Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 32(1): 551–564.
DOI: 10.3934/era.2024027
Received: 07 December 2022
Revised: 03 February 2023
Accepted: 10 February 2023
Published: 04 January 2024

*Research article*

# On some extrapolation in generalized grand Morrey spaces with applications to PDEs

**Eteri Gordadze**[1], **Alexander Meskhi**[1,3] **and Maria Alessandra Ragusa**[2,4,*]

[1] Andrea Razmadze Mathematical Institute, Ivane Javakhishvili Tbilisi State University, Tbilisi 0193, Georgia
[2] Department of mathematics and computer science, University of Catania, Catania 6-95125, Italy
[3] Kutaisi International University, Kutaisi 4600, Georgia
[4] Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Vietnam

* **Correspondence:** Email: maragusa@dmi.unict.it.

**Abstract:** Rubio de Francia's extrapolation in generalized grand Morrey spaces is derived. This result is applied to the investigation of the regularity of solutions for the second order partial differential equations with discontinuous coefficients in the framework of generalized grand Morrey spaces under the Muckenhoupt condition on weights. Density properties for these spaces are also investigated.

**Keywords:** Grand Morrey spaces; weighted extrapolation; weighted estimates; regularity of partial differential equations

## 1. Introduction

Let $(X, \rho, \mu)$ be a quasi–metric measure space with a quasi–metric $\rho$ and a finite doubling measure $\mu$. We deal with Rubio de Francia's extrapolation in generalized weighted grand Morrey spaces $M_w^{p),q,\varphi(\cdot)}(X)$ defined on $(X, \rho, \mu)$, where $w$ is a weight function on $X$. $p$, $q$ and $\varphi(\cdot)$ are appropriate parameters of the space, and the "grandification" of the space is taken with respect to $p$.

Morrey spaces, introduced by Morrey in [1], describe the regularity of solutions of elliptic partial differential equations ($PDE$s) more precisely than Lebesgue spaces.

Let $w$ be a weight function on $X$, i.e., $w$ is a $\mu$- a.e. positive integrable function on $X$. Let $M_w^{p,q}(X)$ be the weighted Morrey space defined with respect to the norm [2]:

$$\|f\|_{M_w^{p,q}(X)} := \sup_B \frac{1}{(w(B))^{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L_w^p(B)} := \sup_B \frac{1}{(w(B))^{\frac{1}{p}-\frac{1}{q}}} \Big( \int_B |f(x)|^p w(x) d\mu(x) \Big)^{\frac{1}{p}},$$

where $1 < p \leq q$, and the supremum is taken over all balls $B$ in $X$. It is easy to notice that if $p = q$, then we have the weighted Lebesgue space denoted by $L^p_w(X)$. For definitions and essential properties of the classical Morrey spaces, we refer, e.g., to the recent monograph [3].

In 1992 Iwaniec and Sbordone [4] introduced new function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. That investigation was related to the integrability problem of the Jacobian on a bounded open set $\Omega$. More general spaces of $L^{p)}(\Omega)$, denoted by $L^{p),\theta}(\Omega)$, appeared first in the work by Greco et al. [5] in 1997 as the appropriate ambient spaces in which some nonlinear *PDE*s have to be considered.

Problems related to Harmonic Analysis in grand Lebesgue spaces and their associate spaces (called *small Lebesgue spaces*), were intensively studied during the last two decades along with various applications. The reader is referred, e.g., to the monographs [6] and references therein.

Denote by $\Phi_p$ the class of non-decreasing functions $\varphi(\cdot)$ on $(0, p-1)$ such that $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = 0$.

Let $w$ be a weight function on $X$, i.e., $w$ is a $\mu-$ a.e. positive integrable function on $X$. We consider the weighted grand Morrey space $M^{p),q,\varphi(\cdot)}_w(X)$ defined by the finite norm:

$$\|f\|_{M^{p),q,\varphi(\cdot)}_w(X)} := \sup_{0<\varepsilon<p-1} \sup_B \frac{\varphi(\varepsilon)}{(w(B))^{\frac{1}{p-\varepsilon}-\frac{1}{q}}}\|f\|_{L^{p-\varepsilon}_w(B)}$$

$$:= \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|f\|_{M^{p-\varepsilon,q}_w(X)},$$

where $1 < p \leq q$, and $\varphi(\cdot) \in \Phi_p$. If $\varphi(t) = t^\theta$, where $\theta > 0$, then we use the notation $M^{p),q,\theta}_w(X)$ for $M^{p),q,\varphi(\cdot)}_w(X)$.

One of our motivations to study the extrapolation problem in $M^{p),q,\varphi(\cdot)}_w(X)$ is related to the investigations carried out in [7] and [8], where the same problem was investigated in $M^{p,q}_w(\mathbb{R}^n)$ and $M^{p),q,\theta}_w(X)$, respectively. Komori and Shirai [2] obtained pioneering results regarding the one-weight problem for Harmonic Analysis operators in weighted classical Morrey spaces with Muckenhoupt $A_p$ weights defined on $\mathbb{R}^n$. Similar problems for sublinear operators involving, for example, maximal, fractional, Calderón-Zygmund integral operators in the spaces $M^{p,q}_w(\mathbb{R}^n)$ with $A_p$ weights were explored in [7, 9–14].

We emphasize that the one-weight estimates for sublinear operators including their commutators in grand Morrey spaces were investigated in [15] and [16]. Extrapolation results in weighted grand Lebesgue spaces were derived in [17].

Historically, unweighted grand Morrey spaces $L^{p),\lambda}(X)$ were introduced and studied in [18]. Later, these spaces were generalized in [19] by introducing grand grand Morrey spaces having the "grandification" not only for $p$, but also for $\lambda$.

## 2. Preliminaries

Let $(X, \rho, \mu)$ be a quasi-metric measure space (*QMMS*, briefly), where $X$ is an abstract set, $\rho$ is a quasi-metric on $X$, and $\mu$ is a measure defined on a $\sigma$- algebra of subsets of $X$. Quasi-metric $\rho$ on $X$ is a non-negative function on $X \times X$ satisfying the following conditions: (a) $\rho(x, y) = 0$ if and only if $x = y$; (b) $\rho(x, y) = \rho(y, x)$, $\forall x, y \in X$; (c) there exists a constant $\kappa \geq 1$ such that $\rho(x, y) \leq \kappa[\rho(x, z) + \rho(z, y)]$, $\forall x, y, z \in X$. Denote by $B(x, R)$ the ball with center $x$ and radius $R$, i.e., $B(x, R) := \{y \in X : \rho(x, y) \leq R\}$. We say that a measure $\mu$ satisfies the doubling condition if there exists a positive constant $C_{dc}$ such that for all $x \in X$ and $r > 0$, $\mu B(x, 2r) \leq C_{dc}\, \mu B(x, r)$. We will deal with a $QMMS$ with doubling measure. Such a $QMMS$ is called a space of homogeneous type ($SHT$, briefly).

