$.

We remind [5] that a *multiserial algebra* is defined to be a finite dimensional algebra $\mathcal{A}$ such that, for every indecomposable projective left and right $\mathcal{A}$-module P, the corresponding radical *rad P* is a sum of uniserial modules (an $\mathcal{A}$- module M is said to be uniserial if it has a unique composition series), i.e., *rad P* $= \sum\limits_{j=1}^{m} V_j$, where for some integer m and uniserial modules V_j, it holds that if $i \neq j$, then $V_i \cap V_j$ is either 0 or a simple $\mathcal{A}$-module.

Remark 3. *Green, Schroll, and Sierra proved the following results regarding the structure of a Brauer configuration algebra $\Lambda_{\mathcal{M}}$ induced by a Brauer configuration $\mathcal{M}$ (see [5] Theorem B, Proposition 2.7, Theorem 3.10, Corollary 3.12. And [29] Theorem 4.9):*

($\mathfrak{B}_1$) *There is a bijective correspondence between the set of indecomposable projective $\Lambda_{\mathcal{M}}$-modules and the polygons in $\mathcal{M}$.*

($\mathfrak{B}_2$) *If P is an indecomposable projective $\Lambda_{\mathcal{M}}$-module corresponding to a polygon V in $\mathcal{M}_1$, then* rad P *is a sum of r indecomposable uniserial modules, where r is the number of (nontruncated) vertices of V and where the intersection of any two of the uniserial modules is a simple $\Lambda_{\mathcal{M}}$-module.*

($\mathfrak{B}_3$) *A Brauer configuration algebra is a multiserial algebra.*

($\mathfrak{B}_4$) *The number of summands in the heart $ht(P) = \mathrm{rad}\, P/\mathrm{soc}\, P$ of an indecomposable projective $\Lambda_{\mathcal{M}}$-module P such that $\mathrm{rad}^2\, P \neq 0$ equals the number of nontruncated vertices of the polygons in $\mathcal{M}$ corresponding to P counting repetitions.*

($\mathfrak{B}_5$) *Let $\Lambda_{\mathcal{M}}$ be the Brauer configuration algebra induced by a connected Brauer configuration $\mathcal{M}$. The algebra $\Lambda_{\mathcal{M}}$ has a length grading induced from the path algebra $kQ_{\mathcal{M}}$, if and only if, there is an $N \in \mathbb{Z}_{>0}$ such that for each nontruncated vertex $m \in M$ $val(m)\mu(m) = N$. In such a case, we say that the algebra $\Lambda_{\mathcal{M}}$ satisfies the length grading property.*

($\mathfrak{B}_6$) *Equation (2.15) gives the dimension of a Brauer configuration algebra* $\Lambda = kQ_{\mathcal{M}}/I_{\mathcal{M}}$ *induced by a Brauer configuration* $\mathcal{M}$ *(see (2.12)).*

$$\dim_k \Lambda_{\mathcal{M}} = 2|M| + \sum_{m_i \in M} val(m_i)(\mu(m_i)val(m_i) - 1) \tag{2.15}$$

($\mathfrak{B}_7$) *The dimension of the center* $Z(\Lambda_{\mathcal{M}})$ *of a Brauer configuration algebra* $\Lambda_{\mathcal{M}}$ *induced by a connected and reduced Brauer configuration* $\mathcal{M}$ *is given by the formula*

$$\dim_k Z(\Lambda_{\mathcal{M}}) = 1 + \sum_{m \in M} \mu(m) + |(Q_{\mathcal{M}})_0| - |M| + \#(Loops\ Q_{\mathcal{M}}) - |\mathcal{C}_{\mathcal{M}}|, \tag{2.16}$$

where $\mathcal{C}_{\mathcal{M}} = \{m \in \mathcal{M} \mid val(m) = 1,$ *and* $\mu(m) > 1\}$ *and the number of vertices* $|(Q_{\mathcal{M}})_0|$ *of the Brauer quiver* $Q_{\mathcal{M}}$ *is given by the number of multisets in* $\mathcal{M}_1$.

Example 5. *According to Remark 3 the analysis of the data (known as Brauer analysis) inducing the Brauer configuration algebra* $\Lambda_{\mathcal{M}_\mathcal{G}}$ *and the Brauer quiver* $Q_{\mathcal{M}_\mathcal{G}}$ *(see Figure 6) gives the following information (see Examples 1–4):*

($\mathfrak{B}_1^{\mathcal{G}}$) $\Lambda_{\mathcal{M}_\mathcal{G}}$ *is biserial. Particularly, if P is an indecomposable projective* $\Lambda_{\mathcal{M}_\mathcal{G}}$*-module then its radical* rad *P is a sum of two indecomposable uniserial modules.*

($\mathfrak{B}_2^{\mathcal{G}}$) $\Lambda_{\mathcal{M}_\mathcal{G}}$ *does not satisfy the length grading property.*

($\mathfrak{B}_3^{\mathcal{G}}$) *The Brauer quiver* $Q_{\mathcal{M}_\mathcal{G}}$ *is connected and has four vertices,* $val(m_1) = 1$*,* $val(m_2) = 3$*,* $val(m_j) = 2$*,* $\mu(m_1) = 2$*, and* $\mu(m_i) = 1$*,* $i \in \{2, 3, 4\}$*,* $j \in \{3, 4\}$.

($\mathfrak{B}_4^{\mathcal{G}}$) $\dim_k \Lambda_{\mathcal{M}_\mathcal{G}} = 2(4) + 1(2-1) + 2(2(1)) + 3(2) = 19.$

($\mathfrak{B}_5^{\mathcal{G}}$) $\#(Loops\ Q_{\mathcal{M}_\mathcal{G}}) = 1.$

($\mathfrak{B}_6^{\mathcal{G}}$) $\dim_k Z(\Lambda_{\mathcal{M}_\mathcal{G}}) = 1 + 5 + 4 - 4 + 1 - 1 = 6.$

2.5. Entropy of a graph

Entropy is a measure of the complexity of a network or graph. Rashevsky introduced the notion of topological information content, which was generalized and studied by Mowshowitz [12, 13, 19, 30].

If a random discrete variable X takes values in the set $\{x_1, x_2, \ldots, x_n\}$ and $\sum_{i=1}^{n} p(x_i) = 1$, where $p(x_i)$ denotes the probability that X takes the value x_i, then the *Shannon's entropy* $H(X)$ of X is given by the following expression:

$$-\sum_{i=1}^{n} p(x_i)\log_2(p(x_i)). \tag{2.17}$$

Shannon's entropy is the most commonly used network complexity measure. However, several different families of graph-based entropy functions have been considered. For instance, if $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G})$ is an arbitrary invariant graph, and τ is an equivalence relation defined on the set of vertices $V_\mathcal{G}$ of the graph $\mathcal{G}$, then τ induces a partition $\{\mathcal{G}_1, \ldots, \mathcal{G}_g\}$ of $V_{\mathcal{G}}$ and the resulting graph entropy $H(\mathcal{G}, \tau)$ is given by the following identity [19, 30]:

$$H(\mathcal{G}, \tau) = -\sum_{i=1}^{g} \frac{|\mathcal{G}_i|}{|V_{\mathcal{G}}|} log_2(\frac{|\mathcal{G}_i|}{|V_{\mathcal{G}}|}). \tag{2.18}$$

Kulkarni et al. [20] defined the entropy of undirected and unweighed networks in their work regarding information content of note transitions in the music of J.S. Bach as follows:

$$H_{\delta_v}(\mathcal{G}) = \frac{1}{2\|\mathcal{G}\|} \sum_{v \in V_{\mathcal{G}}} \delta_v \log_2(\delta_v). \tag{2.19}$$

Another graph entropy is the *vertex degree equality-based information measure* $H_{\mathfrak{d}}(\mathcal{G})$. It is given by the following identity:

$$H_{\mathfrak{d}}(\mathcal{G}) = -\sum_{i=1}^{\overline{\delta_v}} \frac{|N_i^{\delta_v}|}{|V_{\mathcal{G}}|} \log_2(\frac{|N_i^{\delta_v}|}{|V_{\mathcal{G}}|}), \text{ where } |N_i^{\delta_v}| \text{ denotes the number of vertices with degree equal to } i \text{ and}$$
$$\overline{\delta_v} = \max_{v \in V_{\mathcal{G}}} \delta_v, \delta_v \text{ is the degree of } v \in V_{\mathcal{G}}. \tag{2.20}$$

The *entropy* $H(\mathcal{M})$ *of a Brauer configuration* $\mathcal{M}$ (see 2.12) is given by the following identity:

$$H(\mathcal{M}) = -\sum_{\alpha \in M} \frac{\mu(\alpha)val(\alpha)}{\mathfrak{v}} \log_2(\frac{\mu(\alpha)val(\alpha)}{\mathfrak{v}}), \quad \text{where } \mathfrak{v} = \sum_{\alpha \in M} \mu(\alpha)val(\alpha). \tag{2.21}$$

Example 6. *The following Table 3 gives an extended Brauer analysis of the Brauer configuration* $\mathcal{M}_{\mathcal{G}}$ *(see Examples 1–5). In this case, we add degree-based entropies and the entropy* $H(\mathcal{M}_{\mathcal{G}})$ *of the Brauer configuration to the Brauer analysis* $\mathfrak{B}_1^{\mathcal{G}} - \mathfrak{B}_6^{\mathcal{G}}$ *realized before. We let* κ *denote the number of connected components of the covering graph* $\mathfrak{c}(Q_{\mathcal{G}}) = \mathcal{G}$. *Note that* $|2^{H(\mathcal{M}_{\mathcal{G}})+2.25} - \dim_k \Lambda_{\mathcal{M}_{\mathcal{G}}}| < 1$. *This approximation type will be explored more for Brauer configuration algebras induced by Dynkin and Euclidean diagrams.*

Table 3. Extended Brauer analysis of the Brauer configuration $\mathcal{M}_{\mathcal{G}}$ induced by the graph $\mathcal{G}$ (see Examples 1–5).

