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Research article

The approximate lag and anticipating synchronization between two unidirectionally coupled Hindmarsh-Rose neurons with uncertain parameters

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Abstract: This research presents an adaptive synchronization approach crafted to facilitate exact lag synchronization between a pair of unidirectionally linked Hindmarsh-Rose (HR) neurons, taking into account both explicit propagation delays and the existence of uncertain parameters. The precise condition for lag synchronization is deduced analytically, utilizing the Laplace transform and convolution theorem, alongside the iterative approach within the framework of Volterra integral equations theory. The established criterion guarantees robust stability irrespective of the propagation delay's magnitude, facilitating the realization of approximate lag and anticipating synchronization in a pair of HR neurons. The approximate synchronizations are realized in the absence of direct time-delay coupling, with the Taylor series expansion serving as an alternative to the precise time-delay component. Numerical simulations are executed to validate the effectiveness of the suggested approximate synchronization approach. The research demonstrates that employing the current state of an HR neuron, despite having uncertain parameters, enables the accurate prediction of future states and the reconstruction of past states. This study provides a novel perspective for comprehending neural processes and the advantageous attributes inherent in nonlinear and chaotic systems.

Keywords: lag synchronization; anticipating synchronization; Hindmarsh-Rose neuron; Volterra integral equations

1. Introduction

The Hindmarsh-Rose (HR) neuron model [1] continues to be among the most widely used mathematical representations for characterizing the dynamics of an individual neuron. It is an enhancement of Fitzhugh's B.V.P. model [2], distinguished by the feature that each neuronal firing is followed by an extended interspike interval, a characteristic that closely mirrors the behavior observed in actual neurons. The HR model adeptly encapsulates the quintessential features of neuronal activity, encompassing both the genesis of action potentials and the manifestation of oscillations. Additionally, the model incorporates nonlinear components that elucidate the gradual recovery and adaptive processes witnessed in authentic neuronal functioning. Rosa et al. [3] examined a series of unidirectional coupled HR neurons, each exhibiting periodic spiking behavior. They observed that the initial periodic oscillations of certain neurons evolved into localized spiking-bursting chaos along the network, which later transitioned into regular slow oscillations, despite the ongoing chaotic nature of the spiking activity. The HR neuron model can exhibit complex dynamical behaviors, and therefore it has been broadly applied to investigate the dynamics of individual neurons and their reciprocal influences within neural networks [4, 5].

Synchronization is characterized by the alignment or unison of oscillations or behaviors within separate systems or components. It materializes when a pair or more of oscillators or dynamical systems synchronize their operations to achieve a state of phase-locking or to maintain a coherent relationship as time progresses. Synchronization is a key phenomenon that manifests in a variety of natural and man-made systems, with a wide array of applications spanning diverse domains [6]. The synchronization of neuronal activity is integral to the processing of information and the execution of cognitive functions within the brain. Grasping the principles of synchronization in neural systems offers significant insights into brain functionality, cognitive mechanisms, and the nature of neurological disorders. It holds significance for the innovation of new therapeutic approaches, the evolution of brain-inspired computational models, and the progression of brain-machine interface technologies [7].

Lag synchronization and anticipating synchronization are two specialized types of synchronization phenomena that emerge in the context of dynamical systems. Traditional synchronization is characterized by the real-time concordance of dynamical behaviors, whereas lag synchronization and anticipating synchronization entail a temporal offset in the synchronization between the systems involved. Lag synchronization is characterized by the synchronization of two interconnected systems' states, albeit with a temporal delay or lag in their alignment. In this scenario, one system precedes or follows the other by a consistent time delay [8]. The synchronization observed is not instantaneous, but maintains a reliable relationship characterized by a constant phase discrepancy. The phenomenon of lag synchronization is evident in a wide array of both natural and engineered systems, ranging from systems of coupled oscillators and chaotic systems to biological systems, including but not limited to neural networks [9–11]. Anticipating synchronization is distinguished by the capacity of one system to project or foretell the forthcoming dynamics of its counterpart [12, 13]. In this instance, the synchronizing system replicates the dynamics of the reference system, albeit with a temporal lead. The system engaged in anticipating synchronization not only tracks the target system, but also projects its forthcoming states. The occurrence of anticipating synchronization has been identified in setups that incorporate time-delayed feedback within the context of neural networks, as well as in predictive control systems. In the field of neuroscience, the study of anticipating synchronization has been pursued to shed light on the brain's neural coding processes and the mechanisms underlying prediction [14, 15]. The phenomena of lag synchronization and anticipating synchronization exhibit captivating behaviors and have been harnessed in a multitude of fields. They offer profound understanding into the inner workings of intricate systems and pave the way for the creation of innovative technologies that leverage synchronization principles.

Exact synchronization, both in lag and anticipation forms, between a pair of oscillators can be achieved through the application of explicit time delay coupling [11, 14]. Alternatively, approximate lag synchronization can be instigated in reciprocally coupled oscillators that exhibit a disparity in parameters [16]. Specifically, both intermittent and continuous lag synchronizations have manifested as transitional stages en route from phase synchronization to full synchronization as the coupling strength is augmented [17–19]. Anticipating synchronization has also been detected in systems that do not incorporate time delays. This type of phenomenon was initially documented through the classical Voss scheme, which substitutes the actual time-delay term with its first-order approximation [20]. The research has substantiated the feasible existence of a real-time predictor for chaotic dynamics in systems devoid of inherent delays. The Voss scheme facilitates the adjustment of the anticipation time without necessitating any changes to the dynamics of the drive system. However, the synchronization achieved in this manner is not perfectly precise. Pyragienė and Pyragas [21] demonstrated that anticipating synchronization can emerge in unidirectionally coupled chaotic systems that are not identical, possess significantly disparate parameters, and operate without self-feedback of time-delay in the slave system. To transcend the constraint of a brief anticipation time, Pyragienė and Pyragas introduced an innovative coupling scheme devoid of time-delay. This scheme comprises a master system and two slave systems arranged in series, with the use of switching parameters to accomplish anticipating synchronization over an extended anticipation period [22]. In each of the aforementioned scenarios, it is imperative to have prior knowledge of all the parameters of the interconnected systems to ascertain the conditions for lag and anticipating synchronization. Ji et al. [23], Liu et al. [24], and Sun et al. [25] developed adaptive control mechanisms aimed at realizing lag and anticipating synchronization in systems with time delays and uncertain parameters, leveraging the Lyapunov stability theory. However, the synchronization conditions obtained using the Lyapunov method are usually too conservative, which is not conducive to their practical implementation in real-world applications. Actually, relatively few studies have been made available in the literature concerning the approximate lag and anticipating synchronization of interconnected oscillators that encompass uncertain parameters in the absence of time delay.

In this manuscript, we introduce an innovative coupling strategy devoid of time delays tailored for a pair of HR neurons characterized by uncertain parameters, facilitating the achievement of approximate lag and anticipating synchronization. Initially, we devise an adaptive control mechanism and formulate the precise condition for lag synchronization between two HR neurons, factoring in time-delayed coupling and the presence of uncertain parameters. Divergent from other studies, lag synchronization is regarded here as a distinct variant of generalized synchronization, which can be explored through the application of the auxiliary system methodology [26]. We did not use the Lyapunov method, even though there are unknown parameters in the coupled systems. Numerical simulations are conducted to affirm the stability of the synchronization condition across a range of time delay values. A broad spectrum of approximate lag and anticipating synchronization can be attained by substituting the actual time delay term with its Taylor series expansion. This research provides a method that utilizes the present condition of an HR neuron with uncertain parameters and facilitates the precise forecasting of subsequent states and the reconstitution of antecedent states.

The structure of the paper is outlined as follows. In Section 2, we elucidate our proposed coupling scheme, while Section 3 is dedicated to the derivation of analytical conditions for precise lag synchronization. Section 4 focuses on the formulation of parameter update laws, along with numerical illustrations of exact lag synchronization between a pair of HR neurons featuring time delay coupling and an examination of its resilience to variations in time delay. Section 5 delves into the achievements of approximate lag and anticipating synchronization. Finally, Section 6 presents our concluding remarks.

2. A scheme for exact lag synchronization

The form of the HR neuron system is given by

$$\begin{aligned}\dot{u} &= v - au^3 + bu^2 - w + I_{ext},\\ \dot{v} &= c - du^2 - v,\\ \dot{w} &= rs_0(u - x_0) - rw,\end{aligned} \tag{2.1}$$

where u is the membrane potential, v is a recovery variable associated with fast current of N_a^+ or K^+ ions, and w represents a slowly changing adaptation current of, for example, C_a^+ ions. a, b, c, d, s_0, r, and x_0 are uncertain parameters to be identified, and I_{ext} is the external current input.

