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Research article

A generalized quantum cluster algebra of Kronecker type

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Abstract: The notion of generalized quantum cluster algebras was introduced as a natural generalization of Berenstein and Zelevinsky's quantum cluster algebras as well as Chekhov and Shapiro's generalized cluster algebras. In this paper, we focus on a generalized quantum cluster algebra of Kronecker type which possesses infinitely many cluster variables. We obtain the cluster multiplication formulas for this algebra. As an application of these formulas, a positive bar-invariant basis is explicitly constructed. Both results generalize those known for the Kronecker cluster algebra and quantum cluster algebra.

Keywords: generalized quantum cluster algebra; cluster variable; cluster multiplication formula; positive basis; Kronecker quiver

1. Introduction

Cluster algebras were invented by Fomin and Zelevinsky [1, 2] in order to set up an algebraic framework for studying the total positivity and Lusztig's canonical bases. Quantum cluster algebras, as the quantum deformations of cluster algebras, were later introduced by Berenstein and Zelevinsky [3] for studying the dual canonical bases in coordinate rings and their q-deformations. An important feature of (quantum) cluster algebras is the so–called Laurent phenomenon which says that all cluster variables belong to an intersection of certain (may be infinitely many) rings of Laurent polynomials.

Generalized cluster algebras were introduced by Chekhov and Shapiro [4] in order to understand the Teichmüller theory of hyperbolic orbifold surfaces. The exchange relations for cluster variables of generalized cluster algebras are polynomial exchange relations, while the exchange relations for cluster algebras are binomial relations. Generalized cluster algebras also possess the Laurent phenomenon [4] and are studied by many people in a similar way as cluster algebras (see for example [5–9]). As a natural generalization of both quantum cluster algebras and generalized cluster algebras, we defined the generalized quantum cluster algebras [10]. It was not surprising that the Laurent phenomenon also holds in these algebras [11].

One of the most important problems in cluster theory is to construct cluster multiplication formulas. For acyclic cluster algebras, Sherman and Zelevinsky [12] first established the cluster multiplication formulas in rank 2 cluster algebras of finite and affine types. Cerulli [13] generalized this result to rank 3 cluster algebra of affine type $A_2^{(1)}$. Caldero and Keller [14] constructed the cluster multiplication formulas between two cluster characters for simply laced Dynkin quivers, which was generalized to affine types by Hubery in [15] and to acyclic types by Xiao and Xu in [16, 17]. In the quantum case, Ding and Xu [18] first gave the cluster multiplication formulas of the quantum cluster algebra of the Kronecker type. Recently, Chen et al. [19] obtained the cluster multiplication formulas in the acyclic quantum cluster algebras with arbitrary coefficients through some quotients of derived Hall algebras of acyclic valued quivers. Cluster multiplication formulas play an important role in constructing bases of (quantum) cluster algebras with nice properties (see for example [12–14, 18, 20]). In cluster theory, a basis is called positive if its structure constants are positive. Several positive bases such as the atomic bases and the triangular bases of some (quantum) cluster algebras have been found (see [21, 22]). So far, no similar results have been obtained in generalized quantum cluster algebras. It becomes natural to think whether one can give an explicit treatment of the above mentioned problems for generalized quantum cluster algebras.

In this paper, we study a generalized quantum cluster algebra of Kronecker type denoted by $\mathcal{A}_q(2,2)$, in which the exchange relations are trinomial while binomial in the usual quantum cluster algebra of the Kronecker type. We recall the definition of generalized quantum cluster algebras in Section 2, provide the cluster multiplication formulas of $\mathcal{A}_q(2,2)$ in Section 3, and explicitly construct a positive bar-invariant $\mathbb{Z}[q^{\pm\frac{1}{2}}, h]$-basis of $\mathcal{A}_q(2,2)$ in Section 4.

2. Preliminaries

In this section, we mainly review the definition of generalized quantum cluster algebras [10]. Throughout this section, m and n are positive integers with $m \geq n$. Let $\widetilde{B} = (b_{ij})$ be an $m \times n$ integer matrix whose upper $n \times n$ submatrix is denoted by B and $\Lambda = (\lambda_{ij})$ a skew-symmetric $m \times m$ integer matrix.

Definition 2.1. *The pair $(\Lambda, \widetilde{B})$ is called compatible if for any $1 \leq i \leq m$ and $1 \leq j \leq n$, we have*

$$\sum_{k=1}^{m} \lambda_{ki} b_{kj} = \begin{cases} \widetilde{d}_j, & \text{if } i = j; \\ 0, & \text{otherwise}; \end{cases} \tag{2.1}$$

for some positive integers $\widetilde{d}_j$ $(1 \leq j \leq n)$.

Note that the skew-symmetric matrix Λ gives a skew-symmetric bilinear form on $\mathbb{Z}^m$ defined by

$$\Lambda(\mathbf{a}, \mathbf{b}) = \mathbf{a}^T \Lambda \mathbf{b}$$

for any column vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$.

Let q be a formal variable and $\mathbb{Z}[q^{\pm\frac{1}{2}}] \subset \mathbb{Q}(q^{\frac{1}{2}})$ the ring of integer Laurent polynomials in $q^{\frac{1}{2}}$. One can associate to (Λ, q) a quantum torus algebra as follows.

Definition 2.2. *The quantum torus $\mathcal{T}$ over $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ is generated by the symbols $\{X(\mathbf{a}) \mid \mathbf{a} \in \mathbb{Z}^m\}$ subject to the multiplication relations*

$$X(\mathbf{a})X(\mathbf{b}) = q^{\frac{1}{2}\Lambda(\mathbf{a},\mathbf{b})}X(\mathbf{a} + \mathbf{b}), \tag{2.2}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$.

The skew-field of fractions of $\mathcal{T}$ is denoted by $\mathcal{F}$. On the quantum torus $\mathcal{T}$, the $\mathbb{Z}$-linear bar-involution is defined by setting

$$\overline{q^{\frac{r}{2}}X(\mathbf{a})} = q^{-\frac{r}{2}}X(\mathbf{a})$$

for any $r \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{Z}^m$.

