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Theory article

Super-bimodules and $\mathcal{O}$-operators of Bihom-Jordan superalgebras

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Abstract: In this paper, we mainly study super-bimodules on Bihom-Jordan superalgebras and present some interesting constructions from the perspective of super-bimodules. Meanwhile, we give abelian extension through super-bimodules. In addition, we give the representations, $\mathcal{O}$-operators and Rota–Baxter operators of Bihom-Jordan superalgebras. Specially, using $\mathcal{O}$-operators, we characterize Bihom-pre-Jordan superalgebras.

Keywords: Bihom-Jordan superalgebra; super-bimodule; representation; $\mathcal{O}$-operator; Bihom-pre-Jordan superalgebra

1. Introduction

In the early 1930s, in order to generalize the formula of quantum mechanics, Jordan et al. introduced an important commutative non-associative algebra [1], which was initially called "r-order digital system". In 1947, Albert renamed this kind of algebra Jordan algebra and studied their structural theory [2]. Since then, Jordan algebras have attracted extensive attention. Particularly, Jacobson developed the representation theory of Jordan algebras [3, 4]. Jordan superalgebras were first studied by Kac, who classified simple finite dimensional Jordan superalgebras over an algebraically closed field of characteristic zero [5]. Jordan superalgebras also have significant applications in quantum mechanics [6, 7]. More results on Jordan superalgebras are available in [8, 9].

Hom-type algebras were first introduced to study the q-deformation of Witt and Virasoro algebras [10, 11], which played an important role in physics, mainly in conformal field theory. Bihom-type algebras are generalizations of Hom-type algebras, which were presented by Graziani et al. from the categorical point and applied to study certain deformations of quantum groups [12]. Up to now, the (Bi)hom-structures of various algebras have been intensively investigated. The construction relationship between Hom-type algebras and the module structure on them can be found in the literature [13–17]. Naturally, the construction between Bihom-type algebras is studied in the literature [18, 19], and the results of representation and deformation can be found in [20–22]. In this paper, we first generalize bimodules and representations of Bihom-Jordan algebras [23, 24] to Bihom-Jordan superalgebras and then develop the theory of representations and $\mathcal{O}$-operators on Bihom-Jordan superalgebras.

The outline of the paper is presented as follows: In Section 2, we review some basics about Bihom-superalgebras, Bihom-Jordan superalgebras; we study Bihom-super modules and give some easy constructions of Bihom-Jordan superalgebras. In Section 3, we mainly study super-bimodules on Bihom-Jordan superalgebras and obtain some new constructions under the view of module. In Section 4, we study the representation of Bihom-Jordan superalgebra and give the definitions of $\mathcal{O}$-operator and Rota–Baxter operator. At the same time, we also give the definition of Bihom-pre-Jordan superalgebra. Finally, the relationship between $\mathcal{O}$-operator and Bihom-pre-Jordan superalgebra is studied. Actually, on the basis of this section, we can also continue to study cohomology theory.

Throughout the paper, all algebraic systems are supposed to be over a field of characteristic 0. Let A be a linear superspace over K that is a $\mathbb{Z}_2$-graded linear space with a direct sum $A = A_{\bar{0}} \oplus A_{\bar{1}}$. The elements of A_j, $j = 0, 1$, are said to be homogenous and of parity j. The parity of a homogeneous element x is denoted by $|x|$. In the sequel, we will denote by $\mathcal{H}(A)$ the set of all homogeneous elements of A. In this paper, we need to use the elements, all of which are not specified, are homogeneous.

2. Preliminaries

In this section, we recall some basic definitions about Bihom-Jordan superalgebras, provide some construction results. A Bihom-superalgebra is a quadruple (J, μ, α, β), where $\mu : J \otimes J \to J$ is an even bilinear map and $\alpha, \beta : J \to J$ are even linear maps such that $\alpha \circ \mu = \mu \circ \alpha$ and $\beta \circ \mu = \mu \circ \beta$ (multiplicativity).

Definition 2.1. *[25] Let (J, μ, α, β) be a Bihom-superalgebra.*
The Bihom-associator of J is an even trilinear map $as_{\alpha,\beta} : J^{\otimes 3} \longrightarrow J$ defined by

$$as_{\alpha,\beta} = \mu \circ (\mu \otimes \beta - \alpha \otimes \mu). \tag{2.1}$$

For any $\varepsilon, \gamma, \delta \in \mathcal{H}(J)$, $as_{\alpha,\beta}(\varepsilon, \gamma, \delta) = \mu(\mu(\varepsilon, \gamma), \beta(\delta)) - \mu(\alpha(\varepsilon), \mu(\gamma, \delta))$.

In particular, when $\alpha = \beta = \mathrm{Id}$, Bihom-superalgebra is to degenerate to the superalgebra, so is Bihom-associator degenerates to the original associator. If $\alpha = \beta$, Bihom-associator degenerates to the Hom-associator.

Definition 2.2. *Let (J, μ, α, β) be a Bihom-superalgebra. Then*

- *A Bihom-sub-superalgebra of J is a $\mathbb{Z}_2$-graded linear subspace $B \subseteq J$, which satisfies $\mu(\varepsilon, \gamma) \in B$, $\alpha(\varepsilon) \in B$ and $\beta(\varepsilon) \in B$, for all $\varepsilon, \gamma \in \mathcal{H}(J)$. Furthermore, if $\mu(\varepsilon, \gamma), \mu(\gamma, \varepsilon) \in B$, for all $(\varepsilon, \gamma) \in J \times B$, then B is called a two-sided Bihom-ideal of J.*
- *J is regular if α and β are algebra automorphisms.*
- *J is involutive if α and β are two involutions, that is $\alpha^2 = \beta^2 = \mathrm{Id}$.*

Definition 2.3. *Let (J, μ, α, β) and $(J', \mu', \alpha'\beta')$ be two Bihom-superalgebras. If a homomorphism $f : J \to J'$ satisfies the following conditions:*

$$f \circ \mu = \mu' \circ (f \otimes f), \; f \circ \alpha = \alpha' \circ f \text{ and } f \circ \beta = \beta' \circ f.$$

Then f is called Bihom-superalgebra morphism. And we call the set $\Gamma_f = \{\varepsilon + f(\varepsilon) | \varepsilon \in \mathcal{H}(J)\} \subset J \oplus J'$ *the graph of* f.

Proposition 2.1. *Let* $(J, \mu_J, \alpha_J, \beta_J)$ *and* $(B, \mu_B, \alpha_B, \beta_B)$ *be two Bihom-Jordan superalgebras. Then an even linear map* $f : J \to B$ *is a morphism if and only if its graph* Γ_f *is a Bihom-subalgebra of* $(J \oplus B, \mu = \mu_J + \mu_B, \alpha = \alpha_J + \alpha_B, \beta = \beta_J + \beta_B)$.

Proof. Suppose that f is a morphism of Bihom-Jordan superalgebras. Clearly, Γ_f is a subspace of $J \oplus B$, we only need to prove the Γ_f is closed under the μ, α, β. For all $\varepsilon, \gamma \in H(J)$,

$$\mu(\varepsilon + f(\varepsilon), \gamma + f(\gamma)) = \mu_J(\varepsilon, \gamma) + \mu_B(f(\varepsilon), f(\gamma)) = \mu_J(\varepsilon, \gamma) + f\mu_J(\varepsilon, \gamma).$$

Moreover, by $f\alpha_J = \alpha_B f$ and $f\beta_J = \beta_B f$,

$$\alpha(\varepsilon + f(\varepsilon)) = \alpha_J(\varepsilon) + \alpha_B(f(\varepsilon)) = \alpha_J(\varepsilon) + f\alpha_J(\varepsilon),$$
$$\beta(\varepsilon + f(\varepsilon)) = \beta_J(\varepsilon) + \beta_B(f(\varepsilon)) = \beta_J(\varepsilon) + f\beta_J(\varepsilon).$$

It follows that Γ_f is a Bihom-subalgebra of $J \oplus B$.

Conversely, Γ_f is a Bihom-subalgebra of $J \oplus B$, so

$$\mu(\varepsilon + f(\varepsilon), \gamma + f(\gamma)) = \mu_J(\varepsilon, \gamma) + \mu_B(f(\varepsilon), f(\gamma)) \in \Gamma_f,$$

which implies that $\mu_B(f(\varepsilon), f(\gamma)) = f\mu_J(\varepsilon, \gamma)$. Similarly, we also obtain $\alpha_B f = f\alpha_J$ and $\beta_B f = f\beta_J$ from $\alpha(\Gamma_f) \subseteq \Gamma_f$ and $\beta(\Gamma_f) \subseteq \Gamma_f$, respectively. Thus, f is a morphism of Bihom-Jordan superalgebras.

Definition 2.4. *[25] A Bihom-associative superalgebra is a quadruple* (J, μ, α, β)*, where* $\alpha, \beta : J \to J$ *are even linear maps and* $\mu : J \times J \to J$ *is an even bilinear map such that* $\alpha\beta = \beta\alpha, \alpha\mu = \mu\alpha^{\otimes 2}$, $\beta\mu = \mu\beta^{\otimes 2}$ *and satisfying Bihom-associator is zero:*

$$as_{\alpha,\beta}(\varepsilon, \gamma, \delta) = 0, \text{ for all } \varepsilon, \gamma, \delta \in \mathcal{H}(J). \text{ (Bihom-associativity condition)}$$

Clearly, when $\alpha = \beta$, we obtain a Hom-associative superalgebra.

Definition 2.5. *[19] A BiHom superalgebra* (J, μ, α, β) *is called a Bihom-Jordan superalgebra if for all* $\varepsilon, \gamma, \delta, t \in \mathcal{H}(J)$:

(i) $\alpha\beta = \beta\alpha$,
(ii) $\mu(\beta(\varepsilon), \alpha(\gamma)) = (-1)^{|\varepsilon||\gamma|}\mu(\beta(\gamma), \alpha(\varepsilon))$, (Bihom-super commutativity condition)
(iii) $\circlearrowleft_{\varepsilon,\gamma,t} (-1)^{|t|(|\varepsilon|+|\delta|)}\widetilde{as}_{\alpha,\beta}(\mu(\beta^2(\varepsilon), \alpha\beta(\gamma)), \alpha^2\beta(\delta), \alpha^3(t)) = 0.$
(Bihom-Jordan super-identity)

In particular, it is reduced to a Jordan superalgebra when $\alpha = \beta = \text{Id}$.

Next, we give some common construction methods. Let (J, μ, α, β) be a Bihom-superalgebra. Define its plus Bihom-superalgebra as the Bihom-superalgebra $J^+ = (J, *, \alpha, \beta)$, where

$$\varepsilon * \gamma = \frac{1}{2}(\mu(\varepsilon, \gamma) + (-1)^{|\varepsilon||\gamma|}\mu(\alpha^{-1}\beta(\gamma), \alpha\beta^{-1}(\varepsilon))).$$

Note that product $*$ is Bihom-supercommutative. In fact, for all $\varepsilon, \gamma \in \mathcal{H}(J)$,

$$\beta(\varepsilon) * \alpha(\gamma) = \frac{1}{2}(\beta(\varepsilon)\alpha(\gamma) + (-1)^{|\varepsilon||\gamma|}\beta(\gamma)\alpha(\varepsilon))$$
$$= (-1)^{|\varepsilon||\gamma|}\frac{1}{2}(\beta(\gamma)\alpha(\varepsilon) + \beta(\varepsilon)\alpha(\gamma)) = (-1)^{|\varepsilon||\gamma|}\beta(\gamma) * \alpha(\varepsilon).$$

Moreover, the plus Bihom-superalgebra $J^+ = (J, *, \alpha, \beta)$ is a Bihom-Jordan superalgebra. Naturally, we define

$$\varepsilon \diamond \gamma = \mu(\varepsilon, \gamma) + (-1)^{|\varepsilon||\gamma|}\mu(\alpha^{-1}\beta(\gamma), \alpha\beta^{-1}(\varepsilon)),$$

the $\diamond$ is also Bihom-supercommutative. Then $J^\diamond = (J, \diamond, \alpha, \beta)$ is also a Bihom-Jordan superlagebra.

Besides that, Bihom-Jordan superalgebra $(J, \mu_{\alpha,\beta} = \mu(\alpha \otimes \beta), \alpha, \beta)$ can be obtained from Jordan superalgebra (J, μ). We also consider the quotient algebra obtained by modulo Bihom-ideal, given a Bihom-Jordan superalgebra (J, μ, α, β) and I is a Bihom-ideal. Define $\bar{\mu}, \bar{\alpha}, \bar{\beta}$ on J/I as follows:

$$\bar{\mu}(\bar{\varepsilon}, \bar{\gamma}) = \overline{\mu(\varepsilon, \gamma)}, \bar{\alpha}(\bar{\varepsilon}) = \overline{\alpha(\varepsilon)}, \bar{\beta}(\bar{\varepsilon}) = \overline{\beta(\varepsilon)}.$$

Then $(J/I, \bar{\mu}, \bar{\alpha}, \bar{\beta})$ is also a Bihom-Jordan superalgebra.