The HR neuronal model is a widely utilized mathematical framework for depicting the intricate dynamics of neuronal activity. It employs a set of three interconnected first-order differential equations to mimic the fluctuating membrane potentials of neurons, which can result in diverse neuronal activities such as quiescence, spiking, and bursting. However, in real-world scenarios, the HR model's parameters can be subject to uncertainty due to a range of influences, including errors in measurement, simplifications made in the model, and the natural variation present within biological systems. Investigating the synchronization of HR neuronal systems with uncertain parameters is essential for gaining insights and exerting control over the chaotic dynamics of neurons. For system (2.1), we consider all parameters to be uncertain. By implementing an adaptive control strategy and establishing the parameter update laws, we aim to realize in this section exact lag synchronization between a pair of HR neurons, all without the need for precise knowledge of the system's parameters.

If the true values of the uncertain parameters of system (2.1) are chosen as $a = 1.0$, $b = 3.0$, $c = 1.0$, $d = 5.0$, $r = 0.006$, $s_0 = 4.0$, $x_0 = -1.6$, and $I_{ext} = 3.0$, system (2.1) is chaotic bursting. For convenience, by moving the equilibrium of system (2.1) to the origin, it can be rewritten as

$$\begin{aligned}\dot{x}_1 &= -\phi_1 x_1^3 + \phi_5 x_1^2 + \phi_6 x_1 + x_2 - x_3,\\ \dot{x}_2 &= -\phi_2 x_1^2 + \phi_7 x_1 - x_2,\\ \dot{x}_3 &= \phi_3(\phi_4 x_1 - x_3),\end{aligned} \tag{2.2}$$

where $\phi_1 = a$, $\phi_2 = d$, $\phi_3 = r$, $\phi_4 = s_0$, $\phi_5 = -3au_0 + b$, $\phi_6 = -3au_0^2 + 2bu_0$, $\phi_7 = -2du_0$, and u_0 is the unique real root to equation

$$au_0^3 + (d - b)u_0^2 + s_0 u_0 - (s_0 x_0 + c + I_{ext}) = 0. \tag{2.3}$$

In the following, we consider ϕ_j, $j = 1, 2, \cdots, 7$, as uncertain parameters in system (2.2) to be identified. The true values of the uncertain parameters ϕ_j, $j = 1, 2, \cdots, 7$, are taken as ($\phi_{4,5,6}$ are obtained on basis of Eq (2.3))

$$\phi_{01} = 1.0,\ \phi_{02} = 5.0,\ \phi_{03} = 0.006,\ \phi_{04} = 4.0,\ \phi_{05} = 5.3646,\ \phi_{06} = -6.5931,\ \phi_{07} = 7.8822. \tag{2.4}$$

System (2.2) is chosen as the drive system, and the response HR neuron system is described as follows:

$$\begin{aligned}\dot{y}_1 &= -\psi_{11}y_1^3 + \psi_{15}y_1^2 + \psi_{16}y_1 + y_2 - y_3 + k\delta_1,\\ \dot{y}_2 &= -\psi_{12}y_1^2 + \psi_{17}y_1 - y_2,\\ \dot{y}_3 &= \psi_{13}(\psi_{14}y_1 - y_3),\end{aligned} \tag{2.5}$$

where ψ_{1j}, $j = 1, 2, \cdots, 7$, are the estimated values of uncertain parameters ϕ_j, respectively, k is the control parameter, and $\delta_1 = x_1(t-\tau) - y_1$, τ is the time delay in the signal transmission. If there exists

$$\lim_{t\to\infty} \|x_i(t-\tau) - y_i\| = 0, \quad i = 1, 2, 3, \tag{2.6}$$

then lag synchronization is said to be reached for two HR neurons in systems (2.2) and (2.5).

To determine the uncertain parameters in systems (2.2) and (2.5), the parameter update laws are designed as follows:

$$\dot{\psi}_{1j} = g_j(\psi_{1j} - \phi_j, \delta_1), \quad j = 1, 2, \cdots, 7, \tag{2.7}$$

where $g_j : R \to R$ are functions satisfying $g_j(0, 0) = 0$, $j = 1, 2, \cdots, 7$. Our objective is to identify suitable expressions for g_j, $j = 1, 2, \cdots, 7$, in Eq (2.7) and to select an appropriate range for k within Eq (2.5). This selection is intended to facilitate lag synchronization between the systems described by Eqs (2.2) and (2.5). Concurrently, ψ_{1j} in Eqs (2.5) and (2.7) should converge to their respective values ϕ_j, $j = 1, 2, \cdots, 7$.

Indeed, the lag synchronization occurring between systems (2.2) and (2.5) can be classified as a distinct variant of generalized synchronization. This phenomenon can be identified by employing the method of an auxiliary system, as referenced in [26]. Let us contemplate the scenario where systems (2.5) and (2.7) are identical replicas, both being influenced by an identical signal emanating from system (2.2),

$$\begin{aligned}\dot{z}_1 &= -\psi_{21}z_1^3 + \psi_{25}z_1^2 + \psi_{26}z_1 + z_2 - z_3 + k\delta_2,\\ \dot{z}_2 &= -\psi_{22}z_1^2 + \psi_{27}z_1 - z_2,\\ \dot{z}_3 &= \psi_{23}(\psi_{24}z_1 - z_3),\\ \dot{\psi}_{2j} &= g_j(\psi_{2j} - \phi_j, \delta_2), \quad j = 1, 2, \cdots, 7,\end{aligned} \tag{2.8}$$

where ψ_{2j}, $j = 1, 2, \cdots, 7$, are the estimated values of uncertain parameters ϕ_j, respectively, and $\delta_2 = x_1(t-\tau) - z_1$. From the auxiliary system approach, lag synchronization in systems (2.2) and (2.5) can be reached, and ψ_{1j} adapt themselves to the true values $\psi_{1j} = \phi_j$, $j = 1, 2, \cdots, 7$, respectively, if the following conditions are satisfied:

$$\begin{aligned}&\lim_{t\to\infty} \|y_i - z_i\| = 0, \quad i = 1, 2, 3,\\ &\lim_{t\to\infty} \|\psi_{1j} - \psi_{2j}\| = 0, \quad j = 1, 2, \cdots, 7.\end{aligned} \tag{2.9}$$

3. Identification of the exact lag synchronization condition

By letting

$$\begin{aligned}&e_1 = \frac{y_1 - z_1}{2}, \quad e_2 = \frac{y_2 - z_2}{2}, \quad e_3 = \frac{y_3 - z_3}{2}, \quad e_{4j} = \frac{\psi_{1j} - \psi_{2j}}{2},\\ &e_5 = \frac{y_1 + z_1}{2}, \quad e_6 = \frac{y_2 + z_2}{2}, \quad e_7 = \frac{y_3 + z_3}{2}, \quad e_{8j} = \frac{\psi_{1j} + \psi_{2j}}{2}, \quad j = 1, 2, \cdots, 7,\end{aligned}$$

systems (2.2), (2.5), (2.7), and (2.8) are given as

$$
\begin{aligned}
\dot{e}_1 &= (e_{86} - k)e_1 + e_2 - e_3 + (2e_{85}e_5 - 3e_{81}e_5^2)e_1 + (e_{45} - 3e_{41}e_5)e_1^2 - e_{81}e_1^3 + e_{46}e_5 + e_{45}e_5^2 - e_{41}e_5^3,\\
\dot{e}_2 &= (e_{87} - 2e_{82}e_5)e_1 - e_2 + e_{47}e_5 - e_{42}(e_1^2 + e_5^2),\\
\dot{e}_3 &= (e_{43}e_{44} + e_{83}e_{84})e_1 - e_{83}e_3 + (e_{43}e_{84} + e_{44}e_{83})e_5 - e_{43}e_7,\\
\dot{e}_{4j} &= \frac{1}{2}[g_j(e_{8j} + e_{4j} - \phi_j, x_1(t-\tau) - e_5 - e_1) - g_j(e_{8j} - e_{4j} - \phi_j, x_1(t-\tau) - e_5 + e_1)],\\
\dot{e}_5 &= e_{46}e_1 + e_{85}e_1^2 - e_{41}e_1^3 + (e_{86} + 2e_1e_{45} - 3e_{81}e_1^2)e_5 + (e_{85} - 3e_1e_{41})e_5^2 - e_{81}e_5^3 + e_6 - e_7 + k(x_1(t-\tau) - e_5),\\
\dot{e}_6 &= e_{47}e_1 + (e_{87} - 2e_{42}e_1)e_5 - e_6 - e_{82}(e_1^2 + e_5^2),\\
\dot{e}_7 &= (e_{43}e_{44} + e_{83}e_{84})e_5 + (e_{84}e_1 - e_3)e_{43} + e_{83}(e_{44}e_1 - e_7),\\
\dot{e}_{8j} &= \frac{1}{2}[g_j(e_{8j} + e_{4j} - \phi_j, x_1(t-\tau) - e_5 - e_1) + g_j(e_{8j} - e_{4j} - \phi_j, x_1(t-\tau) - e_5 + e_1)], \quad j = 1, 2, \cdots, 7.
\end{aligned}
\tag{3.1}
$$

Condition (2.9) becomes

$$
\begin{aligned}
&\lim_{t\to\infty} \|e_i\| = 0, \quad i = 1, 2, 3,\\
&\lim_{t\to\infty} \|e_{4j}\| = 0, \quad j = 1, 2, \cdots, 7.
\end{aligned}
\tag{3.2}
$$