Extended Brauer analysis of the Brauer configuration $\mathcal{M}_{\mathcal{G}}$							
$\mathcal{M}_{\mathcal{G}}$	$H_{\delta_v}(\mathcal{G})$	$H_{\mathfrak{d}}(\mathcal{G})$	$H(\mathcal{M}_{\mathcal{G}})$	$\dim_k \Lambda_{\mathcal{M}_{\mathcal{G}}}$	$\dim_k Z(\Lambda_{\mathcal{M}_{\mathcal{G}}})$	κ	$\mathfrak{c}(Q_{\mathcal{G}})$
$\mathcal{G}$	1.09436	1.5	1.9749	19	6	1	$\mathcal{G}$

3. Main results

This section provides our main results regarding Dynkin functions based on the data provided by Brauer configurations and graph entropies.

3.1. Covering graphs and Dynkin functions induced by Brauer configurations

This section establishes which conditions satisfies a covering graph that is a Dynkin or Euclidean diagram and defines Dynkin functions based on Brauer configurations.

The following result gives examples of Brauer configurations whose induced covering graphs are Dynkin diagrams.

Theorem 7. *If $\mathcal{M}_1 = \{M_1, M_2, \ldots, M_n\}$ is the collection of all polygons or multisets of a Brauer configuration $\mathcal{M} = (M, \mathcal{M}_1, \mu, O)$ (see Remark 2), $M_i \cap M_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$, and $M_i \cap M_j = \emptyset$ for any $j \neq i+1$, then the covering graph or nerve $\mathfrak{c}(Q_{\mathcal{M}})$ induced by the Brauer configuration $\mathcal{M}$ is isomorphic to a Dynkin diagram of type $\mathbb{A}_n$ (see Figure 2).*

Proof. If $\mathcal{M}_1 = \{M_1, M_2, \ldots, M_n\}$, then for each i, $1 \leq i \leq n-1$, there exists a vertex y_i and a linearly ordered set of type (2.4) with the form $\mathfrak{I}_{y_i} = M_i < M_{i+1}$, then the covering graph $\mathfrak{c}(Q_{\mathcal{M}})$ is a path of the form $\{M_1M_2, M_2M_3, \ldots, M_{n-1}M_n\}$ defined by the sequence of coverings $\mathfrak{I}_{y_i}$. We are done.

Remark 4. *In this paper, we assume that the Brauer configuration $\mathcal{M}_{\mathcal{G}} = (M_0^{\mathcal{G}}, M_1^{\mathcal{G}}, \mu, O)$ induced by a graph $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$ is such that $M_0^{\mathcal{G}} = V_{\mathcal{G}}$, $M_1^{\mathcal{G}} = E_{\mathcal{G}}$, $\mu(\alpha) = 1$ ($\mu(\alpha) = 2$) if $val(\alpha) > 1$ ($val(\alpha) = 1$). The orientation O is defined as in Example 1. That is, if $V_{\mathcal{G}} = \{v_1, v_2, \ldots, v_t\}$, $E_{\mathcal{G}} = \{e_1, e_2, \ldots, e_n\}$, and $e_{i_1}, e_{i_2}, \ldots, e_{i_h}$ are edges at vertex v_j, $\{i_1, i_2, \ldots, i_h\} \subseteq \{1, 2, \ldots, n\}$, $i_1 < i_2 < \cdots < i_h$, then the successor sequence S_{v_j} at v_j has the form $e_{i_1} < e_{i_2} < \cdots < e_{i_h}$ ($S_{v_j} = e_{i_1}$, if $\delta_{v_j} = 1$).*

Henceforth, we will assume a numbering of the form $\{1, 2, 3, \ldots, n\}$ (left to right and top to bottom) for vertices and edges of an n-point Dynkin or Euclidean diagram Δ. The orientation O associated with their Brauer configurations $\mathcal{M}_{\Delta} = (M_0^{\Delta} = V_{\Delta}, M_1^{\Delta} = E_{\Delta}, \mu, O)$ is given bearing in mind that these diagrams are graphs.

For example, in the case $\Delta = \mathbb{D}_n$, we have that $V_{\mathbb{D}_n} = \{v_1, v_2, \ldots, v_{n-2}, v_{n-1}, v_n\}$, $E_{\mathbb{D}_n} = \{e_1, e_2, \ldots, e_{n-1}\}$, with $\delta_{v_1} = \delta_{v_{n-1}} = \delta_{v_n} = 1$, $\delta_{v_{n-2}} = 3$, and $\delta_{v_j} = 2$ for the remaining vertices $v_j \in V_{\mathbb{D}_n}$. Furthermore, $e_j = \{v_j, v_{j+1}\}$, if $1 \leq j \leq n-2$ and $e_{n-1} = \{v_{n-2}, v_n\}$. The successor sequence at v_j is $S_{v_j} = e_j$, if $j \in \{1, n-1, n\}$, and the successor sequence at v_h is $S_{v_h} = e_{h-1} < e_h$, if $2 \leq h \leq n-3$. Finally the successor sequence at v_{n-2} is $S_{v_{n-2}} = e_{n-3} < e_{n-2} < e_{n-1}$.

If $\Delta = \mathbb{E}_6$, we assume that $V_{\mathbb{E}_6} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $E_{\mathbb{E}_6} = \{e_1, e_2, e_3, e_4, e_5\}$, $\delta_{v_i} = 1$, if $i \in \{1, 3, 6\}$, $\delta_{v_4} = 3$, $\delta_{v_j} = 2$ for $j \in \{2, 5\}$. $e_i = \{i, i+1\}$, if $i \in \{1, 3, 4, 5\}$, $e_2 = \{2, 4\}$. The successor sequence at v_i is $S_{v_i} = v_i$, if $i \in \{1, 3, 6\}$, $S_{v_4} = e_2 < e_3 < e_4$, $S_{v_2} = e_1 < e_2$, $S_{v_5} = e_4 < e_5$.

Similar procedures can be used to construct successor sequences associated with Brauer configurations of the remaining Dynkin and Euclidean diagrams.

The following result gives conditions for graphs isomorphic to their induced covering graphs.

Theorem 8. *Let $\mathfrak{c}(Q_{\mathcal{G}})$ be the covering graph of a Brauer configuration $\mathcal{M}_{\mathcal{G}}$ induced by a graph $\mathcal{G}$, then $\mathfrak{c}(Q_{\mathcal{G}})$ is isomorphic to $\mathcal{G}$, if and only if, $\mathcal{G}$ is a finite disjoint union of hair graphs of type $C_n[(v_1, v_2, \ldots, v_n), (s_1, s_2, \ldots, s_n)]$ where C_n is an n-point cycle, $n \geq 3$ and $s_i \geq 1$, $1 \leq i \leq n$.*

Proof. Suppose that $\mathcal{H}$ is a fixed component of graph $\mathcal{G}$, which is a hair graph of a cycle $C_n = \{v_1, v_2, ..., v_n\}$, with $n \geq 3$, then

$$\mathcal{H} = C_n[(v_1, \ldots, v_n); (s_1, \ldots, s_n)].$$

Thus, the set of vertices $V_{P_{s_j}}$ of each path P_{s_j} has the form $V_{P_{s_j}} = \{v_j, v_1^j, v_2^j, \ldots, v_{s_j-1}^j\}$, $j = 1, \ldots, n$. Thereby, the set of edges $E_{\mathcal{H}}$ of $\mathcal{H}$ is the union $E_{\mathcal{H}} = E_{\mathcal{H}_0} \cup E_{\mathcal{H}_1} \cup \cdots \cup E_{\mathcal{H}_n}$, where

(i) $E_{\mathcal{H}_0} = \{\{v_i, v_{i+1}\} \mid i = 1, ..., n \text{ with } v_{n+1} = v_1\}$ and

(ii) $E_{\mathcal{H}_j} = \{\{v_j, v_1^j\}, \{v_1^j, v_2^j\}, \ldots, \{v_{s_j-2}^j, v_{s_j-1}^j\}\}$, if $s_j > 1$, and $j = 1, \ldots, n$. $E_{\mathcal{H}_j} = \varnothing$ if $s_j = 1$.

