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Research article

Local Hölder continuity of nonnegative weak solutions of inverse variation-inequality problems of non-divergence type

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Abstract: Compared to the standard variational inequalities, inverse variational inequalities are more suitable for pricing American options with indefinite payoff. This paper investigated the initial-boundary value problem of inverse variational inequalities constituted by a class of non-divergence type parabolic operators. We established the existence and Hölder continuity of weak solutions. Since the comparison principle in the case of standard variational inequalities is no longer applicable, we constructed an integral inequality using differential inequalities to determine the global upper bound of the solution. By combining it with the continuous method, we obtained the existence of weak solutions. Additionally, by employing truncation factors, we obtained the lower bound of weak solutions in the cylindrical subdomain, thereby obtaining the Hölder continuity.

Keywords: variation-inequality problems; non-divergence parabolic operator; existence; Hölder continuity

1. Introduction and financial background

The variational inequality in the following form has received extensive research attention in recent years:

$$\begin{cases} \min\{Lu, u-u_0\} = 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{1}$$

Here, the operator 'min' is used to control the inequality condition, where Lu represents a linear parabolic operator or a degenerate parabolic operator. The existence of solutions to parabolic variational inequalities has been analyzed using the finite element method in the literature [1–3]. These studies not only provide the discrete schemes but also analyze the convergence between the discrete schemes and the parabolic variational inequalities, thus establishing the existence of solutions to the variational inequality problem. The existence of solutions to parabolic variational inequalities has also been investigated in [4,5] using the Banach fixed-point theorem and the surjectivity theorem, respectively. The uniqueness of solutions to parabolic variational inequalities has been analyzed in [6,7] through energy estimates of weak solution interpolations in Sobolev spaces. Literature [8] estimates the energy of the weak solution's second-order gradient in Sobolev spaces, establishing the higher integrability and regularity of the weak solution. Currently, research on higher integrability and regularity mainly focuses on parabolic initial-boundary value problems [9,10], while it is relatively scarce in the field of variational inequality studies. Literature [11] investigates the Hölder continuity property of variational inequalities. By utilizing the Poincaré inequality and the Gagliardo-Nirenberg inequality, the Caccioppoli inequality is derived, which is then used to establish the Hölder continuity. The structure of this parabolic variational inequality is relatively simple, consisting of a first-order quasi-linear parabolic operator.

In recent years, research in financial theory has found that the inverse variational inequalities,

$$\begin{cases} \min\{-Lu, u-u_0\}=0, & (x,t)\in\Omega_T, \\ u(0,x)=u_0(x), & x\in\Omega, \\ u(t,x)=\frac{\partial u}{\partial \nu}=0, & (x,t)\in\partial\Omega\times(0,T), \end{cases} \tag{2}$$

such as (2) in the Black-Scholes framework, are more suitable for studying the pricing of American options which enables investors to buy or sell the underlying stock at a predetermined price K at any point within the option's time period $[0,T]$. In the Black-Scholes model, the price of American options also satisfies the variational inequality (2) and the parabolic operator (denoted by L_{BS}) satisfies [12,13]

$$L_{BS}u=\partial_t u-\frac{1}{2}\sigma^2\partial_{xx}u-r\partial_x u+ru.$$

Here, S represents the underlying stock of the American option, σ represents the volatility of the stock and r represents the risk-free interest rate in the stock market. For American options, the initial conditions satisfy the following:

$$\text{American call options}: u_0(x)=\max\{e^x-K,0\},$$

$$\text{American put options}: u_0(x)=\max\{K-e^x,0\}.$$

This paper investigates the existence and local Hölder continuity of weak solutions to the variational inequality (2) under the non-divergence degenerate parabolic operator

$$Lu=\partial_t u-u\mathrm{div}(|\nabla u|^{p-2}\nabla u)-\gamma|\nabla u|^p,\ \ p>2,\gamma\in(0,1). \tag{3}$$

Additionally, we assume that u_0 is nonzero in the interior of Ω, otherwise $u_0\equiv 0$ in Ω, which easily leads to $u\equiv 0$ in Ω_T being a solution to (1), rendering the study meaningless.

The motivation behind this research is the lack of documentation on the existence of weak solutions to variational inequality problems in the literature. Specifically, the authors focus on the inverse variational inequality model, where determining the upper bound of weak solutions using the traditional comparison principle is challenging. This motivates the need for new approaches to overcome this difficulty.

This paper introduces two key innovations to address the challenges mentioned above. First, the authors introduce a nonnegative constant M_0 and construct an integral inequality for $(u-M_0)_+$ based on $Lu \leq 0$. This novel approach allows for the determination of an upper bound for weak solutions, which was previously difficult to achieve using the traditional comparison principle. This innovation provides a new perspective on defining weak solutions for variational inequalities. Second, the authors construct an integral inequality using $(u-k)_\pm$ and a nonnegative function ϕ on $W^{1,p}(\Omega)$. By choosing ϕ as a cut-off function, they are able to establish a lower bound for u in a cylindrical subdomain. This lower bound enables the establishment of the Hölder continuity. This contribution is significant as it provides a new method for establishing continuity in variational inequality problems. Overall, this paper makes important contributions to the field by introducing innovative approaches to determine upper bounds for weak solutions and establishing continuity in variational inequality problems. These contributions fill a gap in the existing literature and provide valuable insights for future research in this area.