If condition (3.2) is satisfied, then it is evident that $e_{8j} \to \phi_j$, $j = 1, 2, \cdots, 7$, holds. For sufficiently small $e_{1,2,3}$ and e_{4j}, $j = 1, 2, \cdots, 7$, e_{8j} can be expressed as $e_{8j} = \phi_j + U_{8j}(t)$, $j = 1, 2, \cdots, 7$. Simultaneously, near the origin, the right-hand side of the first four equations in system (3.1) can be written as

$$
\begin{aligned}
\dot{e}_1 &= (\phi_6 - k)e_1 + e_2 - e_3 + P_1,\\
\dot{e}_2 &= \phi_7 e_1 - e_2 + P_2,\\
\dot{e}_3 &= \phi_3\phi_4 e_1 - \phi_3 e_3 + P_3,\\
\dot{e}_{4j} &= \gamma_{1j}e_1 + \gamma_{2j}e_{4j} + F_j, \quad j = 1, 2, \cdots, 7,
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
P_1 &= (2e_{85}e_5 - 3e_{81}e_5^2)e_1 + (e_{45} - 3e_{41}e_5)e_1^2 - e_{81}e_1^3 + e_{46}e_5 + e_{45}e_5^2 - e_{41}e_5^3 + U_{86}(t)e_1,\\
P_2 &= -2e_{82}e_5e_1 + e_{47}e_5 - e_{42}(e_1^2 + e_5^2) + U_{87}(t)e_1,\\
P_3 &= e_{43}e_{44}e_1 + (e_{43}e_{84} + e_{44}e_{83})e_5 - e_{43}e_7 + U_{83}(t)U_{84}(t) - U_{83}(t)e_3,\\
\gamma_{1j} &= -\frac{\partial g_j(0,0)}{\partial \delta_1}, \quad \gamma_{2j} = \frac{\partial g_j(0,0)}{\partial(\psi_{1j} - \phi_j)},\\
F_j &= \frac{1}{(2n+1)!}\sum_{n=1}^{\infty}[\frac{\partial^{(2n+1)} g_j(0,0)}{\partial(\psi_{1j} - \phi_j)^{(2n+1)}} e_{4j}^{(2n+1)} - \frac{\partial^{(2n+1)} g_j(0,0)}{\partial \delta_1^{(2n+1)}} e_1^{(2n+1)}], \qquad j = 1, 2, \cdots, 7.
\end{aligned}
$$

Consider the following definition of the Laplace transform pair:

$$
\begin{aligned}
&E_i(s) = L[e_i] = \int_0^{+\infty} e_i(t)e^{-st}dt, \qquad E_{4j}(s) = L[e_{4j}] = \int_0^{+\infty} e_{4j}(t)e^{-st}dt,\\
&e_i(t) = L^{-1}[E_i] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E_i(s)e^{st}ds, \quad e_{4j}(t) = L^{-1}[E_{4j}] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E_{4j}(s)e^{st}ds,\\
&\qquad\qquad i = 1, 2, 3; \quad j = 1, 2, \cdots, 7.
\end{aligned}
\tag{3.4}
$$

Taking the Laplace transforms on both sides of system (3.3), one has

$$(S-M)[E_1,E_2,E_3,E_{4j}]^T=[e_{10},e_{20},e_{30},e_{4j0}]^T+[W_1,W_2,W_3,W_{4j}]^T, \tag{3.5}$$

where

$$S=\left[\begin{array}{ccc|c} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ \hline 0 & 0 & 0 & sI_7 \end{array}\right],\quad M=\left[\begin{array}{ccc|c} \phi_6-k & 1 & -1 & 0 \\ \phi_7 & -1 & 0 & 0 \\ \phi_3\phi_4 & 0 & -\phi_3 & 0 \\ \hline \gamma_1 & 0 & 0 & \gamma_2 \end{array}\right]\equiv\left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & \gamma_2 \end{array}\right], \tag{3.6}$$

I_7 is the 7×7 real identity matrix; $\gamma_1=[\gamma_{11},\gamma_{12},\cdots,\gamma_{17}]^T$, $\gamma_2=diag\{\gamma_{21},\gamma_{22},\cdots,\gamma_{27}\}$; e_{i0}, $i=1,2,3$, and e_{4j0}, $j=1,2,\cdots,7$, are given initial values of system (3.3); and $W_{1,2,3}$ and W_{4j}, $j=1,2,\cdots,7$, are the Laplace transforms of the nonlinear parts $P_{1,2,3}$ and F_j, $j=1,2,\cdots,7$, in Eq (3.3), respectively, i.e.,

$$W_i=L[P_i],\quad i=1,2,3;\qquad W_{4j}=L[F_j],\quad j=1,2,\cdots,7.$$

Solving Eq (3.5) using Cramer's rule yields

$$\begin{aligned} E_1&=\frac{e_{10}s^2+\beta_4 s+\beta_5}{D_1(s)}+\frac{[s^2+(\phi_3+1)s+\phi_3]W_1}{D_1(s)}+\frac{(s+\phi_3)W_2}{D_1(s)}-\frac{(s+1)W_3}{D_1(s)},\\ E_2&=\frac{e_{20}s^2+\beta_6 s+\beta_7}{D_1(s)}+\frac{\phi_7(s+\phi_3)W_1}{D_1(s)}+\frac{[s^2+(\beta_1-1)s+\beta_8]W_2}{D_1(s)}-\frac{\phi_7 W_3}{D_1(s)},\\ E_3&=\frac{e_{30}s^2+\beta_9 s+\beta_{10}}{D_1(s)}+\frac{\phi_3\phi_4(s+1)W_1}{D_1(s)}+\frac{\phi_3\phi_4 W_2}{D_1(s)}+\frac{[s^2+(\beta_1-\phi_3)s+(k-\phi_6-\phi_7)]W_3}{D_1(s)},\\ E_{4j}&=\frac{e_{4j0}}{s-\gamma_{2j}}+\frac{\gamma_{1j}E_1+W_{4j}}{s-\gamma_{2j}},\quad j=1,2,\cdots,7, \end{aligned} \tag{3.7}$$

where $D_1(s)=s^3+\beta_1 s^2+\beta_2 s+\beta_3$, which is the characteristic polynomial of matrix M_1 in Eq (3.6); $\beta_1=k+\phi_3+1-\phi_6$, $\beta_2=(k+\phi_4+1-\phi_6)\phi_3+k-\phi_6-\phi_7$, $\beta_3=(k+\phi_4-\phi_6-\phi_7)\phi_3$, $\beta_4=e_{10}(\phi_3+1)+e_{20}-e_{30}$, $\beta_5=\phi_3(e_{10}+e_{20})-e_{30}$, $\beta_6=e_{10}\phi_7+e_{20}(k+\phi_3-\phi_6)$, $\beta_7=e_{20}\phi_3(k+\phi_4-\phi_6)+(e_{10}\phi_3-e_{30})\phi_7$, $\beta_8=\phi_3(k+\phi_4-\phi_6)$, $\beta_9=e_{30}(k+1-\phi_6)+e_{10}\phi_3\phi_4$, $\beta_{10}=e_{30}(k-\phi_6-\phi_7)+\phi_3\phi_4(e_{10}+e_{20})$.

Taking the inverse Laplace transforms of Eq (3.7) and applying the convolution theorem produces

$$\begin{aligned} e_1&=\Gamma_1(t)+\int_0^t\Gamma_2(t-\tau)P_1(\tau)d\tau+\int_0^t\Gamma_3(t-\tau)P_2(\tau)d\tau-\int_0^t\Gamma_4(t-\tau)P_3(\tau)d\tau,\\ e_2&=\Gamma_5(t)+\phi_7\int_0^t\Gamma_3(t-\tau)P_1(\tau)d\tau+\int_0^t\Gamma_6(t-\tau)P_2(\tau)d\tau-\phi_7\int_0^t\Gamma_7(t-\tau)P_3(\tau)d\tau,\\ e_3&=\Gamma_8(t)+\phi_3\phi_4\int_0^t\Gamma_4(t-\tau)P_1(\tau)d\tau+\phi_3\phi_4\int_0^t\Gamma_7(t-\tau)P_2(\tau)d\tau+\int_0^t\Gamma_9(t-\tau)P_3(\tau)d\tau,\\ e_{4j}&=e_{4j0}e^{\gamma_{2j}t}+\gamma_{1j}\int_0^t e^{\gamma_{2j}(t-\tau)}e_1(\tau)d\tau+\int_0^t e^{\gamma_{2j}(t-\tau)}F_j(\tau)d\tau,\quad j=1,2,\cdots,7, \end{aligned} \tag{3.8}$$

in which

$$\Gamma_1(t) = L^{-1}[\frac{e_{10}s^2 + \beta_4 s + \beta_5}{D_1(s)}], \quad \Gamma_2(t) = L^{-1}[\frac{s^2 + (\phi_3 + 1)s + \phi_3}{D_1(s)}], \quad \Gamma_3(t) = L^{-1}[\frac{s + \phi_3}{D_1(s)}],$$
$$\Gamma_4(t) = L^{-1}[\frac{s + 1}{D_1(s)}], \quad \Gamma_5(t) = L^{-1}[\frac{e_{20}s^2 + \beta_6 s + \beta_7}{D_1(s)}], \quad \Gamma_6(t) = L^{-1}[\frac{s^2 + (\beta_1 - 1)s + \beta_8}{D_1(s)}],$$
$$\Gamma_7(t) = L^{-1}[\frac{1}{D_1(s)}], \quad \Gamma_8(t) = L^{-1}[\frac{e_{30}s^2 + \beta_9 s + \beta_{10}}{D_1(s)}], \quad \Gamma_9(t) = L^{-1}[\frac{s^2 + (\beta_1 - \phi_3)s + (k - \phi_6 - \phi_7)}{D_1(s)}].$$

