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Blow-up dynamics in nonlinear coupled wave equations with fractional damping and external source

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Abstract: This study focused on a coupled nonlinear wave equation system featuring fractional damping and polynomial source terms within a bounded domain. We demonstrated that, under specific conditions, the solutions exhibited blow-up in a finite time-frame.

Keywords: wave equation; coupled system; blow-up for solution; Caputo's fractional derivative

1. Introduction

In this study, we examine the following problem:

$$\begin{cases} \varphi_{tt} + \partial_t^{\alpha,\eta}\varphi = \Delta\varphi + |\varphi+\psi|^{p-1}(\varphi+\psi) + |\varphi|^{\frac{p-3}{2}}\varphi|\psi|^{\frac{p+1}{2}}, & x\in\Omega, \quad t>0, \\ \psi_{tt} + \partial_t^{\beta,\eta}\psi = \Delta\psi + |\varphi+\psi|^{p-1}(\varphi+\psi) + |\psi|^{\frac{p+1}{2}}\psi|\varphi|^{\frac{p-3}{2}}, & x\in\Omega, \quad t>0, \\ \varphi(x,t) = \psi(x,t) = 0, & x\in\partial\Omega, \ t>0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & x\in\Omega, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with a smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. The constants p and α satisfy

$$\begin{cases} 2 < p < \infty, & n = 1,2, \\ 2 < p \leq \frac{2n}{n-2}, & n \geq 3, \end{cases}$$

and $0 < \alpha, \beta < 1$. The functions $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(x)$, and $\psi_1(x)$ are given.
The symbol $\partial_t^{\alpha,\eta}$ denotes the Caputo fractional derivative of order α with respect to the time variable t, where $0 < \alpha < 1$. It is defined as:

$$\partial_t^{\alpha,\eta} W(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dW}{ds}(s)\, ds, \ \eta \geq 0,$$

which was introduced by Michele Caputo in [1].

Problems of this type arise in material science and physics, particularly in the study of wave propagation in viscoelastic materials, fluid dynamics, and structural mechanics. These models are used to describe the behavior of materials with memory effects, where the damping terms are often nonlocal or fractional in nature. Such systems have been widely studied in the context of their stability, energy decay, and response to external forces (see, for example, [2–4]). The fractional damping terms in the model are used to capture more complex dissipative effects that arise in these physical systems, leading to more accurate descriptions of the material behavior under stress and deformation. Agre and Rammaha [2] established both existence and blow-up results for systems of the form (1.1), specifically when the initial energy is sufficiently small and positive. More recently, Han and Wang [4] extended these results to address systems involving a viscoelastic term, further generalizing the analysis conducted in [2].
In the case of a single wave equation given by

$$\varphi_{tt} - \partial_t^{\alpha,\eta}\varphi = \Delta\varphi + a|\varphi|^{p-1}\varphi,$$

Kirane and Tatar [5] demonstrated that solutions exhibit exponential growth when the initial data is sufficiently large. Matignon et al. [6] explored the case where $a = 0$, achieving results on well-posedness and asymptotic stability by reformulating the problem into a standard form. Tatar [7] employed Fourier transform techniques and the Hardy-Littlewood-Sobolev inequality to establish the finite-time blow-up of solutions. Additionally, [8] showed that finite-time blow-up can occur independently of the time variable T associated with the initial data.
Without internal fractional damping, a polynomial source can lead to finite-time blow-up for solutions with negative initial energy [9–11]. In [12], Georgiev and Todorova demonstrated that the solution remains global if $p \leq m$, whereas solutions blow up in finite time if this condition is not met. Ball [9] established finite-time blow-up in certain scenarios using continuation theorems for ordinary differential equations within Banach spaces. Additional theorems in different contexts are discussed in [13].

Motivated by the aforementioned research, we consider a coupled system with fractional derivatives. This choice is driven by the growing interest in recent years within the scientific community to explore the complex dynamics and practical applications of wave equations. The extension to fractional derivatives is significant because they offer more accurate models for systems exhibiting memory effects and complex dissipation properties, which are commonly encountered in practical scenarios, such as material microstructure analysis. Introducing fractional derivatives into the coupled system is novel in the context of wave equations with external sources, allowing for a more realistic modeling of dissipative effects in various materials and structures. Our results provide new insights into the blow-up behavior of systems with fractional derivatives.

The rest of the paper is organized as follows: In Section 2, we present some preliminary results that will be useful throughout the paper. In Section 3, we provide the main result concerning the blow-up behavior of the solutions of the system and offer a detailed proof.

2. Preliminaries

For a solution (φ, ψ) to the problem (1.1), the energy functional is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx + \frac{1}{2}\int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx - \frac{1}{p+1}\int_\Omega |\varphi+\psi|^{p+1} dx \\ &- \frac{2}{p+1}\int_\Omega |\varphi|^{\frac{p+1}{2}} |\psi|^{\frac{p+1}{2}} dx. \end{aligned} \tag{2.1}$$

In this definition, the classical energy functional $E(t)$ is described in terms of kinetic and potential energy, along with additional interaction terms.
To analyze the behavior of this energy functional, we state and prove the following lemma.

