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Research article

A study on continuous dependence of layered composite materials in binary mixtures on basic data

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Abstract: This paper investigates the continuous dependence of solutions to layered composite materials in binary mixtures on perturbation parameters defined in a semi-infinite cylinder. Due to the fact that the base of the cylinder is easily disturbed by compression, this causes disturbances to the data at the entrance. By introducing auxiliary functions related to the solution of the equations, this article analyzes the impact of these disturbances on the solutions of the binary heat conduction equations and obtains the continuous dependence of the solutions on the base.

Keywords: binary heat conduction equations; base; continuous dependence

1. Introduction

Since Hirsch and Smale [1] proposed the necessity of structural stability, this topic has received sufficient attention from scholars. This type of research focuses on whether small disturbances in the coefficients, initial data, and geometric models in the equations will cause significant disturbances in the solutions. At the beginning, people were mainly keen on dealing with the continuous dependence and convergence of fluid in porous media defined in two-dimensional or three-dimensional bounded regions. Freitas et al. [2] studied the long-term behavior of porous-elastic systems and proved that solutions depend continuously on the initial data. Payne and Straughan [3] established a prior bounds and maximum principles for the solutions and obtained the structural stability of Darcy fluid in porous media, where they assumed that the temperature satisfies Newton's cooling conditions at the boundary. Scott [4] considered the situation where Darcy fluid undergoes exothermic reactions at the boundary and obtained the continuous dependence of the solutions on the boundary parameters. Li et al. [5] studied the interface connection between Brinkman–Forchheimer fluid and Darcy fluid in a bounded region, and obtained the continuous dependence on the heat source and Forchheimer coefficient. For more papers, on can see [6–10].

With the continuous development of technology and progress in the field of engineering, the necessity of studying the structural stability of fluid equations on a semi-infinite cylinder is even more urgent. The semi-infinite cylinder refers to a cylinder whose generatrix is parallel to the coordinate axis and its base is located on the coordinate plane, i.e.,

$$R = \left\{(x_1, x_2, x_3)\middle|(x_1, x_2) \in D,\ x_3 \geq 0\right\},$$

where D is a bounded domain on x_1Ox_2.

Li et al. have already done some work on this topic. Li and Lin [11] proved the continuous dependence on the Forchheimer coefficient of the Brinkman–Forchheimer equations in R. Papers [12] and [13] obtained structural stability for Forchheimer fluid and temperature-dependent bidispersive flow in R, respectively.

In this paper, we introduce a new cylinder with a disturbed base, which has been considered in [14]. Let $D(f)$ represent the disturbed base, i.e.,

$$D(f) = \left\{(x_1, x_2, x_3)\middle|x_3 = f(x_1, x_2) \geq 0,\ (x_1, x_2) \in D\right\},$$

where the given function f satisfies

$$|f(x_1, x_2)| < \epsilon,\ \epsilon > 0.$$

ϵ is called the perturbation parameter. The cylinder with a disturbed base is defined as

$$R(f) = \left\{(x_1, x_2, x_3)\middle|(x_1, x_2) \in D,\ x_3 \geq f(x_1, x_2) \geq 0\right\}.$$

Different from [14], we study the heat conduction equation applicable to the study of layered composite materials in binary mixtures [15]

$$b_1 u_t = k_1 \triangle u - \gamma(u - v),\ \text{in } R \times \{t > 0\}, \tag{1.1}$$

$$b_2 v_t = k_2 \triangle v + \gamma(u - v),\ \text{in } R \times \{t > 0\}, \tag{1.2}$$

$$u = v = 0,\ \text{on } \partial D \times \{x_3 > 0\} \times \{t > 0\}, \tag{1.3}$$

$$u = v = 0,\ \text{in } R \times \{t = 0\}, \tag{1.4}$$

where k_1, k_2, b_1, b_2 and γ are positive constants. u and v are the temperature fields in each constituent. Papers [16–18] further discussed and generalized the application of Eqs (1.1) and (1.2).

In this paper, we shall also use the notations

$$R(z) = \left\{(x_1, x_2, x_3)\middle|(x_1, x_2) \in D, x_3 \geq z \geq 0\right\},$$

$$D(z) = \left\{(x_1, x_2, x_3)\middle|(x_1, x_2) \in D, x_3 = z \geq 0\right\}.$$

The main work of this article investigates the continuous dependence of solutions to Eqs (1.1)–(1.4) on perturbation parameters and base data. Due to many practical constraints, it is very common for the base of the cylinder to experience minor disturbance. Therefore, studying the effects of these disturbances is essential. To this end, we assume that u^* and v^* are perturbed solutions of Eqs (1.1)–(1.4) on $R(f)$, and then prove that the difference between the unperturbed solutions and the perturbed solutions satisfies a first-order differential inequality. By solving this inequality, we can obtain the continuous dependence of the solution.

2. A priori bounds

On the finite end D, we assume that the solutions to (1.1)–(1.4) satisfy

$$u(t,\boldsymbol{x}) = L_{11}(t,x_1,x_2),\ v(t,\boldsymbol{x}) = L_{12}(t,x_1,x_2), t>0,\ x_3=0,\ (x_1,x_2)\in D(0), \tag{2.1}$$

$$u^*(t,\boldsymbol{x}) = L_{21}(t,x_1,x_2),\ v^*(t,\boldsymbol{x}) = L_{22}(t,x_1,x_2), t>0,\ x_3=f(x_1,x_2),\ (x_1,x_2)\in D(0). \tag{2.2}$$

In (2.1) and (2.2), the known functions $L_{ij}(i,j=1,2)$ satisfy the compatibility conditions on ∂D.

We let that $\mathcal{H}_1(t,\boldsymbol{x})$ and $\mathcal{H}_2(t,\boldsymbol{x})$ are specific functions who have the same boundary conditions as u^* and v^*, respectively. That is

$$\mathcal{H}_1(t,\boldsymbol{x}) = L_{21}(t,x_1,x_2)\exp\{-\sigma(x_3-f)\},\ \mathcal{H}_2(t,\boldsymbol{x}) = L_{22}(t,x_1,x_2)\exp\{-\sigma(x_3-f)\}, \tag{2.3}$$

where $\sigma>0$.

We now derive some lemmas.