Example 2.1. *Given a 3-dimensional Jordan superalgebra $(J = J_{\bar{0}} \oplus J_{\bar{1}}, \mu)$ in [5], the bases of $J_{\bar{0}}$ and $J_{\bar{1}}$ are $\{\varepsilon\}$ and $\{u, v\}$, respectively. The nontrivial multiplication is defined as follows:*

$$\mu(\varepsilon, \varepsilon) = \varepsilon,\ \mu(\varepsilon, u) = \frac{1}{2}u,\ \mu(\varepsilon, v) = \frac{1}{2}v,\ \mu(u, v) = \varepsilon.$$

We consider two even endomorphisms α and β, which satisfy $\alpha(\varepsilon) = \varepsilon,\ \alpha(u) = -u,\ \alpha(v) = -v$, and $\beta(\varepsilon) = -\varepsilon,\ \beta(u) = -u,\ \beta(v) = v$. Then we obtain a Bihom-Jordan superalgebra $(J, \mu' = \mu(\alpha \otimes \beta), \alpha, \beta)$.

Example 2.2. *In [5], let $(J = J_{\bar{0}} \oplus J_{\bar{1}}, \mu)$ be a Jordan superalgebra with the nontrivial multiplication as follows:*

$$\mu(\varepsilon, \varepsilon) = 2\varepsilon,\ \mu(\varepsilon, u) = u,\ \mu(\varepsilon, v) = v,\ \mu(u, v) = 1 + kx,$$

$k \in K$ and $k \neq \frac{1}{2}$, where $\{1, \varepsilon\}$ and $\{u, v\}$ are bases of $J_{\bar{0}}$ and $J_{\bar{1}}$, respectively. We define two even endomorphisms α and β satisfies $\alpha(1) = 1,\ \alpha(\varepsilon) = \varepsilon,\ \alpha(u) = -u,\ \alpha(v) = -v$ and $\beta(1) = 1,\ \beta(\varepsilon) = -\varepsilon,\ \beta(u) = u,\ \beta(v) = v$. Then we obtain a Bihom-Jordan superalgebra $(J, \mu' = \mu(\alpha \otimes \beta), \alpha, \beta)$.

Definition 2.6. *Let (J, μ, α, β) be a Bihom-superalgebra.*

1) A Bihom-super-module (V, ϕ, ψ) is called an J-super-bimodule if it is equipped with an even left structure ρ_l and an even right structure map ρ_r on $\mathbb{Z}_2$-graded vector space V, ρ_l and ρ_r are given by

- $\rho_l : (J \otimes V, \alpha \otimes \phi, \beta \otimes \psi) \to (V, \phi, \psi), \rho_l(a, v) = a \cdot v,$
- $\rho_r : (V \otimes J, \phi \otimes \alpha, \psi \otimes \beta) \to (V, \phi, \psi), \rho_r(v, a) = v \cdot a.$

2) An even linear map $f : (V, \phi, \psi, \rho_l, \rho_r) \to (V', \phi', \psi', \rho'_l, \rho'_r)$ is a morphism of the Bihom-super-modules such that the following commutative diagrams

$$\begin{array}{ccc} J \otimes V_{\mathrm{Id}_J \otimes f} & \xrightarrow{\rho_l} & V \\ \downarrow & & \downarrow f \\ J \otimes V' & \xrightarrow{\rho'_l} & V' \end{array} \quad \text{and} \quad \begin{array}{ccc} V \otimes J_{f \otimes \mathrm{Id}_J} & \xrightarrow{\rho_r} & V \\ \downarrow & & \downarrow f \\ V' \otimes J & \xrightarrow{\rho'_r} & V' \end{array} \quad \text{hold.}$$

3) Let $(V, \phi, \psi, \rho_l, \rho_r)$ be an J-super-bimodule. Then the module Bihom-associator $as_{V_{\phi,\psi}}$ of V is defined as:

$$as_{V_{\phi,\psi}} \circ \mathrm{Id}_{V\otimes J\otimes J} = \rho_r \circ (\rho_r \otimes \beta) - \rho_r \circ (\phi \otimes \mu), \tag{2.2}$$

$$as_{V_{\phi,\psi}} \circ \mathrm{Id}_{J\otimes V\otimes J} = \rho_r \circ (\rho_l \otimes \beta) - \rho_l \circ (\alpha \otimes \rho_r), \tag{2.3}$$

$$as_{V_{\phi,\psi}} \circ \mathrm{Id}_{J\otimes J\otimes V} = \rho_l \circ (\mu \otimes \psi) - \rho_l \circ (\alpha \otimes \rho_l). \tag{2.4}$$

Definition 2.7. *Let (J, μ, α, β) be a Bihom-associative superalgebra and (V, ϕ, ψ) be a Bihom-super-module. Then*

1) *A left Bihom-associative J-super-module structure consists of an even morphism $\rho_l : J \otimes V \to V$ satisfies $as_{V_{\phi,\psi}} = 0$ in (2.4).*

2) *A right Bihom-associative J-super-module structure consists of an even morphism $\rho_r : V \otimes J \to V$ satisfies $as_{V_{\phi,\psi}} = 0$ in (2.2).*

3) *A Bihom-associative J-super-bimodule structure consists of an even morphism $\rho_l : J \otimes V \to V$ and an even morphism $\rho_r : V \otimes J \to V$ such that (V, ϕ, ψ, ρ_l) is a left Bihom-associative J-super-module, (V, ϕ, ψ, ρ_r) is a right Bihom-associative J-super-module, and satisfies $as_{V_{\phi,\psi}} = 0$ in (2.3).*

3. Super-bimodules

In this section, we introduce super-bimodules of Bihom-Jordan superalgebras and give some of their constructions. Finally, we define an abelian extension in order to give an application in the next section. For convenience, the sign will subsequently be omitted from the product operation of elements in J.

Definition 3.1. *Let (J, μ, α, β) be a Bihom-Jordan superalgebra. For all $\varepsilon, \gamma, \delta \in \mathcal{H}(J)$, $v \in \mathcal{H}(V)$,*

- *A left Bihom-Jordan J-super-module is a Bihom-super-module (V, ϕ, ψ) that is equipped with an even left structure map $\rho_l : J \otimes V \to V$, $\rho_l(a \otimes v) = a \cdot v$ such that ψ is invertible and the following conditions hold*:

$$\begin{aligned} &\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\gamma||\delta|}\beta^2\alpha^2(\delta) \cdot (\alpha\beta(\varepsilon)\alpha^2(\gamma) \cdot \phi^3(v)) \\ = &\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\varepsilon||\delta|}\alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma) \cdot (\beta\alpha^2(\delta) \cdot \phi^3(v)), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\beta^2\alpha^2(\delta) \cdot (\beta\alpha^2(\gamma) \cdot (\alpha^2(\varepsilon) \cdot \psi^{-1}\phi^3(v))) \\ &+ (-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|+|\gamma||\delta|}\beta^2\alpha^2(\varepsilon) \cdot (\beta\alpha^2(\gamma) \cdot (\alpha^2(\delta) \cdot \psi^{-1}\phi^3(v))) \\ &+ (-1)^{|\varepsilon||\delta|+|\varepsilon||\gamma|}((\beta^2(\varepsilon)\beta\alpha(\delta))\beta\alpha^2(\gamma)) \cdot \phi^3\psi(v) \\ =&(-1)^{|\gamma||\delta|}\beta^2\alpha(\gamma)\beta\alpha^2(\delta) \cdot (\beta\alpha^2(\varepsilon) \cdot \phi^3(v)) \\ &+ (-1)^{|\gamma||\delta|+|\varepsilon||\delta|}\beta^2\alpha(\gamma)\beta\alpha^2(\varepsilon) \cdot (\beta\alpha^2(\delta) \cdot \phi^3(v)) \\ &+ (-1)^{|\varepsilon||\delta|+|\varepsilon||\gamma|}\beta^2\alpha(\varepsilon)\beta\alpha^2(\delta) \cdot (\beta\alpha^2(\gamma) \cdot \phi^3(v)). \end{aligned} \tag{3.2}$$

- *A right Bihom-Jordan J-super-module is a Bihom-super-module (V, ϕ, ψ) that is equipped with an even right structure map $\rho_r : V \otimes J \to V$, $\rho_r(v \otimes a) = v \cdot a$ such that the following conditions hold*:

$$\begin{aligned} &\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\varepsilon||\delta|}(\phi\psi^2(v) \cdot \alpha\beta(\varepsilon)\alpha^2(\gamma)) \cdot \beta\alpha^3(\delta) \\ =& \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\gamma||\delta|}(\phi\psi^2(v) \cdot \beta\alpha^2(\delta)) \cdot \alpha^2\beta(\varepsilon)\alpha^3(\gamma). \end{aligned} \tag{3.3}$$

$$\begin{aligned} &((\psi^2(v) \cdot \beta\alpha(\varepsilon)) \cdot \beta\alpha^2(\gamma)) \cdot \beta\alpha^3(\delta) \\ &+ (-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|+|\gamma||\delta|}((\psi^2(v) \cdot \beta\alpha(\delta)) \cdot \beta\alpha^2(\gamma)) \cdot \beta\alpha^3(\varepsilon) \\ &+ (-1)^{|\gamma||\delta|}\phi^2\psi^2(v) \cdot (\beta\alpha(\varepsilon)\alpha^2(\delta))\alpha^3(\gamma) \\ =&(\phi\psi^2(v) \cdot \beta\alpha^2(\varepsilon)) \cdot \alpha^2\beta(\gamma)\alpha^3(\delta) \\ &+ (-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}(\phi\psi^2(v) \cdot \beta\alpha^2(\delta)) \cdot \alpha^2\beta(\gamma)\alpha^3(\varepsilon) \\ &+ (-1)^{|\varepsilon||\gamma|}(\phi\psi^2(v) \cdot \beta\alpha^2(\gamma)) \cdot \alpha^2\beta(\varepsilon)\alpha^3(\delta). \end{aligned} \tag{3.4}$$

Theorem 3.1. *Let (J, μ, α, β) be a Bihom-Jordan superalgebra, (V, ϕ, ψ) be a Bihom-super-module and $\rho_r : V \otimes J \to V, \rho_r(v \otimes a) = v \cdot a$ be an even linear map, which satisfies the following conditions: for all $\varepsilon, \gamma \in \mathcal{H}(J), v \in \mathcal{H}(V)$,*

$$\phi\rho_r = \rho_r(\phi \otimes \alpha), \ \psi\rho_r = \rho_r(\psi \otimes \beta), \tag{3.5}$$

$$\phi(v) \cdot \beta(\varepsilon)\alpha(\gamma) = (v \cdot \beta(\varepsilon)) \cdot \beta\alpha(\gamma) + (-1)^{|\varepsilon||\gamma|}(v \cdot \beta(\gamma)) \cdot \alpha\beta(\varepsilon). \tag{3.6}$$

Then (V, ϕ, ψ, ρ_r) is a right Bihom-Jordan J-super-module, called a right special Bihom-Jordan J-super-module.