Let us consider the following labeling of edges in $\mathcal{H}$ for $s_j > 1$:

$$\begin{aligned} e_i &= \{v_i, v_{i+1}\}, \quad i = 1, \ldots, n, \text{ with } v_{n+1} = v_1. \\ e_1^j &= \{v_j, v_1^j\}, \quad j = 1, \ldots, n. \\ e_i^j &= \{v_{i-1}^j, v_i^j\}, \quad i = 2, \cdots, s_j - 1, \quad j = 1, \ldots, n. \end{aligned} \tag{3.1}$$

Thereby, the successor sequences defined by vertices of graph $\mathcal{H}$ are defined as follows:

$$\begin{aligned} S_{v_1} &= e_n < e_1, \\ S_{v_i} &= e_{i-1} < e_i, \quad i \in \{2, \ldots, n\}. \\ S_{v_i} &= e_{i-1} < e_i < e_1^j, \quad \text{if} \quad s_{v_i} > 1. \\ S_{v_i^j} &= e_i^j < e_{i+1}^j, \quad i = 1, \ldots, s_j - 2. \\ S_{v_{s_j-1}^j} &= e_{s_j-1}^j. \end{aligned} \tag{3.2}$$

By the definition of the covering graph, it holds that $\mathfrak{c}(Q_{\mathcal{H}}) = C_n[(e_1, ..., e_n); (s_1, ..., s_n)]$, where $C_n = \{e_1, ..., e_n\}$ is an n-point cycle. Similar arguments can be applied to the remaining components of graph $\mathcal{G}$, viewed as hair graphs of cycles, to prove that an isomorphism exists between $\mathcal{G}$ and its corresponding covering graph $\mathfrak{c}(Q_{\mathcal{G}})$. In particular, we note that the covering graph $\mathfrak{c}(Q_{C_n})$ of an n-point cycle C_n is also an n-point cycle whose set of vertices $V_{\mathfrak{c}(Q_{C_n})}$ is in bijective correspondence with E_{C_n}, i.e., $\mathfrak{c}(Q_{C_n}) = C_n[(e_1, e_2, \ldots, e_n); (1, 1, \ldots, 1)]$, if $E_{C_n} = \{e_1, e_2, \ldots, e_n\}$. Thus, the 3-point cycle C_3 is the minimum graph satisfying the condition posed by the theorem.

Conversely, suppose that $\mathcal{G}$ is isomorphic to its covering graph $\mathfrak{c}(Q_{\mathcal{G}})$. Since the isomorphism of non-connected graphs implies isomorphism between their respective components and vice-versa, then it is possible to suppose that graphs $\mathcal{G}$ and $\mathfrak{c}(Q_{\mathcal{G}})$ are also connected without loss of generality.

The isomorphism between $\mathcal{G}$ and its covering graph allows us to conclude that $|V_{\mathcal{G}}| = |V_{\mathfrak{c}(Q_{\mathcal{G}})}| = |E_{\mathcal{G}}|$ and that $\mathcal{G}$ contains a unique cycle, say $C_n = \{v_1, ..., v_n\}$. Thus, $\mathcal{G}$ is obtained from C_n by attaching a suitable tree T_{v_i} to each of the vertices $v_i \in C_n$.

Suppose now that $|V_{\mathcal{G}}| = 4$ and that $\mathcal{G}$ is isomorphic to $\mathfrak{c}(Q_{\mathcal{G}})$, then either $\mathcal{G}$ is a 4-point cycle $\{v_1, v_2, v_3, v_4\}$ or $\mathcal{G}$ contains a 3-point cycle $C_3 = \{v_1, v_2, v_3\}$ with $e_i = \{v_i, v_{i+1}\}$, $e_3 = \{v_1, v_3\}$. Furthermore, there is an edge $e_{i,4}$ connecting one of these vertices v_i, $1 \leq i \leq 3$ to v_4. We can assume that $e_{i,4} = \{v_3, v_4\}$. Thus, $\mathcal{G}$ is a hair graph of the form $C_3[(v_1, v_2, v_3); (1, 1, 2)]$ which is the minimum nontrivial hair graph defined by the vertices of the cycle C_3.

If $|V_{\mathcal{G}}| = 5$ and $\mathcal{G}$ is isomorphic to its covering graph, then either $\mathcal{G}$ is a 5-point cycle, $\mathcal{G}$ contains a 4-point cycle, or $\mathcal{G}$ contains a 3-point cycle. In the first case, $\mathcal{G}$ is the trivial hair graph

$C_5[(v_1, v_2, v_3, v_4, v_5); (1, 1, 1, 1, 1)]$, and in the second case, G contains a cycle $C_4 = \{v_1, v_2, v_3, v_4\}$ with edges $e_i = \{v_i, v_{i+1} \mid 1 \leq i \leq 3\} \cup \{v_1, v_4\}$. It is assumed that v_4 is the only vertex in C_4 for which the degree $\delta_{v_4} = 3$, otherwise $\delta_{v_i} = 2$, $1 \leq i \leq 3$. In such a case, $\mathcal{G}$ is a hair graph of type $[(v_1, v_2, v_3, v_4); (1, 1, 1, 2)]$. In the third case, we assume that $\{v_1, v_2, v_3\}$ is a 3-point cycle C_3 contained in $\mathcal{G}$. Therefore, C_3 has attached a 5-point forest $\mathfrak{F}_{n(5,3)}$, where $n(5, 3) = 2$ is the number of vertices in the forest $\mathfrak{F}_2$ without taking into account the initial vertices of its trees.

The attaching process does not give rise to ramifications in $\mathcal{G}$ with a root vertex $v_i \in C_3$ and two leaves $v_j \notin C_3$. Otherwise $\mathcal{G}$ and its corresponding covering graph would not be isomorphic provided that such a ramification gives rise to a hair graph of the form $C_3[(v_1, v_2, v_3); (1, 1, 3)] = \mathfrak{c}(Q_{\mathcal{G}})$. Therefore, in the third case, $\mathcal{G}$ is a hair graph of one of the following types: $C_3[(v_1, v_2, v_3); (1, 1, 3)]$, $C_3[(v_1, v_2, v_3); (1, 2, 2)]$.

Suppose that $\mathcal{G}$ is a connected graph isomorphic to its covering graph $\mathfrak{c}(Q_{\mathcal{G}})$ with $|V_{\mathcal{G}}| = m > 5$, then $\mathcal{G}$ contains a j-point cycle $C_j = \{v_1, v_2, \ldots, v_j\}$, $3 \leq j \leq m$ which has attached an m-point forest $\mathfrak{F}_{n(m,j)}$. Each vertex $v_i \in C_j$ has attached a tree $T_{v_i} \subseteq \mathfrak{F}_{n(m,j)}$ with $|V_{T_{v_i}}| \geq 1$.

According to the previous arguments (cases, $m = 3, 4, 5$), we note that for each i, $1 \leq i \leq j$, the attached tree $T_{v_i} \subseteq \mathfrak{F}_{n(m,j)}$ has no ramifications with $n \geq 2$ leaves provided that they give rise to $(n-1)$-linear paths in the covering graph. Particularly, the degree $\delta_{v_i} \leq 3$ for any $v_i \in C_j$, then we conclude that $\mathcal{G}$ is a hair graph of type $C_j[(v_1, v_2, v_3, \ldots, v_j); (\lambda_1, \lambda_2, \ldots, \lambda_j)]$, where $\{\lambda_1, \lambda_2, \ldots, \lambda_j\}$ is an integer partition of m into j parts. We are done.

Example 9. *Figure 7 shows a hair graph of type* $\mathcal{G} = C_4[(v_{4,1}, v_{3,1}, v_{2,1}, v_{1,1}); (4, 3, 2, 1)]$ *described in the proof of Theorem 8. Note that* $\lambda = \{4, 3, 2, 1\}$ *is a partition of* $m = 10 = |V_{\mathcal{G}}|$. *The forest* $\mathfrak{F}_{n(10,4)}$ *consists of trees or paths* $T_{v_{1,1}} = \{v_{1,1}\}$, $T_{v_{2,1}} = \{v_{2,1}, v_{2,2}\}$, $T_{v_{3,1}} = \{v_{3,1}, v_{3,2}, v_{3,3}\}$, $T_{v_{4,1}} = \{v_{4,1}, v_{4,2}, v_{4,3}, v_{4,4}\}$.

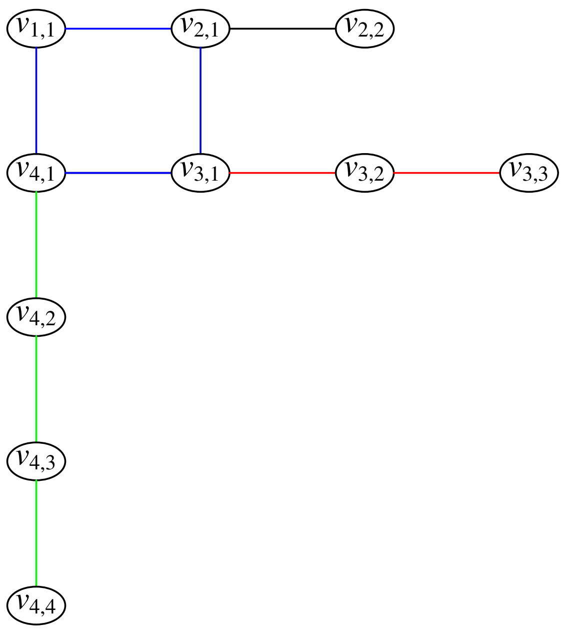

Figure 7. Hair graph $\mathcal{G}$ of type $C_4[(v_{4,1}, v_{3,1}, v_{2,1}, v_{1,1}); (4, 3, 2, 1)]$.

Remark 4 allows us to prove the following result regarding Dynkin functions and their relationships with Brauer configuration algebras.

Theorem 10. *The sequences* $\{5(n-1) \mid n \geq 3\}$, $\{5n \mid n \geq 3\}$, $\{5(m-1) \mid m \geq 4\}$, *and the set* $\{25, 30, 35, 20, 15\}$ *give the values of the Dynkin function defined by map* $\mathfrak{D}_\iota : \Delta \to \iota_\Delta$, *which assigns each Dynkin diagram* $\Delta \in \{\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_m, \mathbb{E}_j, \mathbb{F}_4, \mathbb{G}_2 \mid n \geq 3, m \geq 4, 6 \leq j \leq 8\}$ *the size* ι_Δ *of the fundamental set of relations* $\mathfrak{I}_\Delta$.