Lemma 2.1. If $L_{21},L_{22}\in H^1([0,\infty)\times D(f))$, then

$$\int_0^t \exp\{-\eta_1\tau\}\Big[k_1\|\nabla u^*(\tau)\|^2_{L^2(R(f))} + k_2\|\nabla v^*(\tau)\|^2_{L^2(R(f))}\Big]d\tau \le d_1(t),$$

where

$$\begin{aligned} d_1(t) = &\int_0^t \exp\{-\eta_1\tau\}\Big[k_1\|\nabla\mathcal{H}_1\|^2_{L^2(R(f))} + k_2\|\nabla\mathcal{H}_2\|^2_{L^2(R(f))}\Big]d\tau\\ &+\exp\{-\eta_1 t\}\Big[b_1\|\mathcal{H}_1(t)\|^2_{L^2(R(f))} + b_2\|\mathcal{H}_2(t)\|^2_{L^2(R(f))}\Big]\\ &+\frac{1}{2}\int_0^t \exp\{-\eta_1\tau\}\Big[b_1\eta_1\|\mathcal{H}_{1,\tau}(\tau)\|^2_{L^2(R(f))} + b_2\eta_1\|\mathcal{H}_{2,\tau}(\tau)\|^2_{L^2(R(f))}\Big]d\tau\\ &+\frac{1}{2}\gamma\int_0^t \exp\{-\eta_1\tau\}\|(\mathcal{H}_1-\mathcal{H}_2)(\tau)\|^2_{L^2(R(f))}d\tau. \end{aligned} \tag{2.4}$$

Proof. Using (1.1)–(1.4), we begin with

$$\int_0^t\int_{R(f)} \exp\{-\eta_1\tau\}\Big[b_1u^*_\tau - k_1\triangle u^* + \gamma(u^*-v^*)\Big]u^*dxd\tau = 0,$$

$$\int_0^t\int_{R(f)} \exp\{-\eta_1\tau\}\Big[b_2v^*_\tau - k_2\triangle v^* - \gamma(u^*-v^*)\Big]v^*dxd\tau = 0.$$

We compute

$$\begin{aligned} &\frac{1}{2}\exp\{-\eta_1 t\}\Big[b_1\|u^*(t)\|^2_{L^2(R(f))} + b_2\|v^*(t)\|^2_{L^2(R(f))}\Big]\\ &\quad+\int_0^t \exp\{-\eta_1\tau\}\Big[b_1\eta_1\|u^*(\tau)\|^2_{L^2(R(f))} + b_2\eta_1\|v^*(\tau)\|^2_{L^2(R(f))}\Big]d\tau\\ &\quad+\int_0^t \exp\{-\eta_1\tau\}\Big[k_1\|\nabla u^*(\tau)\|^2_{L^2(R(f))} + k_2\|\nabla v^*(\tau)\|^2_{L^2(R(f))}\Big]d\tau\\ &\quad+\gamma\int_0^t \exp\{-\eta_1\tau\}\|(u^*-v^*)(\tau)\|^2_{L^2(R(f))}d\tau\\ &\quad= -\int_0^t\int_{D(f)} \exp\{-\eta_1\tau\}\Big[k_1\frac{\partial u^*}{\partial x_3}u^* + k_2\frac{\partial v^*}{\partial x_3}v^*\Big]dAd\tau. \end{aligned} \tag{2.5}$$

On the other hand, we use (2.3) to compute

$$
\begin{aligned}
-\int_0^t\int_{D(f)}&\exp\{-\eta_1\tau\}\Big[k_1\frac{\partial u^*}{\partial x_3}u^*+k_2\frac{\partial v^*}{\partial x_3}v^*\Big]dAd\tau\\
&=-\int_0^t\int_{D(f)}\exp\{-\eta_1\tau\}\Big[k_1\frac{\partial u^*}{\partial x_3}\mathcal{H}_1+k_2\frac{\partial v^*}{\partial x_3}\mathcal{H}_2\Big]dAd\tau\\
&=\int_0^t\int_{R(f)}\exp\{-\eta_1\tau\}\Big[k_1\nabla\cdot\Big(\nabla u^*\mathcal{H}_1\Big)+k_2\nabla\cdot\Big(\nabla v^*\mathcal{H}_2\Big)dxd\tau\\
&=\int_0^t\int_{R(f)}\exp\{-\eta_1\tau\}\Big[k_1\nabla u^*\cdot\nabla\mathcal{H}_1+k_2\nabla v^*\cdot\nabla\mathcal{H}_2\Big]dxd\tau\\
&+\exp\{-\eta_1 t\}\int_{R(f)}\Big[b_1u^*\mathcal{H}_1+b_2v^*\mathcal{H}_2\Big]dx\\
&+\eta_1\int_0^t\int_{R(f)}\exp\{-\eta_1\tau\}\Big[b_1u^*\mathcal{H}_{1,\tau}+b_2v^*\mathcal{H}_{2,\tau}\Big]dxd\tau\\
&+\gamma\int_0^t\int_{R(f)}\exp\{-\eta_1\tau\}(u^*-v^*)(\mathcal{H}_1-\mathcal{H}_2)dxd\tau\\
&\doteq\mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3+\mathcal{F}_4.
\end{aligned}\tag{2.6}
$$

An application of the Schwarz inequality leads to

$$
\begin{aligned}
\mathcal{F}_1\leq&\frac{1}{2}\int_0^t\exp\{-\eta_1\tau\}\Big[k_1\|\nabla u^*(\tau)\|^2_{L^2(R(f))}+k_2\|\nabla v^*(\tau)\|^2_{L^2(R(f))}\Big]d\tau\\
&+\frac{1}{2}\int_0^t\exp\{-\eta_1\tau\}\Big[k_1\|\nabla\mathcal{H}_1\|^2_{L^2(R(f))}+k_2\|\nabla\mathcal{H}_2\|^2_{L^2(R(f))}\Big]d\tau,
\end{aligned}\tag{2.7}
$$