Proof. For any $\varepsilon, \gamma, \delta \in \mathcal{H}(J)$, $v \in \mathcal{H}(V)$,

$$\begin{aligned} &\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|}(\phi\psi^2(v) \cdot \beta\alpha^2(\delta)) \cdot \alpha^2\beta(\varepsilon)\alpha^3(\gamma) \\ =& \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|}\phi(\psi^2(v) \cdot \beta\alpha(\delta)) \cdot \alpha^2\beta(\varepsilon)\alpha^3(\gamma) \ \ (by\ (3.5)) \\ =& \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|}((\psi^2(v) \cdot \beta\alpha(\delta)) \cdot \beta\alpha^2(\varepsilon)) \cdot \beta\alpha^3(\gamma) \\ &+ \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|+|\varepsilon||\gamma|}((\psi^2(v) \cdot \beta\alpha(\delta)) \cdot \beta\alpha^2(\gamma)) \cdot \beta\alpha^3(\varepsilon) \ \ (by\ (3.6)) \\ =& \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|}(\phi\psi^2(v) \cdot \beta\alpha(\delta)\alpha^2(\varepsilon)) \cdot \beta\alpha^2(\gamma) \\ &- \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|+|\delta||\varepsilon|}((\psi^2(v) \cdot \beta\alpha(\varepsilon)) \cdot \beta\alpha^2(\delta)) \cdot \beta\alpha^3(\gamma) \\ &+ \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|+|\varepsilon||\gamma|}((\psi^2(v) \cdot \beta\alpha(\delta)) \cdot \beta\alpha^2(\gamma)) \cdot \beta\alpha^3(\varepsilon) \ \ (by\ (3.6)) \\ =& \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\varepsilon||\delta|}(\phi\psi^2(v) \cdot \beta\alpha(\varepsilon)\alpha^2(\gamma)) \cdot \beta\alpha^2(\delta). \end{aligned}$$

So Eq (3.3) holds. On the other hand,

$$\begin{aligned} &(\phi\psi^2(v) \cdot \beta\alpha^2(\varepsilon)) \cdot \beta\alpha^2(\gamma)\alpha^3(\delta) + (-1)^{|\varepsilon||\gamma|}(\phi\psi^2(v) \cdot \beta\alpha^2(\gamma)) \cdot \beta\alpha^2(\varepsilon)\alpha^3(\delta) \\ &+ (-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}(\phi\psi^2(v) \cdot \beta\alpha^2(\delta)) \cdot \beta\alpha^2(\gamma)\alpha^3(\varepsilon) \\ =&\phi(\psi^2(v) \cdot \beta\alpha(\varepsilon)) \cdot \beta\alpha^2(\gamma)\alpha^3(\delta) + (-1)^{|\varepsilon||\gamma|}\phi(\psi^2(v) \cdot \beta\alpha(\gamma)) \cdot \beta\alpha^2(\varepsilon)\alpha^3(\delta) \\ &+ (-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}\phi(\psi^2(v) \cdot \beta\alpha(\delta)) \cdot \beta\alpha^2(\gamma)\alpha^3(\varepsilon) \ \ (by\ (3.5)) \\ =&((\psi^2(v) \cdot \beta\alpha(\varepsilon)) \cdot \beta\alpha^2(\gamma)) \cdot \beta\alpha^3(\delta) + (-1)^{|\gamma||\delta|}((\psi^2(v) \cdot \beta\alpha(\varepsilon)) \cdot \beta\alpha^2(\delta)) \cdot \beta\alpha^3(\gamma) \\ &+ (-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}((\psi^2(v) \cdot \beta\alpha(\delta)) \cdot \beta\alpha^2(\gamma)) \cdot \beta\alpha^3(\varepsilon) \end{aligned}$$

$$
\begin{aligned}
&+(-1)^{|\delta||\varepsilon|+|\delta||\gamma|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^3(\gamma)\\
&+(-1)^{|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\delta))\cdot\beta\alpha^3(\varepsilon)\ \ (by\ (3.6))\\
=&((\psi^2(v)\cdot\beta\alpha(\varepsilon))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\gamma||\delta|}((\phi\psi^2(v)\cdot\beta\alpha(\varepsilon)\alpha^2(\delta))\cdot\alpha^3\beta(\gamma)\\
&-(-1)^{|\gamma||\delta|+|\varepsilon||\delta|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^3(\gamma)\\
&+(-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\varepsilon)\\
&+(-1)^{|\delta||\varepsilon|+|\delta||\gamma|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^3(\gamma)\\
&+(-1)^{|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\delta))\cdot\beta\alpha^3(\varepsilon)\ \ (by\ (3.6))\\
=&((\psi^2(v)\cdot\beta\alpha(\varepsilon))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\gamma||\delta|}((\phi\psi^2(v)\cdot\beta\alpha(\varepsilon)\alpha^2(\delta))\cdot\alpha^3\beta(\gamma)\\
&+(-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\varepsilon)\\
&+(-1)^{|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\delta))\cdot\beta\alpha^3(\varepsilon)\\
=&((\psi^2(v)\cdot\beta\alpha(\varepsilon))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\gamma||\delta|}\phi^2\psi^2(v)\cdot(\beta\alpha(\varepsilon)\alpha^2(\delta))\alpha^3(\delta)\\
&-(-1)^{|\varepsilon||\gamma|}(\phi\psi^2(v)\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^2(\varepsilon)\alpha^3(\delta)\\
&+(-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\varepsilon)\\
&+(-1)^{|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}((\psi^2(v)\cdot\beta\alpha(\gamma))\cdot\beta\alpha^2(\delta))\cdot\beta\alpha^3(\varepsilon)\ \ (by\ (3.6))\\
=&((\psi^2(v)\cdot\beta\alpha(\varepsilon))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\delta)\\
&+(-1)^{|\gamma||\delta|}\phi^2\psi^2(v)\cdot(\beta\alpha(\varepsilon)\alpha^2(\delta))\alpha^3(\delta)\\
&+(-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\varepsilon||\gamma|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\varepsilon)\ \ (by\ (3.6)).
\end{aligned}
$$

It follows Eq (3.4).

Similarly, we have the following result.

Theorem 3.2. *Let (J,μ,α,β) be a Bihom-Jordan superalgebra, (V,ϕ,ψ) be a Bihom-super-module such that ψ is invertible, and $\rho_l : J\otimes V\to V$ be an even linear map given by $\rho_l(a\otimes v)=a\cdot v$ such that the following conditions hold:*

$$\phi\rho_l=\rho_l(\alpha\otimes\phi),\ \ \psi\rho_l=\rho_l(\beta\otimes\psi), \tag{3.7}$$

$$\beta(\varepsilon)\alpha(\gamma)\cdot\psi(v)=\beta\alpha(\varepsilon)\cdot(\alpha(\gamma)\cdot v)+(-1)^{|\varepsilon||\gamma|}\beta\alpha(\gamma)\cdot(\alpha(\varepsilon)\cdot v). \tag{3.8}$$

Then $(V,\phi,,\psi,\rho_l)$ is a left Bihom-Jordan J-super-module called a left special super-module.

Proof. Similar to the proof of Theorem 3.1, the conclusion can be proved by repeatedly using Eqs (3.7) and (3.8).

Now, we give the definition super-bimodule of a BiHom-Jordan superalgebra.

Definition 3.2. *Let (J, μ, α, β) be a Bihom-Jordan superalgebra. A Bihom-Jordan J-super-bimodule is a Bihom-super-module (V, ϕ, ψ) with an even left structure map $\rho_l : J \otimes V \to V$, $\rho_l(a \otimes v) = a \cdot v$ and an even right structure map $\rho_r : V \otimes J \to V$, $\rho_r(v \otimes a) = v \cdot a$ satisfying three conditions:*

$$\rho_l(\beta \otimes \phi) = \rho_r(\psi \otimes \alpha)\tau_1, \tag{3.9}$$

$$\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta|(|\varepsilon|+|v|)} as_{V_{\phi,\psi}}(\mu(\beta^2(\varepsilon), \alpha\beta(\gamma)), \phi^2\psi(v), \alpha^3(\delta)) = 0, \tag{3.10}$$

$$\begin{aligned} &(-1)^{|\gamma||\delta|} as_{V_{\phi,\psi}}(\psi^2(v) \cdot \alpha\beta(\varepsilon), \beta\alpha^2(\gamma), \alpha^3(\delta)) \\ &+ (-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|} as_{V_{\phi,\psi}}(\psi^2(v) \cdot \alpha\beta(\delta), \beta\alpha^2(\gamma), \alpha^3(\varepsilon)) \\ &+ (-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|} as_{V_{\phi,\psi}}(\mu(\beta^2(\varepsilon), \alpha\beta(\delta)), \beta\alpha^2(\gamma), \phi^3(v)) = 0. \end{aligned} \tag{3.11}$$

Remark 3.1. *1) If $\alpha = \beta = \mathrm{Id}_J$ and $\phi = \psi = \mathrm{Id}_V$ then V is reduced to the so-called Jordan supermodule of the Jordan superalgebra (J, μ).*

2) Clearly, a Bihom-Jordan A-super-bimodule is a right Bihom-Jordan super-module. Furthermore, it is a left Bihom-Jordan super-module if ψ is invertible.

Example 3.1. *Here are some examples of Bihom-Jordan super-bimodules.*

1) Let (J, μ, α, β) be a Bihom-Jordan superalgebra. Then (J, α, β) is a Bihom-Jordan J-super-bimodule where the structure maps are $\rho_l = \rho_r = \mu$. More generally, if B is a Bihom-ideal of (J, μ, α, β), then (B, α, β) is a Bihom-Jordan J-super-bimodule where the structure maps are $\rho_l(a, \varepsilon) = \mu(a, \varepsilon) = \mu(\varepsilon, a) = \rho_r(\varepsilon, a)$, for all $(a, \varepsilon) \in \mathcal{H}(J) \times \mathcal{H}(B)$.

2) If (J, μ) is a Jordan superalgebra and M is a Jordan J-super-bimodule in the usual sense, then $(M, \mathrm{Id}_M, \mathrm{Id}_M)$ is a BiHom-Jordan J-super-bimodule where $(J, \mu, \mathrm{Id}_J, \mathrm{Id}_J)$ is a Bihom-Jordan superalgebra.

Theorem 3.3. *Let (J, μ, α, β) be a Bihom-Jordan superalgebra and $(V, \phi, \psi, \rho_l, \rho_r)$ be a Bihom-Jordan J-super-bimodule. Define even linear maps $\tilde{\mu}, \tilde{\alpha}$ and $\tilde{\beta}$ on $J \oplus V$,*

- $\tilde{\mu} : (J \oplus V)^{\otimes 2} \to J \oplus V$, $\tilde{\mu}(\varepsilon + u, \gamma + v) := \mu(\varepsilon, \gamma) + \varepsilon \cdot v + u \cdot \gamma$,
- $\tilde{\alpha}, \tilde{\beta} : (J \oplus V) \to J \oplus V$,
  $\tilde{\alpha}(\varepsilon + u) := \alpha(\varepsilon) + \phi(v)$ *and* $\tilde{\beta}(\varepsilon + u) := \beta(\varepsilon) + \psi(v)$.

Then $(J \oplus V, \tilde{\mu}, \tilde{\alpha}, \tilde{\beta})$ is a Bihom-Jordan superalgebra.

Proof. We omitted the calculation process; it is straightforward to see Bihom-super commutativity condition and Bihom-Jordan super-identity by Definition 3.2.

The next result shows that a special left and right Bihom-Jordan super-module has a Bihom-Jordan super-bimodule structure under a specific condition.

Theorem 3.4. *Let (J, μ, α, β) be a regular Bihom-Jordan superalgebra, (V, ϕ, ψ) be both a left and a right special BiHom-Jordan J-module with the structure maps ρ_1 and ρ_2 respectively, such that ϕ is invertible, and the Bihom-associativity condition holds*

$$\rho_2 \circ (\rho_1 \otimes \beta) = \rho_1 \circ (\alpha \otimes \rho_2). \tag{3.12}$$

Define two even bilinear maps $\rho_l : J \otimes V \to V$ and $\rho_r : V \otimes J \to V$ by

$$\rho_l = \rho_1 + \rho_2(\psi\phi^{-1} \otimes \alpha\beta^{-1}) \circ \tau_1 \text{ and } \rho_r = \rho_1(\beta\alpha^{-1} \otimes \phi\psi^{-1}) \circ \tau_2 + \rho_2. \tag{3.13}$$

Then $(V, \phi, \psi, \rho_l, \rho_r)$ is a Bihom-Jordan J-super-bimodule.