Let e_k be the k-th standard unit vector in $\mathbb{Z}^m$ and set $X_k = X(e_k)$ for $1 \le k \le m$. An easy computation shows that

$$X(\mathbf{a}) = q^{\frac{1}{2}\sum\limits_{i<j}\lambda_{ji}a_ia_j}X_1^{a_1}X_2^{a_2}\dots X_m^{a_m}$$

for $\mathbf{a} = (a_1, a_2, \dots, a_m)^T \in \mathbb{Z}^m$.

For any $1 \le i \le n$, we say that $\widetilde{B}' = (b'_{kl})$ is obtained from the matrix $\widetilde{B} = (b_{kl})$ by the matrix mutation in the direction i if $\widetilde{B}' := \mu_i(\widetilde{B})$ is given by

$$b'_{kl} = \begin{cases} -b_{kl}, & \text{if} \quad k = i \quad \text{or} \quad l = i, \\ b_{kl} + \frac{|b_{ki}|b_{il} + b_{ki}|b_{il}|}{2}, & \text{otherwise.} \end{cases}$$

Denote the function

$$[x]_+ = \begin{cases} x, & \text{if} \quad x \ge 0; \\ 0, & \text{if} \quad x \le 0. \end{cases}$$

For any $1 \le i \le n$ and a sign $\varepsilon \in \{\pm 1\}$, denote by E_ε the $m \times m$ matrix associated to the matrix $\widetilde{B} = (b_{ij})$ with entries as follows

$$(E_\varepsilon)_{kl} = \begin{cases} \delta_{kl}, & \text{if } l \ne i; \\ -1, & \text{if } k = l = i; \\ [-\varepsilon b_{ki}]_+, & \text{if } k \ne l = i. \end{cases}$$

Proposition 2.3 ([3, Proposition 3.4]). *Let $(\Lambda, \widetilde{B})$ be a compatible pair, then the pair $(\Lambda', \widetilde{B}')$ is also compatible and independent of the choice of ε, where $\Lambda' = E_\varepsilon^T \Lambda E_\varepsilon$ and $\widetilde{B}' = \mu_i(\widetilde{B})$.*

We say that the compatible pair $(\Lambda', \widetilde{B}')$ is obtained from the compatible pair $(\Lambda, \widetilde{B})$ by mutation in the direction i and denoted by $\mu_i(\Lambda, \widetilde{B})$. It is known that μ_i is an involution [3, Proposition 3.6].

For each $1 \le i \le n$, let d_i be a positive integer such that $\frac{b_{li}}{d_i}$ are integers for all $1 \le l \le m$ and denote by $\beta^i = \frac{1}{d_i}\mathbf{b}^i$, where $\mathbf{b}^i$ is the i-th column of $\widetilde{B}$. Denote by

$$\mathbf{h}_i = \{h_{i,0}(q^{\frac{1}{2}}), h_{i,1}(q^{\frac{1}{2}}), \dots, h_{i,d_i}(q^{\frac{1}{2}})\}, \ 1 \le i \le n,$$

where $h_{k,l}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ satisfying that $h_{k,l}(q^{\frac{1}{2}}) = h_{k,d_k-l}(q^{\frac{1}{2}})$ and $h_{k,0}(q^{\frac{1}{2}}) = h_{k,d_k}(q^{\frac{1}{2}}) = 1$. We set $\mathbf{h} := (\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n)$.

Definition 2.4. *With the above notations, the quadruple $(X, \mathbf{h}, \Lambda, \widetilde{B})$ is called a quantum seed if the pair $(\Lambda, \widetilde{B})$ is compatible. For a given quantum seed $(X, \mathbf{h}, \Lambda, \widetilde{B})$ and each $1 \le i \le n$, the new quadruple*

$$(X', \mathbf{h}', \Lambda', \widetilde{B}') := \mu_i(X, \mathbf{h}, \Lambda, \widetilde{B})$$

is defined by

$$X'(e_k) = \mu_i(X(e_k)) = \begin{cases} X(e_k), & \text{if } k \neq i; \\ \sum_{r=0}^{d_i} h_{i,r}(q^{\frac{1}{2}})X(r[\beta^i]_+ + (d_i - r)[-\beta^i]_+ - e_i), & \text{if } k = i; \end{cases} \tag{2.3}$$

and

$$\mathbf{h}' = \mu_i(\mathbf{h}) = \mathbf{h} \ \text{ and } \ (\Lambda', \widetilde{B}') = \mu_i(\Lambda, \widetilde{B}).$$

We say that the quadruple $\mu_i(X, \mathbf{h}, \Lambda, \widetilde{B})$ is obtained from $(X, \mathbf{h}, \Lambda, \widetilde{B})$ by mutation in the direction i.

Proposition 2.5 ([10, Proposition 3.6])**.** *Let the quadruple $(X, \mathbf{h}, \Lambda, \widetilde{B})$ be a quantum seed, then the quadruple $\mu_i(X, \mathbf{h}, \Lambda, \widetilde{B})$ is also a quantum seed.*

Note that μ_i is an involution by [10, Proposition 3.7]. Two quantum seeds are said to be mutation-equivalent if they can be obtained from each other by a sequence of seed mutations. Given an initial quantum seed $(X, \mathbf{h}, \Lambda, \widetilde{B})$, let $(X', \mathbf{h}', \Lambda', \widetilde{B}')$ be mutation-equivalent to $(X, \mathbf{h}, \Lambda, \widetilde{B})$. Denote by $X' = \{X'_1, \ldots, X'_m\}$ which is called the extended cluster and the set $\{X'_1, \ldots, X'_n\}$ is called the cluster. The element X'_i is called a cluster variable for any $1 \le i \le n$ and X'_k a frozen variable for any $n+1 \le k \le m$. Note that $X'_k = X_k$ $(n+1 \le k \le m)$. For convenience, let $\mathbb{P}$ denote the multiplicative group generated by $X_{n+1}, \ldots, X_m$ and $q^{\frac{1}{2}}$, and $\mathbb{ZP}$ the ring of the Laurent polynomials in $X_{n+1}, \ldots, X_m$ with coefficients in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Definition 2.6. *Given the initial quantum seed $(X, \boldsymbol{h}, \Lambda, \widetilde{B})$, the associated generalized quantum cluster algebra $\mathcal{A}(X, \boldsymbol{h}, \Lambda, \widetilde{B})$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables from the quantum seeds which are mutation-equivalent to $(X, \boldsymbol{h}, \Lambda, \widetilde{B})$.*

The following Laurent phenomenon is one of the most important results on generalized quantum cluster algebras.