There are many important examples of an $SHT$. We list some of them:

• Carleson (regular) curves on $\mathbb{C}$ with arc-length measure $d\nu$ and Euclidean distance on $\mathbb{C}$;

• nilpotent Lie groups with Haar measure and homogeneous norm (homogeneous groups);

• the triple $(\Omega, \rho, dx)$, where $\Omega$ is a domain in $\mathbb{R}^n$, $\rho$ is the Euclidean metric, and $dx$ is the Lebesgue measure induced to $\Omega$ satisfying the $\mathcal{A}$ condition [20], i.e., there exists a constant $C > 0$ such that for all $x \in \overline{\Omega}$ and $R \in (0, diam(\Omega))$,

$$\mu(\widetilde{B}(x, R)) \geq CR^n, \tag{2.1}$$

where

$$\widetilde{B}(x, R) := \Omega \cap B(x, R). \tag{2.2}$$

Other properties and examples of $SHT$s can be found, e.g., in [21, 22].

Let $1 < s < \infty$. We say that a weight $w$ belongs to the class $A_s(X)$ (Muckenhoupt class of weights) if

$$[w]_{A_s} := \sup_B \left( \mu(B)^{-1} \int_B w(x) d\mu(x) \right) \left( \mu(B)^{-1} \int_B w^{1-s'}(x) d\mu(x) \right)^{s-1} < \infty, \quad s' = \frac{s}{s-1},$$

where the least upper bound is taken over all balls $B \subset X$. In the literature, $[w]_{A_s}$ is called $A_s$ characteristic of the weight $w$.

Furthermore, a weight function $w$ is in the class $A_1(X)$ if $Mw(x) \leq Cw(x)$ a.e., where $Mw$ is the Hardy–Littlewood maximal function of $w$:

$$Mw(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B w(y) d\mu(y) \quad (B \text{ is a ball in } X).$$

In this case, it is assumed that $[w]_{A_1(X)}$ is determined as the essential supremum of $Mw/w$.

Furthermore, the following monotonicity property holds for Muckenhoupt classes:

$$A_r(X) \subset A_s(X), \quad 1 \leq r < s < \infty.$$

Let us recall that the class of weights $A_\infty(X)$ is defined as follows: $A_\infty(X) = \cup_{\ell \geq 1} A_\ell(X)$.

## 3. Density in $M_w^{p),q,\varphi(\cdot)}(X)$

Let $E$ be a Banach space and $F$ be its subset. Let us denote by $[F]_E$ the closure of $F$ in $E$. We are interested in density in $M_w^{p),q,\varphi(\cdot)}(X)$ spaces. In particular we have the following statement [6, 23].

**Proposition 3.1.** *Let $1 < p \leq q$ and let $\varphi(\cdot) \in \Phi_p$. Suppose that $w$ is a weight function on $X$. Then,*

$$\lim_{\varepsilon \to 0} \varphi(\varepsilon) \|f\|_{M_w^{p-\varepsilon,q}(X)} = 0 \tag{3.1}$$

*for $f \in \left[ M_w^{p,q}(X) \right]_{M_w^{p),q,\varphi(\cdot)}(X)}$.*

*Proof.* Let $f \in \left[M_w^{p,q}(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)}$ and $\varepsilon_0 > 0$. Then, there is a function $f_{n_0} \in M_w^{p,q}(X)$ such that $\|f - f_{n_0}\|_{M_w^{p),q,\varphi(\cdot)}(X)} < \varepsilon_0$.

Consequently, for such $f_{n_0}$ and $\varepsilon_0$, in view of the condition $\varphi(\cdot) \in \Phi_p$, we have that for sufficiently small $\varepsilon$, $\varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)} \leq \varepsilon_0$. Hence,

$$\begin{aligned} & \varphi(\varepsilon)\|f\|_{M_w^{p-\varepsilon,q}(X)} = \varphi(\varepsilon) \sup_B w(B)^{-\frac{1}{p-\varepsilon}+\frac{1}{q}} \|f\|_{L_w^{p-\varepsilon}(B)} \\ \leq\ & \varphi(\varepsilon)\|f - f_{n_0}\|_{M_w^{p-\varepsilon,q}(X)} + \varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)} \leq \|f - f_{n_0}\|_{M_w^{p),q,\varphi(\cdot)}(X)} + \varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)} \\ \leq\ & \varepsilon_0 + \varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)} \leq [1 + C_{\varphi,p}]\varepsilon_0 \end{aligned}$$

for sufficiently small $\varepsilon$, where the constant $C_{\varphi,p}$ depends only on $\varphi$ and $p$. Here, we used the embedding

$$M_w^{s,q}(X) \hookrightarrow M_w^{p,q}(X), \quad 1 \leq p \leq s \leq q,$$

which follows from the Hölder inequality and the definition of the weighted Morrey norm. □

**Proposition 3.2.** *Let* $1 < p \leq q$, $\varphi(\cdot) \in \Phi_p$. *Suppose that* $w$ *is a weight function on* $X$. *Then,*

$$\left[L^\infty(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)} = \left\{ f \in M_w^{p),q,\varphi(\cdot)}(X) : \lim_{N\to\infty} \left\|\chi_{\{|f|>N\}} f\right\|_{M_w^{p),q,\varphi(\cdot)}(X)} = 0 \right\}. \tag{3.2}$$

*Proof.* We use arguments from [24]. Initially observe that if $\lim_{N\to\infty} \left\|\chi_{\{|f|>N\}} f\right\|_{M_w^{p),q,\varphi(\cdot)}(X)} = 0$, then $f \in \left[L^\infty(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)}$ because $f = \chi_{\{|f|>N\}} f + \chi_{\{|f|\leq N\}} f$, where $\chi_{\{|f|\leq N\}} f \in L^\infty(X)$.