Proof. ρ_l and ρ_r are even structure maps from ρ_1 and ρ_2. We need to check out (3.9)–(3.11). First, for any $(\varepsilon, v) \in \mathcal{H}(J) \times \mathcal{H}(V)$,

$$\begin{aligned}\rho_l(\beta(\varepsilon), \phi(v)) &= \beta(\varepsilon) \cdot \phi(v) + (-1)^{|a||v|}\psi\phi^{-1}(\phi(v)) \cdot \alpha\beta^{-1}(\beta(\varepsilon)) \\ &= \beta(\varepsilon) \cdot \phi(v) + (-1)^{|a||v|}\psi(v) \cdot \alpha(\varepsilon),\end{aligned}$$

$$\begin{aligned}\rho_r(\psi \otimes \alpha)\tau_1(\varepsilon \otimes v) &= (-1)^{|a||v|}\rho_r(\psi(v), \alpha(\varepsilon)) \\ &= (-1)^{|a||v|}\psi(v) \cdot \alpha(\varepsilon) + \beta\alpha^{-1}(\alpha(\varepsilon)) \cdot \phi\psi^{-1}(\phi(v)) \\ &= \beta(\varepsilon) \cdot \phi(v) + (-1)^{|a||v|}\psi(v) \cdot \alpha(\varepsilon).\end{aligned}$$

So $\rho_l(\beta \otimes \phi) = \rho_r(\psi \otimes \alpha)\tau_1$. Next, for any $\varepsilon, \gamma, \delta \in \mathcal{H}(J),\ v \in \mathcal{H}(V)$

$$\begin{aligned}
&as_{V_{\phi,\psi}}(\mu(\beta^2(\varepsilon), \alpha\beta(\gamma)), \phi^2\psi(v), \alpha^3(\delta)) \\
=&\rho_r(\rho_l(\mu(\beta^2(\varepsilon), \alpha\beta(\gamma)), \phi^2\psi(v)), \beta\alpha^3(\delta)) - \rho_l(\alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma), \rho_r(\phi^2\psi(v), \alpha^3(\delta))) \\
=&\rho_r(\beta^2(\varepsilon)\alpha\beta(\gamma) \cdot \phi^2\psi(v), \beta\alpha^3(\delta)) \\
&+ (-1)^{|\varepsilon||v|+|\gamma||v|}\rho_r(\phi\psi^2(v) \cdot \alpha\beta(\varepsilon)\alpha^2(\gamma), \beta\alpha^3(\delta)) \\
&- (-1)^{|v||\delta|}\rho_l(\alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma), \beta\alpha^2(\delta) \cdot \phi^3(v)) \\
&- \rho_l(\alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma), \phi^2\psi(v) \cdot \alpha^3(\delta)) \ \ (by\ (3.13)) \\
=&(\beta^2(\varepsilon)\alpha\beta(\gamma) \cdot \phi^2\psi(v)) \cdot \beta\alpha^3(\delta) \\
&+ (-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\delta||v|}\alpha^2\beta^2(\delta) \cdot (\alpha\beta(\varepsilon)\alpha^2(\gamma) \cdot \phi^3(v)) \\
&+ (-1)^{|\varepsilon||v|+|\gamma||v|}(\phi\psi^2(v) \cdot \alpha\beta(\varepsilon)\alpha^2(\gamma)) \cdot \beta\alpha^3(\delta) \\
&+ (-1)^{|\varepsilon||v|+|\gamma||v|+|\delta||v|+|\delta||\varepsilon|+|\delta||\gamma|}\alpha^2\beta^2(\delta) \cdot (\phi^2\psi(v) \cdot \alpha^2(\varepsilon)\alpha^3\beta^{-1}(\gamma)) \\
&- (-1)^{|v||\delta|}\alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma) \cdot (\beta\alpha^2(\delta) \cdot \phi^3(v)) \\
&- (-1)^{|v||\delta|+|\delta||\varepsilon|+|\delta||\gamma|+|v||\varepsilon|+|v||\gamma|}(\beta^2\alpha(\delta) \cdot \psi\phi^2(v)) \cdot \alpha^2\beta(\varepsilon)\alpha^3(\gamma) \\
&- \alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma) \cdot (\phi^2\psi(v) \cdot \alpha^3(\delta)) \\
&- (-1)^{|v||\varepsilon|+|v||\gamma|+|\delta||\varepsilon|+|\delta||\gamma|}(\phi\psi^2(v) \cdot \alpha^2\beta(\delta)) \cdot \alpha^2\beta(\varepsilon)\alpha^3(\gamma) \ \ (by\ (3.13)) \\
=&(-1)^{|\varepsilon||v|+|\gamma||v|}(\phi\psi^2(v) \cdot \alpha\beta(\varepsilon)\alpha^2(\gamma)) \cdot \beta\alpha^3(\delta) \\
&- (-1)^{|v||\varepsilon|+|v||\gamma|+|\delta||\varepsilon|+|\delta||\gamma|}(\phi\psi^2(v) \cdot \alpha^2\beta(\delta)) \cdot \alpha^2\beta(\varepsilon)\alpha^3(\gamma) \\
&+ (-1)^{|\delta||\varepsilon|+|\delta||\gamma|+|\delta||v|}\alpha^2\beta^2(\delta) \cdot (\alpha\beta(\varepsilon)\alpha^2(\gamma) \cdot \phi^3(v)) \\
&- (-1)^{|v||\delta|}\alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma) \cdot (\beta\alpha^2(\delta) \cdot \phi^3(v)) \ \ (by\ (3.12)).
\end{aligned}$$

So

$$\begin{aligned}
&\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta|(|\varepsilon|+|v|)}as_{V_{\phi,\psi}}(\beta^2(\varepsilon)\alpha\beta(\gamma), \phi^2\psi(v), \alpha^3(\delta)) \\
=&(-1)^{|v|(|\varepsilon|+|\gamma|+|\delta|)}\{\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\varepsilon||\delta|}(\phi\psi^2(v) \cdot \alpha\beta(\varepsilon)\alpha^2(\gamma)) \cdot \beta\alpha^3(\delta)
\end{aligned}$$

$$
\begin{aligned}
&- \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|}(\phi\psi^2(v)\cdot\beta\alpha^2(\delta))\cdot\alpha^2\beta(\varepsilon)\alpha^3(\gamma)\}\\
&+ \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\gamma|}\alpha^2\beta^2(\delta)\cdot(\alpha\beta(\varepsilon)\alpha^2(\gamma)\cdot\phi^3(v))\\
&- \circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta||\varepsilon|}\alpha\beta^2(\varepsilon)\alpha^2\beta(\gamma)\cdot(\beta\alpha^2(\delta)\cdot\phi^3(v))\\
=&(-1)^{|v|(|\varepsilon|+|\gamma|+|\delta|)}0+0=0.
\end{aligned}
$$

Finally, to prove (3.11), let us compute each of its three terms.

$$
\begin{aligned}
&(-1)^{|\gamma||\delta|}as_{V_{\phi,\psi}}(\rho_r(\psi^2(v),\beta\alpha(\varepsilon)),\beta\alpha^2(\gamma),\alpha^3(\delta))\\
=&(-1)^{|\gamma||\delta|}as_{V_{\phi,\psi}}(\psi^2(v)\cdot\beta\alpha(\varepsilon),\beta\alpha^2(\gamma),\alpha^3(\delta))\\
&+(-1)^{|\gamma||\delta|+|\varepsilon||v|}as_{V_{\phi,\psi}}(\beta^2(\varepsilon)\cdot\phi\psi(v),\beta\alpha^2(\gamma),\alpha^3(\delta))\ \ (by\ (3.13))\\
=&(-1)^{|\gamma||\delta|}\rho_r(\rho_r(\psi^2(v)\cdot\beta\alpha(\varepsilon),\beta\alpha^2(\gamma)),\alpha^3\beta(\delta))\\
&-(-1)^{|\gamma||\delta|}\rho_r(\phi\psi^2(v)\cdot\beta\alpha^2(\varepsilon),\beta\alpha^2(\gamma)\alpha^3(\delta))\\
&+(-1)^{|\gamma||\delta|+|\varepsilon||v|}\rho_r(\rho_r(\beta^2(\varepsilon)\cdot\phi\psi(v),\beta\alpha^2(\gamma)),\alpha^3\beta(\delta))\\
&-(-1)^{|\gamma||\delta|+|\varepsilon||v|}\rho_r(\alpha\beta^2(\varepsilon)\cdot\phi^2\psi(v),\beta\alpha^2(\gamma)\alpha^3(\delta))\\
=&(-1)^{|\gamma||\delta|}\rho_r((\psi^2(v)\cdot\beta\alpha(\varepsilon))\cdot\beta\alpha^2(\gamma),\alpha^3\beta(\delta))\\
&+(-1)^{|\gamma||\delta|+|\gamma||v|+|\varepsilon||\gamma|}\rho_r(\beta^2\alpha(\gamma)\cdot(\phi\psi(v)\cdot\alpha^2(\varepsilon)),\alpha^3\beta(\delta)\\
&-(-1)^{|\gamma||\delta|}(\phi\psi^2(v)\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^2(\gamma)\alpha^3(\delta)\\
&-(-1)^{|\gamma||\delta|+|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\beta^2\alpha(\gamma)\beta\alpha^2(\delta)\cdot(\phi^2\psi(v)\cdot\alpha^3(\varepsilon))\\
&+(-1)^{|\gamma||\delta|+|\varepsilon||v|}\rho_r((\beta^2(\varepsilon)\cdot\phi\psi(v))\cdot\beta\alpha^2(\gamma),\alpha^3\beta(\delta))\\
&+(-1)^{|\gamma||\delta|+|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|}\rho_r(\beta^2\alpha(\gamma)\cdot(\alpha\beta(\varepsilon)\cdot\phi^2(v)),\alpha^3\beta(\delta))\\
&-(-1)^{|\gamma||\delta|+|\varepsilon||v|}(\alpha\beta^2(\varepsilon)\cdot\phi^2\psi(v))\cdot\beta\alpha^2(\gamma)\alpha^3(\delta)\\
&-(-1)^{|\gamma||\delta|+|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|+|\delta||\varepsilon|+|\delta||v|}\beta^2\alpha(\gamma)\beta\alpha^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot\phi^3(v))\ \ (by\ (3.13))\\
=&(-1)^{|\gamma||\delta|}((\psi^2(v)\cdot\beta\alpha(\varepsilon))\cdot\beta\alpha^2(\gamma))\cdot\alpha^3\beta(\delta)\\
&+(-1)^{|\delta||\varepsilon|+|\delta||v|}\alpha^2\beta^2(\delta)\cdot((\phi\psi(v)\cdot\alpha^2(\varepsilon))\cdot\alpha^3(\gamma))\\
&+(-1)^{|\gamma||\delta|+|\gamma||v|+|\varepsilon||\gamma|}(\beta^2\alpha(\gamma)\cdot(\phi\psi(v)\cdot\alpha^2(\varepsilon)))\cdot\alpha^3\beta(\delta)\\
&+(-1)^{|\gamma||v|+|\varepsilon||\gamma|+|\delta||v|+|\delta||\varepsilon|}\alpha^2\beta^2(\delta)\cdot(\beta\alpha^2(\gamma)\cdot(\phi^2(v)\cdot\alpha^3\beta^{-1}(\varepsilon)))\\
&+(-1)^{|\gamma||\delta|+|\varepsilon||v|}((\beta^2(\varepsilon)\cdot\phi\psi(v))\cdot\beta\alpha^2(\gamma))\cdot\alpha^3\beta(\delta)\\
&+(-1)^{|\varepsilon||v|+|\delta||\varepsilon|+|\delta||v|}\alpha^2\beta^2(\delta)\cdot((\alpha\beta(\varepsilon)\cdot\phi^2(v))\alpha^3(\gamma))\\
&+(-1)^{|\gamma||\delta|+|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|}(\beta^2\alpha(\gamma)\cdot(\alpha\beta(\varepsilon)\cdot\phi^2(v)))\cdot\alpha^3\beta(\delta)\\
&+(-1)^{|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|+|\delta||\varepsilon|+|\delta||v|}\alpha^2\beta^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot(\alpha^2(\varepsilon)\cdot\phi^3\psi^{-1}(v)))\\
&-(-1)^{|\gamma||\delta|}(\phi\psi^2(v)\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^2(\gamma)\alpha^3(\delta)\\
&-\underbrace{(-1)^{|\gamma||\delta|+|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\beta^2\alpha(\gamma)\beta\alpha^2(\delta)\cdot(\phi^2\psi(v)\cdot\alpha^3(\varepsilon))}_{B}\\
&-\underbrace{(-1)^{|\gamma||\delta|+|\varepsilon||v|}(\alpha\beta^2(\varepsilon)\cdot\phi^2\psi(v))\cdot\beta\alpha^2(\gamma)\alpha^3(\delta)}_{J}\\
&-(-1)^{|\gamma||\delta|+|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|+|\delta||\varepsilon|+|\delta||v|}\beta^2\alpha(\gamma)\beta\alpha^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot\phi^3(v)).\\
&(by\ (3.13)\ and\ rearranging)
\end{aligned}
$$