Lemma 2.1. *Assume* (φ, ψ) *is a regular solution to problem* (1.1). *The energy functional, as defined by* (2.1)*, satisfies*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\frac{1}{\Gamma(1-\alpha)}\int_\Omega \varphi_t \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \varphi_t(s)\, ds\, dx \\ &- \frac{1}{\Gamma(1-\beta)}\int_\Omega \psi_t \int_0^t (t-s)^{-\beta} e^{-\eta(t-s)} \psi_t(s)\, ds\, dx, \end{aligned} \tag{2.2}$$

and we deduce $E(t) \le E(0), \ \forall t \ge 0.$

Proof. By multiplying the first equation in (1.1) by φ_t and the second equation by ψ_t, then integrating over Ω and applying integration by parts, we derive Eq (2.2). Determining the sign of this functional is not immediately apparent; however, Nohel and Shea demonstrated in [14] that a real function $a(t, x) \in L^1_{\text{loc}}(\mathbb{R}^+; L^\infty(\Omega))$ is of positive type if it satisfies the following condition:

$$\int_0^T \int_\Omega h(t) \int_0^t a(t-s) h(s)\, ds\, d\sigma\, dt \ge 0, \tag{2.3}$$

for all $h \in C(\mathbb{R}^+; H^1(\Omega))$ and for every $T > 0$.

Next, by substituting t with z and integrating Eq (2.2) from 0 to t, we obtain:

$$\begin{aligned} E(t) - E(0) = &-\frac{1}{\Gamma(1-\alpha)}\int_0^t \int_\Omega \varphi_t(z) \int_0^z (z-s)^{-\alpha} e^{-\eta(z-s)} \varphi_t(s)\, ds\, dx\, dz \\ &- \frac{1}{\Gamma(1-\beta)}\int_0^t \int_\Omega \psi_t(z) \int_0^z (z-s)^{-\beta} e^{-\eta(z-s)} \psi_t(s)\, ds\, dx\, dz. \end{aligned} \tag{2.4}$$

Using condition (2.3), we directly deduce the classical energy inequality:

$$\forall t \ge 0, \quad E(0) \ge E(t).$$

□

The following lemma will be beneficial for what follows.

Lemma 2.2. *There exist positive constants c_1 and c_2 such that*

$$c_1\left(|x|^{p+1}+|y|^{p+1}\right)\leq \frac{1}{p+1}|x+y|^{p+1}+\frac{2}{p+1}|xy|^{\frac{p+1}{2}}\leq c_2\left(|x|^{p+1}+|y|^{p+1}\right),$$

for all $x, y\in\mathbb{R}$.

Proof. To start, using the inequality $2ab \leq a^2+b^2$, we obtain:

$$|x|^{\frac{p+1}{2}}|y|^{\frac{p+1}{2}}\leq \frac{1}{2}\left(|x|^{p+1}+|y|^{p+1}\right).$$

Next, we use the fact that $h(x) := |x|^{p+1}$ is convex over $\mathbb{R}_+$, and so

$$|x+y|^{p+1}\leq 2^p\left(|x|^{p+1}+|y|^{p+1}\right).$$

By combining the results above, we obtain:

$$\frac{1}{p+1}|x+y|^{p+1}+\frac{2}{p+1}|xy|^{\frac{p+1}{2}}\leq \frac{2^p+1}{p+1}\left(|x|^{p+1}+|y|^{p+1}\right). \tag{2.5}$$

On the other hand, we can suppose that $|y|\leq|x|$, and so:

$$\begin{aligned}|x|^{p+1}=|x+y-y|^{p+1} &\leq 2^p\left(|x+y|^{p+1}+|y|^{p+1}\right)\\ &\leq 2^p\left(|x+y|^{p+1}+|xy|^{\frac{p+1}{2}}\right).\end{aligned}$$

Therefore,

$$|x|^{p+1}+|y|^{p+1}\leq 2^{p+1}\left(|x+y|^{p+1}+|xy|^{\frac{p+1}{2}}\right). \tag{2.6}$$

Finally, by combining (2.5) and (2.6), we have demonstrated that there exist positive constants c_1 and c_2 such that:

$$c_1\left(|x|^{p+1}+|y|^{p+1}\right)\leq \frac{1}{p+1}|x+y|^{p+1}+\frac{2}{p+1}|xy|^{\frac{p+1}{2}}\leq c_2\left(|x|^{p+1}+|y|^{p+1}\right).$$

□

3. Blow-up of solutions in finite time

Denote by T^* the maximal interval of time for which a solution is defined. When $T^*\neq+\infty$, the solution is said to blow up in finite time. Blow-up means that a singularity forms that prevents the solution from being extended beyond T^*. If we examine the ordinary differential equation related to the one-dimensional (1D) semi-linear wave equation: $u_{tt} = |u|^{p-1}u$, it is known that its solutions exhibit blow-up in finite time. This observation supports our assertion that the solution of our system also experiences blow-up in finite time.

This phenomenon can have two interpretations. If it arises in the modeling of a physical phenomenon for which it has no meaning, it signifies that the model is oversimplified; this is linked with the instability of numerical schemes that one could employ. It can also be of physical relevance (finite time collapse of a star, shocks, etc.).