Let us now take $f \in \left[L^\infty(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)}$ and let $\varepsilon_0 > 0$. We choose $g \in L^\infty(X)$ such that $\|f - g\|_{M_w^{p),q,\varphi(\cdot)}(X)} < \varepsilon_0$. In view of the representation $\left|\chi_{\{|f|>N\}} f\right| \leq |f - g| + \left|\chi_{\left\{|f|>N\}\cap\{|g|\leq \frac{N}{2C_p}\right\}} g\right| + \left|\chi_{\left\{|g|>\frac{N}{2C_p}\right\}} g\right|$, where $N \in \mathbb{N}$ and $C_p = \varphi(p-1)$, we have

$$|g| \leq \frac{N}{2C_p} < \frac{|f|}{2C_p} \leq \frac{|f-g|}{2C_p} + \frac{|g|}{2C_p}, \text{ on the set } \{|f| > N\} \cap \{2C_p|g| \leq N\}.$$

Hence, $|g| \leq C|f - g|$, where the positive constant $C$ is independent of $f$ and $g$. Therefore, if $N > 2C_p\|g\|_{L^\infty(X)}$, we have $\left\|\chi_{\{|f|>N\}} f\right\|_{M_w^{p),q,\varphi(\cdot)}(X)} \leq c\,\|f - g\|_{M_w^{p),q,\varphi(\cdot)}(X)} < c\varepsilon_0$. Finally, we are done. □

## 4. Weighted extrapolation

The main result regarding the extrapolation reads as follows:

**Theorem 4.1.** *Assume that* $1 \leq p_0 < \infty$ *and that* $\mathcal{F}(X)$ *is a family of pairs of non-negative measurable functions defined on* $X$. *Let, for all* $(f, g) \in \mathcal{F}(X)$ *and* $w \in A_{p_0}(X)$, *the inequality*

$$\|g\|_{L_w^{p_0}(X)} \leq N(p_0, [w]_{A_{p_0}(X)})\|f\|_{L_w^{p_0}(X)} \tag{4.1}$$

*hold, where* $N(p_0, [w]_{A_{p_0}(X)})$ *is the positive constant depending only on* $p_0$ *and* $[w]_{A_{p_0}(X)}$ *such that the mapping* $\cdot \mapsto N(p_0, \cdot)$ *is a non-decreasing for a fixed* $p_0$. *Then, for every* $1 < p \leq q$, $\varphi(\cdot) \in \Phi_p$ *and* $w \in A_p(X)$, *the estimate*

$$\|g\|_{M_w^{p),q,\varphi(\cdot)}(X)} \leq C\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}, \quad (f, g) \in \mathcal{F}(X),$$

*holds, where the constant* $C$ *is independent of* $(f, g)$.

Extrapolation result for $A_\infty$ weights is given by the next statement:

**Theorem 4.2.** *Suppose that $\mathcal{F}(X)$ is a class of pairs of functions $(f, g)$, where $f$ and $g$ are $\mu$-measurable functions on $X$. Let $p_0 \in (0, \infty)$ and $l \geq 1$ be fixed parameters. Suppose that there is a function $N : (0, \infty) \times (0, \infty) \mapsto (0, \infty)$, which is non-decreasing with respect to the second variable, such that the inequality*

$$\|g\|_{L^{p_0}_w(X)} \leq N(p_0, [w]_{A_l(X)})\|f\|_{L^{p_0}_w(X)} \tag{4.2}$$

*holds for all $(f, g) \in \mathcal{F}(X)$ and $w \in A_l(X)$. Then, for every $1 < p \leq q$, $\varphi(\cdot) \in \Phi_p$ and all $w \in A_\infty(X)$ the estimate*

$$\|g\|_{M^{p),q,\varphi(\cdot)}_w(X)} \leq C\|f\|_{M^{p),q,\varphi(\cdot)}_w(X)}, \quad (f, g) \in \mathcal{F}(X), \tag{4.3}$$

*is valid, where the constant $C$ does not depend on $(f, g)$.*

These statements for $\varphi(t) = t^\theta$, $t > 0$ were proved in [8].

**Remark 4.1.** *According to Theorem 4.1 and the fact that the Muckenhoupt condition $w \in A_{p_0}(X)$ guarantees the boundedness of Harmonic Analysis operators such as Calderón–Zygmund singular integrals, commutators of singular integrals, fractional integrals and commutators of fractional integrals in $L^{p_0}_w(X)$ spaces [21, 25], we have appropriate one-weight norm estimates for those operators in grand Morrey spaces $M^{p),q,\varphi(\cdot)}_w(X)$ for $w \in A_p(X)$.*

To prove Theorem 4.2 we need some auxiliary statements from [7, 8]:

**Lemma 4.1.** *Let $0 < \gamma < 1$ and let $f$ be a $\mu$-locally integrable function on $X$. Then, $(Mf)^\gamma \in A_1(X)$. Moreover,*

$$\left[(Mf)^\gamma\right]_{A_1} \leq \frac{C_{\kappa,\mu}}{1-\gamma},$$

*where $C_{\kappa,\mu}$ is a structural constant.*

**Lemma 4.2.** *Let $1 \leq \gamma < p < \infty$ and let $w \in A_{p/\gamma}(X)$. Suppose that $p \leq q$. Then, there is $q_0 \in (\gamma, p)$ such that for all $r \in [\gamma, q_0]$, all $s \in (1, s_0(r, w))$, where $s_0(r, w)$ is the constant depending on $r$ and $w$, all balls $B$, sufficiently small numbers $\varepsilon$, and all $h \in L^{(p/r)'}_w(B)$ with $\|h\|_{L^{(p/r)'}_w(B)} = 1$, the inequality*

$$\|f\|_{L^r_{(HW)_{s,B}}(X)} \leq C(w(B))^{\frac{1}{p-\varepsilon}-\frac{1}{q}}\|f\|_{M^{p-\varepsilon,q}_w(X)} \tag{4.4}$$

*holds, where*

$$(HW)_{s,B} := M(h^s w^s \chi_B)^{\frac{1}{s}}, \tag{4.5}$$

*and the constant $C$ does not depend on $f$, $B$ and $\varepsilon$.*

*Proof of Theorem 4.1.* Following [7, 8], initially observe that in view of the Hölder inequality we have for $\sigma < \varepsilon < p - 1$,

$$\frac{1}{w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}}\Big(\int_B g^{p-\varepsilon} w d\mu\Big)^{1/(p-\varepsilon)} \leq \frac{1}{w(B)^{\frac{1}{p-\sigma}-\frac{1}{q}}}\Big(\int_B g^{p-\sigma} w d\mu\Big)^{1/(p-\sigma)}, \quad g \geq 0. \tag{4.6}$$

So, it is enough to show that there is a positive constant $C_{\mu,\sigma,w}$ depending only on $\mu$, $\sigma$, $w$ such that

$$\sup_{0<\varepsilon<\sigma} \frac{\varphi(\varepsilon)}{w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}}\Big(\int\limits_B g^{p-\varepsilon} w d\mu\Big)^{1/(p-\varepsilon)} \leq C_{\mu,\sigma,w}\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}$$

for some sufficiently small positive number $\sigma$.