Observe that

$$\begin{aligned}J =&(-1)^{|\gamma||\delta|+|\varepsilon||v|}\phi(\beta^2(\varepsilon)\cdot\phi\psi(v))\cdot\beta(\alpha^2(\gamma))\alpha(\alpha^2(\delta))\\ =&(-1)^{|\gamma||\delta|+|\varepsilon||v|}((\beta^2(\varepsilon)\cdot\phi\psi(v))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\delta)\\ &+(-1)^{|\varepsilon||v|}((\beta^2(\varepsilon)\cdot\phi\psi(v))\cdot\beta\alpha^2(\delta))\cdot\beta\alpha^3(\gamma)\ \ (by\ (3.6))\\ =&(-1)^{|\gamma||\delta|+|\varepsilon||v|}((\beta^2(\varepsilon)\cdot\phi\psi(v))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\delta)\\ &+(-1)^{|\varepsilon||v|}(\alpha\beta^2(\varepsilon)\cdot(\phi\psi(v)\cdot\alpha^2(\delta)))\cdot\beta\alpha^3(\gamma)\ \ (by\ (3.12))\\ =&\underbrace{(-1)^{|\gamma||\delta|+|\varepsilon||v|}((\beta^2(\varepsilon)\cdot\phi\psi(v))\cdot\beta\alpha^2(\gamma))\cdot\beta\alpha^3(\delta)}_{J_1}\\ &+\underbrace{(-1)^{|\varepsilon||v|}\alpha^2\beta^2(\varepsilon)\cdot((\phi\psi(v)\cdot\alpha^2(\delta))\cdot\alpha^3(\gamma))}_{J_2}.\ \ (by\ (3.12))\end{aligned}$$

$$\begin{aligned}B =&(-1)^{|\gamma||\delta|+|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\beta(\alpha\beta(\gamma))\alpha(\alpha\beta(\delta))\cdot\psi(\phi^2(v)\cdot\alpha^3\beta^{-1}(\varepsilon))\\ =&(-1)^{|\gamma||\delta|+|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\alpha^2\beta^2(\gamma)\cdot(\alpha^2\beta(\delta)\cdot(\phi^2(v)\cdot\alpha^3\beta^{-1}(\varepsilon)))\\ &+(-1)^{|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\alpha^2\beta^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot(\phi^2(v)\cdot\alpha^3\beta^{-1}(\varepsilon)))\ \ (by\ (3.8))\\ =&(-1)^{|\gamma||\delta|+|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\alpha^2\beta^2(\gamma)\cdot((\alpha\beta(\delta)\phi^2(v))\cdot\alpha^3(\varepsilon))\\ &+(-1)^{|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\alpha^2\beta^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot(\phi^2(v)\cdot\alpha^3\beta^{-1}(\varepsilon)))\ \ (by\ (3.12))\\ =&\underbrace{(-1)^{|\gamma||\delta|+|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}(\alpha\beta^2(\gamma)\cdot(\alpha\beta(\delta)\cdot\phi^2(v)))\cdot\alpha^3\beta(\varepsilon)}_{B_1}\\ &+\underbrace{(-1)^{|\gamma||v|+|\gamma||\varepsilon|+|\delta||v|+|\delta||\varepsilon|}\alpha^2\beta^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot(\phi^2(v)\cdot\alpha^3\beta^{-1}(\varepsilon)))}_{B_2}.\ \ (by\ (3.12))\end{aligned}$$

We substitute J_1+J_2 and B_1+B_2 for J and B to obtain

$$\begin{aligned}&(-1)^{|\gamma||\delta|}as_{V_{\phi,\psi}}(\rho_r(\psi^2(v),\beta\alpha(\varepsilon)),\beta\alpha^2(\gamma),\alpha^3(\delta))\\ =&(-1)^{|\gamma||\delta|}((\psi^2(v)\cdot\beta\alpha(\varepsilon))\cdot\beta\alpha^2(\gamma))\cdot\alpha^3\beta(\delta)\\ &+(-1)^{|\delta||\varepsilon|+|\delta||v|}\alpha^2\beta^2(\delta)\cdot((\phi\psi(v)\cdot\alpha^2(\varepsilon))\cdot\alpha^3(\gamma))\\ &+(-1)^{|\gamma||\delta|+|\gamma||v|+|\varepsilon||\gamma|}(\beta^2\alpha(\gamma)\cdot(\phi\psi(v)\cdot\alpha^2(\varepsilon)))\cdot\alpha^3\beta(\delta)\\ &+(-1)^{|\varepsilon||v|+|\delta||\varepsilon|+|\delta||v|}\alpha^2\beta^2(\delta)\cdot((\alpha\beta(\varepsilon)\cdot\phi^2(v))\alpha^3(\gamma))\\ &+(-1)^{|\gamma||\delta|+|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|}(\beta^2\alpha(\gamma)\cdot(\alpha\beta(\varepsilon)\cdot\phi^2(v)))\cdot\alpha^3\beta(\delta)\\ &+(-1)^{|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|+|\delta||\varepsilon|+|\delta||v|}\alpha^2\beta^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot(\alpha^2(\varepsilon)\cdot\phi^3\psi^{-1}(v)))\\ &-(-1)^{|\gamma||\delta|}(\phi\psi^2(v)\cdot\beta\alpha^2(\varepsilon))\cdot\beta\alpha^2(\gamma)\alpha^3(\delta)\\ &-B_1-J_2\\ &-(-1)^{|\gamma||\delta|+|\varepsilon||v|+|\gamma||\varepsilon|+|\gamma||v|+|\delta||\varepsilon|+|\delta||v|}\beta^2\alpha(\gamma)\beta\alpha^2(\delta)\cdot(\alpha^2\beta(\gamma)\cdot\phi^3(v)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}as_{V_{\phi,\psi}}(\rho_r(\psi^2(v)\cdot\beta\alpha(\delta)),\beta\alpha^2(\gamma),\alpha^3(\varepsilon))\\ =&(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}((\psi^2(v)\cdot\beta\alpha(\delta))\cdot\beta\alpha^2(\gamma))\cdot\alpha^3\beta(\varepsilon)\\ &+(-1)^{|\varepsilon||v|}\alpha^2\beta^2(\varepsilon)\cdot((\phi\psi(v)\cdot\alpha^2(\delta))\cdot\alpha^3(\gamma))\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{|\gamma||v|+|\delta||\gamma|+|\varepsilon||\gamma|+|\varepsilon||\delta|}(\beta^2\alpha(\gamma)\cdot(\phi\psi(v)\cdot\alpha^2(\delta)))\cdot\alpha^3\beta(\varepsilon)\\
&+(-1)^{|\delta||v|+|\varepsilon||v|}\alpha^2\beta^2(\varepsilon)\cdot((\alpha\beta(\delta)\phi^2(v))\cdot\alpha^3(\gamma))\\
&+(-1)^{|\delta||v|+|\gamma||\delta|+|\gamma||v|+|\varepsilon||\delta|+|\varepsilon||\gamma|}(\beta^2\alpha(\gamma)\cdot(\alpha\beta(\delta)\cdot\phi^2(v)))\cdot\alpha^3\beta(\varepsilon)\\
&+(-1)^{|\delta||v|+|\gamma||\delta|+|\gamma||v|+|\varepsilon||v|}\alpha^2\beta^2(\varepsilon)\cdot(\alpha^2\beta(\gamma)\cdot(\alpha^2(\delta)\cdot\phi^3\psi^{-1}(v)))\\
&-(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}(\phi\psi^2(v)\cdot\beta\alpha^2(\delta))\cdot\beta\alpha^2(\gamma)\alpha^3(\varepsilon)\\
&-(-1)^{|\varepsilon||v|+|\delta||v|+|\varepsilon||\gamma|+|\varepsilon||\delta|}\alpha^2\beta^2(\delta)\cdot((\phi\psi(v)\cdot\alpha^2(\varepsilon))\cdot\alpha^3(\gamma))\\
&-(-1)^{|\gamma||v|+|\gamma||\delta|+|\varepsilon||v|+|\varepsilon||\gamma|}(\alpha\beta^2(\gamma)\cdot(\alpha\beta(\varepsilon)\cdot\phi^2(v)))\cdot\alpha^3\beta(\delta)\\
&-(-1)^{|\varepsilon||v|+|\delta||v|+|\gamma||\delta|+|\gamma||v|+|\varepsilon||\gamma|}\beta^2\alpha(\gamma)\alpha^2\beta(\varepsilon)\cdot(\alpha^2\beta(\delta)\cdot\phi^3(v)).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}as_{V_{\phi,\psi}}(\beta^2(\varepsilon)\beta\alpha(\delta),\beta\alpha^2(\gamma),\phi^3(v))\\
=&(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}\rho_l((\beta^2(\varepsilon)\beta\alpha(\delta))\beta\alpha^2(\gamma),\phi^3\psi(v))\\
&-(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}\rho_l(\beta^2\alpha(\varepsilon)\beta\alpha^2(\delta),\rho_l(\beta\alpha^2(\gamma),\phi^3(v))\\
=&(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}((\beta^2(\varepsilon)\beta\alpha(\delta))\beta\alpha^2(\gamma))\cdot\phi^3\psi(v)\\
&+\phi^2\psi^2(v)\cdot((\alpha\beta(\varepsilon)\alpha^2(\delta))\alpha^3(\gamma))\\
&-(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}\rho_l(\beta^2\alpha(\varepsilon)\beta\alpha^2(\delta),\beta\alpha^2(\gamma)\cdot\phi^3(v))\\
&-(-1)^{|v||\varepsilon|+|v||\delta|}\rho_l(\beta^2\alpha(\varepsilon)\beta\alpha^2(\delta),\psi\phi^2(v)\cdot\alpha^3(\gamma))\;\;(by\;(3.13))\\
=&(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}((\beta^2(\varepsilon)\beta\alpha(\delta))\beta\alpha^2(\gamma))\cdot\phi^3\psi(v)\\
&+\phi^2\psi^2(v)\cdot((\alpha\beta(\varepsilon)\alpha^2(\delta))\alpha^3(\gamma))\\
&-(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}(\beta^2\alpha(\varepsilon)\beta\alpha^2(\delta))\cdot(\beta\alpha^2(\gamma)\cdot\phi^3(v))\\
&-\underbrace{(-1)^{|v||\gamma|+|\gamma||\varepsilon|+|\gamma||\delta|}(\beta^2\alpha(\gamma)\phi^2\psi(v))\cdot\beta\alpha^2(\varepsilon)\alpha^3(\delta)}_{D}\\
&-\underbrace{(-1)^{|v||\varepsilon|+|v||\delta|}\beta^2\alpha(\varepsilon)\beta\alpha^2(\delta)\cdot(\psi\phi^2(v)\cdot\alpha^3(\gamma))}_{C}\\
&-(-1)^{|\gamma||\varepsilon|+|\gamma||\delta|}(\psi^2\phi(v)\cdot\alpha^2\beta(\gamma))\cdot\alpha^2\beta(\varepsilon)\alpha^3(\delta).\;\;(by\;(3.13))
\end{aligned}
$$

The same way, we replace C and D as follows

$$
\begin{aligned}
C=&(-1)^{|v||\varepsilon|+|v||\delta|}\beta(\alpha\beta(\varepsilon))\alpha(\alpha\beta(\delta))\cdot\psi(\phi^2(v)\cdot\alpha^3\beta^{-1}(\gamma))\\
=&(-1)^{|v||\varepsilon|+|v||\delta|}\beta^2\alpha^2(\varepsilon)\cdot(\alpha^2\beta(\delta)\cdot(\phi^2(v)\cdot\alpha^3\beta^{-1}(\gamma)))\\
&+(-1)^{|v||\varepsilon|+|v||\delta|+|\varepsilon||\delta|}\beta^2\alpha^2(\delta)\cdot(\alpha^2\beta(\varepsilon)\cdot(\phi^2(v)\cdot\alpha^3\beta^{-1}(\gamma)))\;\;(by\;3.8)\\
=&\underbrace{(-1)^{|v||\varepsilon|+|v||\delta|}\beta^2\alpha^2(\varepsilon)\cdot((\alpha\beta(\delta)\cdot\phi^2(v))\cdot\alpha^3(\gamma))}_{C_1}\\
&+\underbrace{(-1)^{|v||\varepsilon|+|v||\delta|+|\varepsilon||\delta|}\beta^2\alpha^2(\delta)\cdot((\alpha\beta(\varepsilon)\cdot\phi^2(v))\cdot\alpha^3(\gamma))}_{C_2},\;\;(by\;3.12)
\end{aligned}
$$

$$
\begin{aligned}
D=&(-1)^{|v||\gamma|+|\gamma||\varepsilon|+|\gamma||\delta|}\phi(\beta^2(\gamma)\cdot\phi\psi(v))\cdot\beta(\alpha^2(\varepsilon))\alpha(\alpha^2(\delta))\\
=&(-1)^{|v||\gamma|+|\gamma||\varepsilon|+|\gamma||\delta|}((\beta^2(\gamma)\cdot\phi\psi(v))\cdot\alpha^2\beta(\varepsilon))\cdot\beta\alpha^3(\delta)
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{|v||\gamma|+|\gamma||\varepsilon|+|\gamma||\delta|+|\varepsilon||\delta|}((\beta^2(\gamma)\cdot\phi\psi(v))\cdot\alpha^2\beta(\delta))\cdot\beta\alpha^3(\varepsilon)\ \ (by\ 3.6)\\
=&\underbrace{(-1)^{|v||\gamma|+|\gamma||\varepsilon|+|\gamma||\delta|}(\alpha\beta^2(\gamma)\cdot(\phi\psi(v)\cdot\alpha^2(\varepsilon)))\cdot\beta\alpha^3(\delta)}_{D_1}\\
&+\underbrace{(-1)^{|v||\gamma|+|\gamma||\varepsilon|+|\gamma||\delta|+|\varepsilon||\delta|}(\alpha\beta^2(\gamma)\cdot(\phi\psi(v)\cdot\alpha^2(\delta)))\cdot\beta\alpha^3(\varepsilon)}_{D_2}.\ \ (by\ 3.12)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&(-1)^{|\gamma||\delta|}as_{V_{\phi,\psi}}(\psi^2(v)\cdot\alpha\beta(\varepsilon),\beta\alpha^2(\gamma),\alpha^3(\delta))\\
&+(-1)^{|\varepsilon||\gamma|+|\varepsilon||\delta|}as_{V_{\phi,\psi}}(\psi^2(v)\cdot\alpha\beta(\delta),\beta\alpha^2(\gamma),\alpha^3(\varepsilon))\\
&+(-1)^{|v||\varepsilon|+|v||\gamma|+|v||\delta|}as_{V_{\phi,\psi}}(\mu(\beta^2(\varepsilon),\alpha\beta(\delta)),\beta\alpha^2(\gamma),\phi^3(v))\\
=&(-1)^{|\gamma||\delta|}(3.2)+(-1)^{|v||\varepsilon|+|\gamma||\varepsilon|+|\gamma||v|+|\delta||\varepsilon|+|\delta||v|}(3.4)=0.
\end{aligned}
$$

Hence, we prove that $(V,\phi,\psi,\rho_l,\rho_r)$ is a Bihom-Jordan J-super-bimodule.