$$
\begin{aligned}
\mathcal{F}_2\leq&\frac{1}{2}\exp\{-\eta_1 t\}\Big[b_1\|u^*(t)\|^2_{L^2(R(f))}+b_2\|v^*(t)\|^2_{L^2(R(f))}\Big]\\
&+\frac{1}{2}\exp\{-\eta_1 t\}\Big[b_1\|\mathcal{H}_1(t)\|^2_{L^2(R(f))}+b_2\|\mathcal{H}_2(t)\|^2_{L^2(R(f))}\Big],
\end{aligned}\tag{2.8}
$$

$$
\begin{aligned}
\mathcal{F}_3\leq&\int_0^t\exp\{-\eta_1\tau\}\Big[b_1\eta_1\|u^*(\tau)\|^2_{L^2(R(f))}+b_2\eta_1\|v^*(\tau)\|^2_{L^2(R(f))}\Big]d\tau\\
&+\frac{1}{4}\int_0^t\exp\{-\eta_1\tau\}\Big[b_1\eta_1\|\mathcal{H}_{1,\tau}(\tau)\|^2_{L^2(R(f))}+b_2\eta_1\|\mathcal{H}_{2,\tau}(\tau)\|^2_{L^2(R(f))}\Big]d\tau,
\end{aligned}\tag{2.9}
$$

$$
\begin{aligned}
\mathcal{F}_4\leq&\gamma\int_0^t\exp\{-\eta_1\tau\}\|(u^*-v^*)(\tau)\|^2_{L^2(R(f))}d\tau\\
&+\frac{1}{4}\gamma\int_0^t\exp\{-\eta_1\tau\}\|(\mathcal{H}_1-\mathcal{H}_2)(\tau)\|^2_{L^2(R(f))}d\tau.
\end{aligned}\tag{2.10}
$$

Inserting Eqs (2.7)–(2.10) into (2.6) and combining (2.5), it can be obtained

$$
\begin{aligned}
&\int_0^t \exp\{-\eta_1\tau\}\Big[k_1\|\nabla u^*(\tau)\|^2_{L^2(R(f))} + k_2\|\nabla v^*(\tau)\|^2_{L^2(R(f))}\Big]d\tau \\
&\qquad\qquad \le \int_0^t \exp\{-\eta_1\tau\}\Big[k_1\|\nabla\mathcal{H}_1\|^2_{L^2(R(f))} + k_2\|\nabla\mathcal{H}_2\|^2_{L^2(R(f))}\Big]d\tau \\
&\qquad\qquad + \exp\{-\eta_1 t\}\Big[b_1\|\mathcal{H}_1(t)\|^2_{L^2(R(f))} + b_2\|\mathcal{H}_2(t)\|^2_{L^2(R(f))}\Big] \\
&\qquad\qquad + \frac{1}{2}\int_0^t \exp\{-\eta_1\tau\}\Big[b_1\eta_1\|\mathcal{H}_{1,\tau}(\tau)\|^2_{L^2(R(f))} + b_2\eta_1\|\mathcal{H}_{2,\tau}(\tau)\|^2_{L^2(R(f))}\Big]d\tau \\
&\qquad\qquad + \frac{1}{2}\gamma\int_0^t \exp\{-\eta_1\tau\}\|(\mathcal{H}_1 - \mathcal{H}_2)(\tau)\|^2_{L^2(R(f))}d\tau.
\end{aligned}
\tag{2.11}
$$

From (2.11), we can conclude that Lemma 2.1 holds.

We not only need a prior bounds for v and v^*, but also for u and u^*. Since u and u^* are undisturbed solutions of Eqs (1.1)–(1.4), in Lemma 2.1 we only need to set $f = 0$ and replace L_{21} and L_{22} with L_{11} and L_{12}, respectively, and then we can obtain the a prior bounds for u and u^*.

Lemma 2.2. If $L_{11}, L_{12} \in H^1([0,\infty) \times D)$, then

$$
\int_0^t \exp\{-\eta_1\tau\}\Big[k_1\|\nabla u(\tau)\|_{L^2(R)} + k_2\|\nabla v(\tau)\|_{L^2(R)}\Big]d\tau \le d_2(t),
$$

where

$$
\begin{aligned}
d_2(t) = &\int_0^t \exp\{-\eta_1\tau\}\Big[k_1\|\nabla\mathcal{H}_3\|^2_{L^2(R)} + k_2\|\nabla\mathcal{H}_4\|^2_{L^2(R)}\Big]d\tau \\
&+ \exp\{-\eta_1 t\}\Big[b_1\|\mathcal{H}_3(t)\|^2_{L^2(R)} + b_2\|\mathcal{H}_4(t)\|^2_{L^2(R)}\Big] \\
&+ \frac{1}{2}\int_0^t \exp\{-\eta_1\tau\}\Big[b_1\eta_1\|\mathcal{H}_{3,\tau}(\tau)\|^2_{L^2(R)} + b_2\eta_1\|\mathcal{H}_{4,\tau}(\tau)\|^2_{L^2(R)}\Big]d\tau \\
&+ \frac{1}{2}\gamma\int_0^t \exp\{-\eta_1\tau\}\|(\mathcal{H}_3 - \mathcal{H}_4)(\tau)\|^2_{L^2(R)}d\tau
\end{aligned}
$$

and

$$
\mathcal{H}_3(t,\boldsymbol{x}) = L_{11}(t,x_1,x_2)\exp\{-\sigma x_3\},\ \mathcal{H}_4(t,\boldsymbol{x}) = L_{12}(t,x_1,x_2)\exp\{-\sigma x_3\}.
$$

Remark 2.1. Lemmas 2.1 and 2.2 will provide a priori estimates for the proof of the lemmas in the next section.