Let $1 < p < \infty$. A classical extrapolation result [17,26] yields that

$$\|g\|_{L^p_w(X)} \leq C\psi([w]_{A_p(X)})\|f\|_{L^p_w(X)}, \quad w \in A_p(X), \tag{4.7}$$

for all $(f, g) \in \mathcal{F}(X)$, where $C$ is the constant independent of $(f, g)$ and $w$, and the mapping $\cdot \mapsto \psi(\cdot)$ is non-decreasing. Furthermore, take $w \in A_p(X)$ and choose $s > 1$ and $r \in (1, p)$ so that inequality (4.4) holds. Introducing the notation $p_\varepsilon := \frac{p-\varepsilon}{r}$, for a ball $B \subset X$, we find that

$$\Big(\int\limits_B g^{p-\varepsilon} w d\mu\Big)^{\frac{1}{p-\varepsilon}} = \Big(\int\limits_B g^{p_\varepsilon r} w d\mu\Big)^{\frac{1}{p_\varepsilon r}} = \sup_{\|h\|_{L_w^{p'_\varepsilon}(X)}=1} \Big(\int\limits_B g^r h w d\mu\Big)^{\frac{1}{r}}.$$

For such an $h$, in view of Lemma 4.1 we see that $[(HW)_{s,B}]_{A_q} \leq [(HW)_{s,B}]_{A_1} \leq \frac{C_\mu}{1-s^{-1}}$. Furthermore, observe that (4.7) implies that

$$\Big(\int_X g^r w d\mu\Big)^{\frac{1}{r}} \leq C_\mu \psi([w]_{A_r(X)})\Big(\int_X f^r w d\mu\Big)^{\frac{1}{r}}$$

for all $w \in A_r(X)$ and all $(f, g) \in \mathcal{F}(X)$, where the mapping $\cdot \mapsto \varphi(\cdot)$ is non-decreasing. Therefore, in view of Lemmas 4.2 and 4.1, we get

$$\begin{aligned}
&\Big(\int_X g^r h w \chi_B d\mu\Big)^{\frac{1}{r}} \leq \Big(\int_X g^r (HW)_{s,B} d\mu\Big)^{\frac{1}{r}} \\
\leq\ & C\psi\Big([(HW)_{s,B}]_{A_r(X)}\Big)\Big(\int_X f^r (HW)_{s,B} d\mu\Big)^{\frac{1}{r}} \\
\leq\ & C\widetilde{C}\varphi\Big(\Big[(HW)_{s,B}\Big]_{A_r(X)}\Big) w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}\|f\|_{M_w^{p-\varepsilon,q}(X)} \\
\leq\ & C\widetilde{C}\psi\Big(\Big[(HW)_{s,B}\Big]_{A_1(X)}\Big) w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}\varphi(\varepsilon)^{-1}\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)} \\
\leq\ & C\widetilde{C}\psi\Big(\frac{C_\mu}{1-s^{-1}}\Big) w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}\varphi(\varepsilon)^{-1}\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)},
\end{aligned}$$

where $\widetilde{C}$ is the constant depending only on $p$, $\sigma$, $w$.

Finally we deduce

$$\frac{\varphi(\varepsilon)}{w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}}\Big(\int_B g^{p-\varepsilon} w d\mu\Big)^{\frac{1}{p-\varepsilon}} \leq C\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}$$

for sufficiently small $\varepsilon$. Since (see (4.6))

$$\|g\|_{M_w^{p,\varphi(\cdot)}(X)} \sim \sup_{0<\varepsilon<\sigma} \frac{\varphi(\varepsilon)}{w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}}\Big(\int_B g^{p-\varepsilon} w d\mu\Big)^{\frac{1}{p-\varepsilon}},$$

where $\sigma \in (0, p-1)$, we are done.

□

*Proof of Theorem* 4.2. Let (4.2) hold for some $p_0 > 0$. Then, the classical $A_\infty$ extrapolation [27,28] gives

$$\|g\|_{L^p_w(X)} \le C_p \psi\Big([w]_{A_p}\Big)\|f\|_{L^p_w(X)} \tag{4.8}$$

for all $1 < p < \infty$ and $w \in A_p$, where $C_p$ is the positive constant depending on $p$, and $\psi(\cdot)$ is a non-decreasing mapping.

Let $1 < p < \infty$ and let $w \in A_\infty$. Now, we will show that (4.3) holds for such a weight $w$ and all $(f, g) \in \mathcal{F}(X)$. If $p \ge r$, then $A_r \subset A_p$, and by (4.8) and Theorem 4.1, we get that (4.3) holds for that $w$ and all $(f, g) \in \mathcal{F}(X)$.