2. Structure of weak solutions and preliminaries

In addition to providing several useful lemmas, this section constructs a weak solution to the inverse variational inequality (2) using the global boundedness of u. Considering that $u_0 \geq 0$ in Ω, we can deduce from (2) that

$$u \geq u_0 \geq 0 \text{ in } \Omega_T. \tag{4}$$

Furthermore, from (1) we also know that $Lu \leq 0$ in Ω_T. Therefore, for any nonnegative fixed constant M_0 and $t \in (0,T]$, multiplying both sides of $Lu \leq 0$ by $(u-M_0)_+$ and integrating over the domain Ω_t, we have

$$\int\int_{\Omega_t} \partial_\tau u \cdot (u-M_0)_+ + u|\nabla(u-M_0)_+|^p + (1-\gamma)(u-M_0)_+|\nabla(u-M_0)_+|^p \mathrm{d}x\mathrm{d}\tau \leq 0. \tag{5}$$

By utilizing the method of integration by parts, we can obtain

$$\int\int_{\Omega_t} \partial_\tau u \cdot (u-M_0)_+ \mathrm{d}x\mathrm{d}\tau = \frac{1}{2}\int\int_{\Omega_t} \partial_\tau (u-M_0)_+^2 \mathrm{d}x\mathrm{d}\tau.$$

Note that $\gamma \in (0,1)$, $u|\nabla(u-M_0)_+|^p$ and $(1-\gamma)(u-M_0)_+|\nabla(u-M_0)_+|^p$ are nonnegative. By removing them in (5), we can obtain

$$\int\int_{\Omega_t} \partial_\tau (u-M_0)_+^2 \mathrm{d}x\mathrm{d}\tau \leq 0,$$

which leads to

$$\int_\Omega (u(\cdot,t)-M_0)_+^2 \mathrm{d}x \leq \int_\Omega (u_0-M_0)_+^2 \mathrm{d}x. \tag{6}$$

Also, since $u_0 \in W_0^{1,p}(\Omega)$, if M_0 is sufficiently large, $\int_\Omega (u_0-M_0)_+^2 \mathrm{d}x = 0$, which implies $\int_\Omega (u(\cdot,t)-M_0)_+^2 \mathrm{d}x = 0$. This means that

$$u \leq M_0 \text{ in } \Omega_T. \tag{7}$$

Next, we define the weak solution of the variational inequality (2). Considering the upper and lower bounds (4) and (7) of the solution to the variational inequality (2) and incorporating the methods from [8], we provide a set of maximal monotone maps

$$G = \{u | u(x) = 0, x > 0; u(x) \in [-M_0, 0], \quad x = 0\}, \tag{8}$$

where M_0 is a positive constant.

Definition 2.1. *A pair (u, ξ) is said to be a generalized solution of the inverse variation-inequality (2) if it satisfies the following conditions: $u \in L^\infty(0, T, W_0^{1,p}(\Omega))$, $\partial_t u \in L^\infty(0, T, L^2(\Omega))$, and $\xi \in G$ for any $(x, t) \in \Omega_T$, (a) $u(x, t) \geq u_0(x)$, $u(x, 0) = u_0(x)$* for any $(x, t) \in \Omega_T$, *(b) for every test function $\varphi \in C^1(\bar{\Omega}_T)$, there exists an equality that holds:*

$$\int\int_{\Omega_T} \partial_t u \cdot \varphi + u|\nabla u|^{p-2}\nabla u \nabla\varphi + (1-\gamma)|\nabla u|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t. \tag{9}$$

Finally, we introduce two lemmas that are utilized in the proof of the Hölder continuity of the weak solution to the inverse variation-inequality (2). The detailed proof can be found in [14,15].

Lemma 2.1. *Suppose that it is a nonnegative sequence satisfying*

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha}, b > 1, \alpha, C > 0.$$

If $Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2}$, then $Y_n \to 0, n \to \infty$.

Lemma 2.2. *There exists a positive constant C depending only on N and p such that*

$$\int\int_{\Omega_T} |u|^p \mathrm{d}x\mathrm{d}t \leq C|\{u > 0\}|^{p/(N+p)} \|u\|^p_{L^p(\Omega_T)}.$$

Lemma 2.1 is used to obtain a lower bound for the weak solution using the limit method, which is then used to prove the Hölder continuity of the weak solution. Lemma 2.2 is used to obtain the conditions required for Lemma 2.1.

3. Existence

To characterize the weak solution defined by $\xi \in G(u - u_0)$, we introduce the penalty function

$$\beta_\varepsilon(z) \leq 0, z \in \mathrm{R};\ \beta_\varepsilon(z) = 0, z \geq \varepsilon;\ \beta_\varepsilon(0) = M_0;\ \beta_\varepsilon \in C(\mathrm{R}), \tag{10}$$

and $\lim\limits_{\varepsilon \to +0} \beta_\varepsilon(z) = \begin{cases} 0 & z > 0, \\ -M_0 & z = 0. \end{cases}$ Consider the following parabolic auxiliary problem

$$\begin{cases} Lu_\varepsilon = \beta_\varepsilon(u_\varepsilon - u_0), & (x, t) \in \Omega_T, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x) = u_0(x) + \varepsilon, & x \in \Omega, \\ u_\varepsilon(t, x) = \frac{\partial u_\varepsilon}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, T). \end{cases} \tag{11}$$

