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Research article

On Liouville-type theorem for the stationary compressible Navier–Stokes equations in $\mathbb{R}^3$

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Abstract: In this paper, we study the Liouville-type theorem for the stationary barotropic compressible Navier–Stokes equations in $\mathbb{R}^3$. Based on a fairly general framework of a kind of local mean oscillations integral and Morrey spaces, we prove that the velocity and the density of the flow are trivial without any integrability assumption on the gradient of the velocity.

Keywords: Liouville-type theorem; compressible Navier–Stokes equaitons; barotropic; local mean oscillations integral; Morrey spaces

1. Introduction

The present paper is concerned with the following three-dimensional steady barotropic compressible Navier–Stokes equations

$$\begin{cases} \operatorname{div}(\rho v) = 0, \\ \operatorname{div}(\rho v \otimes v) - \mu\Delta v - (\lambda + \mu)\nabla \operatorname{div} v + \nabla P = 0, \end{cases} \tag{1.1}$$

where $\rho = \rho(x)$ and $v = (v_1(x), v_2(x), v_3(x))^T$ stand for the density and velocity of the fluid, respectively, $P = P(x)$ is the scalar pressure function given by the so-called γ-law

$$P(\rho) = a\rho^\gamma,\ a > 0,\ \gamma > 1 \tag{1.2}$$

and the constants μ and λ are the shear viscosity and the bulk viscosity, respectively, such that

$$\mu > 0 \quad \text{and} \quad \lambda + \frac{2}{3}\mu > 0.$$

The system (1.1) is the famous Navier–Stokes system, which describes the motion of a compressible viscous barotropic fluid. For more physical backgrounds and explanations of (1.1), we refer the readers to [1–3] and the references therein.

The aim of this paper is to study Liouville-type property of the solutions to the system (1.1), which is mainly inspired by the development of the incompressible Navier–Stokes equations. Recently, the investigation of the Liouville-type theorems for the Navier–Stokes equations has attracted much attention. One can refer to Leray [4] and Galdi [5, Remark X.9.4] for more details on this problem. Though it is still far from complete, there has existed many remarkable results under some additional conditions (see, e.g., [6–8]). Inspired by many works on the regularity of solutions to the stationary compressible Navier–Stokes equations (see, e.g., [9–11]), it is natural to study the Liouville properties of smooth solutions to (1.1). In the following, we will review some related results on the Liouville-type theorem for the compressible Navier–Stokes equations (1.1) to motivate this paper. Under the assumptions $\rho \in L^\infty(\mathbb{R}^3)$ and

$$(v,\ \nabla v) \in L^{\frac{3}{2}}(\mathbb{R}^3) \times L^2(\mathbb{R}^3), \tag{1.3}$$

Chae [12] proved that the smooth solution $(\rho,\ v)$ to (1.1) must satisfy

$$v \equiv 0 \quad \text{and} \quad \rho \equiv \text{constant in } \mathbb{R}^3. \tag{1.4}$$

Later, Li and Yu [13] replaced the intergrability condition (1.3) with

$$(v,\ \nabla v) \in L^{\frac{9}{2}}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$$

to obtain (1.4). Li and Niu [14] demonstrated that (1.4) holds if

$$(v,\ \nabla v) \in L^{p,q}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$$

for $(p,\ q) \in (3,\ \frac{9}{2}) \times [3,\ \infty]$ instead of (1.3). Very recently, Liu [15] improved the result of Li and Niu by assuming that

$$\nabla v \in L^2(\mathbb{R}^3),$$

and there exists a smooth function $\Psi \in C^\infty(\mathbb{R}^3;\ \mathbb{R}^{3\times3})$ such that $v = \operatorname{div}\Psi$ and

$$\sup_{r>1}\left(r^{-4}\int_{B_r} |\Psi - (\Psi)_{B_r}|^6\,\mathrm{d}x\right) < \infty. \tag{1.5}$$

One can refer to [16,17] and the references therein for more different and remarkable results and also to [18–21] for the study of the Liouville property of the solutions to the incompressible\compressible magnetohydrodynamic (MHD) equations and related models. It is not hard to see that the assumption (1.5) is weaker than $v \in L^{p,q}(\mathbb{R}^3)$, considering that the space $\mathrm{BMO}(\mathbb{R}^3)$ (see, e.g., [22, Definition 1.1]) shares similar properties with the space $L^\infty(\mathbb{R}^3) = L^{\infty,\infty}(\mathbb{R}^3)$ and often serves as a substitute for $L^\infty(\mathbb{R}^3)$. A natural question is whether one can weaken the Dirichlet integrability condition $\nabla v \in L^2(\mathbb{R}^3)$. The purpose of this work is to give a positive answer. Inspired by [15, 16, 21], we establish the Liouville-type theorem for the compressible Navier–Stokes equations (1.1) without the assumption $\nabla v \in L^2(\mathbb{R}^3)$.

Before preceding, some notations are introduced as follows. Throughout this paper, we denote B_r the ball with center 0 and radius $r > 0$; that is,

$$B_r := \left\{x \in \mathbb{R}^3 \mid |x| < r\right\}.$$

For each measurable set $\Omega \subset \mathbb{R}^3$ with its Lebesgue measure $|\Omega| > 0$ and for any $g \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, we adopt the standard notation

$$(g)_\Omega := \frac{1}{|\Omega|}\int_\Omega g(x)\,\mathrm{d}x$$

to denote the average of g over Ω.