3. Auxiliary functions

Let w and s represent the difference between the perturbed solutions and the unperturbed solutions, i.e.,

$$
w = u - u^*,\ s = v - v^*, \tag{3.1}
$$

then w and s satisfy

$$b_1 w_t = k_1 \triangle w - \gamma(w-s), \text{ in } R(\epsilon)\times\{t>0\}, \tag{3.2}$$
$$b_2 s_t = k_2 \triangle s + \gamma(w-s), \text{ in } R(\epsilon)\times\{t>0\}, \tag{3.3}$$
$$w = s = 0, \text{ on } \partial D\times\{x_3>\epsilon\}\times\{t>0\}, \tag{3.4}$$
$$w = s = 0, \text{ in } R(\epsilon)\times\{t=0\}. \tag{3.5}$$

To obtain the continuous dependence of the solution on the perturbation parameter, we establish a new energy function

$$V(t,x_3) = \int_0^t \Big[\|w(\tau)\|^2_{L^2(R(x_3))} + \|s(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau,\ \ x_3 \ge \epsilon. \tag{3.6}$$

Noting the definition of $R(x_3)$, we can obtain the derivative of $V(t,x_3)$ as follows:

$$-\frac{\partial}{\partial x_3}V(t,x_3) = \int_0^t \Big[\|w(\tau)\|^2_{L^2(D(x_3))} + \|s(\tau)\|^2_{L^2(D(x_3))}\Big]d\tau.$$

We introduce two auxiliary functions φ and ψ such that

$$b_1\varphi_\tau + k_1\triangle\varphi = -w,\ \ b_2\psi_\tau + k_2\triangle\psi = -s, \text{ in } R(x_3), 0<\tau<t, \tag{3.7}$$
$$\varphi(\tau,x_1,x_2,x_3) = \psi(\tau,x_1,x_2,x_3) = 0, \text{ on } \partial D\times\{x_3\}, 0<\tau<t, \tag{3.8}$$
$$\varphi(\tau,x_1,x_2,x_3) = \psi(\tau,x_1,x_2,x_3) = 0, (x_1,x_2)\in D, 0<\tau<t, \tag{3.9}$$
$$\varphi(t,\boldsymbol{x}) = \psi(t,\boldsymbol{x}) = 0, \text{ in } R(x_3), \tag{3.10}$$
$$\varphi, \nabla\varphi, \psi, \nabla\psi \to 0(\text{uniformly in } x_1,x_2,\tau) \text{ as } x_3\to\infty, \tag{3.11}$$

where $x_3>\epsilon$.

Next, we will derive some necessary properties of the auxiliary functions, which will play a crucial role in proving the continuous dependence of the solutions.

Lemma 3.1. If $\varphi,\psi\in H^1([0,t]\times R(x_3))$, then

$$\int_0^t \Big[b_1\|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))} + b_2\|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau \le a_1 V(t,x_3),\ \ x_3\ge\epsilon,$$

where $a_1 = \max\{b_1^{-1}, b_2^{-1}\}$.

Proof. We begin with

$$\int_0^t\int_{R(x_3)} \varphi_\tau\Big[b_1\varphi_\tau + k_1\triangle\varphi + w\Big]dxd\tau = 0,$$
$$\int_0^t\int_{R(x_3)} \psi_\tau\Big[b_2\psi_\tau + k_2\triangle\psi + s\Big]dxd\tau = 0.$$

Using the divergence theorem $\oint_{\partial R(x_3)} Fds = \int_{R(x_3)} divFdx$ and (3.8)–(3.11), we have

$$
\begin{aligned}
b_1\int_0^t \|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau &= -\frac{1}{2}k_1\|\nabla\varphi(0)\|^2_{L^2(R(x_3))} + \int_0^t\int_{R(x_3)} w\varphi_\tau dxd\tau \\
&\le \Big[\int_0^t \|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau \int_0^t \|w(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}, \qquad (3.12)
\end{aligned}
$$

and

$$
b_2\int_0^t \|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau \le \Big[\int_0^t \|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau \int_0^t \|s(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}. \qquad (3.13)
$$

Using the Schwarz inequality, (3.12) and (3.13), Lemma 3.1 can be obtained.

Lemma 3.2. If $\varphi, \psi \in H^1(R(x_3))$, then

$$
\int_0^t \Big[k_1\|\nabla\varphi(\tau)\|^2_{L^2(R(x_3))} + k_2\|\nabla\psi(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau \le a_2V(t, x_3),
$$

where $a_2 = \frac{1}{\lambda}\max\{k_1^{-1}, k_2^{-1}\}$.
Proof. We begin with

$$
\begin{aligned}
\int_0^t\int_{R(x_3)} \varphi\Big[b_1\varphi_\tau + k_1\triangle\varphi + w\Big]dxd\tau = 0, \\
\int_0^t\int_{R(x_3)} \varphi\Big[b_2\psi_\tau + k_2\triangle\psi + s\Big]dxd\tau = 0.
\end{aligned}
$$

Using the divergence theorem and Lemma 2.2, we have

$$
\begin{aligned}
k_1\int_0^t \|\nabla\varphi(\tau)\|^2_{L^2(R(x_3))}d\tau &= -\frac{1}{2}b_1\|\varphi(0)\|^2_{L^2(R(x_3))} + \int_0^t\int_{R(x_3)} w\varphi dxd\tau \\
&\le \Big[\int_0^t \|\varphi(\tau)\|^2_{L^2(R(x_3))}d\tau \int_0^t \|w(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{\lambda}}\Big[\int_0^t \|\nabla_2\varphi(\tau)\|^2_{L^2(R(x_3))}d\tau \int_0^t \|w(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}} \qquad (3.14)
\end{aligned}
$$

and

$$
k_2\int_0^t \|\nabla\psi(\tau)\|^2_{L^2(R(x_3))}d\tau \le \frac{1}{\sqrt{\lambda}}\Big[\int_0^t \|\nabla_2\psi(\tau)\|^2_{L^2(R(x_3))}d\tau \int_0^t \|s(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}. \qquad (3.15)
$$

Using the following inequality

$$
\sqrt{ab} + \sqrt{cd} \le \sqrt{(a+c)(b+d)}, \text{ for } a, b, c, d > 0, \qquad (3.16)
$$

the Young inequality and Lemma 3.1, we can have from (3.14) and (3.15)

$$\int_0^t \Big[k_1\|\nabla\varphi(\tau)\|^2_{L^2(R(x_3))} + k_2\|\nabla\psi(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau$$
$$\leq \frac{1}{\sqrt{\lambda}}\Big\{\int_0^t \Big[k_1\|\nabla_2\varphi(\tau)\|^2_{L^2(R(x_3))} + k_2\|\nabla_2\psi(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau$$
$$\cdot\int_0^t \Big[k_1^{-1}\|w(\tau)\|^2_{L^2(R(x_3))} + k_2^{-1}\|s(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau\Big\}^{\frac{1}{2}}. \tag{3.17}$$

From (3.17) we can obtain Lemma 3.2.