Lemma 3.1. *Let (J,μ,α,β) be a Bihom-associative superalgebra and (V,ϕ,ψ) be a Bihom-super-module.*

1) If (V,ϕ,ψ) is a right Bihom-associative J-super-module with the structure map ρ_r, then (V,ϕ,ψ) is a right special Bihom-Jordan $J^\diamond$-super-module with the same structure map ρ_r.

2) If (V,ϕ,ψ) is a left Bihom-associative J-super-module with the structure map ρ_l such that ψ is invertible, then (V,ϕ,ψ) is a left special Bihom-Jordan $J^\diamond$-super-module with the same structure map ρ_l.

Proof. It also suffices to prove Eqs (3.6) and (3.8).

1) If (V,ϕ,ψ) is a right Bihom-associative J-super-module with the structure map ρ_r then for all $(\varepsilon,\gamma,v)\in\mathcal{H}(J)\times\mathcal{H}(J)\times\mathcal{H}(V)$. $\phi(v)\cdot(\beta(\varepsilon)\diamond\alpha(\gamma))=\phi(v)\cdot(\beta(\varepsilon)\alpha(\gamma)+(-1)^{|\varepsilon||\gamma|}\beta(\gamma)\alpha(\varepsilon))=(v\cdot\beta(\varepsilon))\cdot\alpha\beta(\gamma)+(-1)^{|\varepsilon||\gamma|}(v\cdot\beta(\gamma))\cdot\alpha\beta(\varepsilon)$. Then (V,ϕ,ψ) is a right special Bihom-Jordan $J^\diamond$-super-module by Theorem 3.1.

2) Similarly, it is easy to obtain by Theorem 3.2.

End of lemma proof.

By Lemma 3.1 and Theorem 3.4, we obtain the following conclusion.

Proposition 3.1. *Let (J,μ,α,β) be a Bihom-associative superalgebra and $(V,\phi,\psi,\rho_1,\rho_2)$ be a Bihom-associative J-super-bimodule such that ϕ and ψ are inversible. Then $(V,\phi,\psi,\rho_l,\rho_r)$ is a Bihom-Jordan $J^\diamond$-super-bimodule where ρ_l and ρ_r are defined as in* Eq (3.13).

That is, a Bihom-associative J-super-bimodule gives rise to a Bihom-Jordan super-bimodule for $J^\diamond$.

Proposition 3.2. *Let (J,μ,α,β) be a Bihom-Jordan superalgebra and $V_{\phi,\psi}=(V,\phi,\psi,\rho_l,\rho_r)$ be a Bihom-Jordan J-super-bimodule. Then for each $n\in\mathbb{N}$ such that $\phi^n=\psi^n=\mathrm{Id}_V$, the maps*

$$\rho_l^{(n)}=\rho_l\circ(\alpha^n\otimes\psi^n),\tag{3.14}$$

and

$$\rho_r^{(n)} = \rho_r \circ (\phi^n \otimes \beta^n). \tag{3.15}$$

as structure maps, $(V, \phi, \psi, \rho_l^{(n)}, \rho_r^{(n)})$ *is given to be a Bihom-Jordan J-super-bimodule. Denoted it by* $V_{\phi,\psi}^{(n)}$.

Proof. ρ_l and ρ_n are easy to prove special left and right super-modules, respectively, which are also left and right super-modules, and Eq (3.9) holds in $V_{\phi,\psi}^{(n)}$. By direct calculation, we can convert $as_{V_{\phi,\psi}^{(n)}}$ in $V_{\phi,\psi}^{(n)}$ to $as_{V_{\phi,\psi}}$ in $V_{\phi,\psi}$, that is

$$as_{V_{\phi,\psi}^{(n)}}(\beta^2(\varepsilon)\alpha\beta(\gamma), \phi^2\psi(v), \alpha^3(\delta)) = as_{V_{\phi,\psi}}(\beta^2(\alpha^n(\varepsilon))\alpha\beta(\alpha^n(\gamma)), \phi^2\psi(v), \alpha^3(\beta^n(\delta)))$$

, furthermore, we have

$$\begin{aligned}
&\circlearrowleft_{\varepsilon,\gamma,\delta} (-1)^{|\delta|(|\varepsilon|+|v|)} as_{V_{\phi,\psi}^{(n)}}(\beta^2(\varepsilon)\alpha\beta(\gamma), \phi^2\psi(v), \alpha^3(\delta)) \\
=& \circlearrowleft_{\alpha^n(\varepsilon),\alpha^n(\gamma),\beta^n(\delta)} (-1)^{|\delta|(|\varepsilon|+|v|)} as_{V_{\phi,\psi}}(\beta^2(\alpha^n(\varepsilon))\alpha\beta(\alpha^n(\gamma)), \phi^2\psi(v), \alpha^3(\beta^n(\delta))) \\
=&0.
\end{aligned}$$

Then we obtain Eq (3.10) in $V_{\phi,\psi}^{(n)}$. Similarly, Eq (3.11) also holds in $V_{\phi,\psi}^{(n)}$, which implies that $V_{\phi,\psi}^{(n)}$ is a Bihom-Jordan J-super-bimodule.

In the sequel, we present some results of Bihom-Jordan super-bimodules constructed by Jordan super-bimodules via endomorphisms.

Theorem 3.5. *Let* (J, μ) *be a Jordan superalgebra,* (V, ρ_l, ρ_r) *be a Jordan J-super-bimodule,* α, β *be endomorphisms of J, which satisfies* $\alpha\beta = \beta\alpha$ *and* ϕ, ψ *be even linear self-maps of V such that* $\phi \circ \rho_l = \rho_l \circ (\alpha \otimes \phi)$, $\phi \circ \rho_r = \rho_r \circ (\phi \otimes \alpha)$, $\psi \circ \rho_l = \rho_l \circ (\beta \otimes \psi)$ *and* $\psi \circ \rho_r = \rho_r \circ (\psi \otimes \beta)$. *Denote* $J_{\alpha,\beta}$ *for the Bihom-Jordan superalgebra* $(J, \mu_{\alpha,\beta} = \mu(\alpha \otimes \beta), \alpha, \beta)$ *and* $V_{\phi,\psi}$ *for the Bihom-super-module* (V, ϕ, ψ). *Define two structure maps as follows:*

$$\tilde{\rho}_l = \rho_l(\alpha \otimes \psi) \text{ and } \tilde{\rho}_r = \rho_r(\phi \otimes \beta). \tag{3.16}$$

Then $V_{\phi,\psi} = (V, \phi, \psi, \tilde{\rho}_l, \tilde{\rho}_r)$ *is a Bihom-Jordan* $J_{\alpha,\beta}$*-super-bimodule.*

Proof. By direct calculation, it is easy to get $as_{V_{\phi,\psi}}(\mu_{\alpha,\beta}(\beta^2(\varepsilon), \alpha\beta(\gamma)), \phi^2\psi(v), \alpha^3(\delta)) = as_V(\alpha^3\beta^2(\varepsilon)\alpha^3\beta^2(\gamma), \phi^3\psi^2(v), \alpha^3\beta^2(\delta))$, So it is clear Eqs (3.10) and (3.11) hold in $V_{\phi,\psi}$. Thus, $(V, \phi, \psi, \tilde{\rho}_l, \tilde{\rho}_r)$ is a Bihom-Jordan $J_{\alpha,\beta}$-super-bimodule.

From Proposition 3.2 and Theorem 3.5, we have the following

Corollary 3.1. *Let* (J, μ) *be a Jordan superalgebra,* (V, ρ_l, ρ_r) *be a Jordan J-super-bimodule,* α, β *be endomorphisms of J, which satify* $\alpha\beta = \beta\alpha$ *and* ϕ, ψ *be even linear self-maps of V such that* $\phi \circ \rho_l = \rho_l \circ (\alpha \otimes \phi)$, $\phi \circ \rho_r = \rho_r \circ (\phi \otimes \alpha)$, $\psi \circ \rho_l = \rho_l \circ (\beta \otimes \psi)$ *and* $\psi \circ \rho_r = \rho_r \circ (\psi \otimes \beta)$. *Denote* $J_{\alpha,\beta}$ *for the Bihom-Jordan superalgebra* $(J, \mu_{\alpha,\beta} = \mu(\alpha \otimes \beta), \alpha, \beta)$ *and* $V_{\phi,\psi}$ *for the Bihom-super-module* (V, ϕ, ψ). *Define two structure maps as follows:*

$$\tilde{\rho}_l^{(n)} = \rho_l(\alpha^{n+1} \otimes \psi) \text{ and } \tilde{\rho}_r^{(n)} = \rho_r(\phi \otimes \beta^{n+1}). \tag{3.17}$$

Then $V_{\phi,\psi} = (V, \phi, \psi, \tilde{\rho}_l^{(n)}, \tilde{\rho}_r^{(n)})$ *is a Bihom-Jordan* $J_{\alpha,\beta}$*-super-bimodule for each* $n \in \mathbb{N}$.

Definition 3.3. *An abelian extension of Bihom-Jordan superalgebra is a short exact sequence of Bihom-Jordan superalgebra:*

$$0 \longrightarrow (V,\phi,\psi) \stackrel{i}{\longrightarrow} (J,\mu_J,\alpha_J,\beta_J) \stackrel{\pi}{\longrightarrow} (B,\mu_B,\alpha_B,\beta_B) \longrightarrow 0.$$

where (V,ϕ,ψ) is a trivial Bihom-Jordan superalgebra, i and π are even morphisms of Bihom-superalgebras. If there exists an even morphism $s:(B,\mu_B,\alpha_B,\beta_B)\to(J,\mu_J,\alpha_J,\beta_J)$ satisfies $\pi\circ s=\mathrm{Id}_B$. Then the abelian extension is said to be split and s is called a section of π.

4. Representation and O-operator

In this section, we study the representation and O-operator. Meanwhile, we characterize Bihom-pre-Jordan superalgebras by using O-operator.