Our first result can be stated as:

Theorem 1.1. *Let* (ρ, v, P) *be a smooth solution to the Eqs* (1.1) *and* (1.2). *Suppose that* $(\rho, v) \in L^\infty(\mathbb{R}^3) \times L^{p,q}(\mathbb{R}^3)$ *with* $(p, q) \in [1, \frac{3}{2}) \times [1, +\infty]$ *or* $p = q = \frac{3}{2}$, *and there exists* $\Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times3})$ *such that* $v = \operatorname{div}\Psi$ *and*

$$\sup_{r>1}\left(r^{-2-\frac{\sigma}{3}}\int_{B_r}|\Psi - (\Psi)_{B_r}|^\sigma\,\mathrm{d}x\right) < \infty \tag{1.6}$$

for some $\sigma \in (3, 6]$, *then* v *vanishes and* ρ *is a constant in* $\mathbb{R}^3$.

Remark 1. *The second author Liu [16] obtained the Liouville-type theorem for the stationary compressible Navier–Stokes equations* (1.1) *and* (1.2) *under the assumptions* $(\rho, v) \in L^\infty(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ *with* $p \in [1, \frac{3}{2}]$ *and there exists* $\Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times3})$ *such that* $v = \operatorname{div}\Psi$ *and the condition* (1.6) *holds with* $\sigma = 6$. *In comparison with the work [16], we establish the Liouville-type theorem in the framework of Lorentz spaces and the growth order for the mean oscillations at infinity. On one hand, we impose the condition* $v \in L^{p,q}(\mathbb{R}^3)$ *with* $(p, q) \in [1, \frac{3}{2}) \times [1, \infty]$ *or* $p = q = \frac{3}{2}$, *which weakens the assumption of* $v \in L^p(\mathbb{R}^3)$ *with* $p \in [1, \frac{3}{2}]$ *in [16]. On the other hand, we carefully discuss the range of parameter* σ *in the condition* (1.6). *Our result can thus be viewed as an extension of the work [16].*

It is well known that a tempered distribution v on $\mathbb{R}^3$ belongs to $\mathrm{BMO}^{-1}(\mathbb{R}^3)$, provided that there exists a function $\Phi \in \mathrm{BMO}(\mathbb{R}^3; \mathbb{R}^{3\times3})$ such that $v = \operatorname{div}\Phi$ (see, e.g., [22, Theorem 1]). Thanks to [23, Corollary, page 144], the condition (1.6) automatically holds under the assumption $v \in \mathrm{BMO}^{-1}(\mathbb{R}^3)$. As a consequence of Theorem 1.1, we obtain:

Corollary 1.1. *Let* (ρ, v, P) *be a smooth solution to the Eqs* (1.1) *and* (1.2). *Suppose that* $\rho \in L^\infty(\mathbb{R}^3)$ *and* $v \in L^{p,q}(\mathbb{R}^3) \cap \mathrm{BMO}^{-1}(\mathbb{R}^3)$ *with* $(p,q) \in [1, \frac{3}{2}) \times [1, +\infty]$ *or* $p = q = \frac{3}{2}$, *then* v *vanishes and* ρ *is a constant in* $\mathbb{R}^3$.

Our second result addresses the case of allowing the velocity v being in the Morrey spaces.

Theorem 1.2. *Let* (ρ, v, P) *be a smooth solution to the Eqs* (1.1) *and* (1.2). *Suppose that* $\rho \in L^\infty(\mathbb{R}^3)$ *and there exists* $\Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times3})$ *such that* $v = \operatorname{div}\Psi$ *and*

$$\sup_{r>1}\left(r^{-2-\frac{\sigma}{3}}\int_{B_r}|\Psi - (\Psi)_{B_r}|^\sigma\,\mathrm{d}x\right) < \infty \tag{1.7}$$

for some $\sigma \in (3, 6]$. *If one of the following conditions of the velocity holds:*
(a) $v \in \dot{M}^{p,\gamma}(\mathbb{R}^3)$ *for* $1 \le p < \gamma < \frac{3}{2}$,
(b) $v \in M^p_\gamma(\mathbb{R}^3)$ *for* $0 \le \gamma < 1 \le p < \frac{3}{2}$ *such that* $2p + \gamma < 3$,
(c) $v \in M^p_{\gamma,0}(\mathbb{R}^3)$ *for* $0 \le \gamma < 1 \le p < \frac{3}{2}$ *such that* $2p + \gamma = 3$,
then v *vanishes and* ρ *is a constant in* $\mathbb{R}^3$.

Remark 2. *Thanks to the embedding relation between the Lorentz spaces and Morrey spaces (see, e.g., [24]):*

$$L^{\gamma}(\mathbb{R}^3) \hookrightarrow L^{\gamma,p_2}(\mathbb{R}^3) \hookrightarrow \dot{M}^{p_1,\gamma}(\mathbb{R}^3), \qquad 1 < p_1 < \gamma \le p_2 \le \infty,$$

our work improves the result of Li and Niu [14] and also extends the result of Liu [16] to the framework of Morrey spaces.

The remaining part of this paper is unfolded as follows. In Section 2, we present the definitions of the Lorentz spaces and the Morrey spaces, then recall some basic inequalities. Section 3 is devoted to the derivation of the Caccioppoli-type inequalities, which will play a vital role in the proof of our main results. The proof of Theorems 1.1 and 1.2 are completed in Section 4.

2. Preliminaries

For the convenience of readers, in this section, we will present the definitions of the Lorentz spaces and the Morrey spaces, and recall some fundamental related facts.