Lemma 3.3. If $\varphi, \psi \in H^1(R(x_3))$, then

$$k_1\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau + k_2\int_0^t \|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau \leq a_3V(t,x_3),$$

where a_3 is a positive constant.
Proof. Letting δ be a positive constant. We compute

$$\int_0^t\int_{R(x_3)}\Big[\frac{\partial\varphi}{\partial x_3} - \delta\varphi_\tau\Big]\Big[b_1\varphi_\tau + k_1\triangle\varphi + w\Big]dxd\tau = 0, \tag{3.18}$$
$$\int_0^t\int_{R(x_3)}\Big[\frac{\partial\psi}{\partial x_3} - \delta\psi_\tau\Big]\Big[b_2\psi_\tau + k_2\triangle\psi + s\Big]dxd\tau = 0. \tag{3.19}$$

Using the divergence theorem and (3.8)–(3.10) in (3.18) and (3.19), we obtain

$$\frac{1}{2}k_1\delta\|\nabla\varphi(0)\|^2_{L^2(R(x_3))}d\tau + b_1\delta\int_0^t \|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau + \frac{1}{2}k_1\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau$$
$$= \int_0^t\int_{R(x_3)}\frac{\partial\varphi}{\partial x_3}\varphi_\tau dxd\tau + \int_0^t\int_{R(x_3)}\Big[\frac{\partial\varphi}{\partial x_3} - \delta\varphi_\tau\Big]wdxd\tau. \tag{3.20}$$

Using the Schwarz inequality, we obtain

$$\int_0^t\int_{R(x_3)}\frac{\partial\varphi}{\partial x_3}\varphi_\tau dxd\tau \leq \Big[\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t \|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}, \tag{3.21}$$
$$\int_0^t\int_{R(x_3)}\frac{\partial\varphi}{\partial x_3}wdxd\tau \leq \Big[\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t \|w(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}, \tag{3.22}$$
$$-\delta\int_0^t\int_{R(x_3)}\varphi_\tau wdxd\tau \leq \delta\Big[\int_0^t \|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t \|w(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}. \tag{3.23}$$

Inserting (3.21)–(3.23) into (3.20) and dropping the first two terms in the left of (3.20), we have

$$\frac{1}{2}k_1\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau \leq \Big[\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t \|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}$$
$$+\Big[\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t \|w(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}$$
$$+\delta\Big[\int_0^t \|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t \|w(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}. \tag{3.24}$$

Similar, we can also have from (3.19)

$$\frac{1}{2}k_2\int_0^t\|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau\le\Big[\int_0^t\|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t\|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}$$
$$+\Big[\int_0^t\|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t\|s(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}$$
$$+\delta\Big[\int_0^t\|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}d\tau\int_0^t\|s(\tau)\|^2_{L^2(R(x_3))}d\tau\Big]^{\frac{1}{2}}. \tag{3.25}$$

Using (3.16) and Lemmas 3.1 and 3.2, we obtain

$$k_1\int_0^t\|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau+k_2\int_0^t\|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau$$
$$\le 2a_1a_2\Big\{\int_0^t\Big[b_1\|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}+b_2\|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau$$
$$\cdot\int_0^t\Big[k_1\|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}+k_2\|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau\Big\}^{\frac{1}{2}}$$
$$+2a_2\Big\{\int_0^t\Big[b_1\|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}+b_2\|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau$$
$$\cdot\int_0^t\Big[\|w(\tau)\|^2_{L^2(R(x_3))}+\|s(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau\Big\}^{\frac{1}{2}}$$
$$+2a_1\delta\Big\{\int_0^t\Big[k_1\|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))}+k_2\|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau$$
$$\cdot\int_0^t\Big[\|w(\tau)\|^2_{L^2(R(x_3))}+\|s(\tau)\|^2_{L^2(R(x_3))}\Big]d\tau\Big\}^{\frac{1}{2}}$$
$$\le a_3V(t,x_3), \tag{3.26}$$

where $a_3=2a_1^2a_2^2+2a_2^2+2a_1^2$.

In the next section, we will use Lemma 3.3 to derive the continuous dependence of the solutions.

4. Main results

In this section, we first derive a bound for $V(t,\epsilon)$. To do this, we define

$$u(t,\boldsymbol{x})=L_{11}(t,x_1,x_2),\ v(t,\boldsymbol{x})=L_{12}(t,x_1,x_2),\ -\epsilon\le x_3\le 0,(x_1,x_2)\in D,t\in[0,+\infty), \tag{4.1}$$
$$u^*(t,\boldsymbol{x})=L_{21}(t,x_1,x_2),\ v^*(t,\boldsymbol{x})=L_{22}(t,x_1,x_2),\ -\epsilon\le x_3\le f(x_1,x_2),(x_1,x_2)\in D,t\in[0,+\infty). \tag{4.2}$$

When $-\epsilon\le x_3\le\epsilon$, we let

$$w(t,\boldsymbol{x})=u(t,\boldsymbol{x})-u^*(t,\boldsymbol{x}),\ s(t,\boldsymbol{x})=v(t,\boldsymbol{x})-v^*(t,\boldsymbol{x}),(x_1,x_2)\in D,t\in[0,+\infty). \tag{4.3}$$

In view of (3.1) and (4.3), using the triangle inequality, it can be obtained that

$$
\begin{aligned}
&k_1\int_0^t\int_{R(-\epsilon)}\Big(\frac{\partial w}{\partial x_3}\Big)^2dxd\tau+k_2\int_0^t\int_{R(-\epsilon)}\Big(\frac{\partial s}{\partial x_3}\Big)^2dxd\tau\\
&\qquad\le\int_0^t\int_{R(-\epsilon)}\Big[k_1\Big(\frac{\partial u}{\partial x_3}\Big)^2+k_2\Big(\frac{\partial v}{\partial x_3}\Big)^2\Big]dxd\tau\\
&\qquad+\int_0^t\int_{R(-\epsilon)}\Big[k_1\Big(\frac{\partial u^*}{\partial x_3}\Big)^2+k_2\Big(\frac{\partial v^*}{\partial x_3}\Big)^2\Big]dxd\tau.
\end{aligned}\tag{4.4}
$$