From definition (10), it can be observed that when $u_\varepsilon \geq u_0 + \varepsilon$, $Lu_\varepsilon = 0$ in Ω_T; and at the same time, when $u_\varepsilon < u_0 + \varepsilon$, $Lu_\varepsilon \leq 0$ in Ω_T. This is exactly the same as the situation in variational inequality (2), which is also the original intention of constructing the auxiliary problem. Additionally, let $t \to 0$; according to the definition of $\beta_\varepsilon(\cdot)$, we have

$$Lu_{0,\varepsilon} = \beta_\varepsilon(u_{0,\varepsilon} - u_0) = \beta_\varepsilon(\varepsilon) = 0 \text{ in } \Omega.$$

On the other hand, by utilizing (11), it can be inferred that

$$Lu_\varepsilon = \beta_\varepsilon(u_\varepsilon - u_0) \leq 0 \,\text{in}\, \Omega_T,$$

which indicates

$$Lu_\varepsilon \leq Lu_{0,\varepsilon} \,\text{in}\, \Omega_T.$$

Furthermore, due to $u_\varepsilon = u_{0,\varepsilon}(x)$ in $\partial\Omega_T$, by utilizing the principle of comparison, it can be inferred that

$$u_\varepsilon \geq u_{0,\varepsilon}(x) \,\text{in}\, \Omega_T. \tag{12}$$

Based on the experience from references [4,8], we provide the weak solution of the auxiliary problem without proof and analyze the boundedness and energy estimation of the auxiliary problem (11) on this basis.

Definition 3.1. *A function u is considered a generalized solution of variation-inequality (1) if it meets the condition*

$$u \in L^\infty(0, T, W^{1,p}(\Omega)),\ \partial_t u \in L^2(\Omega_T),$$

and for any test-function $\varphi \in C^1(\bar{\Omega}_T)$, the equality

$$\int\int_{\Omega_T} \partial_t u_\varepsilon \cdot \varphi + u_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla\varphi + (1-\gamma)|\nabla u_\varepsilon|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \beta_\varepsilon(u_\varepsilon - u_0) \cdot \varphi \mathrm{d}x\mathrm{d}t \tag{13}$$

holds.

Next, we will analyze the properties of the weak solution of the parabolic auxiliary problem (11). Let's start by proving $u_\varepsilon \leq M_0$ in Ω_T. It is important to note that $\beta_\varepsilon(u_\varepsilon - u_0) \leq 0$ in Ω_T. From (11), we can also deduce that $Lu_\varepsilon \leq 0$ in Ω_T, which allows us to repeat the proof process of (6) and obtain

$$\int_\Omega (u_\varepsilon - M_0)_+^2 \mathrm{d}x \leq \int_\Omega (u_{0,\varepsilon} - M_0)_+^2 \mathrm{d}x. \tag{14}$$

When M_0 is sufficiently large, it follows that $\int_\Omega (u_0 - M_0)_+^2 \mathrm{d}x = 0$. This indicates

$$u_\varepsilon \leq M_0 \text{ in } \Omega_T. \tag{15}$$

By combining (12) and (15), it can be shown that there exists a sufficiently large positive constant M_0 such that

$$u_0 \leq u_\varepsilon \leq M_0 \text{ in } \Omega_T. \tag{16}$$

Now, we delve into the estimation of the gradient of u_ε. By selecting u_ε as the basis function in Definition 3.1, we can derive

$$\int\int_{\Omega_T} \partial_t u_\varepsilon \cdot u_\varepsilon + (2-\gamma) u_\varepsilon |\nabla u_\varepsilon|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \beta_\varepsilon(u_\varepsilon - u_0) \cdot u_\varepsilon \mathrm{d}x\mathrm{d}t.$$

It is important to note that $\beta_\varepsilon(u_\varepsilon - u_0) \leq 0$ and $u_\varepsilon \geq 0$ in Ω_T, which enables us to eliminate the nonnegative term $\int\int_{\Omega_T} \beta_\varepsilon(u_\varepsilon - u_0) \cdot u_\varepsilon \mathrm{d}x\mathrm{d}t \leq 0$ and obtain

$$\int\int_{\Omega_T} \partial_t u_\varepsilon \cdot u_\varepsilon + (2-\gamma) u_\varepsilon |\nabla u_\varepsilon|^p \varphi \mathrm{d}x\mathrm{d}t \leq 0. \tag{17}$$

Building upon this, by utilizing the Hölder and Young inequality, we can derive the following result

$$\|u_\varepsilon\|_{L^p(\Omega)}^p \leq C, \int_\Omega |\nabla u_\varepsilon|^p \mathrm{d}x \leq C. \tag{18}$$

For a detailed proof, please refer to [8], as it will not be reiterated here.

By selecting $u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon$ as the basis function, we have

$$\begin{aligned}&\int\int_{\Omega_T} \partial_t u_\varepsilon \cdot u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon + u_\varepsilon|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon) + (1-\gamma)|\nabla u_\varepsilon|^p u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon \mathrm{d}x\mathrm{d}t\\ &= \int\int_{\Omega_T} \beta_\varepsilon(u_\varepsilon - u_0)\cdot u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{19}$$

By utilizing Eq (10) to $\beta_\varepsilon(u_\varepsilon - u_0)$, we obtain

$$\int\int_{\Omega_T} \beta_\varepsilon(u_\varepsilon - u_0)\cdot u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon \mathrm{d}x\mathrm{d}t \leq \frac{1}{\gamma^2}M_0^2 \int_{\Omega_T} |u_{0,\varepsilon}^\gamma|^2 \mathrm{d}x. \tag{20}$$