Theorem 2.7 ([11, Theorem 3.1])**.** *The generalized quantum cluster algebra $\mathcal{A}(X, \boldsymbol{h}, \Lambda, \widetilde{B})$ is a subalgebra of the ring of Laurent polynomials in the cluster variables in any cluster over $\mathbb{ZP}$.*

3. The cluster multiplication formulas of $\mathcal{A}_q(2, 2)$

In the following, we will consider the generalized quantum cluster algebra associated with the initial seed $(X, \mathbf{h}, \Lambda, B)$, where $\mathbf{d} = (2, 2)$, $\mathbf{h}_1 = \mathbf{h}_2 = (1, h, 1)$ with $h \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ and $\overline{h} = h$,

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Note that $\Lambda^T B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and the based quantum torus is

$$\mathcal{T} = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm 1}, X_2^{\pm 1} | X_1X_2 = qX_2X_1].$$

The quiver associated to the matrix B is the Kronecker quiver Q:

$$1\bullet \rightrightarrows \bullet 2$$

We call this algebra a generalized quantum cluster algebra of Kronecker type, denoted by $\mathcal{A}_q(2,2)$. By the definition and the Laurent phenomenon, $\mathcal{A}_q(2,2)$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{T}$ generated by the cluster variables $\{X_k \mid k \in \mathbb{Z}\}$ which are obtained from the following exchange relations:

$$X_{k-1}X_{k+1} = qX_k^2 + q^{\frac{1}{2}}hX_k + 1.$$

Recall that the n-th Chebyshev polynomial of the first kind $F_n(x)$ is defined by

$$F_0(x) = 1, F_1(x) = x, F_2(x) = x^2 - 2, F_{n+1}(x) = F_n(x)x - F_{n-1}(x) \text{ for } n \geq 2,$$

and $F_n(x) = 0$ for $n < 0$.

Denote

$$X_\delta := q^{\frac{1}{2}}X_0X_3 - q^{\frac{1}{2}}(q^{\frac{1}{2}}X_1 + h)(q^{\frac{1}{2}}X_2 + h),$$

thus $X_\delta \in \mathcal{A}_q(2,2)$.

Lemma 3.1. *For each $n \in \mathbb{Z}_{>0}$, $F_n(X_\delta)$ is a bar-invariant element in $\mathcal{A}_q(2,2)$.*

Proof. An direct computation shows that

$$X_\delta = X(-1,-1) + hX(-1,0) + hX(0,-1) + X(-1,1) + X(1,-1),$$

thus X_δ is a bar-invariant element in $\mathcal{A}_q(2,2)$. According to the definition of the n-th Chebyshev polynomial $F_n(x)$, one can deduce that $F_n(X_\delta)$ belong to $\mathbb{Z}[X_\delta]$. Thus the proof is completed. □

We define an automorphism denoted by σ on the generalized quantum cluster algebra $\mathcal{A}_q(2,2)$ as follows

$$\sigma(X_k) = X_{k+1} \text{ and } \sigma(q^{\frac{k}{2}}) = q^{\frac{k}{2}},$$

for any $k \in \mathbb{Z}$. Then we have the following result which will be useful for us to prove the cluster multiplication formulas.

Lemma 3.2. *For each $n \in \mathbb{Z}_{>0}$, $\sigma(F_n(X_\delta)) = F_n(X_\delta)$.*

Proof. Note that

$$\sigma(X_\delta) = q^{\frac{1}{2}}X_1X_4 - q^{\frac{1}{2}}(q^{\frac{1}{2}}X_2 + h)(q^{\frac{1}{2}}X_3 + h),$$

$$X_3 = X(-1,2) + hX(-1,1) + X(-1,0)$$

and

$$\begin{aligned} X_4 =& X(-2,3) + (q^{-\frac{1}{2}} + q^{\frac{1}{2}})hX(-2,2) + (q^{-1} + h^2 + q)X(-2,1) + (q^{-\frac{1}{2}} + q^{\frac{1}{2}})hX(-2,0) \\ &+ X(-2,-1) + hX(-1,1) + h^2X(-1,0) + hX(-1,-1) + X(0,-1). \end{aligned}$$

Thus

$$q^{\frac{1}{2}}X_1X_4 = q^2X(-1,3) + (q + q^2)hX(-1,2) + (1 + qh^2 + q^2)X(-1,1) + (1 + q)hX(-1,0)$$

$$+ X(-1,-1) + qhX(0,1) + q^{\frac{1}{2}}h^2 + hX(0,-1) + X(1,-1)$$

and

$$\begin{aligned} q^{\frac{1}{2}}(q^{\frac{1}{2}}X_2 + h)(q^{\frac{1}{2}}X_3 + h) =& q^2X(-1,3) + (q+q^2)hX(-1,2) + qhX(0,1) \\ &+ (qh^2 + q^2)X(-1,1) + qhX(-1,0) + q^{\frac{1}{2}}h^2. \end{aligned}$$

We obtain that

$$\sigma(X_\delta) = X(-1,1) + hX(-1,0) + X(-1,-1) + hX(0,-1) + X(1,-1) = X_\delta.$$

Then the proof follows from the induction on n and the definition of the n-th Chebyshev polynomial $F_n(x)$. □

For a real number x, denote the floor function by $\lfloor x \rfloor$ and the ceiling function by $\lceil x \rceil$. The following Theorem 3.3 and Remark 3.4 give the explicit cluster multiplication formulas for $\mathcal{A}_q(2,2)$.