We begin with the definition of the Lorentz spaces (see, e.g., [18,25]). For $(p, q) \in [1, \infty] \times [1, \infty]$, the Lorentz space $L^{p,q}(\mathbb{R}^3)$ is the space of measurable functions h defined on $\mathbb{R}^3$ such that the norm $\|h\|_{L^{p,q}(\mathbb{R}^3)}$ is finite, where

$$\|h\|_{L^{p,q}(\mathbb{R}^3)} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} h^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} h^*(t) & \text{if } q = \infty. \end{cases}$$

Here, h^* is the decreasing rearrangement of h given by

$$h^*(t) = \inf\{\tau \ge 0 \mid d_h(\tau) \le t\}$$

with the distribution function d_h of h defined as the Lebesgue measure of the set $\{y \in \mathbb{R}^3 \mid |h(y)| > \tau\}$.

It is well known that $L^{p,q}(\mathbb{R}^3)$ is a quasi-Banach space; that is, $\|\cdot\|_{L^{p,q}(\mathbb{R}^3)}$ satisfies

$$\|g + h\|_{L^{p,q}(\mathbb{R}^3)} \le 2^{1/p} \max\{1, 2^{(1-q)/q}\}(\|g\|_{L^{p,q}(\mathbb{R}^3)} + \|h\|_{L^{p,q}(\mathbb{R}^3)}) \quad \text{for each } g, h \in L^{p,q}(\mathbb{R}^3).$$

One can refer to [25, 26] for more details. In addition, it should be remarked that the usual L^p spaces $L^p(\mathbb{R}^3)$ coincide with the Lorentz spaces $L^{p,p}(\mathbb{R}^3)$ for all $p \in [1, \infty]$, and we also have the continuous embedding

$$L^{p,q_1}(\mathbb{R}^3) \hookrightarrow L^{p,q_2}(\mathbb{R}^3), \qquad 1 \le p \le \infty,\ 1 \le q_1 < q_2 \le \infty.$$

A simple fact we will recall is Hölder's inequality in Lorentz spaces (see, e.g., [26]), which plays a significant role in the proof of our main result.

Lemma 2.1. *Let* $1 \le p_1, p_2, q_1, q_2 \le \infty$. *If* $g \in L^{p_1,q_1}(\mathbb{R}^3)$ *and* $h \in L^{p_2,q_2}(\mathbb{R}^3)$, *then* $gh \in L^{p,q}(\mathbb{R}^3)$ *with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} \le \frac{1}{q_1} + \frac{1}{q_2},$$

and there exists some constant $C > 0$ *such that*

$$\|gh\|_{L^{p,q}(\mathbb{R}^3)} \le C\|g\|_{L^{p_1,q_1}(\mathbb{R}^3)}\|h\|_{L^{p_2,q_2}(\mathbb{R}^3)}.$$

We proceed to review the definitions of Morrey space and local Morrey space (see, e.g., [27]). Given $g \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$ and $1 \le p \le \gamma < \infty$, we define

$$\|g\|_{\dot{M}^{p,\gamma}} = \sup_{r>0,\, x_0\in\mathbb{R}^3} r^{\frac{3}{\gamma}} \left(r^{-3} \int_{B_r(x_0)} |g(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}},$$

where $B_r(x_0)$ is the ball with center x_0 and radius r. The set of all measurable functions g in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ such that $\|g\|_{\dot{M}^{p,\gamma}} < \infty$ is called the homogeneous Morrey space with indices p and γ and denoted by $\dot{M}^{p,\gamma}(\mathbb{R}^3)$. For a function g in $\dot{M}^{p,\gamma}(\mathbb{R}^3)$, it can be readily seen that the average of $\|g\|^p_{L^p(B_r(x_0))}$ over the ball $B_r(x_0)$ admits the decay property for large r, which is characterized by the weight $r^{\frac{3}{\gamma}}$ in the definition.

We shall also consider here the local Morrey space, which describes the average decay of a function in a more general setting. Let $\gamma \ge 0$ and $1 \le p < \infty$. For $g \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$, we define

$$\|g\|_{M^p_\gamma} = \sup_{r\ge 1} \left(r^{-\gamma} \int_{B_r} |g(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

The local Morrey space $M^p_\gamma(\mathbb{R}^3)$ is the space of functions g in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$, such that $\|g\|_{M^p_\gamma}$ is finite. It is obvious that the local Morrey space $M^p_\gamma(\mathbb{R}^3)$ is a Banach space and the parameter γ describes the behavior of the quantity $\|g\|_{L^p(B_r)}$ when r is large. Furthermore, if $\gamma_1 \le \gamma_2$, the following continuous embedding holds

$$M^p_{\gamma_1}(\mathbb{R}^3) \hookrightarrow M^p_{\gamma_2}(\mathbb{R}^3).$$

Consequently, for $1 < p \le \gamma_1 < \infty$, by taking the parameter γ_2 such that $3(1 - \frac{p}{\gamma_1}) < \gamma_2$, we have that

$$\dot{M}^{p,\gamma_1}(\mathbb{R}^3) = M^p_{3(1-\frac{p}{\gamma_1})}(\mathbb{R}^3) \hookrightarrow M^p_{\gamma_2}(\mathbb{R}^3).$$

From this point of view, the local Morrey space $M^p_\gamma(\mathbb{R}^3)$ can be regarded as a generalization of the homogeneous Morrey space $\dot{M}^{p,\gamma_1}(\mathbb{R}^3)$.

We also introduce the space $M^p_{\gamma,0}(\mathbb{R}^3)$, which is the set of functions $g \in M^p_\gamma(\mathbb{R}^3)$ satisfying

$$\lim_{r\to\infty} \left(r^{-\gamma} \int_{B_{\frac{3r}{2}} \setminus Br} |g(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} = 0.$$

In the end of this section, we recall the interpolation inequality in L^p spaces (see, e.g., [28]), which will be utilized frequently later.