Given the setting $\mu = \frac{1}{2}(\gamma+1)$, let us analyze the integration $\int\int_{\Omega_T} \partial_t u_\varepsilon \cdot u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon \mathrm{d}x\mathrm{d}t$. It is important to note that according to (14), we have $\int_\Omega u(\cdot, T)^2 \mathrm{d}x \leq \int_\Omega u_{0,\varepsilon}^2 \mathrm{d}x$, which consequently leads to

$$\int\int_{\Omega_T} \partial_t u_\varepsilon \cdot u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon \mathrm{d}x\mathrm{d}t = \frac{1}{\mu^2}\int\int_{\Omega_T} |\partial_t u_\varepsilon^\mu|^2 \mathrm{d}x\mathrm{d}t \leq \frac{1}{\mu^2}\int_\Omega u_{0,\varepsilon}^2 \mathrm{d}x. \tag{21}$$

By applying the integral transformation to $\int\int_{\Omega_T} u_\varepsilon|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon)\mathrm{d}x\mathrm{d}t$, we can obtain

$$\begin{aligned}&\int\int_{\Omega_T} u_\varepsilon|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(u_\varepsilon^{\gamma-1}\partial_t u_\varepsilon)\mathrm{d}x\mathrm{d}t\\ &= \int\int_{\Omega_T} u_\varepsilon^\gamma|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\partial_t u_\varepsilon)\mathrm{d}x\mathrm{d}t + (\gamma-1)\int\int_{\Omega_T} u_\varepsilon^{\gamma-1}|\nabla u_\varepsilon|^p \partial_t u_\varepsilon \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{22}$$

Please note that $\gamma \in (0,\ 1)$ and $\mu \in (0,\ 1)$. Substituting Eqs (20)–(22) into (19), we have

$$\frac{1}{\mu^2}\int\int_{\Omega_T} |\partial_t u_\varepsilon^\mu|^2 \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} u_\varepsilon^\gamma|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\partial_t u_\varepsilon)\mathrm{d}x\mathrm{d}t \leq \mathrm{C}(\gamma,\mu,M_0)\max\left\{\int_{\Omega_T} |u_{0,\varepsilon}|^2 \mathrm{d}x, 1\right\}. \tag{23}$$

Considering the estimation of $\int\int_{\Omega_T} u_\varepsilon^\gamma|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\partial_t u_\varepsilon)\mathrm{d}x\mathrm{d}t$ in (23), from (16) and (18), we can obtain

$$\begin{aligned}&\left|\int\int_{\Omega_T} u_\varepsilon^\gamma|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\partial_t u_\varepsilon)\mathrm{d}x\mathrm{d}t\right|\\ &= \left|\int\int_{\Omega_T} u_\varepsilon^\gamma \partial_t(|\nabla u_\varepsilon|^p)\mathrm{d}x\mathrm{d}t\right| \leq M_0^\gamma\left|\int\int_{\Omega_T} \partial_t(|\nabla u_\varepsilon|^p)\mathrm{d}x\mathrm{d}t\right|\\ &\leq M_0^\gamma\left(\int_\Omega |\nabla u_{0,\varepsilon}|^p \mathrm{d}x + \int_\Omega |\nabla u_{T,\varepsilon}|^p \mathrm{d}x\right) \leq \infty.\end{aligned} \tag{24}$$

Substituting (24) into (23), we obtain

$$\left\|\partial_t u_\varepsilon^\mu\right\|_{L^2(\Omega_T)} \leq C(p, T, |\Omega|). \tag{25}$$

According to the estimates obtained from (16), (18) and (25), it can be concluded that the set $\{u_\varepsilon, \varepsilon \geq 0\}$ possesses a convergent subsequence and a function u such that

$$u_\varepsilon \to u \text{ a.e. in } \Omega_T \text{ as } \varepsilon \to 0, \tag{26}$$

$$u_\varepsilon \overset{weak}{\to} u \text{ in } L^\infty(0,T;W^{1,p}(\Omega)) \text{ as } \varepsilon \to 0, \tag{27}$$

$$\partial_t u_\varepsilon \overset{weak}{\to} \partial_t u \text{ in } L^2(\Omega_T) \text{ as } \varepsilon \to 0. \tag{28}$$

It is worth noting that (27) also employs Lebesgue's dominated convergence theorem in its proof.

Lemma 3.1. *Let u_ε be a weak solution to the parabolic auxiliary problem (2), then there exists $\xi \in G$ such that*

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \text{ as } \varepsilon \to 0. \tag{29}$$

Proof. According to (16), the sequence $\{\beta_\varepsilon(u_\varepsilon - u_0), \varepsilon \geq 0\}$ has a convergent subsequence and, furthermore,

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \text{ as } \varepsilon \to 0.$$

The following demonstrates the validity of $\xi \in G$. It should be noted that when $u_\varepsilon \geq u_0+\varepsilon, \beta_\varepsilon(u_\varepsilon-u_0) = 0$ and, as a result,

$$\beta_\varepsilon(u_\varepsilon - u_0) \to 0 \text{ as } \varepsilon \to 0.$$

This implies that when $u > u_0$, $\xi = 0$. On the other hand, when $u_\varepsilon \leq u_0 + \varepsilon$, $-M_0 \leq \beta_\varepsilon(u_\varepsilon - u_0) \leq 0$. Based on the boundedness of limits, $-M_0 \leq \xi \leq 0$, which indicates $\xi \in G$.