Proof. Each Dynkin diagram $\Delta \in \{\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2\}$ is a graph which induces a Brauer quiver $Q_{\mathcal{M}_\Delta}$ taking into account conditions given in Remark 4. Thus, vertices of the induced Brauer configurations are nontruncated. Therefore, there is a bijective correspondence between edges in E_Δ and vertices in the Brauer quiver $Q_{\mathcal{M}_\Delta}$ whose arrows (denoted $\alpha_{i_j}^{v_i}$, where $v_i \in V_\Delta$, i_j is a suitable index, and for some $e_i, e_j \in E_\Delta$, it holds that $s(\alpha_{i_j}^{v_i}) = e_i$, $t(\alpha_{i_j}^{v_i}) = e_j$) are given by inequalities of type $e_i < e_j$ in a suitable successor sequence as defined in Remark 4.

Brauer configuration algebras $\Lambda_{\mathcal{M}_\Lambda}$ induced by Dynkin diagrams are bounded path algebras $k\Delta/I_\Delta$, where I_Δ is an appropriate admissible ideal generated by relations of the following forms:

(I_{Δ_0}) $\alpha_{i_1}^{v_{i_1}}\alpha_{i_1}^{v_{i_1}}\alpha_{i_1}^{v_{i_1}} = (\alpha_{i_1}^{v_{i_1}})^3$, if $\delta_{v_i} = 1$.

(I_{Δ_1}) $(\alpha_r^{v_i})^2 - \alpha_1^{v_j}\alpha_2^{v_j}$ or $(\alpha_r^{v_i})^2 - \alpha_2^{v_j}\alpha_1^{v_j}$, if $\delta_{v_i} = 1$, $\delta_{v_j} = 2$, and $\{v_i, v_j\} \in E_\Delta$.

(I_{Δ_2}) $\alpha_r^{v_i}\alpha_s^{v_i} - \alpha_{r'}^{v_j}\alpha_{s'}^{v_j}$, $r \neq s$, $r' \neq s'$, $r, s \in \{1, 2\}$, if $\delta_{v_i} = \delta_{v_j} = 2$, and $\{v_i, v_j\} \in E_\Delta$.

(I_{Δ_3}) $(\alpha_1^{v_i})^2 - \alpha_{i_1}^{v_j}\alpha_{i_2}^{v_j}\alpha_{i_3}^{v_j}$, indices i_s are pairwise different $s \in \{1, 2, 3\}$, if $\delta_{v_i} = 1$, $\delta_{v_j} = 3$, and $\{v_i, v_j\} \in E_\Delta$.

(I_{Δ_4}) $\alpha_{h_1}^{v_i}\alpha_{h_2}^{v_i} - \alpha_{i_1}^{v_j}\alpha_{i_2}^{v_j}\alpha_{i_3}^{v_j}$ if $h_1 \neq h_2$, $h_j \in \{1, 2\}$, indices i_s are pairwise different, $s \in \{1, 2, 3\}$, $\delta_{v_i} = 2$, $\delta_{v_j} = 3$, and $\{v_i, v_j\} \in E_\Delta$.

(I_{Δ_5}) $\alpha_{h_1}^{v_i}\alpha_{h_2}^{v_i}\alpha_{h_3}^{v_i} - \alpha_{i_1}^{v_j}\alpha_{i_2}^{v_j}\alpha_{i_3}^{v_j}$ if indices i_s, h_s are pairwise different, $s \in \{1, 2, 3\}$, $\delta_{v_i} = \delta_{v_j} = 3$, and $\{v_i, v_j\} \in E_\Delta$.

(I_{Δ_6}) $C_{v_i}f$, if f is the first arrow of a special v_i-cycle C_{v_i}, at a given edge e_m, and $\delta_{v_i} > 1$.

(I_{Δ_7}) $\alpha_{h_1}^{v_i}\alpha_{h_2}^{v_j}$, if $i \neq j$.

(ι_{Δ_0}) There are $|\{v_i \in V_\Delta \mid \delta_{v_i} = 1\}|$ relations of type (I_{Δ_0}) in $\mathfrak{I}_\Delta$.

(ι_{Δ_1}) There are $\|E_\Delta\|$ relations of type (I_{Δ_h}), $1 \leq h \leq 5$ in $\mathfrak{I}_\Delta$.

(ι_{Δ_2}) There are $2\|E_\Delta\|$ relations of type ($I_{\Delta_6} \cup I_{\Delta_0}$) in $\mathfrak{I}_\Delta$.

(ι_{Δ_3}) Since $V_\Delta = \{v_1, v_2, \ldots, V_{|\Delta|}\}$, then for each $i \geq 1$ fixed there is a unique $j > i$, such that $\alpha_{m_1}^{v_i}\alpha_{m_2}^{v_j}, \alpha_{m_2}^{v_j}\alpha_{m_1}^{v_i} \in \mathfrak{I}_\Delta$, $m_1 \neq m_2$, $1 \leq m_1, m_2 \leq 3$. Thus, there are $2\|E_\Delta\|$ relations of type (I_{Δ_7}) in $\mathfrak{I}_\Delta$.

Thus the following identities hold:

$$\begin{aligned}
\iota_{\mathbb{A}_n} &= 5(n-1), \quad n \geq 3.\\
\iota_{\mathbb{B}_n} &= \iota_{\mathbb{C}_n} = 5n, \quad n \geq 3.\\
\iota_{\mathbb{D}_n} &= 5(n-1), \quad n \geq 4.\\
\iota_{\mathbb{E}_6} &= 25, \quad \iota_{\mathbb{E}_7} = 30, \quad \iota_{\mathbb{E}_8} = 35.\\
\iota_{\mathbb{F}_4} &= 20.\\
\iota_{\mathbb{G}_2} &= 15.
\end{aligned} \tag{3.3}$$

Figures 8 and 9 show Brauer quivers $Q_{\mathcal{M}_\Delta}$ for each Dynkin diagram Δ. We are done.

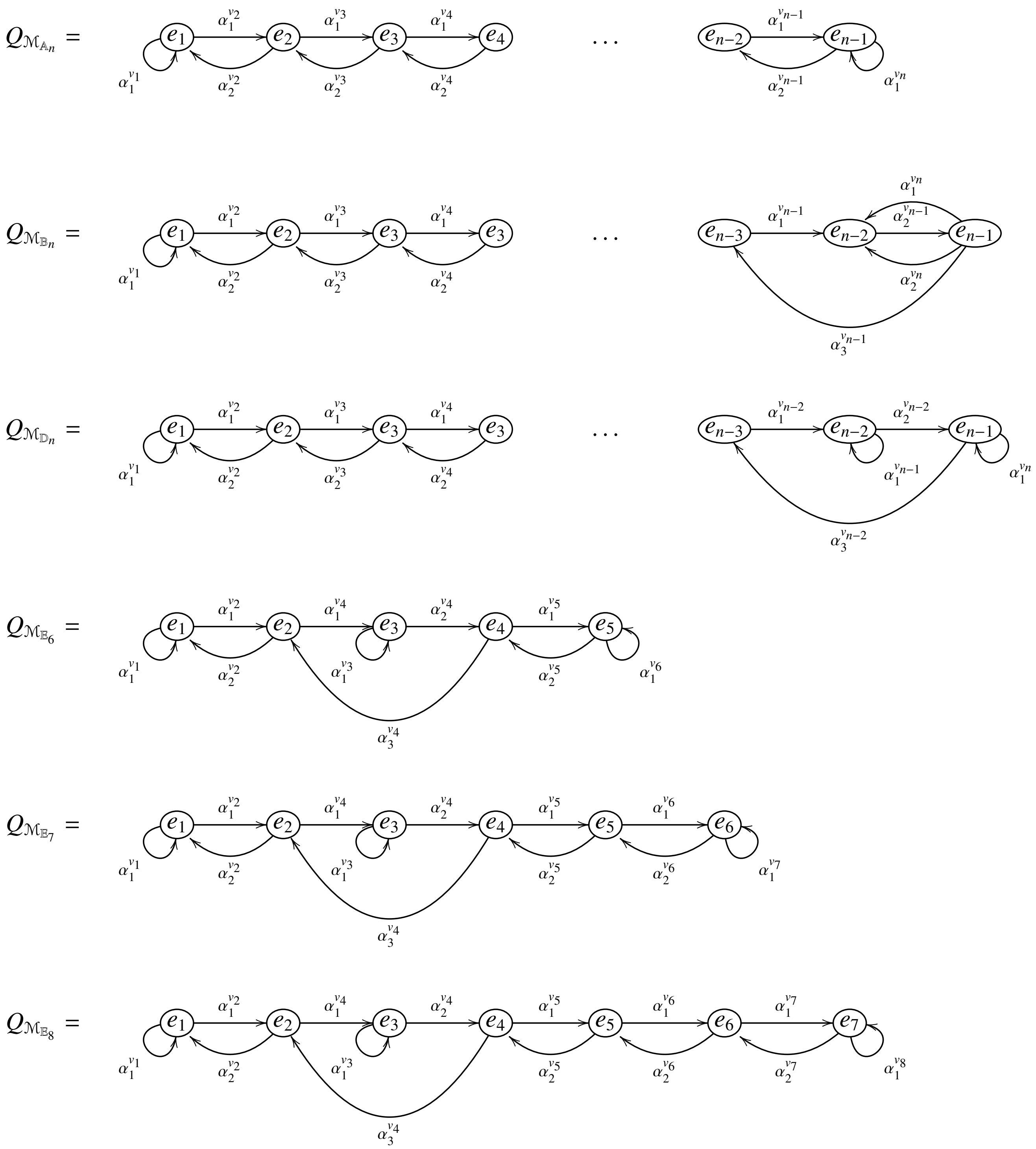

Figure 8. Brauer quivers $Q_{\mathcal{M}_\Delta}$ induced by Dynkin diagrams Δ.