Theorem 3.1. *The necessary condition for $e_i \to 0$, $e_{4j} \to 0$, $i = 1, 2, 3$, $j = 1, 2, \cdots, 7$, with $t \to \infty$ in Eq (3.8) is that all eigenvalues of matrix M_1 defined in Eq (3.6) have negative real parts, and*

$$\gamma_{2j} = \frac{\partial g_j(0, 0)}{\partial(\psi_{1j} - \phi_j)} < 0, \qquad j = 1, 2, \cdots, 7. \tag{3.9}$$

Proof. Without loss of generality, consider the inverse Laplace transform of the true fraction

$$\frac{A_1 s^2 + A_2 s + A_3}{D_1(s)}$$

where A_i, $i = 1, 2, 3$, are constants, and $D_1(s)$ is the characteristic polynomial of matrix M_1 in Eq (3.6). There exist the following four cases:

- $D_1(s)$ has 3 distinct real roots: s_1, s_2, s_3
  $\frac{A_1s^2+A_2s+A_3}{D_1(s)} = \frac{B_1}{s-s_1} + \frac{B_2}{s-s_2} + \frac{B_3}{s-s_3}$,
  where $B_i = \frac{A_1s^2+A_2s+A_3}{D_1(s)}(s - s_i)\Big|_{s=s_i}$, $i = 1, 2, 3.$
  $L^{-1}[\frac{A_1s^2+A_2s+A_3}{D_1(s)}] = B_1e^{s_1t} + B_2e^{s_2t} + B_3e^{s_3t}$.
- $D_1(s)$ has a pair of conjugate complex roots $s_{1,2} = \omega_1 \pm j\omega_2$ and a real root $s_3 = \omega_3$
  $\frac{A_1s^2+A_2s+A_3}{D_1(s)} = \frac{A_1s^2+A_2s+A_3}{(s-\omega_1-j\omega_2)(s-\omega_1+j\omega_2)(s-\omega_3)} = \frac{B_1}{s-\omega_1-j\omega_2} + \frac{B_2}{s-\omega_1+j\omega_2} + \frac{B_3}{s-\omega_3}$,
  where $B_{1,2} = \frac{A_1s^2+A_2s+A_3}{D_1'(s)}\Big|_{s=\omega_1\pm j\omega_2}$, $B_3 = \frac{A_1s^2+A_2s+A_3}{D_1(s)}(s - \omega_3)\Big|_{s=\omega_3}$,
  $L^{-1}[\frac{A_1s^2+A_2s+A_3}{D_1(s)}] = B_1e^{(\omega_1+j\omega_2)t} + B_2e^{(\omega_1-j\omega_2)t} + B_3e^{\omega_3t}$.
- $D_1(s)$ has two repeated real roots $s_{1,2} = s_0$ and one simple real root $s_3 = s_k$
  $\frac{A_1s^2+A_2s+A_3}{D_1(s)} = \frac{B_1}{s-s_0} + \frac{B_2}{(s-s_0)^2} + \frac{B_3}{s-s_k}$,
  where $B_1 = \frac{1}{2}\frac{d^2}{ds^2}[\frac{A_1s^2+A_2s+A_3}{D_1(s)}(s - s_0)^2]\Big|_{s=s_0}$, $B_2 = \frac{A_1s^2+A_2s+A_3}{D_1(s)}(s - s_0)^2\Big|_{s=s_0}$,
  $B_3 = \frac{A_1s^2+A_2s+A_3}{D_1(s)}(s - s_k)\Big|_{s=s_k}$,
  $L^{-1}[\frac{A_1s^2+A_2s+A_3}{D_1(s)}] = (B_1 + B_2t)e^{s_0t} + B_3e^{s_kt}$.
- $D_1(s)$ has a triple real root: $s_{1,2,3} = s_0$
  $\frac{A_1s^2+A_2s+A_3}{D_1(s)} = \frac{B_1}{s-s_0} + \frac{B_2}{(s-s_0)^2} + \frac{B_3}{(s-s_0)^3}$,
  where $B_{(3-i)} = \frac{1}{i!}\frac{d^i}{ds^i}[\frac{A_1s^2+A_2s+A_3}{D_1(s)}(s - s_0)^3]\Big|_{s=s_0}$, $i = 1, 2,$

$B_3 = [\frac{A_1 s^2+A_2 s+A_3}{D_1(s)}(s-s_0)^3]\Big|_{s=s_0}$,

$L^{-1}[\frac{A_1 s^2+A_2 s+A_3}{D_1(s)}] = (B_1 + B_2 t + B_3 t^2)e^{s_0 t}$.

Clearly, from the first three equations in Eq (3.8), the necessary condition for $e_i \to 0$, $i = 1,2,3$, is that all roots of $D_1(s) = 0$ have negative real parts. That is, all eigenvalues of matrix M_1 defined in Eq (3.6) have negative real parts. From the fourth equation in Eq (3.8), it is obvious that $\gamma_{2j} < 0$, $j = 1,2,\cdots,7$, is necessary for $e_{4j} \to 0$, $j = 1,2,\cdots,7$.

Under the conditions given in Theorem 3.1, system (3.8) becomes

$$\begin{aligned} e_1 &= \int_0^t \Gamma_2(t-\tau)P_1(\tau)d\tau + \int_0^t \Gamma_3(t-\tau)P_2(\tau)d\tau - \int_0^t \Gamma_4(t-\tau)P_3(\tau)d\tau,\\ e_2 &= \phi_7 \int_0^t \Gamma_3(t-\tau)P_1(\tau)d\tau + \int_0^t \Gamma_6(t-\tau)P_2(\tau)d\tau - \phi_7 \int_0^t \Gamma_7(t-\tau)P_3(\tau)d\tau,\\ e_3 &= \phi_3\phi_4 \int_0^t \Gamma_4(t-\tau)P_1(\tau)d\tau + \phi_3\phi_4 \int_0^t \Gamma_7(t-\tau)P_2(\tau)d\tau + \int_0^t \Gamma_9(t-\tau)P_3(\tau)d\tau,\\ e_{4j} &= \gamma_{1j} \int_0^t e^{\gamma_{2j}(t-\tau)}e_1(\tau)d\tau + \int_0^t e^{\gamma_{2j}(t-\tau)}F_j(\tau)d\tau, \quad j = 1,2,\cdots,7, \end{aligned} \tag{3.10}$$

Theorem 3.2. $e_i = 0$, $e_{4j} = 0$, $i = 1,2,3$, $j = 1,2,\cdots,7$, *are the unique continuous solutions to Eq (3.10).*

Proof. In fact, system (3.10) comprises a collection of Volterra integral equations which can be resolved through the method of successive approximations, as outlined in [27]. Consider the integral equation of the form

$$y(t) = \Phi(t) + \int_0^t H(t-\tau)f(\tau, y(\tau))d\tau, \tag{3.11}$$

where $y(t) \in R^n$, H is an $n \times n$ matrix, and $\Phi(t)$ and $f(t, y(t))$ are vectors with n components. Moreover, the following conditions are satisfied:

- $|y(t)| < \infty$;
- $\Phi(t)$ and f are continuous for $0 < t < t_0$, in which $0 < t_0 < +\infty$;
- $|H| \in L[0, \epsilon]$ holds for any $0 < \epsilon < t_0$;
- For any $\eta > 0$, there must exist a constant $\rho(\eta) > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \le \rho(\eta)|y_1 - y_2|, \quad (|y_1|, |y_2| \le \eta).$$

From the results given by Nohel [27], the successive approximations

$$\xi_0(t) = 0, \quad \xi_{n+1}(t) = \Phi(t) + \int_0^t H(t-\tau)f(\tau, \xi_n(\tau))d\tau, \quad n = 0,1,2,\cdots,$$

will uniformly converge to the unique continuous solution $y(t) = \xi(t)$ of Eq (3.11).

Upon juxtaposing equations (3.10) and (3.11), it becomes straightforward to verify that $e_i = 0$, $e_{4j} = 0$, $i = 1,2,3$, $j = 1,2,\cdots,7$, are the unique continuous solutions to Eq (3.10).

If condition (3.2) is met, it implies that condition (2.6) is also fulfilled. Meanwhile, ψ_{1j} in Eqs (2.5) and (2.7) converge to their respective values ϕ_j, $j = 1,2,\cdots,7$, respectively. From Theorems 3.1 and 3.2, we have the following result.