In this section, we are ready to state and prove our result. To begin, we present a local existence theorem, which can be established using [12] and easily adapted to the case of a system of equations:

Theorem 3.1. *Let* (φ_0, φ_1) *and* $(\psi_0, \psi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Suppose* p *satisfies*

$$\begin{cases} 2 < p < \infty, & \text{for } n = 1, 2, \\ 2 < p \leq \frac{2n}{n-2}, & \text{for } n \geq 3. \end{cases}$$

Then, the problem (1.1) *has a unique local solution*

$$(\varphi, \psi) \in C([0, T_m); H_0^1(\Omega)), \quad (\varphi_t, \psi_t) \in C([0, T_m); L^2(\Omega)) \cap L^2(\Omega),$$

for some $T_m > 0$.

We now present and prove the main result of our study.

Theorem 3.2. *Let* (φ, ψ) *be the solution of the system* (1.1)*, and assume that the initial data* φ_0 *and* ψ_0 *satisfies*

$$\int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx \neq 0.$$

Then, for any $T > 0$*, there exist* $T^* \leq T$ *and sufficiently large initial data such that* (φ, ψ) *blows up at* T^*.

Proof. To ensure clarity, we will break the proof into several steps.

Step 1. We introduce the functional $H(t)$ defined by:

$$H(t) = -\int_0^t E(s)\, ds + (dt + l) \int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx,$$

where d and l are positive constants whose values will be specified later.
Since the proof relies on a contradiction argument, we will associate the function H with the function Ψ defined in Step 2. We assume that the solution exists up to time T and derive an inequality of the form $\Psi'(t) \geq C\Psi^\delta(t)$ with $\delta > 1$. Integrating this inequality demonstrates that, under certain conditions, the solution cannot exist for all time up to T.

Differentiating $H(t)$ in relation to t, one gets:

$$H'(t) = -E(t) + d \int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx \geq d \int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx - E(0). \tag{3.1}$$

Selecting d ensures that $H'(0) > 0$. It is sufficient to select d such that:

$$H'(0) = -E(0) + d \int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx > 0.$$

By virtue of (2.4) and (3.1), it follows that:

$$\begin{aligned} H'(0) - H'(t) &= E(t) - E(0) \\ &= -\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_\Omega \varphi_t(z) \int_0^z (z-s)^{-\alpha} e^{-\eta(z-s)} \varphi_t(s)\, ds\, dx\, dz \\ &\quad - \frac{1}{\Gamma(1-\beta)} \int_0^t \int_\Omega \psi_t(z) \int_0^z (z-s)^{-\beta} e^{-\eta(z-s)} \psi_t(s)\, ds\, dx\, dz < 0. \end{aligned} \tag{3.2}$$

Step 2. We define the functional

$$\Psi(t) = H^{1-\gamma}(t) + \frac{\varepsilon}{2}\left(\int_\Omega \left(\varphi^2 + \psi^2\right) dx - \int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx\right), \tag{3.3}$$

where $\varepsilon > 0$ and $0 < \gamma < \frac{p-1}{2(p+1)}$.
Initially, we have

$$\Psi(0) = H^{1-\gamma}(0) = \left(l \int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx\right)^{1-\gamma}.$$

Differentiating (3.3) in relation to t gives

$$\Psi'(t) = (1-\gamma)H^{-\gamma}(t)H'(t) + \varepsilon \int_\Omega (\varphi\varphi_t + \psi\psi_t)\, dx. \tag{3.4}$$

Further differentiating (3.4) and then integrating yields:

$$\begin{aligned}\Psi'(t) = {} & (1-\gamma)H^{-\gamma}(t)H'(t) + \varepsilon \int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx \\ & + \varepsilon \int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds \\ & + \varepsilon \int_0^t \int_\Omega (\varphi\varphi_{tt} + \psi\psi_{tt})\, dx\, ds.\end{aligned} \tag{3.5}$$

To assess the last term on the righthand side of (3.5), we multiply the first and second equations of problem (1.1) by φ and ψ, respectively, and integrate over $\Omega \times (0,t)$. This yields:

$$\begin{aligned}\int_0^t \int_\Omega (\varphi\varphi_{tt} + \psi\psi_{tt})\, dx\, ds = {} & -\int_0^t \int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\, ds \\ & + \frac{1}{p+1}\int_0^t \int_\Omega |\varphi + \psi|^{p+1}\, dx\, ds \\ & + \frac{1}{p+1}\int_0^t \int_\Omega |\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\, dx\, ds \\ & - \frac{1}{\Gamma(1-\alpha)}\int_0^t \int_\Omega \varphi \int_0^s (s-z)^{-\alpha}e^{-\eta(s-z)}\varphi_t(z)\, dz\, dx\, ds \\ & - \frac{1}{\Gamma(1-\beta)}\int_0^t \int_\Omega \psi \int_0^s (s-z)^{-\beta}e^{-\eta(s-z)}\psi_t(z)\, dz\, dx\, ds.\end{aligned} \tag{3.6}$$