Definition 4.1. *Let (J,μ,α,β) be a Bihom-Jordan superalgebra, V be a $\mathbb{Z}_2$-graded vector spaces, $\rho: J\to \mathrm{End}(V)$, $\phi,\psi\in \mathrm{Aug}(V)$. Then (V,ρ,ϕ,ψ) is a representation of (J,μ,α,β), if the following conditions hold:*

$$\phi\psi=\psi\phi, \tag{4.1}$$

$$\begin{aligned}
&\rho(\mu(\mu(\beta^2(\varepsilon),\alpha\beta(\gamma)),\alpha^2\beta(\delta)))\phi^3\psi\\
&+(-1)^{|\delta||\gamma|}\rho(\alpha^2\beta^2(\varepsilon))\phi\psi^{-1}\rho(\alpha\beta^2(\delta))\phi\psi^{-1}\rho(\beta^2(\gamma))\phi\psi\\
&+(-1)^{|\gamma||\varepsilon|+|\delta||\varepsilon|}\rho(\alpha^2\beta^2(\gamma))\phi\psi^{-1}\rho(\alpha\beta^2(\delta))\phi\psi^{-1}\rho(\beta^2(\varepsilon))\phi\psi\\
&-\rho(\mu(\alpha\beta^2(\varepsilon),\alpha^2\beta(\gamma)))\rho(\alpha^2\beta(\delta))\phi^3\\
&-(-1)^{|\varepsilon||\delta|+|\delta||\gamma|}\rho(\mu(\alpha\beta^2(\delta),\alpha^2\beta(\varepsilon)))\phi^2\psi^{-1}\rho(\beta^2(\gamma))\phi\psi\\
&-(-1)^{|\varepsilon||\delta|+|\delta||\gamma|+|\varepsilon||\gamma|}\rho(\mu(\alpha\beta^2(\delta),\alpha^2\beta(\gamma)))\phi^2\psi^{-1}\rho(\beta^2(\varepsilon))\phi\psi\\
=&0.
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\circlearrowleft_{\varepsilon,\gamma,\delta}(-1)^{|\varepsilon||\delta|}\rho(\alpha^2\beta^2(\varepsilon))\phi\psi^{-1}\rho(\mu(\beta^2(\gamma),\alpha\beta(\delta)))\phi^2\psi\\
=&\circlearrowleft_{\varepsilon,\gamma,\delta}(-1)^{|\varepsilon||\delta|}\rho(\mu(\alpha\beta^2(\varepsilon),\alpha^2\beta(\gamma)))\rho(\alpha^2\beta(\delta))\phi^3.
\end{aligned} \tag{4.3}$$

Example 4.1. *Let (J,μ,α,β) be a regular Bihom-Jordan superalgebra. Define $\mathrm{ad}: J\to\mathrm{End}(J)$, for any $\varepsilon,\gamma\in\mathcal{H}(J)$, $\mathrm{ad}(\varepsilon)\gamma=\mu(\varepsilon,\gamma)$. Then $(J,\mathrm{ad},\alpha,\beta)$ is a representation of (J,μ,α,β), which is called adjoint representation.*

Proposition 4.1. *Let (J,μ,α,β) be a Bihom-Jordan superalgebra. (V,ρ,ϕ,ψ) be a representation, define an even bilinear map μ and two even linear maps α and β on $J\oplus V$ as follows: for any $\varepsilon,\gamma\in\mathcal{H}(J)$, $a,b\in\mathcal{H}(V)$,*

$$\mu'(\varepsilon+a,\gamma+b)=\mu(\varepsilon,\gamma)+\rho(\varepsilon)b+\rho(\alpha^{-1}\beta(\gamma))\phi\psi^{-1}(a),$$

$$(\alpha+\phi)(\varepsilon+a)=\alpha(\varepsilon)+\phi(a),\ (\beta+\psi)(\varepsilon+a)=\beta(a)+\psi(a).$$

Then $(J\oplus V,\mu',\alpha+\phi,\beta+\psi)$ is a Bihom-Jordan superalgebra, denoted by $J\ltimes V$ and called semidirect product.

Proof. It can be verified directly by Definition 4.1.

We also consider the split null extension on $J \ltimes V$ in Proposition 4.1.

Remark 4.1. *Write elements $a + v$ of $J \oplus V$ as (a, v). There is an injective homomorphism and a surjective homomorphism of Bihom-modules, respectively, as follows:*

- $i : V \to J \oplus V$, $i(v) = (0, v)$,
- $\pi : J \oplus V \to$, $\pi(a, v) = a$.

Moreover, $i(V)$ is a Bihom-ideal of $J \oplus V$ such that $J \oplus V/i(V) \cong J$. On the other hand, there is an even morphisms $\sigma : J \to J \oplus V$ given by $\sigma(a) = (a, 0)$, which is clearly a section of π. Therefore, we obtain the abelian split exact sequence:

$$0 \longrightarrow V \xrightarrow{\ i\ } J \oplus V \underset{f}{\overset{\pi}{\rightleftarrows}} J \longrightarrow 0 .$$

Definition 4.2. *A BiHom superalgebra $(J, \cdot, \alpha, \beta)$ is called a Bihom-pre-Jordan superalgebra if for all $\varepsilon, \gamma, \delta, t \in \mathcal{H}(J)$:*

1) $\alpha\beta = \beta\alpha$, both α and β are reversible,

2)

$$\begin{aligned}
&((\beta^2(\varepsilon) \cdot \alpha\beta(\gamma)) \cdot \alpha^2\beta(\delta)) \cdot \alpha^3\beta(w) + (-1)^{|\varepsilon||\gamma|}((\beta^2(\gamma) \cdot \alpha\beta(\varepsilon)) \cdot \alpha^2\beta(\delta)) \cdot \alpha^3\beta(w)\\
&+ (-1)^{|\delta|(|\varepsilon|+|\gamma|)}(\alpha\beta^2(\delta) \cdot (\alpha\beta(\varepsilon)\alpha^2(\gamma))) \cdot \alpha^3\beta(w)\\
&+ (-1)^{|\delta|(|\varepsilon|+|\gamma|)+|\varepsilon||\gamma|}(\alpha\beta^2(\delta) \cdot (\alpha\beta(\gamma)\alpha^2(\varepsilon))) \cdot \alpha^3\beta(w)\\
&+ (-1)^{|\delta||\gamma|}\alpha^2\beta^2(\varepsilon) \cdot (\alpha^2\beta(\delta) \cdot (\alpha^2(\gamma) \cdot \alpha^3\beta^{-1}(w)))\\
&+ (-1)^{|\delta||\varepsilon|+|\gamma||\varepsilon|}\alpha^2\beta^2(\gamma) \cdot (\alpha^2\beta(\delta) \cdot (\alpha^2(\varepsilon) \cdot \alpha^3\beta^{-1}(w)))\\
&- (\alpha\beta^2(\varepsilon) \cdot \alpha^2\beta(\gamma)) \cdot (\alpha^2\beta(\delta) \cdot \alpha^3(w)) - (-1)^{|\varepsilon||\gamma|}(\alpha\beta^2(\gamma) \cdot \alpha^2\beta(\varepsilon)) \cdot (\alpha^2\beta(\delta) \cdot \alpha^3(w))\\
&- (-1)^{|\varepsilon||\delta|+|\delta||\gamma|}(\alpha\beta^2(\delta) \cdot \alpha^2\beta(\varepsilon)) \cdot (\alpha^2\beta(\gamma) \cdot \alpha^3(w))\\
&- (-1)^{|\delta||\gamma|}(\alpha\beta^2(\varepsilon) \cdot \alpha^2\beta(\delta)) \cdot (\alpha^2\beta(\gamma) \cdot \alpha^3(w))\\
&- (-1)^{|\gamma||\delta|+|\delta||\varepsilon|+|\gamma||\varepsilon|}(\alpha\beta^2(\delta) \cdot \alpha^2\beta(\gamma)) \cdot (\alpha^2\beta(\varepsilon) \cdot \alpha^3(w))\\
&- (-1)^{|\delta||\varepsilon|+|\gamma||\varepsilon|}(\alpha\beta^2(\gamma) \cdot \alpha^2\beta(\delta)) \cdot (\alpha^2\beta(\varepsilon) \cdot \alpha^3(w))\\
=&0,
\end{aligned} \tag{4.4}$$

3)

$$\begin{aligned}
&(-1)^{|w||\gamma|}\alpha^2\beta^2(w) \cdot ((\alpha\beta(\varepsilon) \cdot \alpha^2(\gamma)) \cdot \alpha^3(\delta))\\
&+ (-1)^{|\varepsilon||\gamma|+|w||\gamma|}\alpha^2\beta^2(w) \cdot ((\alpha\beta(\gamma) \cdot \alpha^2(\varepsilon)) \cdot \alpha^3(\delta))\\
&+ (-1)^{|\varepsilon||w|}\alpha^2\beta^2(\varepsilon) \cdot ((\alpha\beta(\gamma) \cdot \alpha^2(w)) \cdot \alpha^3(\delta))\\
&+ (-1)^{|\varepsilon||w|+|\gamma||w|}\alpha^2\beta^2(\varepsilon) \cdot ((\alpha\beta(w) \cdot \alpha^2(\gamma)) \cdot \alpha^3(\delta))\\
&+ (-1)^{|\gamma||\varepsilon|}\alpha^2\beta^2(\gamma) \cdot ((\alpha\beta(w) \cdot \alpha^2(\varepsilon)) \cdot \alpha^3(\delta))\\
&+ (-1)^{|\gamma||\varepsilon|+|w||\varepsilon|}\alpha^2\beta^2(\gamma) \cdot ((\alpha\beta(\varepsilon) \cdot \alpha^2(w)) \cdot \alpha^3(\delta))
\end{aligned}$$

$$
\begin{aligned}
&-(-1)^{|w||\varepsilon|}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\delta))\\
&-(-1)^{|\varepsilon||\gamma|+|w||\varepsilon|}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\delta))\\
&-(-1)^{|\varepsilon||\gamma|}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(\delta))\\
&-(-1)^{|\gamma||w|+|\varepsilon||\gamma|}(\alpha\beta^2(w)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(\delta))\\
&-(-1)^{|\gamma||w|}(\alpha\beta^2(w)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(\delta))\\
&-(-1)^{|w||\varepsilon|+|\gamma||w|}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(\delta)). \qquad (4.5)
\end{aligned}
$$

Actually, condition 3 is equivalent to

$$
\begin{aligned}
&\circlearrowleft_{\varepsilon,\gamma,w}\{(-1)^{|\varepsilon||w|}\alpha^2\beta^2(\varepsilon)\cdot((\alpha\beta(\gamma)\cdot\alpha^2(w))\cdot\alpha^3(\delta))\\
&+(-1)^{|\varepsilon||w|+|\gamma||w|}\alpha^2\beta^2(\varepsilon)\cdot((\alpha\beta(w)\cdot\alpha^2(\gamma))\cdot\alpha^3(\delta))\}\\
=&\circlearrowleft_{\varepsilon,\gamma,w}\{(-1)^{|\varepsilon||w|}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(w)\cdot(\delta))\\
&+(-1)^{|\varepsilon||w|+|\varepsilon||\gamma|}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(w)\cdot(\delta))\}.
\end{aligned}
$$

Theorem 4.1. *Let $(J,\cdot,\alpha,\beta)$ be a Bihom-pre-Jordan superalgebra, define an even bilinear operator μ: for all $\varepsilon,\gamma\in\mathcal{H}(J)$*

$$\mu(\varepsilon,\gamma)=\varepsilon\cdot\gamma+(-1)^{|\varepsilon||\gamma|}\alpha^{-1}\beta(\gamma)\cdot\alpha\beta^{-1}(\varepsilon), \qquad (4.6)$$

then $(J,\bullet,\alpha,\beta)$ is a Bihom-Jordan superalgebra.

Proof. By Eq (4.6), we get

$$
\begin{aligned}
\mu(\beta(\varepsilon),\alpha(\gamma))=&\beta(\varepsilon)\cdot\alpha(\gamma)+(-1)^{|\varepsilon||\gamma|}\beta(\gamma)\cdot\alpha(\varepsilon)\\
=&(-1)^{|\varepsilon||\gamma|}\mu(\beta(\gamma),\alpha(\varepsilon)).
\end{aligned}
$$

That is to say the Bihom-super commutativity condition holds. Next, by direct calculation,