Using Lemmas 2.1 and 2.2, (4.1) and (4.2), from (4.4), we obtain

$$
\begin{aligned}
&k_1\int_0^t\int_{R(-\epsilon)}\Big(\frac{\partial w}{\partial x_3}\Big)^2dxd\tau+k_2\int_0^t\int_{R(-\epsilon)}\Big(\frac{\partial s}{\partial x_3}\Big)^2dxd\tau\\
&\qquad\le\int_0^t\int_{R}\Big[k_1\Big(\frac{\partial u}{\partial x_3}\Big)^2+k_2\Big(\frac{\partial v}{\partial x_3}\Big)^2\Big]dxd\tau\\
&\qquad+\int_0^t\int_{R(f)}\Big[k_1\Big(\frac{\partial u^*}{\partial x_3}\Big)^2+k_2\Big(\frac{\partial v^*}{\partial x_3}\Big)^2\Big]dxd\tau.\\
&\qquad\le e^{\eta_1 t}[d_1(t)+d_2(t)]\doteq d_3(t).
\end{aligned}\tag{4.5}
$$

Now, we write the main theorem as:

Theorem 4.1. If $L_{11},L_{12}\in H^1([0,\infty)\times R), L_{21},L_{22}\in H^1([0,\infty)\times R(f))$ and $t<\frac{\pi}{4a_1\gamma}$, then

$$
\begin{aligned}
V(t,x_3)\le\exp\{-d_4(x_3-\epsilon)\}\Big\{&\frac{32}{d_4\pi}\max\{\frac{1}{k_1},\frac{1}{k_2}\}d_3(t)\epsilon\\
&+d_5\int_0^t\Big[\|(L_{11}-L_{21})(\tau)\|^2_{L^2(D)}+\|(L_{12}-L_{22})(\tau)\|^2_{L^2(D)}\Big]d\tau\Big\}, x_3\ge\epsilon
\end{aligned}
$$

holds, where $d_4=a_3^{-1}\max\{k_1,k_2\}^{-1}$ and $d_5=\frac{d_4\pi}{2}+\frac{2}{d_4}$.

Proof. Let $x_3\ge\epsilon$ be a fixed point on the coordinate axis x_3. Using (3.7)–(3.11) and the divergence theorem, we can have

$$
\begin{aligned}
V(x_3,t)&=-\int_0^t\int_{R(x_3)}w\Big[b_1\varphi_\tau+k_1\triangle\varphi\Big]dxd\tau-\int_0^t\int_{R(x_3)}s\Big[b_2\psi_\tau+k_2\triangle\psi\Big]dxd\tau\\
&=-\int_0^t\int_{R(x_3)}\Big[b_1\varphi_\tau w+b_2\psi_\tau s\Big]dxd\tau+\int_0^t\int_{R(x_3)}\Big[k_1\nabla w\cdot\nabla\varphi+k_2\nabla s\cdot\nabla\psi\Big]dxd\tau\\
&+\int_0^t\int_{D(x_3)}\Big[k_1w\frac{\partial\varphi}{\partial x_3}+k_2s\frac{\partial\psi}{\partial x_3}\Big]dAd\tau\\
&=-\int_0^t\int_{R(x_3)}\Big[b_1\varphi_\tau w+b_2\psi_\tau s\Big]dxd\tau-\int_0^t\int_{R(x_3)}\Big[k_1\triangle w\varphi+k_2\triangle s\psi\Big]dxd\tau\\
&+\int_0^t\int_{D(x_3)}\Big[k_1w\frac{\partial\varphi}{\partial x_3}+k_2s\frac{\partial\psi}{\partial x_3}\Big]dAd\tau\\
&=-\int_0^t\int_{R(x_3)}\Big[b_1\varphi_\tau w+b_2\psi_\tau s\Big]dxd\tau-\int_0^t\int_{R(x_3)}\Big[b_1\varphi w_\tau+b_2\psi s_\tau\Big]dxd\tau\\
&+\int_0^t\int_{D(x_3)}\Big[k_1w\frac{\partial\varphi}{\partial x_3}+k_2s\frac{\partial\psi}{\partial x_3}\Big]dAd\tau-\gamma\int_0^t\int_{R(x_3)}\Big(\varphi-\psi\Big)\Big(w-s\Big)dxd\tau.
\end{aligned}\tag{4.6}
$$

In light of (1.4) and (3.10), it is clear that

$$\int_0^t \int_{R(x_3)} \Big[b_1\varphi_\tau w + b_1\varphi w_\tau\Big]dxd\tau = 0, \ \int_0^t \int_{R(x_3)} \Big[b_2\psi_\tau s + b_2\psi s_\tau\Big]dxd\tau = 0. \tag{4.7}$$

A combination of the Hölder inequality, (3.16) and Lemma 3.3 leads to

$$\begin{aligned}
&\int_0^t \int_{D(x_3)} \Big[k_1 w\frac{\partial\varphi}{\partial x_3} + k_2 s\frac{\partial\psi}{\partial x_3}\Big]dAd\tau \\
&\qquad \le k_1\Big[\int_0^t \|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau \int_0^t \|w(\tau)\|^2_{L^2(D(x_3))}d\tau\Big]^{\frac{1}{2}} \\
&\qquad + k_2\Big[\int_0^t \|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}d\tau \int_0^t \|s(\tau)\|^2_{L^2(D(x_3))}d\tau\Big]^{\frac{1}{2}} \\
&\qquad \le \max\{\sqrt{k_1}, \sqrt{k_2}\}\Big[\int_0^t \Big(k_1\|\frac{\partial\varphi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))} + k_2\|\frac{\partial\psi}{\partial x_3}(\tau)\|^2_{L^2(D(x_3))}\Big)d\tau\Big]^{\frac{1}{2}} \\
&\qquad \cdot\Big[\int_0^t \Big(\|w(\tau)\|^2_{L^2(D(x_3))} + \|s(\tau)\|^2_{L^2(D(x_3))}\Big)d\tau\Big]^{\frac{1}{2}} \\
&\qquad \le \sqrt{a_3}\max\{\sqrt{k_1}, \sqrt{k_2}\}\sqrt{V(t,x_3)}\Big[-\frac{\partial}{\partial x_3}V(t,x_3)\Big]^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