By combining Eqs (26)–(29) and employing the limit method for ε as described in [4], the existence of a weak solution can be obtained.

Theorem 3.1. *Assume that $\gamma \in (0, 1)$ and $u_0 \in W_0^{1,p}(\Omega)$, then (1) admits a solution within the class of Definition 2.1.*

4. Hölder continuity

For any $(t_0, x_0) \in \Omega_T$, let $Q = Q(\rho, \theta) = B_\rho(x_0) \times (t_0 - \theta, t_0)$, where ρ and θ are sufficiently small nonnegative constants to ensure $Q \subset \Omega_T$. In this section, we consider the Hölder continuity of the weak solution u to the inverse variational inequality (2) on Q.

We denote $(u - k)_+$ and $(u - k)_-$ as $(u - k)_\pm$, where k is a positive constant, $(u - k)_+ = \max\{u-k, 0\}$, and $(u - k)_- = \max\{k - u, 0\}$. Let $\phi \in W^{1,p}(\Omega_T)$ be a given function. In the context of (9), we select the test function $w = \phi^p \times (u - k)_\pm$ and set $\phi \geq 0$. It can be readily observed that

$$\begin{aligned} &\int_{t_0-\theta}^{t_0} \int_\Omega \phi^p \times (u - k)_\pm u_t \mathrm{d}x\mathrm{d}t + \int_{t_0-\theta}^{t_0} \int_\Omega u|\nabla u|^{p-2}\nabla u\nabla[\phi^p \times (u - k)_\pm]\mathrm{d}x\mathrm{d}t \\ &+(1 - \gamma) \int_{t_0-\theta}^{t_0} \int_\Omega |\nabla u|^p\phi^p \times (u - k)_\pm\mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \varphi\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{30}$$

It is important to note that $\gamma \in (0, 1)$, ϕ and $(u - k)_\pm$ are nonnegative, which leads to the conclusion of

$$(1 - \gamma) \int_{t_0-\theta}^{t_0} \int_\Omega |\nabla u|^p\phi^p \times (u - k)_\pm\mathrm{d}x\mathrm{d}t \geq 0.$$

On the other hand, it is easily derived from (8) that

$$\int_{t_0-\theta}^{t_0} \int_\Omega \xi \cdot \phi^p \times (u - k)_\pm\mathrm{d}x\mathrm{d}t \leq 0.$$

Removing the nonpositive term $(1 - \gamma) \int_{t_0-\theta}^{t_0} \int_\Omega |\nabla u|^p\phi^p \times (u - k)_\pm\mathrm{d}x\mathrm{d}t$ and the nonnegative term $\int_{t_0-\theta}^{t_0} \int_\Omega \xi \cdot \phi^p \times (u - k)_\pm\mathrm{d}x\mathrm{d}t$, we have

$$\int_{t_0-\theta}^{t_0} \int_\Omega \phi^p \times (u - k)_\pm u_t\mathrm{d}x\mathrm{d}t + \int_{t_0-\theta}^{t_0} \int_\Omega u|\nabla u|^{p-2}\nabla u\nabla(\phi^p \times (u - k)_\pm)\mathrm{d}x\mathrm{d}t \leq 0. \tag{31}$$

Integrate the temporal gradient term $\int_\Omega \partial_t(\phi^p \times (u-k)_\pm^2)\mathrm{d}x$ with respect to time, yielding

$$\int_\Omega \partial_t(\phi^p \times (u-k)_\pm^2)\mathrm{d}x = 2\int_\Omega \phi^p \times (u-k)_\pm u_t \mathrm{d}x\mathrm{d}t + p\int_\Omega \phi^{p-1} \times \partial_t\phi \times (u-k)_\pm^2 \mathrm{d}x. \tag{32}$$

Upon integrating the spatial gradient term $\int_\Omega u|\nabla u|^{p-2}\nabla u\nabla(\phi^p \times (u-k)_\pm)\mathrm{d}x$, we obtain

$$\begin{aligned}&\int_\Omega u|\nabla u|^{p-2}\nabla u\nabla(\phi^p \times (u-k)_\pm)\mathrm{d}x\\&= \int_\Omega u|\nabla(u-k)_\pm|^p \times \phi^p\mathrm{d}x + \int_\Omega u|\nabla u|^{p-2}\nabla u \times (u-k)_\pm\nabla\phi^p\mathrm{d}x.\end{aligned} \tag{33}$$

Further analysis of $\int_\Omega u|\nabla u|^{p-2}\nabla u \times (u-k)_\pm\nabla\phi^p\mathrm{d}x$ in (33) reveals that the Hölder and Young inequalities can be employed to obtain

$$\begin{aligned}&\int_\Omega u|\nabla u|^{p-2}\nabla u \times (u-k)_\pm\nabla\phi^p\mathrm{d}x\\&\le \tfrac{p-1}{p}\int_\Omega u|\nabla(u-k)_\pm|^p \times \phi^p\mathrm{d}x + \tfrac{1}{p}\int_\Omega u^{\frac{p-1}{p}}|(u-k)_\pm^p|\nabla\phi|^p\mathrm{d}x.\end{aligned} \tag{34}$$

By combining Eqs (33) and (34), and substituting them together with Eq (32) into (31), we obtain the following result, which serves as the cornerstone for proving the weak solution's Hölder continuity.