Theorem 3.3. *Lag synchronization can be accomplished between systems (2.2) and (2.5) with parameter update laws (2.7), and the variables ψ_{1j} within Eqs (2.5) and (2.7) will adjust to align with the values of ϕ_j for $j = 1, 2, \cdots, 7$, provided that every eigenvalue of the matrices γ_2 and M_1 (as specified in Eq (3.6)) possesses a negative real component.*

Remark 1. *The condition stipulated in Theorem (3.3) is independent of the time delay τ, indicating its universal applicability regardless of the time delay's magnitude. As a result, the stability of lag synchronization between systems (2.2) and (2.5), when governed by the parameter update laws (2.7), is maintained for any time delay value.*

4. The design of parameter update laws and numerical demonstrations

According to the expressions of γ_{1j} and F_j, $j = 1, 2, \cdots, 7$, given in Eq (3.3), if one chooses appropriate parameter update laws so that

$$\frac{\partial^n g_j(0,0)}{\partial \delta_1^n} = 0, \quad n = 1, 3, 5, \cdots, \tag{4.1}$$

then under the above circumstances, e_{4j}, $j = 1, 2, \cdots, 7$, is independent of $e_{1,2,3}$. At this juncture, provided that condition (3.9) is met, the estimated values of the uncertain parameters, denoted by ψ_{1j} for $j = 1, 2, \cdots, 7$ in Eq (2.5), will invariably converge to their respective true values of ϕ_j over time. This convergence is assured, irrespective of whether lag synchronization is achieved between the systems represented by Eqs (2.2) and (2.5). To elucidate the aforementioned discoveries, we consider that the parameter update laws in Eq (2.7) have the following form:

$$\dot{\psi}_{1j} = g_j(\psi_{1j} - \phi_j, \delta_1) = -(\psi_j - \phi_j) + 1 - \cos(\delta_1), \qquad j = 1, 2, \cdots, 7. \tag{4.2}$$

From Eq (3.3), one can check that $\gamma_{2j} = -1 < 0$, for $j = 1, 2, \cdots, 7$. Additionally,

$$\frac{\partial^n g_j(\psi_{1j} - \phi_j, \delta_1)}{\partial \delta_1^n} = \begin{cases} (-1)^{\frac{n-1}{2}} \sin(\delta_1), & n = 1, 3, 5, \cdots \\ (-1)^{\frac{n+2}{2}} \cos(\delta_1), & n = 2, 4, 6, \cdots \end{cases}. \tag{4.3}$$

Obviously, the parameter update laws designed in Eq (4.2) satisfy condition (4.1). Let us assume that in system (4.2) the values of ψ_{1j} individually converge to ψ_{1j0} for each j ranging from 1 to 7 after an adequate duration has passed. Based on Theorem 3.3, lag synchronization between systems (2.2) and (2.5) will be achieved if all eigenvalues of the following matrix have negative real parts

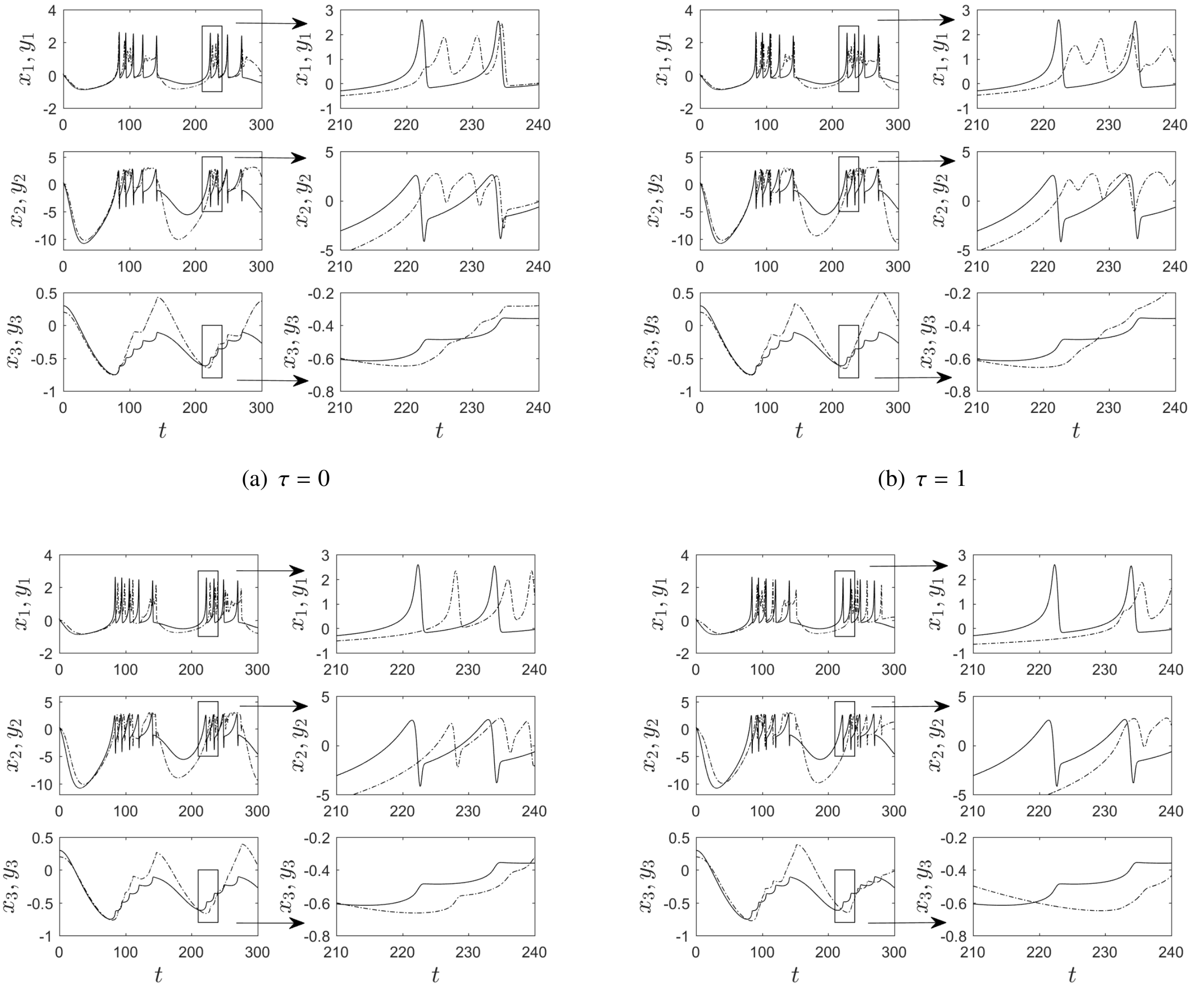
$$M_1 = \begin{bmatrix} \psi_{160} - k & 1 & -1 \\ \psi_{170} & -1 & 0 \\ \psi_{130}\psi_{140} & 0 & -\psi_{130} \end{bmatrix}. \tag{4.4}$$

On the basis of the Routh-Hurwitz criterion, the range of k can be determined through the following inequality:

$$k > \psi_{160} - \psi_{130} - 1, \qquad k > \frac{(-\psi_{140} + \psi_{160} - 1)\,\psi_{130} + \psi_{160} + \psi_{170}}{1 + \psi_{130}},$$
$$k > -\psi_{140} + \psi_{160} + \psi_{170}, \qquad (\psi_{130} + 1)k^2 + \theta_1 k + \theta_2 > 0, \tag{4.5}$$

where

$$\theta_1 = {\psi_{130}}^2 + (\psi_{140} - 2\,\psi_{160} + 2)\,\psi_{130} - 2\,\psi_{160} - \psi_{170} + 1,$$
$$\theta_2 = (\psi_{140} - \psi_{160} + 1)\,{\psi_{130}}^2 + ({\psi_{160}}^2 - (\psi_{140} + 2)\psi_{160}) + 1)\psi_{130} + {\psi_{160}}^2 + (\psi_{170} - 1)\,\psi_{160} - \psi_{170}.$$



(a) $\tau = 0$ (b) $\tau = 1$ (c) $\tau = 5$ (d) $\tau = 10$

Figure 1. Lag synchronization between systems (2.2) and (2.5) with parameter update laws (4.2) can not be achieved for any value of τ when the control parameter is set to $k = 0.5$ (a) $\tau = 0$ (b) $\tau = 1$ (c) $\tau = 5$ (d) $\tau = 10$. The initial conditions are taken as $x_1(t) = 0.1$, $x_2(t) = 0.2$, $x_3(t) = 0.3$ for $t \in (-\tau, 0]$, and $y_1(0) = 0.1$, $y_2(0) = 0.2$, $y_3(0) = 0.2$, $\psi_{2i}(0) = 2.0$, $i = 1, 2, \cdots, 7$. $x_{1,2,3}$ (solid line $-$), $y_{1,2,3}$ (dot dash line $-.$).