For the fourth term in the right of (4.6), we compute

$$\begin{aligned}
&-\gamma\int_0^t \int_{R(x_3)} \Big(\varphi - \psi\Big)\Big(w - s\Big)dxd\tau \\
&\qquad \le \gamma\Big[\int_0^t \Big(\|\varphi(\tau)\|^2_{L^2(R(x_3))} + \|\psi(\tau)\|^2_{L^2(R(x_3))}\Big)d\tau \\
&\qquad \cdot\int_0^t \Big(\|w(\tau)\|^2_{L^2(R(x_3))} + \|s(\tau)\|^2_{L^2(R(x_3))}\Big)d\tau\Big]^{\frac{1}{2}}.
\end{aligned} \tag{4.9}$$

Using the inequality (see p182 in [19])

$$\int_0^1 \phi^2 dx \le \frac{4}{\pi^2}\int_0^1 (\phi')^2 dx, \ \text{for } \phi(0) = 0, \tag{4.10}$$

we have from (4.9)

$$\begin{aligned}
&-\gamma\int_0^t \int_{R(x_3)} \Big(\varphi - \psi\Big)\Big(w - s\Big)dxd\tau \\
&\qquad \le \gamma\frac{2t}{\pi}\Big[\int_0^t \Big(\|\varphi_\tau(\tau)\|^2_{L^2(R(x_3))} + \|\psi_\tau(\tau)\|^2_{L^2(R(x_3))}\Big)d\tau V(t,x_3)\Big]^{\frac{1}{2}} \\
&\qquad \le \gamma\frac{2t}{\pi}a_1 V(t,x_3),
\end{aligned} \tag{4.11}$$

where we have also used Lemma 3.1. Combining (4.6), (4.7), (4.14) and (4.11) and choosing $t < \frac{\pi}{4a_1\gamma}$, we can have

$$V(t,x_3) \le -\frac{1}{d_4}\frac{\partial}{\partial x_3}V(t,x_3), x_3 > \epsilon. \tag{4.12}$$

Integrating (4.12) from ϵ to x_3, we have

$$V(t,x_3) \le V(t,\epsilon)\exp\{-d_4(x_3-\epsilon)\}, x_3 \ge \epsilon. \tag{4.13}$$

Equation (4.13) only indicates that the solutions to (1.1)–(1.4) decay exponentially as $x_3 \to \infty$. This decay result is not rigorous because we do not yet know whether $V(t,\epsilon)$ depends on the perturbation parameter ϵ. Therefore, we derive the explicit bound of $V(t,\epsilon)$ in terms of ϵ and $L_{ij}(ij = 1, 2)$.

After letting $x_3 = \epsilon$ in (4.12), we have

$$\begin{aligned}
V(\epsilon,t) \le{}& \frac{1}{d_4}\int_0^t \Big[\|w(\tau)\|^2_{L^2(D(\epsilon))} + \|s(\tau)\|^2_{L^2(D(\epsilon))}\Big]dAd\tau \\
={}& \frac{2}{d_4}\int_0^t\int_{-\epsilon}^{\epsilon}\int_{D(x_3)}\Big[w\frac{\partial w}{\partial x_3} + s\frac{\partial s}{\partial x_3}\Big]dxd\tau \\
&+ \frac{2}{d_4}\int_0^t\Big[\|(L_{11}-L_{21})(\tau)\|^2_{L^2(D)} + \|(L_{12}-L_{22})(\tau)\|^2_{L^2(D)}\Big]d\tau \\
\le{}& \frac{2}{d_4}\Big[\int_0^t \|w(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau\int_0^t\|\frac{\partial w}{\partial x_3}(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau\Big]^{\frac{1}{2}} \\
&+ \frac{2}{d_4}\Big[\int_0^t \|s(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau\int_0^t\|\frac{\partial s}{\partial x_3}(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau\Big]^{\frac{1}{2}} \\
&+ \frac{2}{d_4}\int_0^t\Big[\|(L_{11}-L_{21})(\tau)\|^2_{L^2(D)} + \|(L_{12}-L_{22})(\tau)\|^2_{L^2(D)}\Big]d\tau.
\end{aligned} \tag{4.14}$$

Using (4.10) again, we have

$$\begin{aligned}
\int_0^t \|w(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau \le{}& \frac{16\epsilon^2}{\pi^2}\int_0^t\|\frac{\partial w}{\partial x_3}(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau \\
&+ 2\epsilon\int_0^t\|(L_{11}-L_{21})(\tau)\|^2_{L^2(D)}d\tau,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\int_0^t \|s(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau \le{}& \frac{16\epsilon^2}{\pi^2}\int_0^t\|\frac{\partial s}{\partial x_3}(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}d\tau \\
&+ 2\epsilon\int_0^t\|(L_{12}-L_{22})(\tau)\|^2_{L^2(D)}d\tau.
\end{aligned} \tag{4.16}$$

Inserting (4.15) into (4.16) and combining the Schwarz inequality, we obtain

$$\begin{aligned}
V(\epsilon,t) \le{}& \frac{32}{d_4\pi}\epsilon\int_0^t\Big[\|\frac{\partial w}{\partial x_3}(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])} + \|\frac{\partial s}{\partial x_3}(\tau)\|^2_{L^2(D(x_3)\times[-\epsilon,\epsilon])}\Big]d\tau \\
&+ \Big[\frac{d_4\pi}{2} + \frac{2}{d_4}\Big]\int_0^t\Big[\|(L_{11}-L_{21})(\tau)\|^2_{L^2(D)} + \|(L_{12}-L_{22})(\tau)\|^2_{L^2(D)}\Big]d\tau \\
\le{}& \frac{32}{d_4\pi}\max\{\frac{1}{k_1},\frac{1}{k_2}\}\epsilon\int_0^t\Big[k_1\|\frac{\partial w}{\partial x_3}(\tau)\|^2_{L^2(R(-\epsilon))} + k_2\|\frac{\partial s}{\partial x_3}(\tau)\|^2_{L^2(R(-\epsilon))}\Big]d\tau \\
&+ \Big[\frac{d_4\pi}{2} + \frac{2}{d_4}\Big]\int_0^t\Big[\|(L_{11}-L_{21})(\tau)\|^2_{L^2(D)} + \|(L_{12}-L_{22})(\tau)\|^2_{L^2(D)}\Big]d\tau.
\end{aligned} \tag{4.17}$$

In view of (4.5) and (4.13), from (4.17) we have Theorem 4.1.