Inserting the relation (3.6) into (3.5), one gets

$$\begin{aligned}\Psi'(t) = {} & (1-\gamma)H^{-\gamma}(t)H'(t) + \varepsilon \int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx \\ & + \varepsilon \int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds - \varepsilon \int_0^t \int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\, ds \\ & + \frac{\varepsilon}{p+1}\int_0^t \int_\Omega |\varphi + \psi|^{p+1}\, dx\, ds + \frac{2\varepsilon}{p+1}\int_0^t \int_\Omega |\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\, dx\, ds \\ & - \frac{\varepsilon}{\Gamma(1-\alpha)}\int_0^t \int_\Omega \varphi \int_0^s (s-z)^{-\alpha}e^{-\eta(s-z)}\varphi_t(z)\, dz\, dx\, ds \\ & - \frac{\varepsilon}{\Gamma(1-\beta)}\int_0^t \int_\Omega \psi \int_0^s (s-z)^{-\beta}e^{-\eta(s-z)}\psi_t(z)\, dz\, dx\, ds.\end{aligned} \tag{3.7}$$

We introduce the extension operators over the entire domain in the following manner:

$$L\omega(s) = \begin{cases} \omega & \text{if } s \in [0,t], \\ 0 & \text{if } s \in \mathbb{R} \setminus [0,t], \end{cases}$$

and

$$L\kappa_\beta(s) = \begin{cases} \kappa_\beta(s) & \text{if } s > 0, \\ 0 & \text{if } s < 0, \end{cases}$$

where the kernel is defined by

$$\kappa_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(1-\gamma)}, \quad 0 < \gamma < 1.$$

We start with the following equation:

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \varphi(s) \int_0^s (s-z)^{-\alpha} \varphi_t(z)\, dz\, ds = \int_{\mathbb{R}} L\varphi(s) \int_{\mathbb{R}} L\kappa_\alpha(s-z)(L\varphi_t)(z)\, dz\, ds. \tag{3.8}$$

Using Parseval's theorem, we rewrite the left-hand side of (3.8) as:

$$\int_{\mathbb{R}} L\varphi(s) \int_{\mathbb{R}} L\kappa_\alpha(s-z)(L\varphi_t)(z)\, dz\, ds = \int_{\mathbb{R}} F(L\varphi)(\sigma) \overline{F(L\kappa_\alpha \star L\varphi_t)}(\sigma)\, d\sigma,$$

where $F(f)$ represents the standard Fourier transform of f.
Using the convolution property, we obtain:

$$\kappa_{\gamma+\eta}(t) = \kappa_\gamma(t) \star \kappa_\eta(t), \quad 0 < \gamma, \eta < 1.$$

Applying the Cauchy-Schwarz and Young inequalities, we obtain:

$$\begin{aligned} \int_{\mathbb{R}} L\varphi(s) \int_{\mathbb{R}} L\kappa_\alpha(s-z)(L\varphi_t)(z)\, dz\, ds &\le \left(\int_{\mathbb{R}} |F(L\kappa_{\frac{\alpha}{2}}) F(L\varphi)|^2\, d\sigma \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |F(L\kappa_{\frac{\alpha}{2}}) F(L\varphi_t)|^2\, d\sigma \right)^{\frac{1}{2}} \\ &\le \delta \int_{\mathbb{R}} |F(L\kappa_{\frac{\alpha}{2}}) F(L\varphi_t)|^2\, d\sigma + \frac{1}{4\delta} \int_{\mathbb{R}} |F(L\kappa_{\frac{\alpha}{2}}) F(L\varphi)|^2\, d\sigma, \end{aligned} \tag{3.9}$$

where $\delta > 0$. From (3.9), as stated in [15], we infer:

$$\begin{aligned} \int_{\mathbb{R}} L\varphi(s) \int_{\mathbb{R}} L\kappa_\alpha(s-z)(L\varphi_t)(z)\, dz\, ds \le \frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)} \Bigg[&\delta \int_{\mathbb{R}} L\varphi_t(s)(L\kappa_\alpha \star L\varphi_t)(s)\, ds \\ &+ \frac{1}{4\delta} \int_{\mathbb{R}} L\varphi(s)(L\kappa_\alpha \star L\varphi)(s)\, ds \Bigg]. \end{aligned} \tag{3.10}$$

Similarly, we have:

$$\begin{aligned} \int_{\mathbb{R}} L\psi(s) \int_{\mathbb{R}} L\kappa_\beta(s-z)(L\psi_t)(z)\, dz\, ds \le \frac{1}{\cos\left(\frac{\beta\pi}{2}\right)} \Bigg[&\delta \int_{\mathbb{R}} L\psi_t(s)(L\kappa_\beta \star L\psi_t)(s)\, ds \\ &+ \frac{1}{4\delta} \int_{\mathbb{R}} L\psi(s)(L\kappa_\beta \star L\psi)(s)\, ds \Bigg]. \end{aligned} \tag{3.11}$$