Lemma 2.2. *Let* $1 \le p_0 < p_\theta < p_1 \le \infty$ *and* $\theta \in (0, 1)$ *satisfy*

$$\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Then, for all $f \in L^{p_0}(\mathbb{R}^3) \cap L^{p_1}(\mathbb{R}^3)$,

$$\|f\|_{L^{p_\theta}(\mathbb{R}^3)} \le \|f\|^\theta_{L^{p_0}(\mathbb{R}^3)} \|f\|^{1-\theta}_{L^{p_1}(\mathbb{R}^3)}.$$

3. A priori estimates

This section is devoted to deriving the Caccioppoli-type inequalities, which will play a crucial role in the proof of our main results.

Proposition 3.1. *Let* (ρ, v, P) *be a smooth solution to* (1.1) *and* (1.2). *Suppose that* $\rho \in L^\infty(\mathbb{R}^3)$ *and there exists* $\Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times 3})$, *such that* $v = \operatorname{div}\Psi$ *and*

$$\sup_{r>1}\left(r^{-2-\frac{\sigma}{3}}\int_{B_r}|\Psi-(\Psi)_{B_r}|^\sigma\,\mathrm{d}x\right)<\infty$$

for some $\sigma \in (3, 6]$, *then*

$$\int_{B_r}|\nabla v|^2\,\mathrm{d}x \le C\left(1+r^{-\frac{1}{3}-\frac{2}{\sigma}}+r^{-1}\,\|v\|_{L^1(B_{\frac{3r}{2}}\setminus B_r)}\right) \tag{3.1}$$

for any $r>1$.

Proof. Let $r \in (1, +\infty)$. Throughout the rest of this paper, C is a positive constant independent of r, which may be different on different lines. The proofs are split into two steps.

Step 1. Local estimate of ∇v.

Select two positive numbers r_1 and r_2 such that

$$r \le r_1 < r_2 \le \frac{3r}{2}, \tag{3.2}$$

and choose a radial smooth function $\varphi \in C_c^\infty(\mathbb{R}^3)$ satisfying

$$\varphi(x)=\begin{cases}1 & \text{in } B_{r_1},\\ 0 & \text{in } \mathbb{R}^3\setminus B_{r_2},\end{cases}$$

$0 \le \varphi \le 1$ and $\|\nabla^k\varphi\|_{L^\infty} \le C(r_2-r_1)^{-k}$ $(k \in \mathbb{N}^+)$.

Taking the L^2-inner product of the second equation in (1.1) with $\varphi^2 v$ and integrating by parts, we have

$$\begin{aligned}
&\mu\int_{B_{r_2}}\varphi^2|\nabla v|^2\,\mathrm{d}x+(\lambda+\mu)\int_{B_{r_2}}\varphi^2|\operatorname{div}v|^2\,\mathrm{d}x\\
=&-\mu\int_{B_{r_2}}\nabla v:\left(v\otimes\nabla(\varphi^2)\right)\mathrm{d}x-(\lambda+\mu)\int_{B_{r_2}}\left(v\cdot\nabla(\varphi^2)\right)\operatorname{div}v\,\mathrm{d}x\\
&-\int_{B_{r_2}}\operatorname{div}(\rho v\otimes v)\cdot\varphi^2 v\,\mathrm{d}x-\int_{B_{r_2}}\varphi^2 v\cdot\nabla P\,\mathrm{d}x\\
:=&\,\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.
\end{aligned} \tag{3.3}$$

We will estimate the four terms $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ one by one.

For $\mathcal{I}_1$, by Hölder's inequality and Young's inequality, we see

$$\begin{aligned}\mathcal{I}_1 &\le 2\mu \int_{B_{r_2}} |\varphi||\nabla v||v||\nabla\varphi| \,\mathrm{d}x \\ &\le \frac{\mu}{8}\int_{B_{r_2}} |\nabla v|^2 \,\mathrm{d}x + \frac{C}{(r_2-r_1)^2}\int_{B_{r_2}\setminus B_{r_1}} |v|^2 \,\mathrm{d}x.\end{aligned} \tag{3.4}$$

Similar to (3.4), we observe

$$\begin{aligned}\mathcal{I}_2 &\le 2(\lambda+\mu) \int_{B_{r_2}} |\varphi||\operatorname{div} v||v||\nabla\varphi| \,\mathrm{d}x \\ &\le \frac{\mu}{8}\int_{B_{r_2}} |\nabla v|^2 \,\mathrm{d}x + \frac{C}{(r_2-r_1)^2}\int_{B_{r_2}\setminus B_{r_1}} |v|^2 \,\mathrm{d}x.\end{aligned} \tag{3.5}$$

For $\mathcal{I}_3$, utilizing the first equation in (1.1) and integrating by parts, we obtain

$$\begin{aligned}\mathcal{I}_3 &= -\int_{B_{r_2}} \rho v\cdot\nabla v\cdot\varphi^2 v \,\mathrm{d}x = \frac{1}{2}\int_{B_{r_2}} |v|^2 \operatorname{div}\left(\varphi^2\rho v\right) \mathrm{d}x \\ &= \int_{B_{r_2}} \varphi\rho|v|^2 v\cdot\nabla\varphi \,\mathrm{d}x,\end{aligned}$$

which implies

$$\mathcal{I}_3 \le \frac{C}{r_2-r_1}\int_{B_{r_2}\setminus B_{r_1}} |v|^3 \,\mathrm{d}x. \tag{3.6}$$