Theorem 4.1. *Let $k \ge 0$ and $\phi \in W^{1,p}(\Omega_T)$ be any nonnegative constants. If $(t_0-\theta, t_0) \subset (0,T)$ holds for any nonnegative constant θ , then*

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in(t_0-\theta,t_0)}\int_\Omega (\phi^p \times (u-k)_\pm^2)\mathrm{d}x + \tfrac{1}{p}\int_{t_0-\theta}^{t_0}\int_\Omega u|\nabla(u-k)_\pm|^p \times \phi^p\mathrm{d}x\mathrm{d}t\\&\le p\int_\Omega \phi^{p-1} \times |\partial_t\phi| \times (u-k)_\pm^2\mathrm{d}x + \int_\Omega (\phi^p(x,t_0-\theta) \times (u(x,t_0-\theta)-k)_\pm^2)\mathrm{d}x\\&\quad+\tfrac{1}{p}\int_{t_0-\theta}^{t_0}\int_\Omega u^{\frac{p-1}{p}}|(u-k)_\pm^p|\nabla\phi|^p\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{35}$$

For any given $(t_0,x_0) \in \Omega_T$, select R to be sufficiently small such that $Q_n = Q(R_n, R_n^p) \in \Omega_T$. Furthermore, let us define $\mu^+ = \operatorname*{ess\,sup}_{Q(R,R^p)} u$, $\mu^- = \operatorname*{ess\,inf}_{Q(R,R^p)} u$, $\omega = \operatorname*{osc}_{Q(R,R^p)} u = \mu^+ - \mu^-$ and also utilize the symbol $R_n = \frac{1}{2}R + \frac{1}{2^{n+1}}R$.

Lemma 4.1. *Given the definitions of $k_n = \mu^- + \frac{1}{2^{s_*+1}}\omega + \frac{1}{2^{s_*+n+1}}\omega$, $n = 1,2,3,\cdots$, it follows that*

$$(u-k_n)_-^2 \ge \left(\frac{2^{s_*}}{\omega}\right)^{p-2}(u-k_n)_-^p. \tag{36}$$

Proof. According to the definition of k_n, when u takes μ^-, $\left(\frac{2^{s_*}}{\omega}\right)^{p-2}(u-k_n)_-^p$ reaches its maximum; thus,

$$\left(\frac{2^{s_0}}{\omega}\right)^{p-2}(u-k_n)_-^p \le \left(\frac{2^{s_0}}{\omega}\right)^{p-2}\left(\frac{\omega}{2^{s_0}}\right)^p = \left(\frac{\omega}{2^{s_0}}\right)^2. \tag{37}$$

At this point, $(u-k_n)_-^2$ satisfies

$$(u-k_n)_-^2 = \left(\frac{1}{2^{s_0+1}}\omega + \frac{1}{2^{s_0+n+1}}\omega\right)^2 = \left(\frac{\omega}{2^{s_0}}\right)^2. \tag{38}$$

By combining Eqs (37) and (38), the result is proven to hold.

Next, we analyze the weak solution's Hölder continuity. Let $s_* > 1$ be set and define the truncation function

$$\phi_n(x,t) = \begin{cases} 0, & (x,t) \in \partial Q_n, \\ 1, & (x,t) \in Q_{n+1}. \end{cases} \tag{39}$$

Additionally, assume that ϕ_n satisfies the condition

$$|\nabla \phi_n(x,t)| \leq \frac{2^n}{R_n},\ |\partial_t \phi_n(x,t)| \leq \frac{2^{pn}}{R_n^p}. \tag{40}$$

Choose $\phi = \phi_n(x,t)$ and $k = k_n$. Furthermore, due to $\int_{B_n} (\phi_n^p(x, t_0 - R_n^p) \times (u(x, t_0 - R_n^p) - k_n)_-^2)\mathrm{d}x = 0$, from (35) we conclude that

$$\begin{aligned} &\underset{t\in(t_0-R_n^p,t_0)}{ess\sup} \int_{B_n} (\phi_n^p \times (u-k_n)_-^2)\mathrm{d}x + \tfrac{1}{p}\int_{t_0-R_n^p}^{t_0}\int_{B_n} |\nabla(u-k_n)_-|^p \times \phi^p \mathrm{d}x\mathrm{d}t \\ &\leq p\int_{t_0-R_n^p}^{t_0}\int_{B_n} \phi^{p-1} \times |\partial_t\phi| \times (u-k_n)_-^2 \mathrm{d}x\mathrm{d}t + \tfrac{1}{p}\int_{t_0-R_n^p}^{t_0}\int_{B_n} |(u-k_n)_-^p|\nabla\phi|^p \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{41}$$

After organizing, we have

$$\begin{aligned} &\underset{t\in(t_0-R_n^p,t_0)}{ess\sup} \int_{B_n} (\phi_n^p \times (u-k_n)_-^2)\mathrm{d}x + \tfrac{1}{p}\int_{t_0-R_n^p}^{t_0}\int_{B_n} u|\nabla(u-k_n)_-|^p \times \phi^p \mathrm{d}x\mathrm{d}t \\ &\leq p\tfrac{2^{pn}}{R^p}\left(\int_{t_0-R_n^p}^{t_0}\int_{B_n} \phi^{p-1} \times (u-k_n)_-^2 \mathrm{d}x\mathrm{d}t + \tfrac{1}{p^2}\int_{t_0-R_n^p}^{t_0}\int_{B_n} u^{\frac{p-1}{p}}|(u-k_n)_-^p|\mathrm{d}x\mathrm{d}t\right) \\ &\leq p\tfrac{2^{pn}}{R^p}\left(\int_{t_0-R_n^p}^{t_0}\int_{B_n} \phi^{p-1} \times (u-k_n)_-^2 \mathrm{d}x\mathrm{d}t + \tfrac{1}{p^2}M_0^{\frac{p-1}{p}}\int_{t_0-R_n^p}^{t_0}\int_{B_n} u^{\frac{p-1}{p}}|(u-k_n)_-^p|\mathrm{d}x\mathrm{d}t\right). \end{aligned} \tag{42}$$