Inserting the estimates (3.10) and (3.11) into (3.7), we deduce

$$\begin{aligned}\Psi'(t) \geq& (1-\gamma) H^{-\gamma}(t)H'(t) + \varepsilon \int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx + \varepsilon \int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds \\ &+ \varepsilon \int_0^t \int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\, ds + \frac{\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi+\psi|^{p+1}\, dx\, ds + \frac{2\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi|^{\frac{p+1}{2}} |\psi|^{\frac{p+1}{2}}\, dx\, ds \\ &- \frac{\varepsilon}{\cos\left(\frac{\alpha\pi}{2}\right)} \left[\delta \int_\Omega \int_{\mathbb{R}} L\varphi_t(s)\, (L\kappa_\alpha \star L\varphi_t)(s)\, ds\, dx + \frac{1}{4\delta} \int_\Omega \int_{\mathbb{R}} L\varphi(s)\, (L\kappa_\alpha \star L\varphi)(s)\, ds\, dx\right] \\ &- \frac{\varepsilon}{\cos\left(\frac{\beta\pi}{2}\right)} \left[\delta \int_\Omega \int_{\mathbb{R}} L\psi_t(s) \left(L\kappa_\beta \star L\psi_t\right)(s)\, ds\, dx + \frac{1}{4\delta} \int_\Omega \int_{\mathbb{R}} L\psi(s) \left(L\kappa_\beta \star L\psi\right)(s)\, ds\, dx\right].\end{aligned} \tag{3.12}$$

If we define $k = \min\left(\cos\left(\frac{\alpha\pi}{2}\right), \cos\left(\frac{\beta\pi}{2}\right)\right)$, from (3.2) we see that

$$\begin{aligned}&- \frac{\varepsilon\delta}{\cos\left(\frac{\alpha\pi}{2}\right)} \int_\Omega \int_{\mathbb{R}} L\varphi_t(s)\, (L\kappa_\alpha \star L\varphi_t)(s)\, ds\, dx - \frac{\varepsilon\delta}{\cos\left(\frac{\beta\pi}{2}\right)} \int_\Omega \int_{\mathbb{R}} L\psi_t(s) \left(L\kappa_\beta \star L\psi_t\right)(s)\, ds\, dx \\ &\geq \frac{\varepsilon\delta}{k} [H'(0) - H'(t)].\end{aligned} \tag{3.13}$$

Therefore, (3.12) and (3.13) imply:

$$\begin{aligned}\Psi'(t) \geq& (1-\gamma) H^{-\gamma}(t)H'(t) + \varepsilon \int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx + \varepsilon \int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds \\ &+ \varepsilon \int_0^t \int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\, ds + \frac{\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi+\psi|^{p+1}\, dx\, ds + \frac{2\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi|^{\frac{p+1}{2}} |\psi|^{\frac{p+1}{2}}\, dx\, ds \\ &+ \frac{\varepsilon\delta}{k} [H'(0) - H'(t)] - \frac{\varepsilon}{4\delta k} \int_\Omega \int_{\mathbb{R}} L\varphi(s)\, (L\kappa_\alpha \star L\varphi)(s)\, ds\, dx \\ &- \frac{\varepsilon}{4\delta k} \int_\Omega \int_{\mathbb{R}} L\psi(s) \left(L\kappa_\beta \star L\psi\right)(s)\, ds\, dx.\end{aligned} \tag{3.14}$$

We now estimate the last two terms in (3.14). Using the Cauchy-Schwarz inequality, we have:

$$\int_\Omega \int_{\mathbb{R}} L\varphi(s)\, (L\kappa_\alpha \star L\varphi)(s)\, ds\, dx \leq \int_\Omega \left(\int_{\mathbb{R}} |L\varphi|^2\, ds\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |L\kappa_\alpha \star L\varphi|^2(s)\, ds\right)^{\frac{1}{2}} dx.$$

Then, by Young's inequality (see [8]), we obtain:

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}} L\varphi(s)\, (L\kappa_\alpha \star L\varphi)(s)\, ds\, dx &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t s^{\alpha-1}\, ds\right) \left(\int_\Omega \int_{\mathbb{R}} |L\varphi|^2\, ds\, dx\right) \\ &\leq \frac{t^\alpha}{\Gamma(1+\alpha)} \int_\Omega \int_{\mathbb{R}} |L\varphi|^2\, ds\, dx.\end{aligned} \tag{3.15}$$

Similarly,

$$\begin{aligned}\int_\Omega \int_0^t L\varphi(s) \left(L\kappa_\beta \star L\psi\right)(s)\, ds\, dx &\leq \frac{1}{\Gamma(\beta)} \left(\int_0^t s^{\beta-1}\, ds\right) \left(\int_\Omega \int_{\mathbb{R}} |L\psi|^2\, ds\, dx\right) \\ &\leq \frac{t^\beta}{\Gamma(1+\beta)} \int_\Omega \int_{\mathbb{R}} |L\psi|^2\, ds\, dx.\end{aligned} \tag{3.16}$$

Substituting relations (3.15) and (3.16) into (3.14), one gets:

$$\begin{aligned}\Psi'(t) \geq & \left[(1-\gamma) H^{-\gamma}(t) - \frac{\varepsilon\delta}{k}\right] H'(t) + \frac{\varepsilon\delta}{k} H'(0) + \varepsilon \int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx \\ & + \varepsilon \int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds + \varepsilon \int_0^t \int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\, ds \\ & + \frac{\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi + \psi|^{p+1}\, dx\, ds + \frac{2\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi|^{\frac{p+1}{2}} |\psi|^{\frac{p+1}{2}}\, dx\, ds \\ & - \frac{\varepsilon t^\alpha}{4\delta k \Gamma(\alpha+1)} \int_\Omega \int_0^t |\varphi|^2\, ds\, dx - \frac{\varepsilon t^\beta}{4\delta k \Gamma(\beta+1)} \int_\Omega \int_0^t |\psi|^2\, ds\, dx.\end{aligned}$$