Remark 4.1. Theorem 4.1 indicates that $V(t, x_3)$ continuously depends on ϵ and the base data. That is, when ϵ approaches 0, then $u(t, x_3)$ and $v(t, x_3)$ approach 0. If $\epsilon = 0$, Theorem 4.1 is the Saint-Venant's principle type decay result.

Remark 4.2. In any cross-section of R, the continuous dependence result can still be obtained. We compute

$$\begin{aligned}
&\int_0^t \Big[\|w(\tau)\|^2_{L^2(D(x_3))} + \|s(\tau)\|^2_{L^2(D(x_3))}\Big]d\tau \\
&\qquad = -2\int_0^t \int_{R(x_3)} \Big[w\frac{\partial w}{\partial x_3} + s\frac{\partial s}{\partial x_3}\Big]dxd\tau \\
&\qquad \leq 2\sqrt{V(x_3)}\Big[\int_0^t \Big[k_1\|\frac{\partial w}{\partial x_3}(\tau)\|^2_{L^2(R(-\epsilon))} + k_2\|\frac{\partial s}{\partial x_3}(\tau)\|^2_{L^2(R(-\epsilon))}\Big]d\tau\Big]^{\frac{1}{2}}.
\end{aligned} \tag{4.18}$$

Using (4.18) and Theorem 4.1, we can obtain the continuous dependence result.

5. Conclusions

This article adopts the methods of the a prior estimates and energy estimate to obtain the continuous dependence of the solution on the base. This method can be further extended to other linear partial differential equation systems, such as pseudo-parabolic equation

$$u_t = \Delta u + \delta\Delta u_t,$$

where δ is a positive constant. However, for nonlinear equations (e.g., the Darcy equations), due to the inability to control nonlinear terms and derive a prior bounds for nonlinear terms, Lemma 3.3 will be difficult to obtain. This is a difficult problem we need to solve next.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

References

1. M. W. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

2. M. M. Freitas, M. L. Santos, J. A. Langa, Porous elastic system with nonlinear damping and sources terms, *J. Differ. Equations*, **264** (2018), 2970–3051. https://doi.org/10.1016/j.jde.2017.11.006

3. L. E. Payne, B. Straughan, Structural stability for the Darcy equations of flow in porous media, *Proc. R. Soc. A*, **454** (1998), 1691–1698. https://doi.org/10.1098/rspa.1998.0227

4. N. L. Scott, Continuous dependence on boundary reaction terms in a porous medium of Darcy type, *J. Math. Anal. Appl.*, **399** (2013), 667–675. https://doi.org/10.1016/j.jmaa.2012.10.054

5. Y. F. Li, X. J. Chen, J. C. Shi, Structural stability in resonant penetrative convection in a Brinkman-Forchheimer fluid interfacing with a Darcy fluid, *Appl. Math. Optim.*, **84** (2021), 979–999. https://doi.org/10.1007/s00245-021-09791-7

6. Y. Liu, Continuous dependence for a thermal convection model with temperaturedependent solubitity, *Appl. Math. Comput.*, **308** (2017), 18–30. https://doi.org/10.1016/j.amc.2017.03.004

7. Y. Liu, S. Z. Xiao, Y. W. Lin, Continuous dependence for the Brinkman-Forchheimer fluid interfacing with a Darcy fluid in a bounded domain, *Math. Comput. Simul.*, **150** (2018), 66–82. https://doi.org/10.1016/j.matcom.2018.02.009

8. F. Franchi, R. Nibbi, B. Straughan, Continuous dependence on boundary and Soret coefficients in double diffusive bidispersive convection, *Math. Methods Appl. Sci.*, **43** (2020), 8882–8893. https://doi.org/10.1002/mma.6581

9. Y. F. Li, S. Z. Xiao, Continuous dependence of 2D large scale primitive equations on the boundary conditions in oceanic dynamics, *Appl. Math.*, **67** (2022), 103–124. https://doi.org/10.21136/AM.2021.0076-20

10. J. B. Han, R. Z. Xu, C. Yang, Continuous dependence on initial data and high energy blow-up time estimate for porous elastic system, *Commun. Anal. Mech.*, **15** (2023), 214–244. https://doi.org/10.3934/cam.2023012

11. Y. F. Li, C. H. Lin, Continuous dependence for the nonhomogeneous Brinkman-Forchheimer equations in a semi-infinite pipe, *Appl. Math. Comput.*, **244** (2014), 201–208. https://doi.org/10.1016/j.amc.2014.06.082

12. Z. Q. Li, W. B. Zhang, Y. F. Li, Structural stability for Forchheimer fluid in a semi-infinite pipe, *Electron. Res. Arch.*, **31** (2023), 1466–1484. https://doi.org/10.3934/era.2023074

13. Y. F. Li, X. J. Chen, Structural stability for temperature-dependent bidispersive flow in a semi-infinite pipe, *Lith. Math. J.*, **63** (2023), 337–366. https://doi.org/10.1007/s10986-023-09600-4

14. R. J. Knops, L. E. Payne, Continuous dependence on base data in an Elastic prismatic cylinder, *J. Elast.*, **64** (2001), 179–190. https://doi.org/10.1023/A:1015214420596

15. A. H. Nayfeh, A continuum mixture theory of heat conduction in laminated composite, *ASME J. Appl. Mech.*, **42** (1975), 399–404. https://doi.org/10.1115/1.3423589

16. D. Iesan, A theory of mixtures with different constituent temperatures, *J. Therm. Stresses*, **20** (1997), 147–167. https://doi.org/10.1080/01495739708956096

17. D. Iesan, R. Quintanilla, On the problem of propagation of heat in mixtures, *Appl. Mech. Eng.*, **4** (1999), 529–551.

18. R. Quintanilla, Study of the solutions of the propagation of heat in mixtures, *Dyn. Contin. Discrete Impulsive Syst. - Ser. B*, **8** (2001), 15–28.

19. G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

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