The validity of the last inequality in the above equation is ensured by utilizing (15). By applying (31) to $\int_{t_0-R_n^p}^{t_0}\int_{B_n} (u-k_n)_-^2 \mathrm{d}x\mathrm{d}t + \frac{1}{p^2}M_0^{\frac{p-1}{p}}\int_{t_0-R_n^p}^{t_0}\int_{B_n} |(u-k_n)_-^p|\mathrm{d}x\mathrm{d}t$, the result is

$$\begin{aligned} &\int_{t_0-R_n^p}^{t_0}\int_{B_n} (u-k_n)_-^2 \mathrm{d}x\mathrm{d}t + \tfrac{1}{p^2}M_0^{\frac{p-1}{p}}\int_{t_0-R_n^p}^{t_0}\int_{B_n} |(u-k_n)_-^p|\mathrm{d}x\mathrm{d}t \\ &\leq \left[1 + \tfrac{1}{p^2}M_0^{\frac{p-1}{p}}\left(\tfrac{\omega}{2^{s_*}}\right)^{p-2}\right]\int_{t_0-R_n^p}^{t_0}\int_{B_n} (u-k_n)_-^2 \mathrm{d}x\mathrm{d}t, \end{aligned}$$

which allows (42) to be rewritten as

$$\begin{aligned} &\underset{t\in(t_0-R_n^p,t_0)}{ess\sup} \int_{B_n} (\phi_n^p \times (u-k_n)_-^2)\mathrm{d}x + \tfrac{1}{p}\int_{t_0-R_n^p}^{t_0}\int_{B_n} u|\nabla(u-k_n)_-|^p \times \phi_n^p \mathrm{d}x\mathrm{d}t \\ &\leq p\tfrac{2^{pn}}{R^p}\left(\tfrac{\omega}{2^{s_*}}\right)^2\left[1 + \tfrac{1}{p^2}M_0^{\frac{p-1}{p}}\left(\tfrac{\omega}{2^{s_*}}\right)^{p-2}\right]\int_{t_0-R_n^p}^{t_0}\int_{B_n} \chi_{(u-k_n)_->0}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{43}$$

Here, it is easy to deduce from (15) that there exists a nonnegative constant C such that

$$\int_{B_n} u|\nabla(u-k_n)_-|^p \times \phi_n^p \mathrm{d}x \geq C\int_{B_n} |\nabla(u-k_n)_-|^p \times \phi_n^p \mathrm{d}x. \tag{44}$$

If not, for any small nonnegative constant C_1, we have

$$\int_{B_n} u|\nabla(u-k_n)_-|^p \times \phi_n^p \mathrm{d}x \leq C_1\int_{B_n} |\nabla(u-k_n)_-|^p \times \phi_n^p \mathrm{d}x.$$

This implies that $u \equiv 0$ or $|\nabla(u-k_n)_-|^p \times \phi_n^p \equiv 0$ in B_n. If $|\nabla(u-k_n)_-|^p \times \phi_n^p \equiv 0$, the continuity result for Hölder holds directly. If $u \equiv 0$, then from (16) it can be seen that this contradicts the fact that u_0 is not zero everywhere inside Ω . Combining (43) and (44), we have

$$\begin{aligned} &ess \sup_{t\in(t_0-R_n^p,t_0)} \int_{B_n} (\phi_n^p \times (u-k_n)_-^2)\mathrm{d}x + \tfrac{C}{p}\int_{t_0-R_n^p}^{t_0}\int_{B_n} |\nabla(u-k_n)_-|^p \times \phi_n^p \mathrm{d}x\mathrm{d}t \\ &\leq p\tfrac{2^{pn}}{R^p}\left(\tfrac{\omega}{2^{s_*}}\right)^2\left[1+\tfrac{1}{p^2}M_0^{\frac{p-1}{p}}\left(\tfrac{\omega}{2^{s_*}}\right)^{p-2}\right]\int_{t_0-R_n^p}^{t_0}\int_{B_n}\chi_{(u-k_n)_->0}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{45}$$

To facilitate the discussion, define $A_n = \{x \in B_n | u \leq k_n\}$, and it can be derived from Eqs (43) and (45) that

$$\|(u-k_n)_-\phi_n\|_{L^p(Q_n)}^p \leq p\frac{2^{pn}}{R^p}\left(\frac{\omega}{2^{s_*}}\right)^2\left[1+\frac{1}{p^2}\left(\frac{\omega}{2^{s_*}}\right)^{p-2}\right]\int_{t_0-R_n^p}^{t_0}|A_n|\mathrm{d}t. \tag{46}$$

By applying Lemma 2.2 to $\|(u-k)_-\phi_n\|_{L^p(Q_n)}^p$, we obtain

$$\|(u-k_n)_-\|_{L^p(Q_n)}^p \leq \|(u-k_n)_-\phi_n\|_{L^p(Q_n)}^p\left(\int_{t_0-R_n^p}^{t_0}|A_n|\mathrm{d}t\right)^{\frac{p}{N+p}}. \tag{47}$$