Choosing $\delta = MkH^{-\gamma}(t)$ for some $M > 0$, we obtain:

$$\begin{aligned}\Psi'(t) \geq & \left[(1-\gamma) - M\varepsilon\right] H^{-\gamma}(t) H'(t) + \varepsilon M H^{-\gamma}(t) H'(0) + \varepsilon \int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx \\ & + \varepsilon \int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds + \varepsilon \int_0^t \int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\, ds \\ & + \frac{\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi + \psi|^{p+1}\, dx\, ds + \frac{2\varepsilon}{p+1} \int_0^t \int_\Omega |\varphi|^{\frac{p+1}{2}} |\psi|^{\frac{p+1}{2}}\, dx\, ds \\ & - \frac{\varepsilon t^\alpha H^\gamma(t)}{4Mk^2 \Gamma(\alpha+1)} \int_\Omega \int_0^t |\varphi|^2\, ds\, dx - \frac{\varepsilon t^\beta H^\gamma(t)}{4Mk^2 \Gamma(\beta+1)} \int_\Omega \int_0^t |\psi|^2\, ds\, dx.\end{aligned} \tag{3.17}$$

To bound the last two terms in (3.17), we apply Hölder's inequality and Lemma 2.2:

$$\begin{aligned}\int_0^t \int_\Omega |\varphi|^2\, ds\, dx & \leq |\Omega|^{\frac{p-1}{p+1}} t^{\frac{p-1}{p+1}} \left(\int_0^t \int_\Omega |\varphi|^{p+1}\, dx\, ds\right)^{\frac{2}{p+1}} \\ & \leq |\Omega|^{\frac{p-1}{p+1}} T^{\frac{p-1}{p+1}} \left(\int_0^t \int_\Omega \left(|\varphi|^{p+1} + |\psi|^{p+1}\right) dx\, ds\right)^{\frac{2}{p+1}} \\ & \leq C_1 \left(\int_0^t \int_\Omega \left(\frac{1}{p+1} |\varphi + \psi|^{p+1} + \frac{2}{p+1} |\varphi|^{\frac{p+1}{2}} |\psi|^{\frac{p+1}{2}}\right) dx\, ds\right)^{\frac{2}{p+1}},\end{aligned} \tag{3.18}$$

where $C_1 = |\Omega|^{\frac{p-1}{p+1}} T^{\frac{p-1}{p+1}}$. Similarly, we have:

$$\int_0^t \int_\Omega |\psi|^2\, ds\, dx \leq C_1 \left(\int_0^t \int_\Omega \left(\frac{1}{p+1} |\varphi + \psi|^{p+1} + \frac{2}{p+1} |\varphi|^{\frac{p+1}{2}} |\psi|^{\frac{p+1}{2}}\right) dx\, ds\right)^{\frac{2}{p+1}}. \tag{3.19}$$

By recalling the definition of $H^\gamma(t)$, we obtain:

$$
\begin{aligned}
I &= H^\gamma(t)\left(\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right)^{\frac{2}{p+1}}\\
&\le\left[\frac{1}{p+1}\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds+(dt+l)\int_\Omega(\varphi_0^2+\psi_0^2)\,dx\right]^\gamma\\
&\quad\times\left(\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right)^{\frac{2}{p+1}}\\
&\le\left[\frac{1}{(p+1)^\gamma}\left(\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right)^\gamma\right.\\
&\quad\left.+(dt+l)^\gamma\left(\int_\Omega(\varphi_0^2+\psi_0^2)\,dx\right)^\gamma\right]\times\left(\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right)^{\frac{2}{p+1}}\\
&\le c_1\left(\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right)^{\gamma+\frac{2}{p+1}}+(dt+l)^\gamma\left(\int_\Omega(\varphi_0^2+\psi_0^2)\,dx\right)^\gamma\\
&\quad\times\left(\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right)^{\frac{2}{p+1}},
\end{aligned}
\tag{3.20}
$$

where $c_1=\frac{1}{(p+1)^\gamma}$. We choose $\gamma<1-\frac{2}{p+1}$, and one gets:

$$
\begin{aligned}
I &\le c_1\left[1+\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right]+(dT+l)^\gamma\left(\int_\Omega(\varphi_0^2+\psi_0^2)\,dx\right)^\gamma\\
&\times\left[1+\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right].
\end{aligned}
\tag{3.21}
$$