Theorem 3.3. *Let m and n be integers.*

(1) For any $m > n \geq 1$, we have

$$F_m(X_\delta)F_n(X_\delta) = F_{m+n}(X_\delta) + F_{m-n}(X_\delta),\ F_n(X_\delta)F_n(X_\delta) = F_{2n}(X_\delta) + 2. \tag{3.1}$$

(2) For any $n \geq 1$, we have

$$X_mF_n(X_\delta) = q^{-\frac{n}{2}}X_{m-n} + q^{\frac{n}{2}}X_{m+n} + \sum_{k=1}^{n}(\sum_{l=1}^{k} q^{-\frac{k+1}{2}+l})hF_{n-k}(X_\delta). \tag{3.2}$$

(3) For any $n \geq 2$, we have

$$\begin{aligned} X_mX_{m+n} =& q^{\lfloor \frac{n}{2} \rfloor}X_{\lfloor m+\frac{n}{2} \rfloor}X_{\lceil m+\frac{n}{2} \rceil} + \sum_{k=1}^{n-1}(\sum_{l=1}^{min(k,n-k)} q^{-\frac{1}{2}+l})hX_{m+n-k} \\ &+ \sum_{l=1}^{n-1} q^{-\frac{n-1-l}{2}}c_lF_{n-1-l}(X_\delta), \end{aligned} \tag{3.3}$$

where $c_1 = 1$, $c_2 = h^2$ and for $k \geq 2$,

$$c_{2k} = [\sum_{i=1}^{k-1} a_i(q^{-(k-i)} + q^{k-i}) + a_k]h^2$$

and

$$c_{2k-1} = 2[\sum_{i=1}^{k-1} b_i(q^{-(k-i)} + q^{k-i}) + b_k]h^2 + \begin{cases} \sum_{i=1}^{\frac{k}{2}}(q^{-(k+1-2i)} + q^{k+1-2i}), & \textit{if } k \textit{ is even}; \\ \sum_{i=1}^{\frac{k-1}{2}}(q^{-(k+1-2i)} + q^{k+1-2i}) + 1, & \textit{if } k \textit{ is odd}; \end{cases}$$

with $a_i = \frac{i(i+1)}{2}$ *and*

$$b_i = \begin{cases} \dfrac{i^2-1}{4}, & \textit{if } i \textit{ is odd;} \\ \dfrac{i^2}{4}, & \textit{if } i \textit{ is even.} \end{cases}$$

Proof. (1) The proof is immediately from the definition of the n-th Chebyshev polynomial $F_n(x)$.

(2) By using the automorphism σ repeatedly, it suffices to prove the following equation

$$X_1F_n(X_\delta) = q^{-\frac{n}{2}}X_{1-n} + q^{\frac{n}{2}}X_{1+n} + \sum_{k=1}^{n}(\sum_{l=1}^{k} q^{-\frac{k+1}{2}+l})hF_{n-k}(X_\delta),$$

for $n \geq 1$.

When $n = 1$,

$$\begin{aligned} X_1X_\delta =& X(1,0)(X(-1,-1) + hX(-1,0) + hX(0,-1) + X(-1,1) + X(1,-1)) \\ =& q^{-\frac{1}{2}}X(0,-1) + h + q^{-\frac{1}{2}}hX(1,-1) + q^{\frac{1}{2}}X(0,1) + q^{-\frac{1}{2}}X(2,-1). \end{aligned}$$

Note that $X_0 = X(2,-1) + hX(1,-1) + X(0,-1)$. Thus $X_1X_\delta = q^{-\frac{1}{2}}X_0 + q^{\frac{1}{2}}X_2 + h$. It follows that

$$X_mX_\delta = q^{-\frac{1}{2}}X_{m-1} + q^{\frac{1}{2}}X_{m+1} + h$$

for all $m \in \mathbb{Z}$.

When $n = 2$,

$$\begin{aligned} X_1F_2(X_\delta) =& X_1(X_\delta^2 - 2) = q^{-\frac{1}{2}}X_0X_\delta + q^{\frac{1}{2}}X_2X_\delta + hX_\delta - 2X_1 \\ =& q^{-1}X_{-1} + qX_3 + (q^{-\frac{1}{2}} + q^{\frac{1}{2}})h + hX_\delta. \end{aligned}$$

When $n \geq 3$, assume that $X_1F_t(X_\delta) = q^{-\frac{t}{2}}X_{1-t} + q^{\frac{t}{2}}X_{1+t} + \sum_{k=1}^{t}(\sum_{l=1}^{k} q^{-\frac{k+1}{2}+l})hF_{n-k}(X_\delta)$ for $t \leq n-1$.

If $t = n$, then

$$X_1F_n(X_\delta) = X_1(F_{n-1}(X_\delta)X_\delta - F_{n-2}(X_\delta)) = X_1F_{n-1}(X_\delta)X_\delta - X_1F_{n-2}(X_\delta).$$

By induction, we have

$$\begin{aligned} & X_1F_{n-1}(X_\delta)X_\delta \\ =& q^{-\frac{n-1}{2}}X_{2-n}X_\delta + q^{\frac{n-1}{2}}X_nX_\delta + \sum_{k=1}^{n-1}(\sum_{l=1}^{k} q^{-\frac{k+1}{2}+l})hF_{n-1-k}(X_\delta)X_\delta \\ =& q^{-\frac{n}{2}}X_{1-n} + q^{1-\frac{n}{2}}X_{3-n} + q^{\frac{n}{2}-1}X_{n-1} + q^{\frac{n}{2}}X_{n+1} + (q^{-\frac{n-1}{2}} + q^{\frac{n-1}{2}})h \\ &+ \sum_{k=1}^{n-1}(\sum_{l=1}^{k} q^{-\frac{k+1}{2}+l})hF_{n-1-k}(X_\delta)X_\delta, \end{aligned}$$

and $X_1F_{n-2}(X_\delta)=q^{1-\frac{n}{2}}X_{3-n}+q^{\frac{n}{2}-1}X_{n-1}+\sum\limits_{k=1}^{n-2}(\sum\limits_{l=1}^{k}q^{-\frac{k+1}{2}+l})hF_{n-2-k}(X_\delta)$. Note that