Furthermore, from $(u-k_n)_-^2$ we can also obtain

$$\|(u-k_n)_-\|_{L^p(Q_{n+1})}^p \geq |k_n-k_{n+1}|^p\int_{t_0-R_n^p}^{t_0}|A_{n+1}|\mathrm{d}t \geq \frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_*}}\right)^p\int_{t_0-R_n^p}^{t_0}|A_{n+1}|\mathrm{d}t. \tag{48}$$

By combining (47) and (48) and substituting the result into Eq (46), we obtain

$$\frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_*}}\right)^p\int_{t_0-R_n^p}^{t_0}|A_{n+1}|\mathrm{d}t \leq p\frac{2^{pn}}{R^p}\left(\frac{\omega}{2^{s_*}}\right)^2\left[1+\frac{1}{p^2}\left(\frac{\omega}{2^{s_*}}\right)^{p-2}\right]\left(\int_{t_0-R_n^p}^{t_0}|A_n|\mathrm{d}t\right)^{\frac{p}{N+p}}. \tag{49}$$

Simplifying Eq (49) leads to the inequality of

$$\int_{t_0-R_n^p}^{t_0}|A_{n+1}|\mathrm{d}t \leq 2p\frac{4^{pn}}{R^p}\left(\int_{t_0-R_n^p}^{t_0}|A_n|\mathrm{d}t\right)^{1+\frac{p}{N+p}}. \tag{50}$$

By utilizing Lemma 2.1, we can obtain $\int_{t_0-R_n^p}^{t_0}|A_n|\mathrm{d}t \to 0$ as $n\to\infty$ if $Y_0 \leq \left(\frac{2p}{R^p}\right)^{\frac{N+p}{p}} 4^{\frac{(N+p)^2}{p}}$. Consequently, we can draw the following conclusion.

Theorem 4.2. *If s_* is sufficiently large,*

$$u \geq \mu^- + \frac{\omega}{2^{s_*+1}} \text{ a.e. } (x,t) \in Q(\frac{1}{2}R,(\frac{1}{2}R)^p). \tag{51}$$

Theorem 4.3. *The weak solution of the variational inequality problem (1) possesses Hölder continuity, i.e., there exists a nonnegative constant σ such that*

$$\underset{Q(\frac{1}{2}R,(\frac{1}{2}R)^p)}{osc} u \leq \sigma\omega.$$

Proof. Due to the presence of $\mu^+ = \underset{Q(R,R^p)}{ess\sup} u$ and $\mu^- = \underset{Q(R,R^p)}{ess\inf} u$,

$$\underset{Q(\frac{1}{2}R,(\frac{1}{2}R)^p)}{osc} u = \underset{Q(\frac{1}{2}R,(\frac{1}{2}R)^p)}{ess\sup} u - \underset{Q(\frac{1}{2}R,(\frac{1}{2}R)^p)}{ess\inf} u \leq \mu^+ - \underset{Q(\frac{1}{2}R,(\frac{1}{2}R)^p)}{ess\inf}. \tag{52}$$

Substituting (52) into (51) and selecting $\sigma = (1 - \frac{1}{2^{s_*+1}})$, the theorem proposition holds.

$$\underset{Q(\frac{1}{2}R,(\frac{1}{2}R)^p)}{osc} u \leq \sigma\omega.$$

5. Conclusions

In recent years, numerous scholars have conducted theoretical research on variational inequalities. Variational inequalities of the form (1) are convenient for scholars to use the comparison principle to obtain upper bounds for the solution u, thereby constructing weak solutions through the use of the maximal operator. In the case of the inverse variational inequality (2), since we can only obtain $Lu \leq 0$ in Ω_T, the use of the comparison principle can only demonstrate that the solution of the inverse variational inequality (2) has a nonnegative lower bound, thus limiting the study of inverse variational inequalities.

The present study starts by considering $Lu \leq 0$ and $(u - M_0)_+$ and obtains an energy estimate for $(u - M_0)_+$. It is demonstrated that when M_0 is sufficiently large, the upper bound of this estimate is 0, thereby obtaining a global upper bound for the solution of the inverse variation inequality (2). Subsequently, a continuous method is employed to prove the existence of weak solutions for the inverse variation inequality (2). Finally, we analyzed the Hölder continuity of the weak solution to the inverse variation inequality (2). Combining the global upper and lower bounds of the weak solution to the inverse variation inequality (2), we obtained an integral inequality involving $\phi^p \times (u - k)_\pm$ starting from the weak solution, as shown in Theorem 4.1. We then chose ϕ as the cut-off factor for the subdomain $Q(\frac{1}{2}R, (\frac{1}{2}R)^p)$, and by using Hölder and Young inequalities as well as a sequence convergence result (see Lemma 2.1), we established the Hölder continuity of the weak solution.

So far, there are still some limitations in this paper: (i) Regarding the parameter γ, the proof in (15) relies on the constraint $\gamma \in (0, 1)$, and the proof in (17) relies on the constraint $\gamma < 2$; thus, the paper consistently assumes $\gamma \in (0, 1)$. (ii) Regarding the parameter p, the existence of weak solutions and the Hölder continuity are both repeatedly used under the condition $p \geq 2$ using Hölder and Young inequalities. Therefore, the paper also consistently assumes $p \geq 2$. The author intends to overcome these limitations in future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no confict of interest.

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