For $\mathcal{I}_4$, we first deduce from (1.2) that

$$\nabla P = \frac{a\gamma}{\gamma-1}\rho\nabla\left(\rho^{\gamma-1}\right),$$

then making use of the integration by parts and utilizing $(1.1)_1$, we find

$$\begin{aligned}\mathcal{I}_4 &= \frac{a\gamma}{\gamma-1}\int_{B_{r_2}} \rho^{\gamma-1}\operatorname{div}\left(\varphi^2\rho v\right)\mathrm{d}x = \frac{2a\gamma}{\gamma-1}\int_{B_{r_2}} \rho^{\gamma}\varphi v\cdot\nabla\varphi \,\mathrm{d}x \\ &\le \frac{C}{r_2-r_1}\int_{B_{r_2}\setminus B_{r_1}} |v| \,\mathrm{d}x.\end{aligned} \tag{3.7}$$

Plugging (3.4)–(3.7) into (3.3), we arrive at

$$\begin{aligned}\int_{B_{r_1}} |\nabla v|^2 \,\mathrm{d}x + \frac{\lambda+\mu}{\mu}\int_{B_{r_1}} |\operatorname{div} v|^2 \,\mathrm{d}x &\le \frac{1}{4}\int_{B_{r_2}} |\nabla v|^2 \,\mathrm{d}x + \frac{C}{(r_2-r_1)^2}\int_{B_{r_2}\setminus B_{r_1}} |v|^2 \,\mathrm{d}x \\ &\quad + \frac{C}{r_2-r_1}\int_{B_{r_2}\setminus B_{r_1}} |v|^3 \,\mathrm{d}x + \frac{C}{r_2-r_1}\int_{B_{r_2}\setminus B_{r_1}} |v| \,\mathrm{d}x.\end{aligned} \tag{3.8}$$

Step 2. Caccioppoli type inequality.

Select a radial smooth function $\zeta \in C_c^\infty(\mathbb{R}^3)$ satisfying

$$\zeta(x) = \begin{cases} 1 & \text{if } x \in B_{r_2}, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus B_{2r_2-r_1}, \end{cases}$$

$0 \le \zeta \le 1$ and $\|\nabla^k \zeta\|_{L^\infty} \le C(r_2 - r_1)^{-k}$ $(k \in \mathbb{N}^+)$.

From (3.2), it can be readily verified that

$$r \le r_1 < r_2 < 2r_2 - r_1 \le 2r. \tag{3.9}$$

According to $v = \operatorname{div} \Psi$, we have

$$\int_{B_{r_2} \backslash B_{r_1}} |v|^2 \,\mathrm{d}x \le \int_{B_{2r_2 - r_1}} |\zeta v|^2 \,\mathrm{d}x = \int_{B_{2r}} \operatorname{div} \left(\Psi - (\Psi)_{B_{2r}}\right) \cdot \zeta^2 v \,\mathrm{d}x. \tag{3.10}$$

Integrating by parts and using Hölder's inequality and (1.6), we can get

$$\begin{aligned}
\int_{B_{2r}} |\zeta v|^2 \,\mathrm{d}x &= -\int_{B_{2r}} \left(\Psi - (\Psi)_{B_{2r}}\right) : \nabla \left(\zeta^2 v\right) \mathrm{d}x \\
&\le C r^{\frac{3}{2} - \frac{3}{\sigma}} \left(\int_{B_{2r}} |\Psi - (\Psi)_{B_{2r}}|^\sigma \,\mathrm{d}x\right)^{\frac{1}{\sigma}} \left(\int_{B_{2r}} \left|\nabla \left(\zeta^2 v\right)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C r^{\frac{11}{6} - \frac{1}{\sigma}} \left(\int_{B_{2r}} |\zeta^2 \nabla v + 2\zeta \nabla \zeta \otimes v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C r^{\frac{11}{6} - \frac{1}{\sigma}} \left(\int_{B_{2r}} |\zeta^2 \nabla v|^2 \,\mathrm{d}x + \int_{B_{2r}} |\zeta \nabla \zeta \otimes v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}$$

In view of Young's inequality, we find

$$\begin{aligned}
\int_{B_{2r}} |\zeta v|^2 \,\mathrm{d}x &\le C r^{\frac{11}{6} - \frac{1}{\sigma}} \left(\int_{B_{2r}} |\zeta^2 \nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \frac{C r^{\frac{11}{6} - \frac{1}{\sigma}}}{r_2 - r_1} \left(\int_{B_{2r}} |\zeta v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C r^{\frac{11}{6} - \frac{1}{\sigma}} \left(\int_{B_{2r_2 - r_1}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \frac{C r^{\frac{11}{3} - \frac{2}{\sigma}}}{(r_2 - r_1)^2} + \frac{1}{2} \int_{B_{2r}} |\zeta v|^2 \,\mathrm{d}x.
\end{aligned}$$

By the fact that ζ is supported in $B_{2r_2 - r_1}$ and (3.9), we have

$$\int_{B_{2r_2 - r_1}} |\zeta v|^2 \,\mathrm{d}x \le C r^{\frac{11}{6} - \frac{1}{\sigma}} \left(\int_{B_{2r_2 - r_1}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \frac{C r^{\frac{11}{3} - \frac{2}{\sigma}}}{(r_2 - r_1)^2}, \tag{3.11}$$

which ensures

$$\begin{aligned}
\frac{C}{(r_2 - r_1)^2} \int_{B_{r_2} \backslash B_{r_1}} |v|^2 \,\mathrm{d}x &\le \frac{C}{(r_2 - r_1)^2} \int_{B_{2r_2 - r_1}} |\zeta v|^2 \,\mathrm{d}x \\
&\le \frac{1}{12} \int_{B_{2r_2 - r_1}} |\nabla v|^2 \,\mathrm{d}x + \frac{C r^{\frac{11}{3} - \frac{2}{\sigma}}}{(r_2 - r_1)^4}.
\end{aligned} \tag{3.12}$$