$$\begin{aligned}&\sum_{k=1}^{n-1}(\sum_{l=1}^{k}q^{-\frac{k+1}{2}+l})hF_{n-1-k}(X_\delta)X_\delta-\sum_{k=1}^{n-2}(\sum_{l=1}^{k}q^{-\frac{k+1}{2}+l})hF_{n-2-k}(X_\delta)\\&=\sum_{k=1}^{n-3}(\sum_{l=1}^{k}q^{-\frac{k+1}{2}+l})h(F_{n-1-k}(X_\delta)X_\delta-F_{n-2-k}(X_\delta))+(\sum_{l=1}^{n-2}q^{-\frac{n-1}{2}+l})h(X_\delta^2-2)\\&\quad+(\sum_{l=1}^{n-2}q^{-\frac{n-1}{2}+l})h+(\sum_{l=1}^{n-1}q^{-\frac{n}{2}+l})hX_\delta\\&=\sum_{k=1}^{n-1}(\sum_{l=1}^{k}q^{-\frac{k+1}{2}+l})hF_{n-k}(X_\delta)+(\sum_{l=1}^{n-2}q^{-\frac{n-1}{2}+l})h\end{aligned}$$

and $\sum\limits_{l=1}^{n-2}q^{-\frac{n-1}{2}+l}h+(q^{-\frac{n-1}{2}}+q^{\frac{n-1}{2}})h=\sum\limits_{l=1}^{n}q^{-\frac{n+1}{2}+l}h$.

It follows that $X_1F_n(X_\delta)=q^{-\frac{n}{2}}X_{1-n}+q^{\frac{n}{2}}X_{n+1}+\sum\limits_{k=1}^{n}(\sum\limits_{l=1}^{k}q^{-\frac{k+1}{2}+l})hF_{n-k}(X_\delta)$.

(3) In order to prove (3.3), it suffices to show that

$$X_1X_{1+n}=q^{\lfloor\frac{n}{2}\rfloor}X_{\lfloor 1+\frac{n}{2}\rfloor}X_{\lceil 1+\frac{n}{2}\rceil}+\sum_{k=1}^{n-1}(\sum_{l=1}^{\min(k,n-k)}q^{-\frac{1}{2}+l})hX_{1+n-k}+\sum_{l=1}^{n-1}q^{-\frac{n-1-l}{2}}c_lF_{n-1-l}(X_\delta)$$

for $n\geq 1$.

When $n=2$, it is the exchange relation. When $n=3$, by (3.2), we have that

$$\begin{aligned}&X_1X_4\\=&X_1(q^{-\frac{1}{2}}X_3X_\delta-q^{-1}X_2-q^{-\frac{1}{2}}h)\\=&q^{-\frac{1}{2}}(qX_2^2+q^{\frac{1}{2}}hX_2+1)X_\delta-q^{-1}X_1X_2-q^{-\frac{1}{2}}hX_1\\=&q^{\frac{1}{2}}X_2(q^{-\frac{1}{2}}X_1+q^{\frac{1}{2}}X_3+h)+h(q^{-\frac{1}{2}}X_1+q^{\frac{1}{2}}X_3+h)+q^{-\frac{1}{2}}X_\delta-q^{-1}X_1X_2-q^{-\frac{1}{2}}hX_1\\=&qX_2X_3+q^{\frac{1}{2}}hX_2+q^{\frac{1}{2}}hX_3+q^{-\frac{1}{2}}X_\delta+h^2.\end{aligned}$$

Assume that

$$X_1X_{1+t}=q^{\lfloor\frac{t}{2}\rfloor}X_{\lfloor 1+\frac{t}{2}\rfloor}X_{\lceil 1+\frac{t}{2}\rceil}+\sum_{k=1}^{t-1}(\sum_{l=1}^{\min(k,t-k)}q^{-\frac{1}{2}+l})hX_{1+t-k}+\sum_{l=1}^{t-1}q^{-\frac{t-1-l}{2}}c_lF_{t-1-l}(X_\delta)$$

for all $t\leq n-1$.

Note that $X_1X_{n+1}=q^{-\frac{1}{2}}X_1X_nX_\delta-q^{-1}X_1X_{n-1}-q^{-\frac{1}{2}}hX_1$.

When n is even and $n\geq 4$, then

$$X_1X_n=q^{\frac{n}{2}-1}X_{\frac{n}{2}}X_{\frac{n}{2}+1}+\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)}q^{-\frac{1}{2}+l})hX_{n-k}+\sum_{l=1}^{n-2}q^{-\frac{n-2-l}{2}}c_lF_{n-2-l}(X_\delta),$$

$$q^{-1}X_1X_{n-1} = q^{\frac{n}{2}-2}X^2_{\frac{n}{2}} + \sum_{k=1}^{n-3}(\sum_{l=1}^{\min(k,n-2-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} + \sum_{l=1}^{n-3} q^{-\frac{n-1-l}{2}}c_lF_{n-3-l}(X_\delta)$$

and

$$\begin{aligned}
&q^{-\frac{1}{2}}X_1X_nX_\delta \\
=&q^{\frac{n-3}{2}}X_{\frac{n}{2}}X_{\frac{n}{2}+1}X_\delta + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})hX_{n-k}X_\delta + \sum_{l=1}^{n-2} q^{-\frac{n-1-l}{2}}c_lF_{n-2-l}(X_\delta)X_\delta \\
=&q^{\frac{n}{2}-2}X^2_{\frac{n}{2}} + q^{\frac{n}{2}}X^2_{\frac{n}{2}+1} + q^{\frac{n-1}{2}}hX_{\frac{n}{2}+1} + q^{\frac{n}{2}-1} + q^{\frac{n-3}{2}}hX_{\frac{n}{2}} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} \\
&+ \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{1}{2}+l})hX_{n+1-k} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2 \\
&+ \sum_{l=1}^{n-2} q^{-\frac{n-1-l}{2}}c_lF_{n-2-l}(X_\delta)X_\delta.
\end{aligned}$$