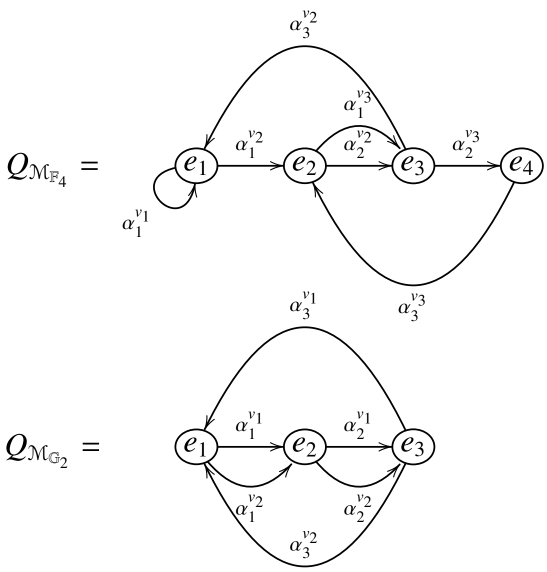

Figure 9. Brauer quivers $Q_{\mathcal{M}_{\mathbb{F}_4}}$ and $Q_{\mathcal{M}_{\mathbb{G}_2}}$ induced by Dynkin diagrams $\mathbb{F}_4$ and $\mathbb{G}_2$ respectively.

The following results are consequences of Remarks 3 and 4.

Corollary 1. *For $n \geq 3$, the sequence $\mathbb{A}_n$ gives the values of the Dynkin function defined by the map $\mathfrak{c}$, which assigns each Dynkin diagram Δ its covering graph $\mathfrak{c}(Q_\Delta)$.*

Proof. According to Remark 4, it holds that for $n \geq 3$, $\mathfrak{c}(\Delta) = \mathbb{A}_{n-h}$, where $h = 0$, if $\Delta \in \{\mathbb{B}_n, \mathbb{C}_n\}$; $h = 1$, if $\Delta \in \{\mathbb{A}_n, \mathbb{D}_n\}$. $\mathfrak{c}(\mathbb{E}_s) = \mathbb{A}_{s-1}$, if $s \in \{6, 7, 8\}$. $\mathfrak{c}(\mathbb{F}_4) = \mathbb{A}_4$, and $\mathfrak{c}(\mathbb{G}_2) = \mathbb{A}_3$. We are done.

The following result gives values of Dynkin functions defined by dimensions of Brauer configuration algebras.

Corollary 2. *The Dynkin functions* dim *and* $\dim Z$*, which assign the dimensions* $\dim_k \Lambda_\Delta$ *and* $\dim_k Z(\Lambda_\Delta)$ *to each Dynkin diagram, take the following values*

1) $\dim_k \Lambda_{\mathbb{A}_n} = 4n - 4, \quad \dim_k Z(\Lambda_{\mathbb{A}_n}) = n + 2.$

2) $\dim_k \Lambda_{\mathbb{B}_n} = \dim_k \Lambda_{\mathbb{C}_n} = 4n + 3, \quad \dim_k Z(\Lambda_{\mathbb{B}_n}) = \dim_k Z(\Lambda_{\mathbb{C}_n}) = n + 2.$

3) $\dim_k \Lambda_{\mathbb{D}_n} = 4n - 1, \quad \dim_k Z(\Lambda_{\mathbb{D}_n}) = n + 3.$

4) $\dim_k \Lambda_{\mathbb{E}_6} = 23, \quad \dim_k Z(\Lambda_{\mathbb{E}_6}) = 9.$

5) $\dim_k \Lambda_{\mathbb{E}_7} = 27, \quad \dim_k Z(\Lambda_{\mathbb{E}_7}) = 10.$

6) $\dim_k \Lambda_{\mathbb{E}_8} = 31, \quad \dim_k Z(\Lambda_{\mathbb{E}_7}) = 11.$

7) $\dim_k \Lambda_{\mathbb{F}_4} = 22, \quad \dim_k Z(\Lambda_{\mathbb{F}_4}) = 7.$

8) $\dim_k \Lambda_{\mathbb{G}_2} = 18, \quad \dim_k Z(\Lambda_{\mathbb{G}_2}) = 4.$

Proof. Given a Dynkin diagram Δ, we note that the degree δ_{v_i} of each vertex $v_i \in V_\Delta$ gives its valency as vertex of the corresponding Brauer configuration $\mathcal{M}_\Delta$ in the following form:

1) If $\Delta = \mathbb{A}_n$ then $\delta_{v_1} = \delta_{v_n} = 1$ and $\delta_{v_i} = 2$, for the remaining vertices $v_i \in V_{\mathbb{A}_n}$.

2) If $\Delta = \mathbb{B}_n$ or $\Delta = \mathbb{C}_n$ then $\delta_{v_1} = 1$, $\delta_{v_n} = 2$, $\delta_{v_{n-1}} = 3$ and $\delta_{v_i} = 2$, for the remaining vertices $v_i \in V_{\mathbb{B}_n}$ or $v_i \in V_{\mathbb{C}_n}$.

3) If $\Delta = \mathbb{D}_n$ then $\delta_{v_1} = \delta_{v_{n-1}} = \delta_{v_n} = 1$, $\delta_{v_n} = 2$, for the remaining vertices $v_i \in V_{\mathbb{D}_n}$.

4) If $\Delta = \mathbb{E}_s$ then $\delta_{v_1} = \delta_{v_3} = \delta_{v_{|\mathbb{E}_s|}} = 1$, $\delta_{v_4} = 3$, and $\delta_{v_i} = 2$ for the remaining vertices $v_i \in V_{\mathbb{E}_s}$, $s \in \{6, 7, 8\}$.

5) If $\Delta = \mathbb{F}_4$ then $\delta_{v_1} = \delta_{v_4} = 1$, $\delta_{v_2} = \delta_{v_3} = 2$.

6) If $\Delta = \mathbb{G}_2$ then $\delta_{v_1} = \delta_{v_2} = 3$.

Since for each Dynkin diagram Δ, it holds that the number of edges $\|E_\Delta\|$ gives the number of polygons in the Brauer configuration $\mathcal{M}_\Delta$ and the loops in the induced Brauer quiver $Q_{\mathcal{M}_\Delta}$ are given by vertices v_i with $\delta_{v_i} = 1$. We are done.

3.2. Graph entropies and their use to define Dynkin functions

This section compares the entropies of different graph families with the entropy defined by a Brauer configuration (see (2.21)) and uses them to construct Dynkin functions.

The following result establishes relationships between Brauer configuration algebras that satisfy the length grading property (see Remark [5]) and graph entropy.

Theorem 11. *If the Brauer configuration $\mathcal{M} = (M, \mathcal{M}_1, \mu, O)$ induces a Brauer configuration algebra $\Lambda_\mathcal{M}$ which satisfies the length grading property then $H(\mathcal{M}) \leq \log_2 |M|$, where $H(\mathcal{M})$ is the entropy defined by the Brauer configuration $\mathcal{M}$.*

Proof. We note that by definition there exists a positive integer N such that $val(m)\mu(m) = N$ for any $m \in M$, provided that vertices in Γ_0 are nontruncated. Therefore, $H(\mathcal{M}) = -\sum\limits_{m\in M} \frac{val(m)\mu(m)}{N|M|} \log_2 \frac{val(m)\mu(m)}{N|M|} = -\sum\limits_{m\in M} \frac{1}{|M|} \log_2 \frac{1}{|M|} \leq \log_2 \sum\limits_{m\in M} \frac{|M|}{|M|} = \log_2 |M|$.

For the next results, we assume the notation $H(\mathcal{G}^1)$, $H(\mathcal{G}^2)$, and $H(\mathcal{G}^3)$ for entropies of type $H_{\delta_v}(\mathcal{G})$, $H_\mathfrak{d}(\mathcal{G})$, and $H(\mathcal{M})$, respectively (see identities 2.19, 2.20, and 2.21).