Substituting (3.18), (3.19), (3.20), and (3.21) into (3.17) yields:

$$
\begin{aligned}
\Psi'(t) &\ge[(1-\gamma)-M\varepsilon]\,H^{-\gamma}(t)H'(t)+\varepsilon MH^{-\gamma}(t)H'(0)+\varepsilon\int_\Omega(\varphi_0\varphi_1+\psi_0\psi_1)\,dx\\
&+\varepsilon\int_0^t\int_\Omega\left(\varphi_t^2+\psi_t^2\right)dx\,ds+\varepsilon\int_0^t\int_\Omega\left(|\nabla\varphi|^2+|\nabla\psi|^2\right)dx\,ds\\
&+\varepsilon\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\\
&-\frac{\varepsilon B}{Mk^2}\left[1+\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right],
\end{aligned}
\tag{3.22}
$$

where $B=\frac{1}{4}\left[\frac{c_1T^\alpha}{\Gamma(\alpha+1)}+\frac{c_1T^\beta}{\Gamma(\beta+1)}\right]\left[c_3+(dT+l)^\gamma\left(\int_\Omega(\varphi_0^2+\psi_0^2)\,dx\right)^\gamma\right]$.
We select $\varepsilon>0$ such that $\varepsilon\le\frac{1-\gamma}{M}$. Then, from (3.22), we deduce:

$$
\begin{aligned}
\Psi'(t) &\ge\varepsilon\int_\Omega(\varphi_0\varphi_1+\psi_0\psi_1)\,dx+\varepsilon\int_0^t\int_\Omega\left(\varphi_t^2+\psi_t^2\right)dx\,ds\\
&+\varepsilon\left(1-\frac{B}{MK^2}\right)\int_0^t\int_\Omega\left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds-\frac{\varepsilon B}{Mk^2}.
\end{aligned}
$$

Moreover, by adding and subtracting $\lambda\varepsilon H(t)$ with $0 < \lambda < p+1$, we obtain:

$$\begin{aligned}\Psi'(t) \geq & \lambda\varepsilon H(t) + \varepsilon \int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx + \varepsilon\left(1 + \frac{\lambda}{2}\right)\int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds \\ & + \lambda\frac{\varepsilon}{2}\int_0^t \int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\, ds \\ & + \varepsilon\left(1 - \frac{B}{MK^2} - \lambda\right)\int_0^t \int_\Omega \left(\frac{1}{p+1}|\varphi+\psi|^{p+1} + \frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right) dx\, ds \\ & - \lambda\varepsilon(dt + l)\int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx - \frac{\varepsilon B}{Mk^2}.\end{aligned} \tag{3.23}$$

We choose $\varphi_0, \varphi_1, \psi_0$, and ψ_1 such that

$$\int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx - \lambda l \int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx > 0,$$

and M is sufficiently large so that

$$\int_\Omega (\varphi_0\varphi_1 + \psi_0\psi_1)\, dx - \lambda\,(dT + l)\int_\Omega \left(\varphi_0^2 + \psi_0^2\right) dx > \frac{B}{MK^2}.$$

Next, we select $\widetilde{b}$ such that $0 < \widetilde{b} < 1 - \frac{B}{MK^2} - \lambda$. Inequality (3.23) takes the form

$$\begin{aligned}\Psi'(t) \geq & \lambda\varepsilon H(t) + \varepsilon\left(\frac{\lambda+2}{2}\right)\int_0^t \int_\Omega \left(\varphi_t^2 + \psi_t^2\right) dx\, ds \\ & + \varepsilon\widetilde{b}\int_0^t \int_\Omega \left(\frac{1}{p+1}|\varphi+\psi|^{p+1} + \frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right) dx\, ds.\end{aligned} \tag{3.24}$$

Step 3. Alternatively, we have:

$$\begin{aligned}& \Psi^{\frac{1}{1-\gamma}}(t) \leq 2^{\frac{1}{1-\gamma}}\left[H(t) + \varepsilon^{\frac{1}{1-\gamma}}\left(\int_0^t \int_\Omega (\varphi\varphi_t + \psi_t)\, dx\, ds\right)^{\frac{1}{1-\gamma}}\right], \\ & \text{and} \\ & \left(\int_0^t \int_\Omega (\varphi\varphi_t + \psi\psi_t)\, dx\, ds\right)^{\frac{1}{1-\gamma}} \leq \left(\int_0^t \int_\Omega \varphi\varphi_t\, dx\, ds\right)^{\frac{1}{1-\gamma}} + \left(\int_0^t \int_\Omega \psi\psi_t\, dx\, ds\right)^{\frac{1}{1-\gamma}}.\end{aligned} \tag{3.25}$$

Utilizing the Cauchy-Schwarz and Hölder's inequalities, one observes:

$$\begin{aligned}\left(\int_0^t \int_\Omega \varphi\varphi_t\, dx\, ds\right)^{\frac{1}{1-\gamma}} & \leq \left(\int_0^t \left(\int_\Omega |\varphi|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\varphi_t|^2\, dx\right)^{\frac{1}{2}} ds\right)^{\frac{1}{1-\gamma}} \\ & \leq c_3\left(\int_0^t \left(\int_\Omega |\varphi|^{p+1}\, dx\right)^{\frac{1}{p+1}} \left(\int_\Omega |\varphi_t|^2\, dx\right)^{\frac{1}{2}} ds\right)^{\frac{1}{1-\gamma}} \\ & \leq c_3\left(\int_0^t \left(\int_\Omega |\varphi|^{p+1}\, dx\right)^{\frac{2}{p+1}} ds\right)^{\frac{1}{2(1-\gamma)}} \left(\int_0^t \int_\Omega |\varphi_t|^2\, dx\, ds\right)^{\frac{1}{2(1-\gamma)}},\end{aligned}$$