Considering $\sigma \in (3,\ 6]$, by the integration by parts and Hölder's inequality, it follows that

$$
\begin{aligned}
\int_{B_{r_2}\setminus B_{r_1}} |v|^3\,\mathrm{d}x &\le \int_{B_{2r}} |\zeta^2 v|^3\,\mathrm{d}x \\
&= \int_{B_{2r}} \operatorname{div}\left(\Psi - (\Psi)_{B_{2r}}\right)\cdot |\zeta^3 v|\zeta^3 v\,\mathrm{d}x \\
&= -\int_{B_{2r}} \left(\Psi - (\Psi)_{B_{2r}}\right) : \nabla\left(|\zeta^3 v|\zeta^3 v\right)\mathrm{d}x \\
&\le \left(\int_{B_{2r}} \left|\Psi - (\Psi)_{B_{2r}}\right|^{\sigma}\mathrm{d}x\right)^{\frac{1}{\sigma}} \left(\int_{B_{2r}} \left|\nabla\left(\zeta^3 v\right)\right|^2\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^{\frac{2\sigma}{\sigma-2}}\,\mathrm{d}x\right)^{\frac{\sigma-2}{2\sigma}},
\end{aligned}
$$

which together with (1.6), Lemma 2.2 and Young's inequality implies

$$
\begin{aligned}
\int_{B_{2r}} |\zeta^2 v|^3\,\mathrm{d}x &\le Cr^{\frac{1}{3}+\frac{2}{\sigma}} \left(\int_{B_{2r}} |\zeta^3\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^6\,\mathrm{d}x\right)^{\frac{1}{\sigma}-\frac{1}{6}} \\
&\quad + Cr^{\frac{1}{3}+\frac{2}{\sigma}} \left(\int_{B_{2r}} (|\zeta^2 v||\nabla\zeta|)^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^6\,\mathrm{d}x\right)^{\frac{1}{\sigma}-\frac{1}{6}} \\
&\le \frac{1}{2}\int_{B_{2r}} |\zeta^2 v|^3\,\mathrm{d}x + Cr\left(\int_{B_{2r}} |\zeta^3\nabla v|^2\,\mathrm{d}x\right)^{\frac{3\sigma}{2(\sigma+6)}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^6\,\mathrm{d}x\right)^{\frac{6-\sigma}{2(\sigma+6)}} \\
&\quad + Cr\left(\frac{1}{(r_2-r_1)^2}\int_{B_{2r}} |\zeta^2 v|^2\,\mathrm{d}x\right)^{\frac{3\sigma}{2(\sigma+6)}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^6\,\mathrm{d}x\right)^{\frac{6-\sigma}{2(\sigma+6)}},
\end{aligned}
$$

namely,

$$
\begin{aligned}
\int_{B_{2r}} |\zeta^2 v|^3\,\mathrm{d}x &\le Cr\left(\int_{B_{2r}} |\zeta^3\nabla v|^2\,\mathrm{d}x\right)^{\frac{3\sigma}{2(\sigma+6)}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^6\,\mathrm{d}x\right)^{\frac{6-\sigma}{2(\sigma+6)}} \\
&\quad + Cr\left(\frac{1}{(r_2-r_1)^2}\int_{B_{2r}} |\zeta^2 v|^2\,\mathrm{d}x\right)^{\frac{3\sigma}{2(\sigma+6)}} \left(\int_{\mathbb{R}^3} |\zeta^3 v|^6\,\mathrm{d}x\right)^{\frac{6-\sigma}{2(\sigma+6)}}.
\end{aligned}
\tag{3.13}
$$

Making use of the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ (see, e.g., [29]), one observes

$$
\begin{aligned}
\left(\int_{\mathbb{R}^3} |\zeta^3 v|^6\,\mathrm{d}x\right)^{\frac{6-\sigma}{2(\sigma+6)}} &\le C\left(\int_{\mathbb{R}^3} \left|\nabla\left(\zeta^3 v\right)\right|^2\mathrm{d}x\right)^{\frac{18-3\sigma}{2(\sigma+6)}} \\
&\le C\left(\int_{\mathbb{R}^3} |\zeta^3\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} |\zeta^2\nabla\zeta\otimes v|^2\,\mathrm{d}x\right)^{\frac{18-3\sigma}{2(\sigma+6)}} \\
&\le C\left(\int_{\mathbb{R}^3} |\zeta^3\nabla v|^2\,\mathrm{d}x\right)^{\frac{18-3\sigma}{2(\sigma+6)}} + C\left(\frac{1}{(r_2-r_1)^2}\int_{\mathbb{R}^3} |\zeta^2 v|^2\,\mathrm{d}x\right)^{\frac{18-3\sigma}{2(\sigma+6)}}.
\end{aligned}
\tag{3.14}
$$

Inserting (3.14) into (3.13) leads to

$$\begin{aligned}\int_{B_{2r}} |\zeta^2 v|^3 \,\mathrm{d}x \le{}& Cr\left(\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + Cr\left(\frac{1}{(r_2-r_1)^2}\int_{B_{2r_2-r_1}} |\zeta^2 v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}}\\
&+ Cr\left(\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{3\sigma}{2(\sigma+6)}}\left(\frac{1}{(r_2-r_1)^2}\int_{B_{2r_2-r_1}} |\zeta^2 v|^2 \,\mathrm{d}x\right)^{\frac{18-3\sigma}{2(\sigma+6)}}\\
&+ Cr\left(\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{18-3\sigma}{2(\sigma+6)}}\left(\frac{1}{(r_2-r_1)^2}\int_{B_{2r_2-r_1}} |\zeta^2 v|^2 \,\mathrm{d}x\right)^{\frac{3\sigma}{2(\sigma+6)}}\\
\le{}& Cr\left(\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + Cr\left(\frac{1}{(r_2-r_1)^2}\int_{B_{2r_2-r_1}} |\zeta^2 v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}}.\end{aligned}$$