Note that

$$\begin{cases}
k \leq n-2-k, \text{ if } 1 \leq k \leq \frac{n}{2}-1; \\
k > n-2-k, \text{ if } \frac{n}{2} \leq k \leq n-3; \\
k < n-1-k, \text{ if } 1 \leq k \leq \frac{n}{2}-1; \\
k > n-1-k, \text{ if } \frac{n}{2} \leq k \leq n-3; \\
k < n-k, \quad \text{ if } 1 \leq k \leq \frac{n}{2}-1; \\
k \geq n-k, \quad \text{ if } \frac{n}{2} \leq k \leq n-1.
\end{cases}$$

It follows that

$$\begin{aligned}
&\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} - \sum_{k=1}^{n-3}(\sum_{l=1}^{\min(k,n-2-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} - q^{-\frac{1}{2}}hX_1 \\
=&\sum_{k=\frac{n}{2}}^{n-3} q^{-\frac{5}{2}+n-k}hX_{n-1-k} = \sum_{k=\frac{n}{2}+2}^{n-1} q^{-\frac{1}{2}+n-k}hX_{n+1-k}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{n-1}(\sum_{l=1}^{\min(k,n-k)} q^{-\frac{1}{2}+l})hX_{n+1-k} \\
=&\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{1}{2}+l})hX_{n+1-k} + q^{\frac{n-1}{2}}hX_{\frac{n}{2}+1}
\end{aligned}$$

$$+ q^{\frac{n-3}{2}} hX_{\frac{n}{2}} - \sum_{k=1}^{n-3}(\sum_{l=1}^{\min(k,n-2-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} - q^{-\frac{1}{2}}hX_1.$$

Note that

$$\begin{aligned}
&q^{\frac{n}{2}-1} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2 + \sum_{l=1}^{n-2} q^{-\frac{n-1-l}{2}} c_l F_{n-2-l}(X_\delta)X_\delta \\
&- \sum_{l=1}^{n-3} q^{-\frac{n-1-l}{2}} c_l F_{n-3-l}(X_\delta) \\
=&q^{\frac{n}{2}-1} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2 + q^{-1}c_{n-3} + \sum_{l=1}^{n-2} q^{-\frac{n-1-l}{2}} c_l F_{n-1-l}(X_\delta),
\end{aligned}$$

it suffices to prove that $q^{\frac{n}{2}-1} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2 + q^{-1}c_{n-3} = c_{n-1}$.

We have that

$$c_{n-1} = [\sum_{i=1}^{\frac{n}{2}-1} b_i(q^{-(\frac{n}{2}-i)} + q^{\frac{n}{2}-i}) + b_{\frac{n}{2}}]2h^2 + (q^{-(\frac{n}{2}-1)} + q^{-(\frac{n}{2}-3)} + \ldots + q^{\frac{n}{2}-3} + q^{\frac{n}{2}-1}),$$

$$\begin{aligned}
q^{-1}c_{n-3} =&\Big[\sum_{i=1}^{\frac{n}{2}-2} b_i(q^{-(\frac{n}{2}-i)} + q^{\frac{n}{2}-2-i}) + b_{\frac{n}{2}-1}q^{-1}\Big]2h^2 \\
&+ (q^{-(\frac{n}{2}-1)} + q^{-(\frac{n}{2}-3)} + \ldots + q^{\frac{n}{2}-5} + q^{\frac{n}{2}-3})
\end{aligned}$$

and $b_k - b_{k-2} = k-1$. Thus

$$c_{n-1} - q^{-1}c_{n-3} - q^{\frac{n}{2}-1} = \Big[\sum_{k=1}^{\frac{n}{2}}(k-1)q^{\frac{n}{2}-k}\Big]2h^2 = \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2.$$

Therefore

$$\begin{aligned}
&q^{\frac{n}{2}-1} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2 + q^{-1}c_{n-3} + \sum_{l=1}^{n-2} q^{-\frac{n-1-l}{2}} c_l F_{n-1-l}(X_\delta) \\
=&\sum_{l=1}^{n-1} q^{-\frac{n-1-l}{2}} c_l F_{n-1-l}(X_\delta)
\end{aligned}$$

and $X_1X_{1+n} = q^{\frac{n}{2}}X^2_{\frac{n}{2}+1} + \sum_{k=1}^{n-1}(\sum_{l=1}^{\min(k,n-k)} q^{-\frac{1}{2}+l})hX_{n+1-k} + \sum_{l=1}^{n-1} q^{-\frac{n-1-l}{2}} c_l F_{n-1-l}(X_\delta)$.

When n is odd and $n \geq 5$, we have

$$X_1X_n = q^{\frac{n-1}{2}}X^2_{\frac{n+1}{2}} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{1}{2}+l})hX_{n-k} + \sum_{l=1}^{n-2} q^{-\frac{n-2-l}{2}} c_l F_{n-2-l}(X_\delta)$$

and

$$\begin{aligned}&q^{-1}X_1X_{n-1}\\ =&q^{\frac{n-5}{2}}X_{\frac{n-1}{2}}X_{\frac{n+1}{2}}+\sum_{k=1}^{n-3}(\sum_{l=1}^{\min(k,n-2-k)}q^{-\frac{3}{2}+l})hX_{n-1-k}+\sum_{l=1}^{n-3}q^{-\frac{n-1-l}{2}}c_lF_{n-3-l}(X_\delta).\end{aligned}$$

Then

$$\begin{aligned}&q^{-\frac{1}{2}}X_1X_nX_\delta\\ =&q^{\frac{n}{2}-1}X_{\frac{n+1}{2}}(q^{-\frac{1}{2}}X_{\frac{n-1}{2}}+q^{\frac{1}{2}}X_{\frac{n+3}{2}}+h)+\sum_{l=1}^{n-2}q^{-\frac{n-1-l}{2}}c_lF_{n-2-l}(X_\delta)X_\delta\\ &+\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)}q^{-1+l})h(q^{-\frac{1}{2}}X_{n-1-k}+q^{\frac{1}{2}}X_{n+1-k}+h)\end{aligned}$$

$$\begin{aligned}=&q^{\frac{n-3}{2}}X_{\frac{n+1}{2}}X_{\frac{n-1}{2}}+q^{\frac{n-1}{2}}X_{\frac{n+1}{2}}X_{\frac{n+3}{2}}+q^{\frac{n}{2}-1}hX_{\frac{n+1}{2}}+\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)}q^{-\frac{3}{2}+l})hX_{n-1-k}\\ &+\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)}q^{-\frac{1}{2}+l})hX_{n+1-k}+\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)}q^{-1+l})h^2\\ &+\sum_{l=1}^{n-2}q^{-\frac{n-1-l}{2}}c_lF_{n-2-l}(X_\delta)X_\delta.\end{aligned}$$