for some $c_3 > 0$. By applying Young's inequality and Hölder's inequality again, we obtain

$$\begin{aligned}\left(\int_0^t\int_\Omega \varphi\varphi_t\,dx\,ds\right)^{\frac{1}{1-\gamma}} &\le c_4\left[\int_0^t\int_\Omega |\varphi_t|^2\,dx\,ds+\left(\int_0^t\left(\int_\Omega |\varphi|^{p+1}\,dx\right)^{\frac{2}{p+1}}ds\right)^{\frac{1}{1-2\gamma}}\right]\\ &\le c_4\left[\int_0^t\int_\Omega |\varphi_t|^2\,dx\,ds+\left(\int_0^t ds\right)^{\mu}\int_0^t\int_\Omega |\varphi|^{p+1}\,dx\,ds\right],\end{aligned} \tag{3.26}$$

for some $c_4 > 0$ and $\mu = \frac{p-1}{(p+1)(1-2\gamma)}$. Similarly,

$$\left(\int_0^t\int_\Omega \psi\psi_t\,dx\,ds\right)^{\frac{1}{1-\gamma}} \le c_5\left[\int_0^t\int_\Omega |\psi_t|^2\,dx\,ds+T^{\mu}\int_0^t\int_\Omega |\psi|^{p+1}\,dx\,ds\right]. \tag{3.27}$$

From (3.26) and (3.27), we have

$$\begin{aligned}&\left(\int_0^t\int_\Omega (\varphi\varphi_t+\psi\psi_t)\,dx\,ds\right)^{\frac{1}{1-\gamma}}\\ \le\ &\lambda\left[\int_0^t\int_\Omega \left(\varphi_t^2+\psi_t^2\right)dx\,ds+T^{\mu}\int_0^t\int_\Omega \left(|\varphi|^{p+1}+|\psi|^{p+1}\right)dx\,ds\right]\\ \le\ &\lambda\left[\int_0^t\int_\Omega \left(\varphi_t^2+\psi_t^2\right)dx\,ds+T^{\mu}\int_0^t\int_\Omega \left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\right].\end{aligned} \tag{3.28}$$

Substituting (3.28) in (3.25), we get:

$$\begin{aligned}\Psi^{\frac{1}{1-\gamma}}(t) &\le 2^{\frac{1}{1-\gamma}}H(t)+2^{\frac{1}{1-\gamma}}\varepsilon^{\frac{1}{1-\gamma}}\lambda\Big[\int_0^t\int_\Omega \left(\varphi_t^2+\psi_t^2\right)dx\,ds\\ &+T^{\mu}\int_0^t\int_\Omega \left(\frac{1}{p+1}|\varphi+\psi|^{p+1}+\frac{2}{p+1}|\varphi|^{\frac{p+1}{2}}|\psi|^{\frac{p+1}{2}}\right)dx\,ds\Big].\end{aligned} \tag{3.29}$$

Obviously, from (3.24) and (3.29), the inequality

$$\Psi^{\frac{1}{1-\gamma}}(t) \le R\Psi'(t), \tag{3.30}$$

holds for sufficiently large R. Integrating (3.30) implies

$$\Psi^{\frac{1}{1-\gamma}}(t) \ge \frac{1}{\Psi^{-\frac{1}{1-\gamma}}(0)-\frac{\gamma t}{R(1-\gamma)}}, \tag{3.31}$$

provided that the expression $\Psi^{-\frac{1}{1-\gamma}}(0) - \dfrac{\gamma t}{R(1-\gamma)}$ is positive. Therefore, $\psi(t)$ blows up at a certain moment

$$T^{\star} \le \frac{R(1-\gamma)}{\gamma\Psi^{\frac{1}{1-\gamma}}(0)} < T.$$

Noting that T^* depends on the fractional orders α and β, we observe that R, which is essential in determining T^*, is related to $\tilde{b}$ as expressed in Eq (3.24). Moreover, $\tilde{b}$ depends on B, where B is explicitly a function of α and β, as defined after Eq (3.22). This establishes the connection between the blow-up time and the fractional parameters, further reinforcing the influence of these orders on the solution's behavior. □

4. Conclusions

In this study, we have investigated the blow-up behavior of solutions to a system of nonlinear wave equations with fractional damping and source terms. Our analysis demonstrates that the introduction of fractional derivatives significantly impacts the stability and blow-up characteristics of the system. We provided a detailed examination of how fractional orders influence the blow-up time T^*, revealing that T^* is intricately related to the fractional orders α and β. Specifically, we established that T^* depends on the parameters involved in the system, including those introduced in the energy functional.

Our results offer new insights into the dynamics of systems with fractional damping, which are often encountered in real-world applications involving complex dissipation and memory effects. The approach of combining classical analysis with fractional calculus not only enriches the theoretical understanding but also opens new avenues for practical applications in material science and engineering.

Future research could explore further implications of fractional derivatives in other nonlinear systems and extend the analysis to higher dimensions or different types of sources and damping mechanisms. Overall, this work contributes to a deeper understanding of fractional wave equations and their behavior, providing a foundation for subsequent studies in this evolving field.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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