Noting that $\frac{9}{\sigma+6} < 1$ and utilizing Young's inequality, we then obtain

$$\begin{aligned}\frac{C}{r_2-r_1}\int_{B_{2r_2-r_1}} |\zeta^2 v|^3 \,\mathrm{d}x ={}& \frac{C}{r_2-r_1}\int_{B_{2r}} |\zeta^2 v|^3 \,\mathrm{d}x\\
\le{}& \frac{Cr}{r_2-r_1}\left(\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + \frac{Cr}{r_2-r_1}\left(\frac{1}{(r_2-r_1)^2}\int_{B_{2r_2-r_1}} |\zeta^2 v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}}\\
\le{}& \frac{1}{12}\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{(r_2-r_1)^2}\int_{B_{2r_2-r_1}} |\zeta^2 v|^2 \,\mathrm{d}x + \frac{Cr^{\frac{\sigma+6}{\sigma-3}}}{(r_2-r_1)^{\frac{\sigma+6}{\sigma-3}}},\end{aligned}$$

which along with (3.12) implies

$$\frac{C}{r_2-r_1}\int_{B_{2r_2-r_1}} |\zeta^2 v|^3 \,\mathrm{d}x \le \frac{1}{6}\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x + \frac{Cr^{\frac{11}{3}-\frac{2}{\sigma}}}{(r_2-r_1)^4} + \frac{Cr^{\frac{\sigma+6}{\sigma-3}}}{(r_2-r_1)^{\frac{\sigma+6}{\sigma-3}}}.$$

Therefore,

$$\begin{aligned}\frac{C}{r_2-r_1}\int_{B_{r_2}\setminus B_{r_1}} |v|^3 \,\mathrm{d}x \le{}& \frac{C}{r_2-r_1}\int_{B_{2r_2-r_1}} |\zeta^2 v|^3 \,\mathrm{d}x\\
\le{}& \frac{1}{6}\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x + \frac{Cr^{\frac{11}{3}-\frac{2}{\sigma}}}{(r_2-r_1)^4} + \frac{Cr^{\frac{\sigma+6}{\sigma-3}}}{(r_2-r_1)^{\frac{\sigma+6}{\sigma-3}}}.\end{aligned} \tag{3.15}$$

Since $r_2 \le \frac{3r}{2}$, plugging (3.12) and (3.15) into (3.8), one sees

$$\begin{aligned}\int_{B_{r_1}} |\nabla v|^2 \,\mathrm{d}x \le{}& \frac{1}{2}\int_{B_{2r_2-r_1}} |\nabla v|^2 \,\mathrm{d}x\\
&+ C\left(\frac{r^{\frac{11}{3}-\frac{2}{\sigma}}}{(r_2-r_1)^4} + \frac{r^{\frac{\sigma+6}{\sigma-3}}}{(r_2-r_1)^{\frac{\sigma+6}{\sigma-3}}} + \frac{1}{r_2-r_1}\int_{B_{\frac{3r}{2}}\setminus B_r} |v| \,\mathrm{d}x\right).\end{aligned} \tag{3.16}$$

From (3.9) and (3.16), we can deduce by the standard iteration argument (see, e.g., [30, Lemma 3.1, page 161]) that

$$\int_{B_r} |\nabla v|^2 \,\mathrm{d}x \le C\left(1 + r^{-\frac{1}{3}-\frac{2}{\sigma}} + \frac{1}{r}\int_{B_{\frac{3r}{2}}\setminus B_r} |v| \,\mathrm{d}x\right),$$

which is consistent with (3.1).

4. Proof of the main theorems

In this section, we will utilize the Caccioppoli-type inequalities established in Section 3 to prove Theorems 1.1 and 1.2. We begin with some estimates in the framework of Lebesgue spaces.

Proposition 4.1. *Let $v \in C^\infty(\mathbb{R}^3)$ satisfy $\nabla v \in L^2(\mathbb{R}^3)$. Suppose that there is $\Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times 3})$, such that $v = \operatorname{div}\Psi$ and*

$$\sup_{r>1}\left(r^{-2-\frac{\sigma}{3}}\int_{B_r}|\Psi-(\Psi)_{B_r}|^\sigma\,\mathrm{d}x\right)<\infty$$

with $\sigma \in (3, 6]$, then we have

$$\frac{1}{r^2}\int_{B_{\frac{3r}{2}}\setminus B_r}|v|^2\,\mathrm{d}x \to 0 \quad as\ \ r\to+\infty$$

and

$$\frac{1}{r}\int_{B_{\frac{3r}{2}}\setminus B_r}|v|^3\,\mathrm{d}x \to 0 \quad as\ \ r\to+\infty.$$

Proof. Let $r>1$, then choose a radial smooth function $\chi\in C_c^\infty(\mathbb{R}^3)$ satisfying

$$\chi(x)=\begin{cases}1 & \text{in } B_{\frac{3r}{2}}\setminus B_r,\\ 0 & \text{in } B_{\frac{r}{2}}\cup\left(\mathbb{R}^3\setminus B_{2r}\right),\end{cases}$$

$0\le\chi\le1$ and $\|\nabla^k\chi\|_{L^\infty}\le Cr^{-k}$ $(k\in\mathbb{N}^+)$.