Note that

$$\begin{cases}k<n-2-k, \text{ if } 1\le k\le \dfrac{n-3}{2};\\ k>n-2-k, \text{ if } \dfrac{n-1}{2}\le k\le n-3;\\ k\le n-1-k, \text{ if } 1\le k\le \dfrac{n-1}{2};\\ k>n-1-k, \text{ if } \dfrac{n+1}{2}\le k\le n-3;\\ k<n-k, \quad\quad \text{if } 1\le k\le \dfrac{n-1}{2};\\ k>n-k, \quad\quad \text{if } \dfrac{n+1}{2}\le k\le n.\end{cases}$$

Hence

$$\begin{aligned}&\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)}q^{-\frac{3}{2}+l})hX_{n-1-k}-\sum_{k=1}^{n-3}(\sum_{l=1}^{\min(k,n-2-k)}q^{-\frac{3}{2}+l})hX_{n-1-k}\\ =&q^{-\frac{1}{2}}hX_1+\sum_{k=\frac{n+1}{2}}^{n-3}q^{n-\frac{5}{2}-k}hX_{n-1-k}+q^{\frac{n}{2}-2}hX_{\frac{n-1}{2}}\end{aligned}$$

$$=q^{-\frac{1}{2}}hX_1 + \sum_{k=\frac{n-1}{2}}^{n-3} q^{n-\frac{5}{2}-k}hX_{n-1-k} = \sum_{k=\frac{n+3}{2}}^{n} q^{n-\frac{1}{2}-k}hX_{n+1-k}.$$

Note that

$$q^{\frac{n}{2}-1}hX_{\frac{n+1}{2}} + \sum_{l=1}^{\frac{n-3}{2}} q^{-\frac{1}{2}+l}hX_{\frac{n+1}{2}} = \sum_{l=1}^{\frac{n-1}{2}} q^{-\frac{1}{2}+l}hX_{\frac{n+1}{2}}$$

and

$$\sum_{k=\frac{n-1}{2}}^{n-3} q^{n-\frac{5}{2}-k}hX_{n-1-k} = \sum_{k=\frac{n+3}{2}}^{n-1} q^{n-\frac{1}{2}-k}hX_{n-1-k},$$

then we obtain that

$$\begin{aligned}
&q^{\frac{n}{2}-1}hX_{\frac{n+1}{2}} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-\frac{1}{2}+l})hX_{n+1-k}\\
&-\sum_{k=1}^{n-3}(\sum_{l=1}^{\min(k,n-2-k)} q^{-\frac{3}{2}+l})hX_{n-1-k} - q^{-\frac{1}{2}}hX_1\\
=&\sum_{k=1}^{n-1}(\sum_{l=1}^{\min(k,n-k)} q^{-\frac{1}{2}+l})hX_{n+1-k}.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{l=1}^{n-2} q^{-\frac{n-1-l}{2}}c_lF_{n-2-l}(X_\delta)X_\delta - \sum_{l=1}^{n-3} q^{-\frac{n-1-l}{2}}c_lF_{n-3-l}(X_\delta)\\
=&\sum_{l=1}^{n-2} q^{-\frac{n-1-l}{2}}c_lF_{n-1-l}(X_\delta) + q^{-1}c_{n-3},
\end{aligned}$$

we only need to show that $q^{-1}c_{n-3} + \sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2 = c_{n-1}$.

Note that $a_k - a_{k-2} = 2k-1$ for $k \geq 3$, then

$$\begin{aligned}
&c_{n-1} - q^{-1}c_{n-3}\\
=&\Big[(n-2) + (n-4)q + (n-6)q^2 + \ldots + 5q^{\frac{n-7}{2}} + 3q^{\frac{n-5}{2}} + q^{\frac{n-3}{2}}\Big]h^2\\
=&\sum_{k=1}^{n-2}(\sum_{l=1}^{\min(k,n-1-k)} q^{-1+l})h^2.
\end{aligned}$$

Therefore

$$X_1X_{1+n} = q^{\frac{n-1}{2}}X_{\frac{n+1}{2}}X_{\frac{n+3}{2}} + \sum_{k=1}^{n-1}(\sum_{l=1}^{\min(k,n-k)} q^{-\frac{1}{2}+l})hX_{n+1-k} + \sum_{l=1}^{n-1} q^{-\frac{n-1-l}{2}}c_lF_{n-1-l}(X_\delta).$$

The proof is completed. □

Remark 3.4. *According to [10, Proposition 4.6] and Lemma 3.1, all cluster variables and $F_n(X_\delta)$ ($n \in \mathbb{Z}_{>0}$) are bar-invariant. Therefore, the cluster multiplication formulas for $F_n(X_\delta)F_m(X_\delta)$, $F_n(X_\delta)X_m$ and $X_{m+n}X_m$ can be obtained by applying the bar-involution to all formulas in Theorem 3.3.*

4. A positive bar-invariant basis of $\mathcal{A}_q(2,2)$

In this section, we will explicitly construct a positive bar-invariant $\mathbb{Z}[q^{\pm\frac{1}{2}},h]$-basis of $\mathcal{A}_q(2,2)$.