Suppose now that $p < r$. Since $w \in A_r$, by the openness property of Muckenhoupt classes [21] we have that $w \in A_{r-\sigma}$ for some small positive $\sigma$ (the exact value of $\sigma$ can be found in [29]). Consequently, by the monotonicity property of Muckenhoupt classes, $w \in A_{r-\eta}$ for all $\eta$ satisfying $0 < \eta < \sigma$. Hence, in view of (4.8), we find that

$$\left\| |g|^{\frac{p-\varepsilon}{r-\eta}} \right\|_{M^{r-\eta, \frac{r(p-\varepsilon)}{r-\eta}}(X)} \le C_{p,r,\varepsilon,\eta} \psi\Big(\big[w\big]_{A_{r-\eta}}\Big) \left\| |f|^{\frac{p-\varepsilon}{r-\eta}} \right\|_{M^{r-\eta, \frac{r(p-\varepsilon)}{r-\eta}}(X)}, \tag{4.9}$$

were $C_{p,r,\varepsilon,\eta}$ is the positive constant depending only on $p, r, \varepsilon, \eta$, and $\psi$ is a non-decreasing function. Since $\big[w\big]_{A_{r-\eta}} \le \big[w\big]_{A_{r-\sigma}}$ and $\sup_{\varepsilon,\eta} C_{p,r,\varepsilon,\eta} < \infty$ (see also the proof of Theorem 4.1 for this fact), we have that

$$\sup_{\varepsilon,\eta} C_{p,r,\varepsilon,\eta} \psi\Big(\big[w\big]_{A_{r-\eta}}\Big) < \infty,$$

where the least upper bound is taken over all sufficiently small $\eta$ and $\varepsilon$. Due to (4.9) we see that

$$\left\| g \right\|^{\frac{p-\varepsilon}{r-\eta}}_{M^{p-\varepsilon,q}(X)} \le C_{p,r,\varepsilon,\eta} \psi\Big(\big[w\big]_{A_{r-\eta}}\Big) \left\| f \right\|^{\frac{p-\varepsilon}{r-\eta}}_{M^{p-\varepsilon,q}(X)}. \tag{4.10}$$

Raising both sides of (4.10) to the power $\frac{r-\eta}{p-\varepsilon}$, multiplying them by $\varphi(\varepsilon)$ and taking the supremum with respect to $\varepsilon$, we are done.

□

## 5. Applications to *PDE*s

During the last three decades a quite large number of papers explored local and global regularity problems for strong solutions to elliptic *PDE*s with discontinuous coefficients. To be evident, we take the second order *PDE*

$$\mathcal{L}u(x) \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{x_i x_j} u(x) = f(x) \quad \text{for almost all } x \in \Omega, \tag{5.1}$$

where $\mathcal{L}$ denotes a uniformly elliptic operator on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$.

Suppose that $\Omega$ is a domain in $\mathbb{R}^n$. As we know, the triple $(\Omega, \rho, dx)$ satisfying the condition $\mathcal{A}$ (see (2.1) for this condition), where $\rho$ is the Euclidean metric, and $dx$ is the Lebesgue measure induced to $\Omega$, is an example of an $SHT$. Hence, the previous statements are true for such domains.

The regularizing property of $\mathcal{L}$ in Hölder spaces (i.e., $\mathcal{L}u \in C^{\alpha}(\bar{\Omega})$ implies $u \in C^{2+\alpha}(\bar{\Omega})$) has been intensively investigated for the case of Hölder continuous coefficients $a_{ij}$. Also, we emphasize that unique classical solvability of the Dirichlet problem for (5.1) has been obtained in this case (we refer to [30] and references therein). For uniformly continuous coefficients $a_{ij}$, an $L^p$-Schauder theory has been elaborated for the operator $\mathcal{L}$ [30, 31]. In particular, $\mathcal{L}u \in L^p(\Omega)$ implies that the strong solution to (5.1) belongs to the Sobolev space $W^{2,p}(\Omega)$ for each $p \in (1, \infty)$. However, the situation becomes more complicated if we try to allow discontinuity at the principal coefficients of $\mathcal{L}$. In general, it is known (cf. [32]) that discontinuity of the coefficients $a_{ij}$ implies that the $L^p$-theory of $\mathcal{L}$ and the strong solvability of the Dirichlet problem for (5.1) fail. A considerable exception of that rule is the two-dimensional case ($\Omega \subset \mathbb{R}^2$). Talenti [33] proved that the solely condition on measurability and boundedness of the $a_{ij}$'s guarantees isomorphic properties for $\mathcal{L}$ as a function from $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ to $L^2(\Omega)$. For the multidimensional case, i.e., when $n \geq 3$, except the uniform ellipticity, some additional properties on the coefficients $a_{ij}$ are assumed in order to ensure that $\mathcal{L}$ possesses the regularizing property in Sobolev functional scales. In particular, if $a_{ij}$ belong to $W^{1,n}(\Omega)$ (cf. [34]), or if the difference between the largest and the smallest eigenvalues of $\{a_{ij}\}$ is sufficiently small (the Cordes condition), then $\mathcal{L}u \in L^2(\Omega)$ yields $u \in W^{2,2}(\Omega)$, and these results can be extended to $W^{2,p}(\Omega)$ for $p \in (2 - \varepsilon, 2 + \varepsilon)$ with sufficiently small $\varepsilon$.

Later, the Sarason class of functions with vanishing mean oscillation (denoted by $VMO$) was applied in the investigation of local and global Sobolev regularity of the strong solutions for (5.1).

Furthermore, let us define the space $BMO$ of functions of bounded mean oscillation, and the smaller class of functions of vanishing mean oscillation denoted by $VMO$, where we consider coefficients $a_{ij}$ and later that one where we consider the known term $f$.

In the sequel, we will assume that $\Omega$ is an open bounded set in $\mathbb{R}^n$.

**Definition 5.1.** *For $f \in L^1_{\text{loc}}(\Omega)$, define the integral mean $f_{x,R}$ by the formula*

$$f_{x,R} := |B(x,R)|^{-1} \int_{\widetilde{B}(x,R)} f(y)dy,$$

*where $\widetilde{B}(x,R)$ is defined by* (2.2).

If there is no need to specify the center, we just use the symbol $B_R$ for $B(x,R)$.

We now recall the definition of the class of functions with bounded mean oscillation functions (denoted by $BMO$) that appeared for the first time in the publication by John and Nirenberg [35].

**Definition 5.2.** *For $f \in L^1_{\text{loc}}(\Omega)$, we say that $f$ belongs to $BMO(\Omega)$ if $\|f\|_* < \infty$, where*

$$\|f\|_* := \sup_{B(x,R)} |B(x,R)|^{-1} \int_{\widetilde{B}(x,R)} |f(y) - f_{x,R}| dy.$$

Next, we consider the class of functions with Vanishing Mean Oscillation ($VMO$), introduced by Sarason [36].

**Definition 5.3.** *Let $f \in BMO(\Omega)$ and define*

$$\eta(f,R) := \sup_{\rho \leq R} |B_\rho|^{-1} \int_{\widetilde{B}_\rho} |f(y) - f_\rho| dy.$$

*Furthermore, a function $f$ belongs to the class $VMO(\Omega)$ if* $\lim_{R\to 0}\eta(f,R) = 0$.