Lemma 1. *If $\mathcal{G}$ is a regular finite connected graph, then $H(\mathcal{G}^2) = 0$.*

Proof. If $\mathcal{G}$ is an r-regular finite connected graph, then $|N_r^{\delta_v}| = |V_\mathcal{G}|$. □

Theorem 12. *If Δ is a Dynkin or Euclidean diagram then $H(\Delta^j) < H(\widetilde{\mathbb{E}}_8^3) < 3.2$, with $j \in \{1, 2\}$. In particular, the following identities hold:*

$$\operatorname*{Lim}_{n\to\infty} H(\mathbb{A}_n^1) = \operatorname*{Lim}_{n\to\infty} H(\mathbb{B}_n^1) = \operatorname*{Lim}_{n\to\infty} H(\mathbb{C}_n^1) = \operatorname*{Lim}_{n\to\infty} H(\mathbb{D}_n^1) = \operatorname*{Lim}_{n\to\infty} H(\widetilde{\mathbb{A}}_n^1) = \operatorname*{Lim}_{n\to\infty} H(\widetilde{\mathbb{D}}_n^1) = 1. \quad (3.4)$$

Furthermore,

$$\operatorname*{Lim}_{n\to\infty} H(\mathbb{A}_n^2) = \operatorname*{Lim}_{n\to\infty} H(\mathbb{B}_n^2) = \operatorname*{Lim}_{n\to\infty} H(\mathbb{C}_n^2) = \operatorname*{Lim}_{n\to\infty} H(\mathbb{D}_n^2) = \operatorname*{Lim}_{n\to\infty} H(\widetilde{\mathbb{D}}_n^2) = 0. \quad (3.5)$$

$$\underset{n\to\infty}{\text{Lim}} H(\mathbb{A}_n^3) = \underset{n\to\infty}{\text{Lim}} H(\mathbb{B}_n^3) = \underset{n\to\infty}{\text{Lim}} H(\mathbb{C}_n^3) = \underset{n\to\infty}{\text{Lim}} H(\widetilde{\mathbb{D}}_n^3) = \infty. \tag{3.6}$$

$$\begin{aligned} &\underset{n\to\infty}{\text{Lim}} H(\mathbb{A}_n^3) - H(\mathbb{D}_n^3) = \underset{n\to\infty}{\text{Lim}} H(\mathbb{A}_n^3) - H(\mathbb{B}_n^3) = \underset{n\to\infty}{\text{Lim}} H(\mathbb{A}_n^3) - H(\mathbb{C}_n^3) = \\ &\underset{n\to\infty}{\text{Lim}} H(\mathbb{B}_n^3) - H(\mathbb{D}_n^3) = \underset{n\to\infty}{\text{Lim}} H(\widetilde{\mathbb{A}}_n^3) - H(\widetilde{\mathbb{D}}_n^3) = \underset{n\to\infty}{\text{Lim}}\, H(\widetilde{\mathbb{A}}_n^3) - H(\mathbb{A}_n^3) = 0. \end{aligned} \tag{3.7}$$

Proof. We note that $2.5 \le H(X_j) < 3.2$ if $X_j \in \{E_j, \widetilde{\mathbb{E}}_j \mid 6 \le j \le 8\}$. The result follows from the following identities and inequalities:

$$\begin{aligned} &H(\mathbb{A}_n^1) = 1 - \frac{1}{n-1}. \\ &H(\mathbb{B}_n^1) = H(\mathbb{C}_n^1) = \frac{n-2}{n} + \frac{3\log_2(3)}{2n}. \\ &H(\mathbb{D}_n^1) = \frac{n-4}{n-1} + \frac{3}{n-1}\log_2(3) \\ &H(\mathbb{A}_n^2) = \log_2(n) - \frac{n-2}{n}\log_2(n-2) - \frac{2}{n}. \\ &H(\mathbb{B}_n^2) = H(\mathbb{C}_n^2) = \frac{2}{n}\log_2(n) - \frac{n-2}{n}\log_2(\frac{n-2}{n}). \\ &H(\mathbb{D}_n^2) = -[\frac{3}{n}\log_2(\frac{3}{n}) + \frac{n-4}{n}\log_2(\frac{n-4}{n}) + \frac{1}{n}\log_2(\frac{1}{n})]. \end{aligned} \tag{3.8}$$

$$\begin{aligned} &H(\mathbb{A}_n^3) = \log_2(n). \\ &H(\mathbb{B}_n^3) = H(\mathbb{C}_n^3) = -[\frac{2n-2}{2n+1}\log_2(2) + \frac{3}{2n+1}\log_2(3)] + \log_2(2n+1). \\ &H(\mathbb{D}_n^3) = -[\frac{2n-2}{2n+1}\log_2(2) + \frac{3}{2n+1}\log_2(3)] + \log_2(2n+1). \\ &H(\mathbb{E}_j^1) \le H(\mathbb{E}_j^2) \le H(\mathbb{E}_j^3) \le H(\mathbb{E}_8^3) \le 2.98424. \end{aligned} \tag{3.9}$$

Furthermore,

$$\begin{aligned} &H(\widetilde{\mathbb{A}}_n^1) = 1. \\ &H(\widetilde{\mathbb{D}}_n^1) = \frac{1}{2(n+1)}[6\log_2(3) + 2(n-4)]. \\ &H(\widetilde{\mathbb{A}}_n^2) = 0. \\ &H(\widetilde{\mathbb{D}}_n^2) = -[\frac{4}{n+2} + \frac{6}{n+2}\log_2(\frac{2}{n+2}) + \frac{n-4}{n+2}\log_2(\frac{n-4}{n+2})]. \\ &H(\widetilde{\mathbb{A}}_n^3) = \log_2(n+1). \\ &H(\widetilde{\mathbb{D}}_n^3) = -[\frac{2n}{2n+6}\log_2(\frac{2}{2n+6}) + \frac{6}{2n+6}\log_2(\frac{3}{2n+6})]. \\ &H(\widetilde{\mathbb{E}}_j^1) \le H(\mathbb{E}_j^2) \le H(\widetilde{\mathbb{E}}_j^3) \le H(\widetilde{\mathbb{E}}_8^3) \le 3.15558. \end{aligned} \tag{3.10}$$

The following result proves that entropies $H_{\delta_v} = H(\Delta^1)$, $H_{\mathfrak{d}}(\Delta) = H(\Delta^2)$, and $H(\mathcal{M}_\Delta) = H(\Delta^3)$ define Dynkin functions.

Corollary 3. *Table 4 gives the values of the Dynkin functions defined by maps $\mathfrak{H}^i$, $i \in \{1, 2, 3\}$ which assign to each Dynkin diagram Δ the entropy $H(\Delta^i)$.*

Table 4. Dynkin functions defined by functions $\mathfrak{H}^i$, $i \in \{1, 2, 3\}$.

Dynkin diagrams	Extended Brauer analysis		
	$H_{\delta_v}(\Delta)$	$H_{\mathfrak{d}}(\Delta)$	$H(\mathcal{M}_\Delta)$
$\mathbb{A}_n$	$1 - \frac{1}{n-1}$	$\log_2(n) - \frac{n-2}{n}\log_2(n-2) - \frac{2}{n}$	$\log_2(n)$
$\mathbb{B}_n$	$\frac{n-2}{n} + \frac{3\log_2(3)}{2n}$	$\frac{2}{n}\log_2(n) - \frac{n-2}{n}\log_2(\frac{n-2}{n})$	$-[\frac{2n-2}{2n+1}\log_2(2) + \frac{3}{2n+1}\log_2(3)] + \log_2(2n+1)$
$\mathbb{D}_n$	$\frac{1}{2(n-1)}[3\log_2(3) + 2(n-4)]$	$-[\frac{3}{n}\log_2(\frac{3}{n}) + \frac{n-4}{n}\log_2(\frac{n-4}{n}) + \frac{1}{n}\log_2(\frac{1}{n})]$	$-[\frac{2n-2}{2n+1}\log_2(2) + \frac{3}{2n+1}\log_2(3)] + \log_2(2n+1)$
$\mathbb{E}_6$	0.875489	1.45915	2.56545
$\mathbb{E}_7$	0.896241	1.44882	2.7899
$\mathbb{E}_8$	0.911063	1.40564	2.98423
$\mathbb{F}_4$	1.18872	1	1.56128
$\mathbb{G}_2$	1.68496	0	1

Proof. The sequences $H(\Delta^i)$ with $\Delta \in \{\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n\}$ shown in Table 4 are given in the proof of Theorem 12 (see identities (3.8) and (3.9)) since straight computations give rise to the remaining values of $H(\Delta^i)$. We are done.

The following theorem proves that Euclidean diagrams give rise to similar results as those presented for Dynkin diagrams.

Theorem 13. *Let $\Lambda_{\widetilde{\Delta}}$ be the Brauer configuration algebra induced by a Euclidean diagram which is built according to Remark 4, then the following results hold:*

(E_1) $\dim_k \Lambda_{\widetilde{\mathbb{A}}_n} = 4(n+1)$, $\dim_k \Lambda_{\widetilde{\mathbb{D}}_n} = 4n + 10$, $\dim_k \Lambda_{\widetilde{\mathbb{E}}_i} = 4i + 3$, $6 \leq i \leq 8$.

(E_2) $\dim_k Z(\Lambda_{\widetilde{\mathbb{A}}_n}) = n + 2$, $\dim_k Z(\Lambda_{\widetilde{\mathbb{D}}_n}) = n + 6$, $\dim_k Z(\Lambda_{\widetilde{\mathbb{E}}_i}) = i + 4$, $6 \leq i \leq 8$.

(E_3) *If $\mathfrak{c}(Q_{\widetilde{\Delta}})$ denotes the covering graph of a Euclidean diagram, then* $\mathfrak{c}(Q_{\widetilde{\mathbb{A}}_i}) = \widetilde{\mathbb{A}}_i$, $i \geq 1$; $\mathfrak{c}(Q_{\widetilde{\mathbb{D}}_j}) = \mathbb{A}_{j+1}$, $j \geq 4$; $\mathfrak{c}(Q_{\widetilde{\mathbb{E}}_6}) = \mathbb{E}_6$, *and* $\mathfrak{c}(Q_{\widetilde{\mathbb{E}}_i}) = \mathbb{A}_i$, $i \in \{7, 8\}$.

Proof.