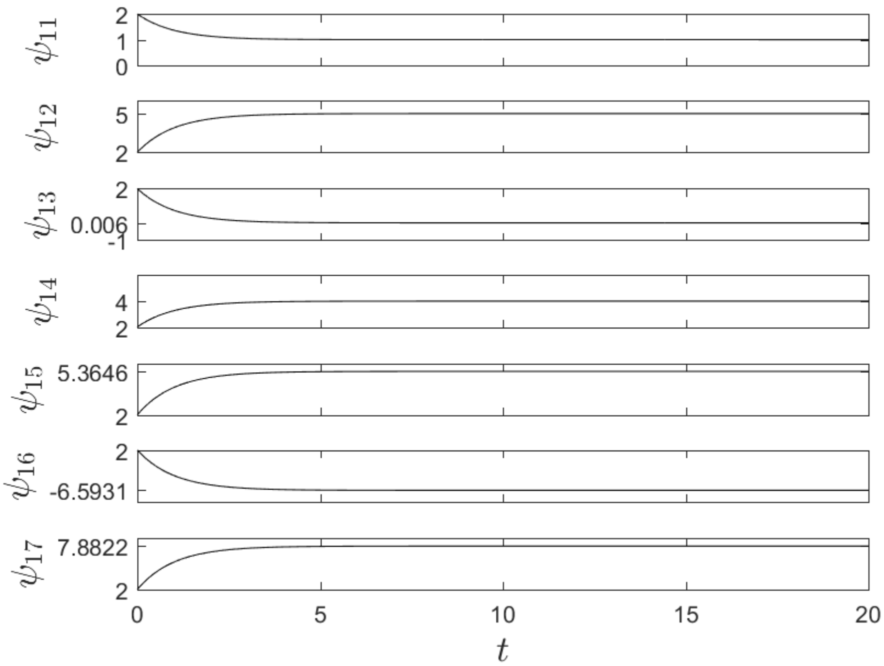
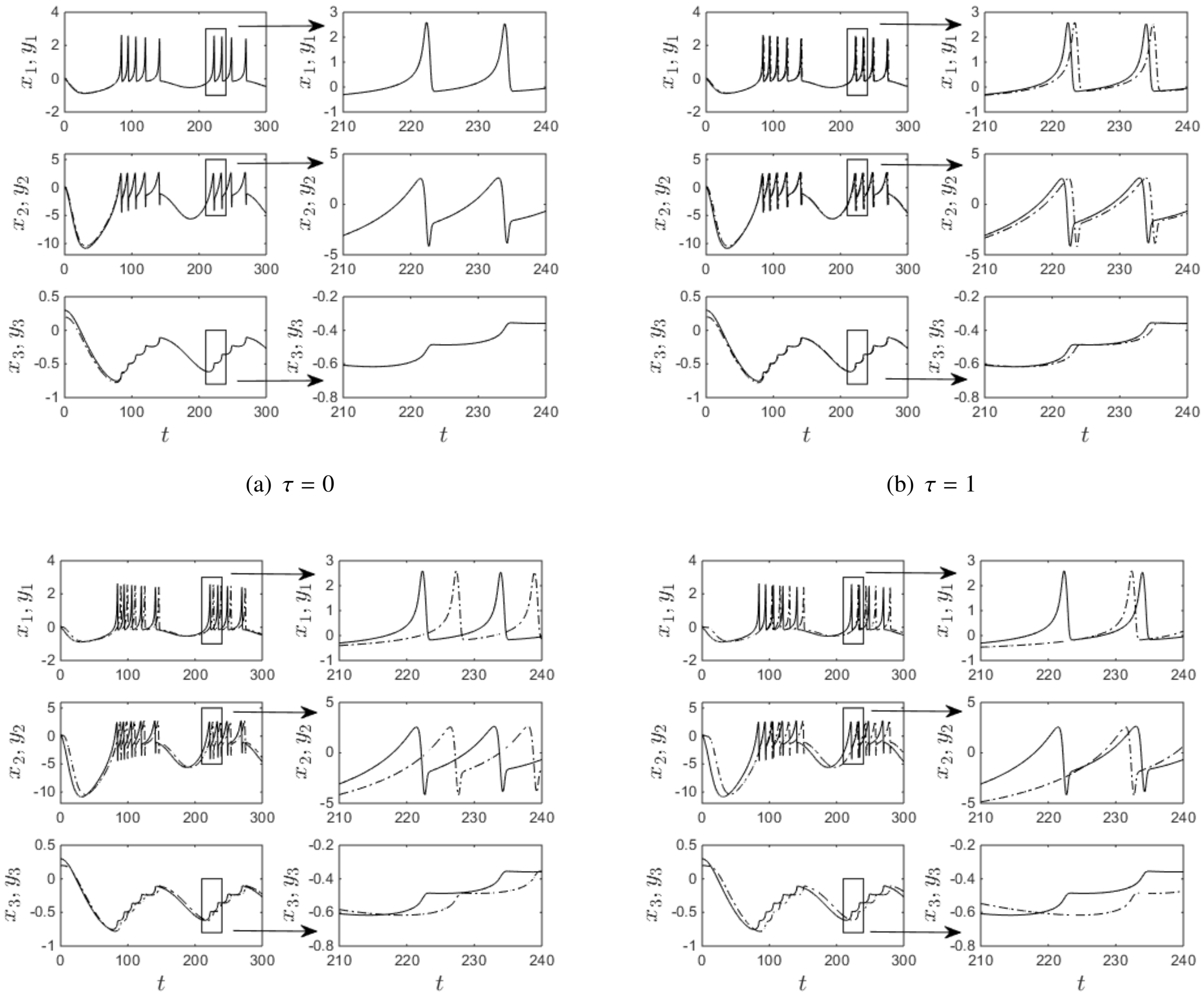


Figure 2. The estimates of the uncertain parameters of systems (2.2) and (2.5) with parameter update laws (4.2). The initial conditions are chosen as $x_1(t) = 0.1$, $x_2(t) = 0.2$, $x_3(t) = 0.3$ for $t \in (-\tau, 0]$ and $y_1(0) = 0.1$, $y_2(0) = 0.2$, $y_3(0) = 0.2$, $\psi_{1j}(0) = 2.0$, $j = 1, 2, \cdots, 7$. The control parameter is set to $k = 0.5$.

We conduct numerical simulations on systems (2.2) and (2.5) employing the parameter update laws specified in Eq (4.2) to validate the accuracy of the aforementioned analytical findings. The true values of the uncertain parameters ϕ_j, $j = 1, 2, \cdots, 7$, are presented in Eq (2.4). The initial conditions are chosen as $x_1(t) = 0.1$, $x_2(t) = 0.2$, $x_3(t) = 0.3$ for $t \in (-\tau, 0]$ and $y_1(0) = 0.1$, $y_2(0) = y_3(0) = 0.2$, $\psi_{1i}(0) = 2.0$, $i = 1, 2, \cdots, 7$. The control parameter is arbitrarily set to $k = 0.5$. Despite the absence of lag synchronization between systems (2.2) and (2.5), as depicted in Figure 1, the estimated parameters ψ_{1j} within these systems continue to align with the true values of ϕ_j, for $j = 1, 2, \cdots, 7$, i.e., $\psi_{110} = 1.0$, $\psi_{120} = 5.0$, $\psi_{130} = 0.006$, $\psi_{140} = 4.0$, $\psi_{150} = 5.3646$, $\psi_{160} = -6.5931$, and $\psi_{170} = 7.8822$, as shown in Figure 2.

To fulfill the requirement of condition (4.5), we opt for $k = 1.5$ and proceed with the numerical simulations for systems (2.2) and (2.5), utilizing the parameter update laws as outlined in Eq (4.2). The parameters and initial conditions, aside from the value of k being adjusted, are consistent with those depicted in Figure 1. The numerical findings, as illustrated in Figure 3, substantiate the effectiveness of the lag synchronization criteria established in the preceding section.



(a) $\tau = 0$ (b) $\tau = 1$

(c) $\tau = 5$ (d) $\tau = 10$

Figure 3. Lag synchronization between systems (2.2) and (2.5) with parameter update laws (4.2) can be achieved for any value of τ when the control parameter is set to $k = 1.5$ (a) $\tau = 0$ (b) $\tau = 1$ (c) $\tau = 5$ (d) $\tau = 10$. The initial conditions are taken as $x_1(t) = 0.1$, $x_2(t) = 0.2$, $x_3(t) = 0.3$ for $t \in (-\tau, 0]$ and $y_1(0) = 0.1$, $y_2(0) = 0.2$, $y_3(0) = 0.2$, $\psi_{1j}(0) = 2.0$, $j = 1, 2, \cdots, 7$. $x_{1,2,3}$ (solid line $-$), $y_{1,2,3}$ (dot dash line $-.$).