In fact, the class $VMO$ is the subspace of $BMO$ whose $BMO$ norm over a ball vanishes when the radius of balls goes to zero. From this property it follows that a number of good features of functions from $VMO$ are not shared by $BMO$ functions; for example, functions from this class can be approximated by smooth functions. The $VMO$ class was studied by various authors from different viewpoints. It is worth mentioning the work by Chiarenza et al. [37], in which the authors answer a question that arose thirty years before Miranda [34]. In the latter work the author considered linear elliptic $PDE$s, in which the coefficients $a_{ij}$ with the higher order derivatives belong to the class $W^{1,n}(\Omega)$, and, moreover, he asked whether the gradient of the solution is bounded, if $p > n$. In the work [37] the authors supposed that $a_{ij} \in VMO$ and proved that $Du$ is Hölder continuous.

Furthermore, it is possible to see that bounded uniformly functions belong to the class $VMO$ as well as functions belonging to fractional Sobolev spaces $W^{\theta,\frac{n}{\theta}}$, $\theta \in (0, 1)$.

The investigation of Sobolev regularity of strong solutions of (5.1) was initiated in 1991 by the pioneering work of Chiarenza et al. [38]. In that work it was proved that, if $a_{ij} \in VMO \cap L^\infty(\Omega)$ and $\mathcal{L}u \in L^p(\Omega)$, then $u \in W^{2,p}(\Omega)$ for *each* value of $p \in (1,\infty)$. Moreover, well-posedness of the Dirichlet problem for (5.1) in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ was obtained. As a consequence, if the exponent $p$ is sufficiently large, then it follows Hölder continuity for the strong solution or for its gradient.

By virtue of the fundamental accessibility of the works [37,39], many other authors have used $VMO$ class to obtain regularity results for $PDE$s and systems with discontinuous coefficients.

It can be checked that Hölder continuity can be inferred for small $p$ if one has more information on $\mathcal{L}u$, such as, for example, its belonging to suitable Morrey class $L^{p,\lambda}(\Omega)$.

We denote by $L^{p,\lambda}(\Omega)$ the Morrey space defined on a domain $\Omega \subset \mathbb{R}^n$ which is determined by the following norm:

$$\|f\|_{p,\lambda} := \sup_{\substack{x\in\Omega \\ 0<R<diam\,(\Omega)}} \left(\frac{1}{R^\lambda}\int_{\widetilde{B}(x,R)} |f(y)|^p\, dy\right)^{1/p},$$

where $\widetilde{B}(x,R)$ is defined by (2.2).

The exponent $\lambda$ can take values outside $(0,n)$ but, as usual, the unique case of real interest is that one for which $\lambda \in (0,n)$. Indeed, from the definition we easily see that $L^{p,\lambda}(\Omega) = L^p(\Omega)$, if $\lambda \le 0$. It is also clear that $L^{p,0}(\Omega) = L^p(\Omega)$.
Moreover, if $\lambda = n$, by using the Lebesgue differentiation theorem, we find that

$$\lim_{\rho\to 0^+}\rho^{-n}\int_{\widetilde{B}(x,\rho)} |f(y)|^p\, dy = \lim_{\rho\to 0^+}\rho^{-n}\int_{B(x,\rho)} |f(y)|^p dy = C|f(x)|^p$$

for every Lebesgue point $x \in \Omega$. Then, $f(x) \in L^{p,n}(\Omega)$ if and only if $f$ is bounded. This means that $L^{p,n}(\Omega) = L^\infty(\Omega)$. Furthermore, if $\lambda > n$, then $L^{p,\lambda}(\Omega) = \{0\}$.

In view of the spaces defined above, a natural problem arises when one studies the regularizing properties of the operator $\mathcal{L}$ in Morrey spaces for the case of $VMO$ principal coefficients. In [40] it was proved that each $W^{2,p}$-viscosity solution to (5.1) lies in $C^{1+\alpha}(\Omega)$ if $f$ belongs to $L^{n,n\alpha}(\Omega)$ with $\alpha \in (0, 1)$.

One of the main results of this note is to obtain local regularity in generalized grand Morrey spaces $M^{p),q,\varphi(\cdot)}_w(X)$, for highest order derivatives of solutions of elliptic $PDE$s in non-divergence form with coefficients, which might be discontinuous.

We recall the work by Agmon et al. [31] in which the appropriate results were obtained for the case of continuous coefficients of the above kind of equation. Later, discontinuous coefficients were considered also by Campanato [41].

In this paper we continue the study of the $L^p$ regularity of solutions of second order elliptic *PDE*s to the maximum order derivatives of the solutions to a certain class of linear elliptic *PDE*s in nondivergence form having discontinuous coefficients [8].

We consider the second order differential operator

$$\mathcal{L} \equiv \sum_{i,j=1}^{n} a_{ij} D_{ij}, \qquad D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}.$$

Here, we have adopted the usual summation convention on repeated indices.

We will also need the following regularity and ellipticity assumptions for the coefficients of $\mathcal{L}$, $\forall i, j = 1 \ldots n$ :

$$\begin{cases} a_{ij} \in L^\infty(\Omega) \cap VMO, \\ a_{ij}(x) = a_{ji}(x), \quad \text{for a.e. } x \in \Omega, \\ \exists\, \kappa > 0 : \ \frac{1}{\kappa}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \ \text{for a.e. } x \in \Omega. \end{cases} \tag{5.2}$$

Set $\eta_{ij}$ for the $VMO$-modulus of the function $a_{ij}$ and suppose that $\eta = \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{1/2}$. In this case the normalized fundamental solution is given by the formula

$$\Gamma(x,\xi) = \frac{1}{n(2-n)\omega_n \sqrt{\det\{a_{ij}(x)\}}} \left( \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \right)^{(2-n)/2}, \ \forall \xi \in \mathbb{R}^n \setminus \{0\} \text{and a.e. } x,$$

where $A_{ij}(x)$ are the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\ldots,n}$, and $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. We set

$$\Gamma_i(x,\xi) = \frac{\partial}{\partial \xi_i}\Gamma(x,\xi), \quad \Gamma_{ij}(x,\xi) = \frac{\partial}{\partial \xi_i \partial \xi_j}\Gamma(x,\xi),$$

$$M = \max_{i,j=1,\ldots,n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(\cdot,\xi)}{\partial \xi^\alpha} \right\|_{L^\infty(\Omega\times\Sigma)}.$$

It is well known that $\Gamma_{ij}(x,\xi)$ are Calderón–Zygmund kernels with respect to the variable $\xi$.