$(E_1), (E_2)$ According to the numbering defined in Remark 4, it holds that

$$|\{v_i \in V_\Delta \mid \delta_{v_i} = val(v_i) = 1\}| = \#Loops(Q_\Delta) = \begin{cases} 0, & \text{if } \Delta = \widetilde{\mathbb{A}}_n, \\ 4, & \text{if } \Delta = \widetilde{\mathbb{D}}_n, \\ 3, & \text{if } \Delta \in \{\widetilde{\mathbb{E}}_s \mid s = 6, 7, 8\}. \end{cases}$$

$$|\{v_i \in V_\Delta \mid \delta_{v_i} = val(v_i) = 3\}| = \begin{cases} 0, & \text{if } \Delta = \widetilde{\mathbb{A}}_n, \\ 2, & \text{if } \Delta = \widetilde{\mathbb{D}}_n, \\ 1, & \text{if } \Delta \in \{\widetilde{\mathbb{E}}_s \mid s = 6, 7, 8\}. \end{cases}$$

Moreover, for any vertex $v_i \in V_{\widetilde{\Delta}}$ and any Euclidean diagram, it holds that if $\delta_{v_i} \neq 1$ and $\delta_{v_i} \neq 3$, then $\delta v_i = 2$. Thus,

$$\sum_{v_i \in V_\Delta} \mu(v_i) = \begin{cases} n+1, & \text{if } \Delta = \widetilde{\mathbb{A}}_n, \\ n+4, & \text{if } \Delta = \widetilde{\mathbb{D}}_n, \\ |E_s| + 3, & \text{if } \Delta \in \{\widetilde{\mathbb{E}}_s \mid s = 6,7,8\}. \end{cases}$$

$\dim_k \Lambda_\Delta$ and $\dim_k Z(\Lambda_\Delta)$ can be easily obtained by applying these data to the corresponding formulas (2.15) and (2.16) in Remark 3).

(E_3) The successor sequence S_{v_i} induced by a vertex $v_i \in V_\Delta$, where Δ is a fixed Euclidean diagram Δ has one of the following forms:

$$\begin{aligned} &e_j, \text{ if } v_i \in e_i, \text{ for some edge } e_i \in E_\Delta \text{ and } \delta_{v_i} = 1. \\ &e_{j-1} < e_j, \text{ if } v_i \in e_{j-1} \cap e_j \text{ for some edges } e_i, e_j \in E_\Delta, \text{ and } \delta_{v_i} = 2. \\ &e_{j_1} < e_{j_2} < e_{j_3}, \text{ if } v_i \in e_{j_1} \cap e_{j_2} \cap e_{j_3}, \text{ for some edges } e_{j_1}, e_{j_2}, e_{j_3} \in E_\Delta, \text{ and } \delta_{v_i} = 3, \end{aligned} \tag{3.11}$$

then, it holds that $\mathfrak{c}(Q_{\widetilde{\mathbb{A}}_n}) = \widetilde{\mathbb{A}}_n$ as a consequence of Theorem 8. $\mathfrak{c}(Q_{\widetilde{\mathbb{D}}_n})$ arises from successor sequences (3.12) induced by $\widetilde{\mathbb{D}}_n$. Bearing in mind the numbering described in Remark 4,

$$\begin{aligned} &S_{v_1} = e_1, \quad S_{v_3} = e_2, \quad S_{v_{n-1}} = e_{n-1}, \quad S_{v_n} = e_n. \\ &S_{v_2} = e_1 < e_2 < e_3, \quad S_{v_n} = e_{n-1} < e_n < e_{n+1}. \\ &S_{v_i} = e_i < e_{i+1}, \text{ for } 4 \leq i \leq n-1. \end{aligned} \tag{3.12}$$

Therefore, the covering sequence $e_1 < e_2 < e_3 < e_4 < e_5 < \ldots e_{n-3} < e_{n-1} < e_n < e_{n+1}$ induces the Dynkin diagram $\mathbb{A}_{n+1}$.

Since similar constructions can be adapted to construct the covering graphs $\mathfrak{c}(Q_{\widetilde{\mathbb{E}}_s})$ induced by Euclidean diagrams $\widetilde{\mathbb{E}}_s$, $s \in \{6,7,8\}$ (Figure 10 shows Brauer quivers $Q_{\widetilde{\Delta}}$ induced by Euclidean diagrams). We are done.

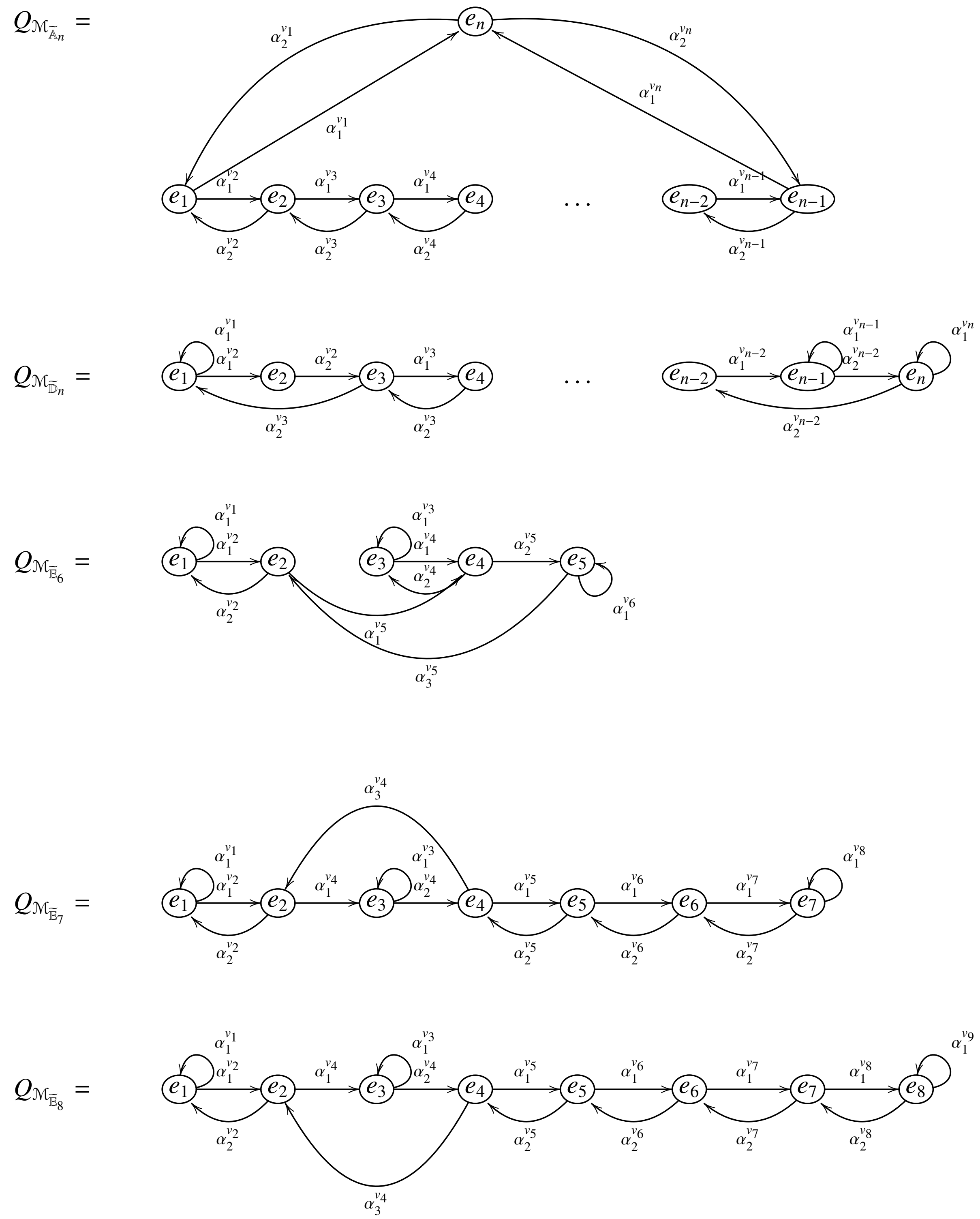

Figure 10. Brauer quivers $Q_{\mathcal{M}_{\widetilde{\Delta}}}$ induced by Euclidean diagrams $\widetilde{\Delta}$.

The following result establishes a straight connection between algebras and entropies induced by Brauer configurations.

Theorem 14. *For Brauer configuration algebras Λ_Δ induced by diagrams $\Delta \in \{\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n\}$, it holds that*

$$\mathfrak{d}_\Delta = \operatorname*{Lim}_{n\to\infty} \log_2(\dim_k \Lambda_\Delta) - H(\Delta^3) = 2. \tag{3.13}$$

Proof.

1) $\log_2(\dim_k \Lambda_{\mathbb{A}_n}) = \log_2(4n-4)$ and $H(\mathbb{A}_n^3) = \log_2(n)$, then $\mathfrak{d}_{\mathbb{A}_n} = \operatorname*{Lim}_{n\to\infty} \log_2(\frac{4n-4}{n}) = 2$.

2) $\log_2(\dim_k \Lambda_{\mathbb{B}_n}) = \log_2(4n+3)$ and $H(\mathbb{B}_n^3) = -[\frac{2n-2}{2n+1}\log_2(\frac{2}{2n+1}) + \frac{3}{2n+1}\log_2(\frac{3}{2n+1})] = -[\frac{2n-2}{2n+1}\log_2(2) + \frac{3}{2n+1}\log_2(3)] + \log_2(2n+1)$, then $\mathfrak{d}_{\mathbb{B}_n} = \operatorname*{Lim}_{n\to\infty} \log_2(\frac{4n+3}{2n+1}) + 1 = 2$.