$$
\begin{aligned}
&(-1)^{|w|(|\varepsilon|+|\delta|)}\widetilde{as}_{\alpha,\beta}(\mu(\beta^2(\varepsilon),\alpha\beta(\gamma)),\alpha^2\beta(\delta),\alpha^3(w))\\
=&\underline{(-1)^{|w|(|\varepsilon|+|\delta|)}((\beta^2(\varepsilon)\cdot\alpha\beta(\gamma))\cdot\alpha^2\beta(\delta))\cdot\alpha^3\beta(w)}_{①}\\
&\underbrace{+(-1)^{|w||\gamma|}\alpha^2\beta^2(w)\cdot((\alpha\beta(\varepsilon)\cdot\alpha^2(\gamma))\cdot\alpha^3(\delta))}_{1'}\\
&\underline{+(-1)^{|w|(|\varepsilon|+|\delta|)+|\delta|(|\varepsilon|+|\gamma|)}(\alpha\beta^2(\delta)\cdot(\alpha\beta(\varepsilon)\cdot\alpha^2(\gamma)))\cdot\alpha^3\beta(w)}_{②}\\
&+(-1)^{|w||\gamma|+|\delta|(|\varepsilon|+|\gamma|)}\alpha^2\beta^2(w)\cdot(\alpha^2\beta(\delta)\cdot(\alpha^2(\varepsilon)\cdot\alpha^3\beta^{-1}(\gamma)))\\
&\underline{-(-1)^{|w|(|\varepsilon|+|\delta|)}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(\delta)\cdot\alpha^3(w))}_{③}\\
&-(-1)^{|w|(|\delta|+|\gamma|)+|\delta|(|\varepsilon|+|\gamma|)}(\alpha\beta^2(\delta)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(\gamma))\\
&\underbrace{-(-1)^{|w||\varepsilon|}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\delta))}_{2'}\\
&-(-1)^{|w||\gamma|+|\delta|(|\varepsilon|+|\gamma|)}(\alpha\beta^2(w)\cdot\alpha^2\beta(\delta))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(\gamma))\\
&\underline{+(-1)^{|w|(|\varepsilon|+|\delta|)+|\varepsilon||\gamma|}((\beta^2(\gamma)\cdot\alpha\beta(\varepsilon))\cdot\alpha^2\beta(\delta))\cdot\alpha^3\beta(w)}_{④}\\
&\underbrace{+(-1)^{|w||\gamma|+|\varepsilon||\gamma|}\alpha^2\beta^2(w)\cdot((\alpha\beta(\gamma)\cdot\alpha^2(\varepsilon))\cdot\alpha^3(\delta))}_{3'}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{+(-1)^{|w|(|\varepsilon|+|\delta|)+|\delta|(|\varepsilon|+|\gamma|)+|\varepsilon||\gamma|}(\alpha\beta^2(\delta)\cdot(\alpha\beta(\gamma)\cdot\alpha^2(\varepsilon)))\cdot\alpha^3\beta(w)}\textcircled{5}\\
&+(-1)^{|w||\gamma|+|\delta|(|\varepsilon|+|\gamma|)+|\varepsilon||\gamma|}\alpha^2\beta^2(w)\cdot(\alpha^2\beta(\delta)\cdot(\alpha^2(\gamma)\cdot\alpha^3\beta^{-1}(\varepsilon)))\\
&\underline{-(-1)^{|w|(|\varepsilon|+|\delta|)+|\varepsilon||\gamma|}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(\delta)\cdot\alpha^3(w))}\textcircled{6}\\
&-(-1)^{|w|(|\delta|+|\gamma|)+|\delta|(|\varepsilon|+|\gamma|)+|\varepsilon||\gamma|}(\alpha\beta^2(\delta)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(\varepsilon))\\
&\underbrace{-(-1)^{|w||\varepsilon|+|\varepsilon||\gamma|}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\delta))}_{4'}\\
&-(-1)^{|w||\gamma|+|\delta|(|\varepsilon|+|\gamma|)+|\varepsilon||\gamma|}(\alpha\beta^2(w)\cdot\alpha^2\beta(\delta))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(\varepsilon)),
\end{aligned}
$$

$$
\begin{aligned}
&(-1)^{|\varepsilon|(|\gamma|+|\delta|)}\widetilde{as}_{\alpha,\beta}(\mu(\beta^2(\gamma),\alpha\beta(w)),\alpha^2\beta(\delta),\alpha^3(\varepsilon))\\
=&(-1)^{|\varepsilon|(|\gamma|+|\delta|)}((\beta^2(\gamma)\cdot\alpha\beta(w))\cdot\alpha^2\beta(\delta))\cdot\alpha^3\beta(\varepsilon)\\
&\underbrace{+(-1)^{|\varepsilon||w|}\alpha^2\beta^2(\varepsilon)\cdot((\alpha\beta(\gamma)\cdot\alpha^2(w))\cdot\alpha^3(\delta))}_{5'}\\
&+(-1)^{|\varepsilon|(|\gamma|+|\delta|)+|\delta|(|w|+|\gamma|)}(\alpha\beta^2(\delta)\cdot(\alpha\beta(\gamma)\cdot\alpha^2(w)))\cdot\alpha^3\beta(\varepsilon)\\
&\underline{+(-1)^{|\varepsilon||w|+|\delta|(|\gamma|+|w|)}\alpha^2\beta^2(\varepsilon)\cdot(\alpha^2\beta(\delta)\cdot(\alpha^2(\gamma)\cdot\alpha^3\beta^{-1}(w)))}\textcircled{7}\\
&-(-1)^{|\varepsilon|(|\gamma|+|\delta|)}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\delta)\cdot\alpha^3(\varepsilon))\\
&\underline{-(-1)^{|\varepsilon|(|\delta|+|w|)+|\delta|(|\gamma|+|w|)}(\alpha\beta^2(\delta)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(w))}\textcircled{8}\\
&\underbrace{-(-1)^{|\varepsilon||\gamma|}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(\delta))}_{6'}\\
&\underline{-(-1)^{|\varepsilon||w|+|\delta|(|\gamma|+|w|)}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(\delta))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(w))}\textcircled{9}\\
&+(-1)^{|\varepsilon|(|\gamma|+|\delta|)+|\gamma||w|}((\beta^2(w)\cdot\alpha\beta(\gamma))\cdot\alpha^2\beta(\delta))\cdot\alpha^3\beta(\varepsilon)\\
&\underbrace{+(-1)^{|\varepsilon||w|+|\gamma||w|}\alpha^2\beta^2(\varepsilon)\cdot((\alpha\beta(w)\cdot\alpha^2(\gamma))\cdot\alpha^3(\delta))}_{7'}\\
&+(-1)^{|\varepsilon|(|\gamma|+|\delta|)+|\delta|(|\gamma|+|w|)+|\gamma||w|}(\alpha\beta^2(\delta)\cdot(\alpha\beta(w)\cdot\alpha^2(\gamma)))\cdot\alpha^3\beta(\varepsilon)\\
&+(-1)^{|w||\gamma|+|\delta|(|\gamma|+|w|)+|\varepsilon||w|}\alpha^2\beta^2(\varepsilon)\cdot(\alpha^2\beta(\delta)\cdot(\alpha^2(w)\cdot\alpha^3\beta^{-1}(\gamma)))\\
&-(-1)^{|\varepsilon|(|\gamma|+|\delta|)+|\gamma||w|}(\alpha\beta^2(w)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(\delta)\cdot\alpha^3(\varepsilon))\\
&-(-1)^{|\varepsilon|(|\delta|+|w|)+|\delta|(|\gamma|+|w|)+|\gamma||w|}(\alpha\beta^2(\delta)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\gamma))\\
&\underbrace{-(-1)^{|w||\gamma|+|\varepsilon||\gamma|}(\alpha\beta^2(w)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(\delta))}_{8'}\\
&-(-1)^{|w||\gamma|+|\delta|(|\gamma|+|w|)+|\varepsilon||w|}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(\delta))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\gamma)),
\end{aligned}
$$

$$
\begin{aligned}
&(-1)^{|\gamma|(|w|+|\delta|)}\widetilde{as}_{\alpha,\beta}(\mu(\beta^2(w),\alpha\beta(\varepsilon)),\alpha^2\beta(\delta),\alpha^3(\gamma))\\
=&(-1)^{|\gamma|(|w|+|\delta|)}((\beta^2(w)\cdot\alpha\beta(\varepsilon))\cdot\alpha^2\beta(\delta))\cdot\alpha^3\beta(\gamma)\\
&\underbrace{+(-1)^{|\gamma||\varepsilon|}\alpha^2\beta^2(\gamma)\cdot((\alpha\beta(w)\cdot\alpha^2(\varepsilon))\cdot\alpha^3(\delta))}_{9'}\\
&+(-1)^{|\gamma|(|w|+|\delta|)+|\delta|(|w|+|\varepsilon|)}(\alpha\beta^2(\delta)\cdot(\alpha\beta(w)\cdot\alpha^2(\varepsilon)))\cdot\alpha^3\beta(\gamma)\\
&+(-1)^{|\gamma||\varepsilon|+|\delta|(|w|+|\varepsilon|)}\alpha^2\beta^2(\gamma)\cdot(\alpha^2\beta(\delta)\cdot(\alpha^2(w)\cdot\alpha^3\beta^{-1}(\varepsilon)))
\end{aligned}
$$

$$- (-1)^{|\gamma|(|w|+|\delta|)}(\alpha\beta^2(w)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(\delta)\cdot\alpha^3(\gamma))$$
$$- (-1)^{|\gamma|(|\delta|+|\varepsilon|)+|\delta|(|w|+|\varepsilon|)}(\alpha\beta^2(\delta)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\varepsilon))$$
$$\underbrace{-(-1)^{|\gamma||w|}(\alpha\beta^2(w)\cdot\alpha^2\beta(\varepsilon))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(\delta))}_{10'}$$
$$- (-1)^{|\gamma||\varepsilon|+|\delta|(|w|+|\varepsilon|)}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(\delta))\cdot(\alpha^2\beta(w)\cdot\alpha^3(\varepsilon))$$
$$+ (-1)^{|\gamma|(|w|+|\delta|)+|w||\varepsilon|}((\beta^2(\varepsilon)\cdot\alpha\beta(w))\cdot\alpha^2\beta(\delta))\cdot\alpha^3\beta(\gamma)$$
$$\underbrace{+(-1)^{|w||\varepsilon|+|\varepsilon||\gamma|}\alpha^2\beta^2(\gamma)\cdot((\alpha\beta(\varepsilon)\cdot\alpha^2(w))\cdot\alpha^3(\delta))}_{11'}$$
$$+ (-1)^{|\gamma|(|w|+|\delta|)+|\delta|(|w|+|\varepsilon|)+|w||\varepsilon|}(\alpha\beta^2(\delta)\cdot(\alpha\beta(\varepsilon)\cdot\alpha^2(w)))\cdot\alpha^3\beta(\gamma)$$
$$\underline{+(-1)^{|w||\varepsilon|+|\delta|(|w|+|\varepsilon|)+|\varepsilon||\gamma|}\alpha^2\beta^2(\gamma)\cdot(\alpha^2\beta(\delta)\cdot(\alpha^2(\varepsilon)\cdot\alpha^3\beta^{-1}(w)))}_{⑩}$$
$$- (-1)^{|\gamma|(|w|+|\delta|)+|w||\varepsilon|}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\delta)\cdot\alpha^3(\gamma))$$
$$\underline{-(-1)^{|\gamma|(|\delta|+|\varepsilon|)+|\delta|(|w|+|\varepsilon|)+|w||\varepsilon|}(\alpha\beta^2(\delta)\cdot\alpha^2\beta(\gamma))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(w))}_{⑪}$$
$$\underbrace{-(-1)^{|w||\varepsilon|+|\gamma||w|}(\alpha\beta^2(\varepsilon)\cdot\alpha^2\beta(w))\cdot(\alpha^2\beta(\gamma)\cdot\alpha^3(\delta))}_{12'}$$
$$\underline{-(-1)^{|w||\varepsilon|+|\delta|(|w|+|\varepsilon|)+|\varepsilon||\gamma|}(\alpha\beta^2(\gamma)\cdot\alpha^2\beta(\delta))\cdot(\alpha^2\beta(\varepsilon)\cdot\alpha^3(w))}_{⑫},$$

By Eq (4.4), we have ① + ⋯ + ⑫ = 0, and by Eq (4.5), $1' + \cdots + 12' = 0$, Analogously, the conclusion that the sum is zero can be obtained by recombining the remaining unmarked formulas, which implies

$$\circlearrowleft_{\varepsilon,\gamma,w} (-1)^{|w|(|\varepsilon|+|\delta|)}\widetilde{as}_{\alpha,\beta}(\mu(\beta^2(\varepsilon),\alpha\beta(\gamma)),\alpha^2\beta(\delta),\alpha^3(w)) = 0.$$

This completes the proof.

Definition 4.3. *Let (J,μ,α,β) be a Bihom-Jordan superalgebra, and (V,ρ,ϕ,ψ) be its representation. If even the linear map $T : J \to V$ satisfies the following conditions: for all $a, b \in \mathcal{H}(V)$,*

$$\mu(T(a),T(b)) = T(\rho(T(a))b + (-1)^{|a||b|}\rho(T(\phi^{-1}\psi(b)))\phi\psi^{-1}(a)),$$

$$T\circ\phi = \alpha\circ T,\ T\circ\psi = \beta\circ T,$$

then T is called O-operator with respect to representation.

Definition 4.4. *Let (J,μ,α,β) be a Bihom-Jordan superalgebra and α,β be reversible, $R \in \mathrm{gl}(J)$, R is called Rota–Baxter operator on J, if for all $\varepsilon,\gamma \in \mathcal{H}(J)$, the following conditions hold:*

$$\mu(R(\varepsilon),R(\gamma)) = R(\mu(R(\varepsilon),\gamma) + (-1)^{|\varepsilon||\gamma|}\mu(R(\alpha^{-1}\beta(\gamma)),\alpha\beta^{-1}(\varepsilon))),$$

$$R\circ\alpha = \alpha\circ R,\ R\circ\beta = \beta\circ R.$$

Theorem 4.2. *Let (J,μ,α,β) be a Bihom-Jordan superalgebra, (V,ρ,ϕ,ψ) be its representation, and T be an O-operator with respect to representation. Define bilinear operation $\cdot$ on V:*

$$a\cdot b = \rho(T(a))b,\ \forall a,b \in \mathcal{H}(V).$$

Then $(V,\cdot,\phi,\psi)$ is a Bihom-pre-Jordan superalgebra.

Proof. Actually, it can be calculated directly from Definition 4.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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