Recall that since the condition $\mathcal{A}$ for $\Omega$ is satisfied, $\Omega$ with the Euclidean distance and the Lebesgue measure induced on $\Omega$ is a special case of $SHT$.

**Theorem 5.1.** *Suppose that* (5.2) *holds,* $1 < p \le q < \infty$, $\varphi(\cdot) \in \Phi_p$. *Let* $\Omega$ *be a domain satisfying* $\mathcal{A}$ *condition (see* (2.1)*) and let* $w$ *be a weight on* $\Omega$ *such that* $w \in A_p(\Omega)$. *Then, for every ball* $B_\rho \subset\subset \Omega$, *and every* $u \in W_0^{2,p}(B_\rho)$ *with* $\mathcal{L}u \in M_w^{p),q,\varphi(\cdot)}(B_\rho)$, *we have* $D_{ij}u \in M_w^{p),q,\varphi(\cdot)}(B_\rho)$, *and moreover, there exist positive constants* $c = c(n,\kappa,p,q,\varphi(\cdot),M,w)$ *such that the estimate*

$$\|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \le c\|\mathcal{L}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)}, \quad \forall i, j = 1, \ldots, n \tag{5.3}$$

*holds.*

*Proof.* Initially observe that the representation for the second order derivatives of functions in $W_0^{2,p}(B)$, where $B$ is an open ball in $\mathbb{R}^n$, is given by the formula: [38]:

$$D_{ij}u(x) = \text{P.V.}\int_B \Gamma_{ij}(x, x-y)\sum_{h,k=1}^{n}(a_{hk}(x) - a_{hk}(y))D_{hk}u(y)dy \tag{5.4}$$
$$+ \text{P.V.}\int_B \Gamma_{ij}(x, x-y)\mathcal{L}u(y)dy + \mathcal{L}u(x)\int_{|\xi|=1}\Gamma_i(x,\xi)\xi_j d\sigma_\xi.$$

Let us remark that

i) The first and the second integrals appearing in (5.4) are Principal Value ones (In fact, they are commutators of the Calderón–Zygmund singular integrals. The reader is referred, e.g., to [42], Ch. 7, [25], and references therein for appropriate weighted inequalities), and we can use Theorem 4.1 together with Remark 4.1 and Condition (2.1) to obtain the appropriate weighted inequality in $M_w^{p),q,\varphi(\cdot)}(\Omega)$, where $w$ is the Muckenhoupt weight.

ii) $\int_{|\xi|=1}\Gamma_i(\cdot,\xi)\xi_j\, d\sigma_\xi \in L^\infty(B_\rho)$ with a bound independent of $\rho$.

Now, taking the $M_w^{p),q,\varphi(\cdot)}(B_\rho)$ norms of both sides in (5.4), applying Theorem 4.1 and taking into account Remark 4.1 and Condition (2.1), we get

$$\|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \le c\left(\eta(\rho)\|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} + \|\mathcal{L}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)}\right).$$

This way, in view of the $VMO$ assumption on the coefficients $a_{ij}(x)$, it is possible to choose $\rho_0$ so small that $c\eta(\rho_0) \le 1/2$ and then

$$\|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \le c\|\mathcal{L}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \qquad \text{for each } \rho < \rho_0.$$

□

## 6. Conclusions

The authors obtained regularity results for solutions of second order $PDE$s having discontinuous coefficients in the framework of generalized grand Morrey spaces under the Muckenhoupt condition on weights. In the future it will be possible to extend the obtained properties to other kinds of equations, making use of density properties and extrapolation in generalized weighted grand Morrey spaces, that are proved in the present paper.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work has been supported by Piano della Ricerca di Ateneo 2020-2022-PIACERI: Project MO.S.A.I.C. "Monitoraggio satellitare, modellazioni matematiche e soluzioni architettoniche e urbane per lo studio, la previsione e la mitigazione delle isole di calore urbano," Project EEEP&DLaD. The third author likes to thank Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Vietnam, for the opportunity to work in it. The second and third authors express their gratitude to the organizers of the International Conference on Function Spaces and Applications Apolda/Thür. (Germany) October 1–7, 2022, and in particular, to Prof. Dorothee Haroske, for their hospitality, invitation and giving an excellent opportunity to discuss the problems studied in this paper.

## Conflict of interest

The authors declare there is no conflicts of interest.

## References

1. C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Am. Math. Soc.*, **43** (1938), 126–166.
2. Y. Komori, S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.,* **282** (2009), 219–231. https://doi.org/10.1002/mana.200610733
3. Y. Sawano, G. Di Fazio, D. I. Hakim, *Morrey Spaces*, CRC Press, New York, 2020. https://doi.org/10.1201/9781003042341
4. T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, *Arch. Rational Mech. Anal.,* **119** (1992), 129–143. https://doi.org/10.1007/BF00375119
5. L. Greco, T. Iwaniec, C. Sbordone, Inverting the $p$-harmonic operator, *Manuscripta Math.,* **92** (1997), 249–258. https://doi.org/10.1007/BF02678192
6. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-Standard Function Spaces,* Springer International Publishing, Switzerland, 2016. https://doi.org/10.1007/978-3-319-21018-6
7. J. Duoandikietxea, M. Rosental, Extension and boundedness of operators on Morrey spaces from extrapolation techniques and embeddings, *J. Geom. Anal.,* **28** (2018), 3081–3108. https://doi.org/10.1007/s12220-017-9946-5
8. V. Kokilashvili, A. Meskhi, M. A .Ragusa, Weighted extrapolation in Grand Morrey spaces and applications to partial differential equations, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.*, **30** (2019), 67–92. https://doi.org/10.4171/rlm/836
9. V. Kokilashvili, A. Meskhi, The boundedness of sublinear operators in weighted morrey spaces defined on spaces of homogeneous type, in *Function Spaces and Inequalities,* (2017), 193–211. https://doi.org/10.1007/978-981-10-6119-6_9
10. R. Mustafayev, On boundedness of sublinear operators in weighted Morrey spaces, *Azerb. J. Math.* **2** (2012), 63–75.
11. M. Rosental, H. Schmeisser, The boundedness of operators in Muckenhoupt weighted Morrey spaces via extrapolation tecjniques and duality, *Rev. Mat. Complut.,* **29** (2016), 623–657. https://doi.org/10.1007/s13163-016-0208-z