5. The approximation of lag and anticipating synchronization

In this section, we delve into the investigation of approximate lag and anticipating synchronization phenomena between systems (2.2) and (2.5), considering a scenario without an explicit time delay. In the preceding section, we examined the stability of lag synchronization between systems (2.2) and (2.5) under parameter update laws (4.2), which is maintained for all values of τ. To achieve approximate lag and anticipating synchronization, we propose substituting the actual time-delay term $x_1(t - \tau)$ with its

Taylor series expansion. Near $\tau = 0$, $x_1(t-\tau)$ in system (2.5) can be approximately expanded as

$$x_1(t-\tau) = x_1 - \frac{dx_1}{dt}\tau + \frac{1}{2!}\frac{d^2x_1}{dt^2}\tau^2 - \frac{1}{3!}\frac{d^3x_1}{dt^3}\tau^3 + \cdots, \tag{5.1}$$

where d^ix_1/dt^i, $i = 1, 2, 3$, can be determined by taking the derivatives with respect to t on both sides of the first equation in system (2.2), in which the uncertain parameters ϕ_i are replaced by using their estimated values ψ_{1j}, $i = 1, 2, \cdots, 7$, respectively,

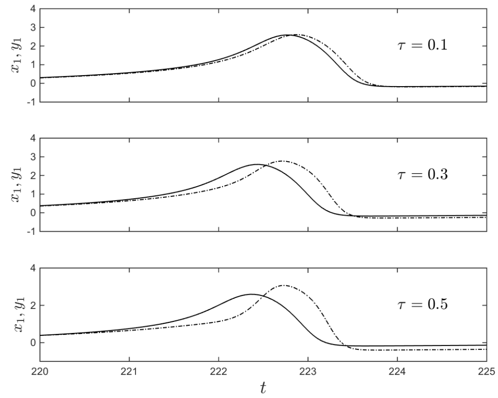
$$\begin{aligned}
\frac{dx_1}{dt} &= -\psi_{110}x_1^3 + \psi_{150}x_1^2 + \psi_{160}x_1 + x_2 - x_3,\\
\frac{d^2x_1}{dt^2} &= (-3\psi_{110}x_1^2 + 2\psi_{150}x_1 + \psi_{160})\frac{dx_1}{dt} - \psi_{120}x_1^2 + \psi_{170}x_1 - x_2 - \psi_{130}(\psi_{140}x_1 - x_3),\\
\frac{d^3x_1}{dt^3} &= (-3\psi_{110}x_1^2 + 2\psi_{150}x_1 + \psi_{160})\frac{d^2x_1}{dt^2} + 2(\psi_{150} - 3\psi_{110}x_1)(\frac{dx_1}{dt})^2 + (\psi_{170} - 2\psi_{120}x_1 - \psi_{130}\psi_{140})\frac{dx_1}{dt}\\
&\quad + \psi_{130}^2(\psi_{140}x_1 - x_3) + \psi_{120}x_1^2 - \psi_{170}x_1 + x_2.
\end{aligned} \tag{5.2}$$

Define the following functions of τ:

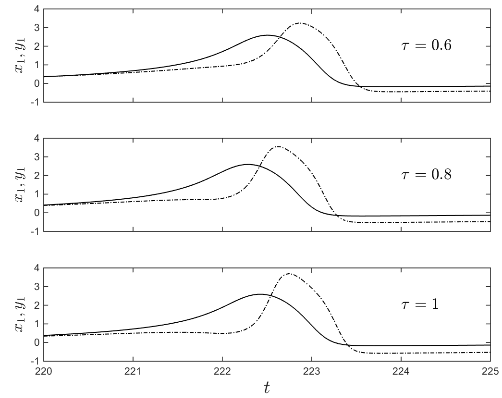
$$\begin{aligned}
Q_1(\tau) &= x_1 - \frac{dx_1}{dt}\tau,\\
Q_2(\tau) &= x_1 - \frac{dx_1}{dt}\tau + \frac{1}{2!}\frac{d^2x_1}{dt^2}\tau^2,\\
Q_3(\tau) &= x_1 - \frac{dx_1}{dt}\tau + \frac{1}{2!}\frac{d^2x_1}{dt^2}\tau^2 - \frac{1}{3!}\frac{d^3x_1}{dt^3}\tau^3.
\end{aligned} \tag{5.3}$$

$Q_{1,2,3}$ are the first, second, and third-order approximate expressions of $x_1(t-\tau)$, respectively. From the analysis in previous sections, for small time delay τ the response of system (2.5) with $x_1(t-\tau) = Q_i$, $i = 1, 2, 3$, allows an approximate time-shifted synchronization that lags or anticipates the drive of system (2.2) with the lag τ. Actually, system (2.5) with $x_1(t-\tau) = Q_i$, $i = 1, 2, 3$, is an intentionally mismatched response system that yields lag ($\tau > 0$) and anticipating ($\tau < 0$) synchronization. Numerical simulations are conducted for systems (2.2) and (2.5), incorporating parameter update laws (4.2) and the approximations $x_1(t-\tau) = Q_i$, for $i = 1, 2, 3$. These simulations are presented in Figures 4 and 5, showcasing the approximate time-shifted synchronizations achieved without the use of time delay coupling. The control parameter is taken as $k = 1.5$ and the initial conditions are chosen as $x_1(0) = y_1(0) = 0.1$, $x_2(0) = y_2(0) = y_3(0) = 0.2$, $x_3(0) = 0.3$, and $\psi_{1j}(0) = 2.0$, $j = 1, 2, \cdots, 7$.

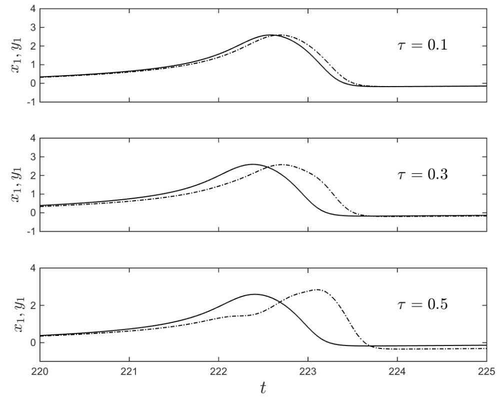
Figures 4 and 5 demonstrate that a broad spectrum of approximate lag and anticipation synchronization is attainable in the two HR neurons, even in the absence of time delay coupling. Overall, employing a higher-order approximation for the actual time delay $x_1(t-\tau)$ can enhance the precision of the approximation. Consequently, the scope of approximate lag and anticipation can be expanded by incorporating additional higher-order derivatives from the Taylor series expansion. The investigation presented in this section suggests that an HR neuron is capable of precisely approximating both past and future signals based solely on its current signal.



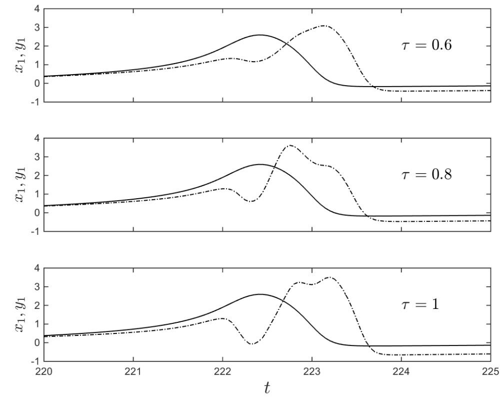
(a) replace $x_1(t-\tau)$ with Q_1, $\tau = 0.1, 0.3, 0.5$



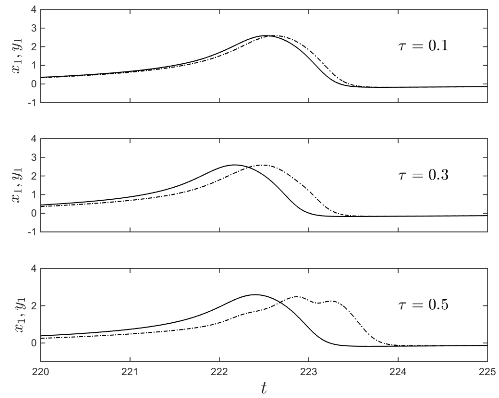
(b) replace $x_1(t-\tau)$ with Q_1, $\tau = 0.6, 0.8, 1$



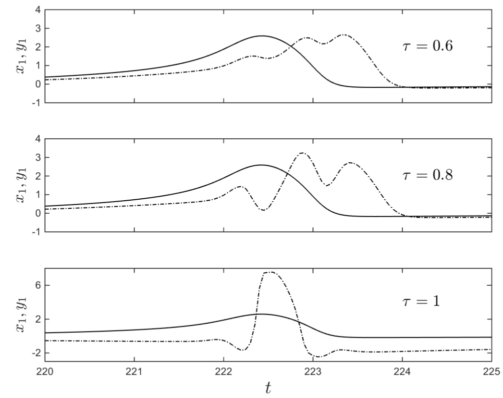
(c) replace $x_1(t-\tau)$ with Q_2, $\tau = 0.1, 0.3, 0.5$