3) $\log_2(\dim_k \Lambda_{\mathbb{D}_n}) = \log_2(4n-1)$ and $H(\mathbb{D}_n^3) = -[\frac{2n-2}{2n+1}\log_2(2) + \frac{3}{2n+1}\log_2(3)] + \log_2(2n+1)$, then $\mathfrak{d}_{\mathbb{D}_n} = \operatorname*{Lim}_{n\to\infty} \log_2(\frac{4n-1}{2n+1}) + 1 = 2$.

4) $\log_2(\dim_k \Lambda_{\widetilde{\mathbb{A}}_n}) = \log_2(4n)$ and $H(\widetilde{\mathbb{A}}_n^3) = \log_2(n+1)$, then $\mathfrak{d}_{\widetilde{\mathbb{A}}_n} = \operatorname*{Lim}_{n\to\infty} \log_2(\frac{4n}{n+1}) = 2$.

5) $\log_2(\dim_k \Lambda_{\widetilde{\mathbb{D}}_n}) = \log_2(4n+10)$ and $H(\widetilde{\mathbb{D}}_n^3) = -[\frac{2n}{2n+6}\log_2(\frac{2}{2n+6}) + \frac{6}{2n+6}\log_2(\frac{3}{2n+6})]$, then $\mathfrak{d}_{\widetilde{\mathbb{D}}_n} = \operatorname*{Lim}_{n\to\infty} -\frac{2n}{2n+6}\log_2(\frac{2}{2n+6}) + \log_2(4n+10) = 2$.

Since computations for $\mathbb{B}_n$ are the same for $\mathbb{C}_n$, we are done. Similar results can be found for the remaining Dynkin and Euclidean diagrams (see Tables 7 and 4).

4. Discussion

Tables 4–7 make extended Brauer analysis to Dynkin and Euclidean diagrams. They show the Dynkin functions $\mathfrak{c}$ (Corollary 1), dim, dim Z (Corollary 2), $\mathfrak{D}_\iota$ (Theorem 10), and $\mathfrak{H}^i$, $i \in \{1,2,3\}$ (Corollary 3) defined in this paper.

As a consequence of Theorems 8, 12, and Corollary 2, we note that the covering graph induced by a Dynkin diagram Δ is isomorphic to a suitable Dynkin diagram $\mathbb{A}_j$, and that since $\kappa = 1$, the induced Brauer configuration algebras are indecomposable as algebras. Similar conclusions are obtained for covering graphs induced by Euclidean diagrams (see Theorem 8). Identities (4.2) give the asymptotic behavior of the entropy values $H_{\delta_v}(\Delta) = H(\Delta^1)$, $H_{\mathfrak{d}}(\Delta) = H(\Delta^2)$, and $H(\mathcal{M}_\Delta) = H(\Delta^3)$ for Dynkin and Euclidean diagrams Δ and $\widetilde{\Delta}$. Note that,

$$\dim_k \Lambda_{\mathcal{M}_\Psi} \sim 2^{H(\mathcal{M}_\Psi)+2}, \text{ for any Dynkin or Euclidean diagram } \Psi \in \{\Delta, \widetilde{\Delta}\}. \tag{4.1}$$

Identities (4.2) show the asymptotic behavior of entropies $H(\Delta^j)$, for Dynkin and Euclidean diagrams.

$$\begin{aligned}
\operatorname*{Lim}_{n\to\infty} H_{\delta_v}(\mathbb{A}_n) &= \operatorname*{Lim}_{n\to\infty} H_{\delta_v}(\mathbb{D}_n) = 1.\\
\operatorname*{Lim}_{n\to\infty} H_{\mathfrak{d}}(\widetilde{\mathbb{A}}_n) &= \operatorname*{Lim}_{n\to\infty} H_{\mathfrak{d}}(\widetilde{\mathbb{D}}_n) = 0.\\
\operatorname*{Lim}_{n\to\infty} H_{\mathfrak{d}}(\mathbb{A}_n) &= \operatorname*{Lim}_{n\to\infty} H_{\mathfrak{d}}(\mathbb{D}_n) = 0.\\
\operatorname*{Lim}_{n\to\infty} H(\mathcal{M}_{\mathbb{A}_n}) &= \operatorname*{Lim}_{n\to\infty} H(\mathcal{M}_{\mathbb{D}_n}) = \infty.\\
\operatorname*{Lim}_{n\to\infty} H(\mathcal{M}_{\widetilde{\mathbb{A}}_n}) &= \operatorname*{Lim}_{n\to\infty} H(\mathcal{M}_{\widetilde{\mathbb{D}}_n}) = \infty.
\end{aligned} \tag{4.2}$$

5. Concluding remarks and future work

Since its introduction, Brauer analysis has been a helpful tool for investigating different fields of mathematics and science. To date, such analysis has been focused on using the structure of the indecomposable projective modules and the dimensions of the algebra and its center without paying attention to the entropy of the graphs involved in the procedures, i.e., in the topological content information of the graphs involved in the procedures.

This paper adds two new tools to Brauer analysis. On the one hand, it introduces the entropy of a Brauer configuration and compares its values with degree-based entropies of Dynkin and Euclidean diagrams. Such analysis gives rise to Dynkin functions associated with Brauer configuration algebras induced by Dynkin diagrams. The introduced covering graphs of a Brauer configuration is another way to define Dynkin functions.

The following are interesting problems to be addressed in the future:

1) Our aim in the future is to establish additional properties of covering graphs and determine the bounds of the entropy of the Brauer configurations induced by significant family of graphs or networks distinct from Dynkin and Euclidean diagrams (e.g. Butterfly, star, and pancake networks).

2) To define new Dynkin functions based on Brauer configurations associated with Dynkin or Euclidean diagrams.

3) To give the size $\iota_{\mathcal{G}}$ of the fundamental set of relations induced by any graph $\mathcal{G}$ in a significant class of graphs.

4) To study the behavior of the entropy $H(\mathfrak{c}^3(Q_{\mathcal{M}}))$ of the covering graph induced by a Brauer configuration $\mathcal{M}$. Establishing for which of its values it holds that $2^{H(\mathcal{G}^3)} = \dim_k \Lambda_{\mathcal{M}}$, where $\Lambda_{\mathcal{M}}$ denotes the induced Brauer configuration algebra.

Table 5. Extended Brauer analysis of Dynkin diagrams.

Dynkin diagrams	Extended Brauer analysis				
	$\dim_k \Lambda_{\mathcal{M}_\Delta}$	$\dim_k Z(\Lambda_{\mathcal{M}_\Delta})$	ι_Δ	κ	$\mathfrak{c}(Q_\Delta)$
$\mathbb{A}_n$	$4n-4$	$n+2$	$5(n-1)$	1	$\mathbb{A}_{n-1}$
$\mathbb{B}_n$	$4n+3$	$n+2$	$5n$	1	$\mathbb{A}_n$
$\mathbb{D}_n$	$4n-1$	$n+3$	$5(n-1)$	1	$\mathbb{A}_{n-1}$
$\mathbb{E}_6$	23	9	25	1	$\mathbb{A}_5$
$\mathbb{E}_7$	27	10	30	1	$\mathbb{A}_6$
$\mathbb{E}_8$	31	11	35	1	$\mathbb{A}_7$
$\mathbb{F}_4$	22	7	20	1	$\mathbb{A}_4$
$\mathbb{G}_2$	18	4	15	1	$\mathbb{A}_3$

Table 6. Extended Brauer analysis of Euclidean diagrams.

Euclidean diagrams	Extended Brauer analysis			
	$\dim_k \Lambda_{\mathcal{M}_{\widetilde{\Delta}}}$	$\dim_k Z(\Lambda_{\mathcal{M}_{\widetilde{\Delta}}})$	κ	$\mathfrak{c}(Q_{\widetilde{\Delta}})$
$\widetilde{\mathbb{A}}_n$	$4n+4$	$n+2$	1	$\mathbb{A}_n$
$\widetilde{\mathbb{D}}_n$	$4n+10$	$n+6$	1	$\mathbb{A}_{n+1}$
$\widetilde{\mathbb{E}}_6$	27	10	1	$\mathbb{E}_6$
$\widetilde{\mathbb{E}}_7$	31	11	1	$\mathbb{A}_7$
$\widetilde{\mathbb{E}}_8$	35	12	1	$\mathbb{A}_8$

Table 7. Graph entropies given by an extended Brauer analysis of Euclidean diagrams.

Euclidean diagrams	Extended Brauer analysis		
	$H_{\delta_v}(\widetilde{\Delta})$	$H_{\mathfrak{d}}(\widetilde{\Delta})$	$H(\mathcal{M}_{\widetilde{\Delta}})$
$\widetilde{\mathbb{A}}_n$	1	0	$\log_2(n+1)$
$\widetilde{\mathbb{D}}_n$	$\frac{1}{2(n+1)}[6\log_2(3)+2(n-4)]$	$-[\frac{4}{n+2}+\frac{6}{n+2}\log_2(\frac{2}{n+2})+\frac{n-4}{n+2}\log_2(\frac{n-4}{n+2})]$	$-[\frac{2n}{2n+6}\log_2(\frac{2}{2n+6})+\frac{6}{2n+6}\log_2(\frac{3}{2n+6})]$
$\widetilde{\mathbb{E}}_6$	0.896241	1.44882	2.7899
$\widetilde{\mathbb{E}}_7$	0.911063	1.40564	2.95237
$\widetilde{\mathbb{E}}_8$	0.92218	1.35164	3.15557

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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