12. N. Samko, On a Muckenhoupt-type condition for Morrey spaces, *Mediterr. J. Math.,* **10** (2013), 941–951. https://doi.org/10.1007/s00009-012-0208-2
13. S. Shi, Z. Fu, F. Zhao, Estimates for operators on weighted Morrey spaces and their applications to nondivergence elliptic equations, *J. Inequal. Appl.,* **390** (2013), https://doi.org/10.1186/1029-242X-2013-390
14. S. Nakamura, Y. Sawano, The singular integral operator and its commutator on weighted Morrey spaces, *Collect. Math.,* **68** (2017), 145–174. https://doi.org/10.1007/s13348-017-0193-7
15. V. Kokilashvili, A. Meskhi, H. Rafeiro, Commutators of sublinear operators in grand Morrey spaces, *Stud. Sci. Math. Hung.,* **56** (2019), 211–232 https://doi.org/10.1556/012.2019.56.2.1425
16. V. Kokilashvili, A. Meskhi, H. Rafeiro, Boundedness of sublinear operators in weighted grand Morrey spaces, *Math. Notes*, **102** (2017), 664–676. https://doi.org/10.1134/S0001434617110062
17. V. Kokilashvili, A. Meskhi, Weighted extrapolation in Iwaniec-Sbordone spaces. Applications to integral operators and theory of approximation, in *Proceedings of the Steklov Institute of Mathematics,* **293** (2016), 161–185. https://doi.org/10.1134/S008154381604012X
18. A. Meskhi, Maximal functions, potentials and singular integrals in grand Morrey spaces, *Complex Var. Elliptic Equations,* **56** (2011), 1003–1019. https://doi.org/10.1080/17476933.2010.534793
19. H. Rafeiro, A note on boundedness of operators in Grand Grand Morrey spaces, in *Advances in Harmonic Analysis and Operator Theory*, (2013), 349–356. https://doi.org/10.1007/978-3-0348-0516-2_19
20. L. Pick, A. Kufner, O. John, S. Fucík, *Function Spaces,* De Gruyter academic publishing, Berlin, 2013. https://doi.org/10.1515/9783110250428
21. J. Strömberg, A. Torchinsky, *Weighted Hardy Spaces,* Springer, Berlin, 1989. https://doi.org/10.1007/BFb0091154
22. R. R. Coifman, G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Springer-Verlag, Berlin, 1971. https://doi.org/10.1007/BFb0058946
23. A. Meskhi, Y. Sawano, Density, duality and preduality in grand variable exponent lebesgue and morrey spaces, *Mediterr. J. Math.,* **15** (2018). https://doi.org/10.1007/s00009-018-1145-5
24. D. I. Hakim, M. Izuki, Y. Sawano, Complex interpolation of grand Lebesgue spaces, *Monatsh. Math.,* **184** (2017), 245–272. https://doi.org/10.1007/s00605-017-1022-5
25. G. Pradolini, O. Salinas, Commutators of singular integrals on spaces of homogeneous type, *Czech. Math. J.,* **57** (2007), 75–93. https://doi.org/10.1007/s10587-007-0045-9
26. J. Duoandikoetxea, Extrapolation of weights revisited: New proofs and sharp bounds, *J. Funct. Anal.,* **260** (2011), 1886–1901. https://doi.org/10.1016/j.jfa.2010.12.015
27. D. Cruz-Uribe, J. M. Martell, C. Perez, Extrapolation from $A_\infty$ weights and applications, *J. Funct. Anal.*, **213** (2004), 412–439. https://doi.org/10.1016/j.jfa.2003.09.002
28. V. Kokilashvili A. Meskhi, Extrapolation in grand Lebesgue spaces with $A_\infty$ weights, *Math. Notes,* **104** (2018), 518-–529. https://doi.org/10.1134/S0001434618090195
29. T. P. Hytönen, C. Pérez, E. Rela, Sharp reverse Hölder property for $A_\infty$ weights on spaces of homogeneous type, *J. Funct. Anal.,* **263** (2012), 3883–3899. https://doi.org/10.1016/j.jfa.2012.09.013

30. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* $2^{nd}$ edition, Springer–Verlag, Berlin, 1983.

31. S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, *Commun. Pure Appl. Math.,* **12** (1959), 623–727. https://doi.org/10.1002/cpa.3160120405

32. N. Meyers, An $L^p$ estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Sc. Norm. Super. Pisa-classe Sci.,* **17** (1963), 189–206.

33. G. Talenti, Equazioni lineari ellittiche in due variabili, *Matematiche*, **21** (1966), 339–376.

34. C. Miranda, Sulle equazioni ellittiche del secondo ordine di tipo non variazionale a coefficienti discontinui, *Ann. Mat. Pura Appl.,* **63** (1963), 353–386. https://doi.org/10.1007/BF02412185

35. F. John, L. Nirenberg, On functions of bounded mean oscillation, *Commun. Pure Appl. Math.,* **14** (1961), 415–426. https://doi.org/10.1002/cpa.3160140317

36. D. Sarason, Functions of vanishing mean oscillation, *Trans. Am. Math. Soc.*, **207** (1975), 391–405.

37. F. Chiarenza, M. Frasca, P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with $VMO$ coefficients, *Trans. Am. Math. Soc.,* **336** (1993), 841–853.

38. F. Chiarenza, M. Frasca, P. Longo, Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients, *Ricerche Mat.*, **40** (1991), 149–168.

39. F. Chiarenza, M. Franciosi, M. Frasca, $L^p$-estimates for linear elliptic systems with discontinuous coefficients, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl.*, **5** (1994), 27–32.

40. L. Caffarelli, Elliptic second order equations, *Seminario Mat. e. Fis. di Milano*, **58** (1988), 253–284. https://doi.org/10.1007/BF02925245

41. S. Campanato, Sistemi parabolici del secondo ordine, non variazionali a coefficienti discontinui, *Ann. Univ. Ferrara*, **23** (1977), 169–187. https://doi.org/10.1007/BF02825996

42. L. Grafakos, *Classical Fourier Analysis,* Springer, New York, 2014. https://doi.org/10.1007/978-1-4939-1194-3

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)