(d) replace $x_1(t-\tau)$ with Q_2, $\tau = 0.6, 0.8, 1$

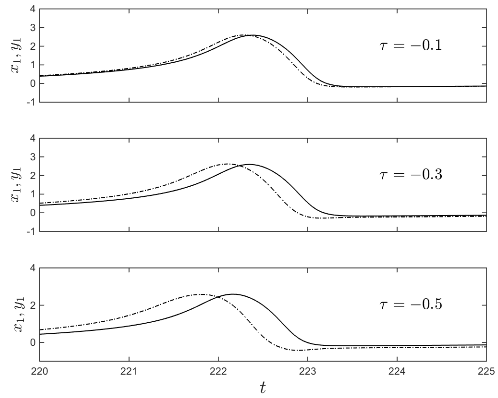


(e) replace $x_1(t-\tau)$ with Q_3, $\tau = 0.1, 0.3, 0.5$

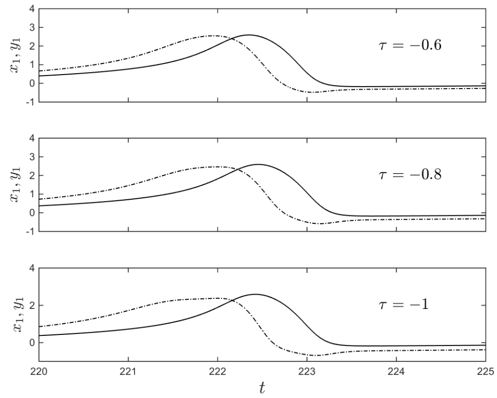


(f) replace $x_1(t-\tau)$ with Q_3, $\tau = 0.6, 0.8, 1$

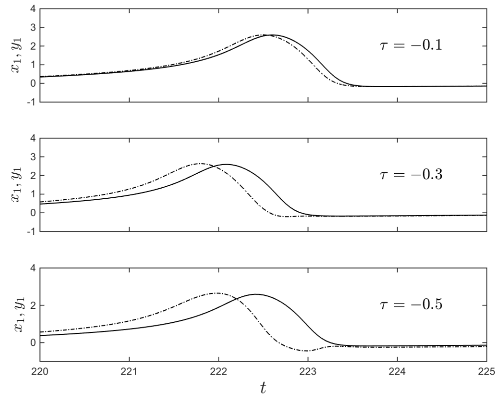
Figure 4. The approximate lag synchronization between systems (2.2) and (2.5) with the parameter update laws (4.2) for different values of τ. $x_1(t-\tau)$ in system (2.5) is replaced with Q_i, $i = 1, 2, 3$, respectively. The control parameter is set to $k = 1.5$. The initial conditions are taken as $x_1(0) = y_1(0) = 0.1$, $x_2(0) = y_2(0) = y_3(0) = 0.2$, $x_3(0) = 0.3$, and $\psi_{1j}(0) = 2.0$, $j = 1, 2, \cdots, 7$. x_1 (solid line $-$), y_1 (dot dash line $-.$).



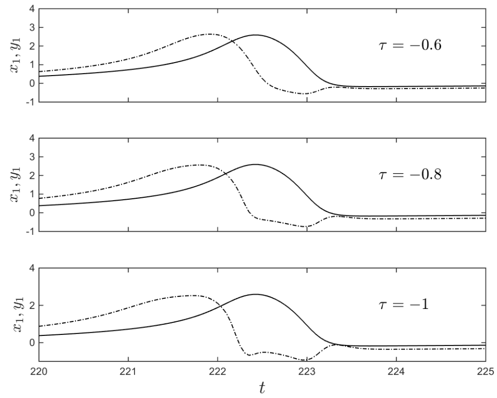
(a) replace $x_1(t - \tau)$ with Q_1, $\tau = -0.1, -0.3, -0.5$



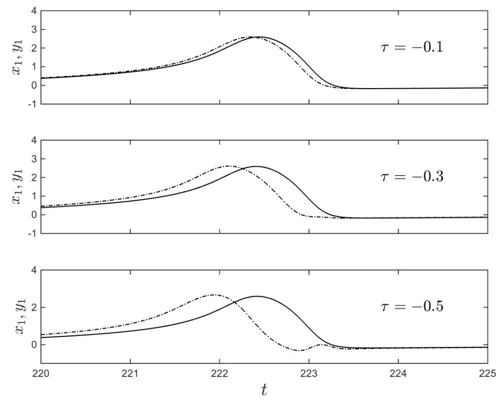
(b) replace $x_1(t - \tau)$ with Q_1, $\tau = -0.6, -0.8, -1$



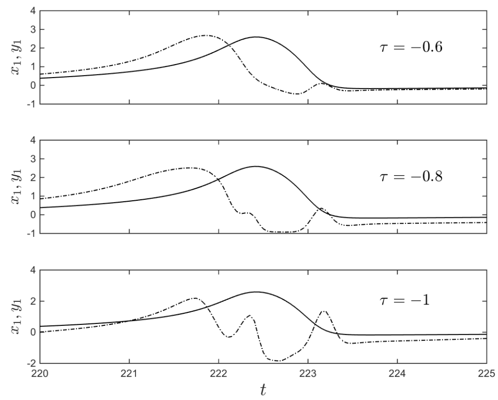
(c) replace $x_1(t - \tau)$ with Q_2, $\tau = -0.1, -0.3, -0.5$



(d) replace $x_1(t - \tau)$ with Q_2, $\tau = -0.6, -0.8, -1$



(e) replace $x_1(t - \tau)$ with Q_3, $\tau = -0.1, -0.3, -0.5$



(f) replace $x_1(t - \tau)$ with Q_3, $\tau = -0.6, -0.8, -1$

Figure 5. The approximate anticipating synchronization between systems (2.2) and (2.5) with parameter update laws (4.2) for different values of τ. $x_1(t - \tau)$ in system (2.5) is replaced with Q_i, $i = 1, 2, 3$, respectively. The control parameter is set to $k = 1.5$. The initial conditions are taken as $x_1(0) = y_1(0) = 0.1$, $x_2(0) = y_2(0) = y_3(0) = 0.2$, $x_3(0) = 0.3$, and $\psi_{1j}(0) = 2.0$, $j = 1, 2, \cdots, 7$. x_1 (solid line −), y_1 (dot dash line −.).

6. Conclusions

In this manuscript, we introduce an adaptive control scheme and a synchronization technique aimed at achieving lag synchronization between two unidirectionally coupled HR neurons. This approach is applicable even with explicit propagation delays and in the presence of complete uncertainty regarding the system parameters. Utilizing this approach, the response system can accurately track the driver system with a time lag, while simultaneously enabling the identification of the unknown parameters. Distinguished from prior research, the lag synchronization condition is derived here by employing the auxiliary system methodology, recognizing that lag synchronization can be considered a unique form of generalized synchronization. Ultimately, an analytical criterion is formulated to ascertain the emergence of lag synchronization in the two HR neurons. This formulation is grounded in the Laplace transform and the convolution theorem, alongside the iterative techniques within Volterra integral equation theory. The principal benefit of the lag synchronization scheme introduced in this paper is the ability to devise straightforward controllers and parameter update laws, circumventing the need to construct Lyapunov functions, which simplifies its physical implementation. Numerical simulations are employed to substantiate the efficacy of the proposed lag synchronization strategy.

The precise lag synchronization criteria established herein demonstrate robust stability for any magnitude of propagation delay. This breakthrough introduces a pioneering strategy that harnesses the Taylor series expansion for the intrinsic time-delay component. By employing this technique, it becomes feasible to achieve approximate lag synchronization and predictive synchronization in a pair of HR neurons that are interconnected in a one-way manner, eliminating the necessity for direct time-delay linkage. Numerical simulations confirm the feasibility of attaining a broad spectrum of approximate lag and anticipation synchronization in two unidirectionally coupled HR neurons through the application of this approach. Furthermore, the precision of the approximation can be enhanced by incorporating higher-order derivatives from the Taylor series expansion of the actual time-delay term. This paper introduces a straightforward, memoryless method for achieving approximate lag and anticipating synchronization between two unidirectionally coupled HR neurons, even in the presence of uncertain parameters. For an HR neuron characterized by uncertain parameters, it is feasible to accurately anticipate future states or reconstruct past states based solely on the current state information. The approach outlined herein offers an innovative perspective on understanding the dynamics of neural processes, as well as the beneficial attributes of systems that exhibit nonlinearity and chaos. Our findings additionally present a straightforward method to mitigate the adverse impacts of time delays in signal transmission between the two interconnected HR neurons.

Biological neural networks consist of an extensive array of neurons intricately linked together, which underscores the significance of studying neuronal synchronization phenomena at the network scale. The self-organizing capability of multiple neurons refers to the ability of a group of neurons to spontaneously form patterns and structures without external guidance. This phenomenon is a key aspect of complex systems in neuroscience, where the interactions between neurons can lead to emergent properties that are not present in individual neurons. One of our future research priorities is to explore how our proposed method can be adapted for analyzing synchronization among a large ensemble of neurons. It has been observed that anticipating synchronization can sometimes be characterized by the more fundamental notion of negative group delay in a filter, which could potentially be generalized to lag synchronization or positive group delay. In this context, the driven system or its linear approximation serves as a filter, and every stable filter exhibits both positive and negative group delays for specific frequencies. It would be intriguing to interpret the large scale system in this broader framework, especially considering that the group delay approach does not inherently require explicit coupling delays.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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