Definition 4.1. *A basis of $\mathcal{A}_q(2,2)$ is called a positive $\mathbb{Z}[q^{\pm\frac{1}{2}},h]$-basis if its structure constants belong to $\mathbb{Z}_{\geq 0}[q^{\pm\frac{1}{2}},h]$.*

Denote

$$\mathcal{B} = \{q^{-\frac{a_1a_2}{2}}X_m^{a_1}X_{m+1}^{a_2}|m \in \mathbb{Z}, (a_1,a_2) \in \mathbb{Z}^2_{\geq 0}\} \sqcup \{F_n(X_\delta)|n \in \mathbb{Z}_{>0}\}.$$

Lemma 4.2. *All elements in $\mathcal{B}$ are bar-invariant.*

Proof. According to [10, Lemma 4.3, Proposition 4.6], the following equations hold for any $m \in \mathbb{Z}$:

$$X_mX_{m+1} = qX_{m+1}X_m,\ \overline{X_m} = X_m.$$

Thus, for any $m \in \mathbb{Z}$ and $(a_1,a_2) \in \mathbb{Z}^2_{\geq 0}$, we have

$$\overline{q^{-\frac{a_1a_2}{2}}X_m^{a_1}X_{m+1}^{a_2}} = q^{\frac{a_1a_2}{2}}X_{m+1}^{a_2}X_m^{a_1} = q^{-\frac{a_1a_2}{2}}X_m^{a_1}X_{m+1}^{a_2}$$

which assert that all elements in the set $\{q^{-\frac{a_1a_2}{2}}X_m^{a_1}X_{m+1}^{a_2}|m \in \mathbb{Z}, (a_1,a_2) \in \mathbb{Z}^2_{\geq 0}\}$ are bar-invariant. Together with Lemma 3.1, we know that any element in $\mathcal{B}$ is bar-invariant. □

In order to prove that the elements in $\mathcal{B}$ are $\mathbb{Z}[q^{\pm\frac{1}{2}},h]$-independent, we need the following definition which gives a partial order $\leq$ on $\mathbb{Z}^2$.

Definition 4.3. *Let (r_1,r_2) and $(s_1,s_2) \in \mathbb{Z}^2$. If $r_i \leq s_i$ for each $1 \leq i \leq 2$, we write $(r_1,r_2) \leq (s_1,s_2)$. Furthermore, if $r_i < s_i$ for some i, we write $(r_1,r_2) < (s_1,s_2)$.*

Theorem 4.4. *The set $\mathcal{B}$ is a positive bar-invariant $\mathbb{Z}[q^{\pm\frac{1}{2}},h]$-basis of $\mathcal{A}_q(2,2)$.*

Proof. According to Theorem 3.3 and Remark 3.4, we can deduce that the generalized quantum cluster algebra $\mathcal{A}_q(2,2)$ is $\mathbb{Z}[q^{\pm\frac{1}{2}},h]$-spanned by the elements in $\mathcal{B}$.

Note that X_δ has the minimal non-zero term $X(-1,-1)$ associated to the partial order in Definition 4.3, and thus by Theorem 3.3, we deduce that the element $F_n(X_\delta)$ has the minimal non-zero term $X(-n,-n)$ for each $n \in \mathbb{Z}_{>0}$. According to Theorem 3.3 (2), we have $X_nX_\delta = q^{\frac{1}{2}}X_{n+1}+q^{-\frac{1}{2}}X_{n-1}+h$. Thus, for each $n \geq 2$, we obtain that the cluster variable X_n has the minimal non-zero term $a_nX(-n+2,-n+3)$ where $a_n \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$, and for each $n \geq -1$, the cluster variable X_{-n} has the minimal non-zero term $b_nX(-n,-n-1)$ where $b_n \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Hence, there exists a bijection between the set of all minimal non-zero terms in cluster variables and $F_n(X_\delta)$ $(n \in \mathbb{Z}_{>0})$ and almost positive roots associated to the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Using the same discussion as [12, Proposition 3.1], we have that there exists a bijection between the set of all minimal non-zero terms in the elements in $\mathcal{B}$ and $\mathbb{Z}^2$, which implies that the elements in $\mathcal{B}$ are $\mathbb{Z}[q^{\pm\frac{1}{2}},h]$-independent.

It is easy to see that the structure constants of the cluster multiplication formulas in Theorem 3.3 and Remark 3.4 belong to $\mathbb{Z}_{\geq 0}[q^{\pm\frac{1}{2}},h]$, i.e., positive. Thus by using Theorem 3.3 and Remark 3.4 repeatedly, one can deduce that the structure constants of the basis elements are positive. Together with Lemma 4.2, the proof is completed. □

Remark 4.5. *If we set $h = 0$ and $q = 1$, then the set $\mathcal{B}$ is exactly the canonical basis of the cluster algebra of Kronecker quiver obtained in [12].*

Definition 4.6. *An element in $\mathcal{A}_q(2,2)$ is called positive if the coefficients of its Laurent expansion associated to any cluster belong to $\mathbb{Z}_{\geq 0}[q^{\pm\frac{1}{2}}, h]$.*

Remark 4.7. *Using the same arguments as [24, Corollary 8.3.3], it is not difficult to see that every element in $\mathcal{B}$ is positive: According to Theorem 2.7, for any element $b \in \mathcal{B}$ and any cluster (X_n, X_{n+1}), we have*

$$bX_n^{d_1}X_{n+1}^{d_2} = \sum_{(m_1,m_2)} b_{m_1,m_2}X_n^{m_1}X_{n+1}^{m_2}$$

where d_1, d_2, m_1, m_2 are nonnegative integers and the coefficients $b_{m_1,m_2} \in \mathbb{Z}[q^{\pm\frac{1}{2}}, h]$. Note that $\mathcal{B}$ is a positive basis by Theorem 4.4, thus $b_{m_1,m_2} \in \mathbb{Z}_{\geq 0}[q^{\pm\frac{1}{2}}, h]$. In particular, we obtain that all cluster variables of $\mathcal{A}_q(2,2)$ are positive, which is a special case in [23].

5. Conclusions

We study a generalized quantum cluster algebra of Kronecker type $\mathcal{A}_q(2,2)$. We prove the cluster multiplication formulas of $\mathcal{A}_q(2,2)$. For this, we define the element X_δ in $\mathcal{A}_q(2,2)$, and then use the n-th Chebyshev polynomial of the first kind $F_n(x)$ ($n \in \mathbb{Z}_{\geq 0}$) which naturally arises in cluster theory associated to quivers of affine type and surface type. As an application of the cluster multiplication formulas, a positive bar-invariant basis of this algebra is explicitly constructed. We hope the combinatorics developed here will be used to study generalized quantum cluster algebras for any rank in a future study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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