Making use of the assumption $v=\operatorname{div}\Psi$, Hölder's inequality and Young's inequality, integrating by parts and repeating the previous estimation process of (3.10) and (3.11) in Section 3, we can obtain

$$\int_{B_{2r}}|v\chi|^2\,\mathrm{d}x \le Cr^{\frac{11}{6}-\frac{1}{\sigma}}\left(\int_{B_{2r}\setminus B_{\frac{r}{2}}}|\nabla v|^2\,\mathrm{d}x\right)^{\frac12}+Cr^{\frac53-\frac{2}{\sigma}}. \tag{4.1}$$

Therefore,

$$\begin{aligned}\frac{1}{r^2}\int_{B_{\frac{3r}{2}}\setminus B_r}|v|^2\,\mathrm{d}x &\le \frac{1}{r^2}\int_{B_{2r}}|v\chi|^2\,\mathrm{d}x\\ &\le Cr^{-\frac16-\frac1\sigma}\left(\int_{B_{2r}\setminus B_{\frac r2}}|\nabla v|^2\,\mathrm{d}x\right)^{\frac12}+Cr^{-\frac13-\frac2\sigma}\\ &\le \int_{B_{2r}\setminus B_{\frac r2}}|\nabla v|^2\,\mathrm{d}x+Cr^{-\frac13-\frac2\sigma},\end{aligned}$$

which together with the assumption $\nabla v\in L^2(\mathbb{R}^3)$ ensures

$$\frac{1}{r^2}\int_{B_{\frac{3r}{2}}\setminus B_r}|v|^2\,\mathrm{d}x \to 0 \quad \text{as}\ \ r\to+\infty.$$

Considering $v = \operatorname{div}\Psi$ and integrating by parts, we derive that

$$
\begin{aligned}
\int_{B_{\frac{3r}{2}}\setminus B_r} |v|^3 \,\mathrm{d}x &\le \int_{B_{2r}\setminus B_{\frac{r}{2}}} |v\chi^3|^3 \,\mathrm{d}x \\
&= \int_{B_{2r}\setminus B_{\frac{r}{2}}} \chi^9 |v| v \cdot \operatorname{div}\left(\Psi - (\Psi)_{B_{2r}}\right) \mathrm{d}x \\
&\le \frac{C}{r}\int_{B_{2r}\setminus B_{\frac{r}{2}}} \chi^2 \left|\Psi - (\Psi)_{B_{2r}}\right| \left|v\chi^3\right|^2 \mathrm{d}x + \int_{B_{2r}\setminus B_{\frac{r}{2}}} \chi^2 \left|\Psi - (\Psi)_{B_{2r}}\right| \left|\nabla v \chi\right| \left|v\chi^3\right| \mathrm{d}x \\
&:= \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}
\tag{4.2}
$$

In what follows, we estimate $\mathcal{J}_1$ and $\mathcal{J}_2$ separately.

For $\mathcal{J}_1$, by the assumption (1.6) and Hölder's inequality, we get

$$
\begin{aligned}
\mathcal{J}_1 &= \frac{C}{r}\int_{B_{2r}\setminus B_{\frac{r}{2}}} \chi^2 \left|\Psi - (\Psi)_{B_{2r}}\right| \left|v\chi^3\right|^2 \mathrm{d}x \\
&\le \frac{C}{r}\left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} \left|\Psi - (\Psi)_{B_{2r}}\right|^\sigma \mathrm{d}x\right)^{\frac{1}{\sigma}} \left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} |v\chi^3|^3 \,\mathrm{d}x\right)^{\frac{2}{3}} r^{1-\frac{3}{\sigma}} \\
&\le C r^{\frac{1}{3}-\frac{1}{\sigma}} \left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} |v\chi^3|^3 \,\mathrm{d}x\right)^{\frac{2}{3}},
\end{aligned}
$$

which together with Young's inequality yields

$$
\mathcal{J}_1 \le C r^{1-\frac{3}{\sigma}} + \frac{1}{4}\int_{B_{2r}\setminus B_{\frac{r}{2}}} |v\chi^3|^3 \,\mathrm{d}x. \tag{4.3}
$$

For $\mathcal{J}_2$, by the assumption (1.6) and by applying the Hölder inequality, we get

$$
\begin{aligned}
\mathcal{J}_2 &= \int_{B_{2r}\setminus B_{\frac{r}{2}}} \chi^2 \left|\Psi - (\Psi)_{B_{2r}}\right| \left|\nabla v \chi\right| \left|v\chi^3\right| \mathrm{d}x \\
&\le \left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} \left|\Psi - (\Psi)_{B_{2r}}\right|^\sigma \mathrm{d}x\right)^{\frac{1}{\sigma}} \left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |v\chi^3|^{\frac{2\sigma}{\sigma-2}} \,\mathrm{d}x\right)^{\frac{\sigma-2}{2\sigma}} \\
&\le C r^{\frac{1}{3}+\frac{2}{\sigma}} \left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |v\chi^3|^{\frac{2\sigma}{\sigma-2}} \,\mathrm{d}x\right)^{\frac{\sigma-2}{2\sigma}}.
\end{aligned}
\tag{4.4}
$$

By Lemma 2.2 and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we can see

$$
\begin{aligned}
\left(\int_{\mathbb{R}^3} |v\chi^3|^{\frac{2\sigma}{\sigma-2}}\,\mathrm{d}x\right)^{\frac{\sigma-2}{2\sigma}} &\le \left(\int_{\mathbb{R}^3} |v\chi^3|^6\,\mathrm{d}x\right)^{\frac{1}{\sigma}-\frac{1}{6}} \left(\int_{\mathbb{R}^3} |v\chi^3|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \\
&\le C\left(\left(\int_{\mathbb{R}^3} |\nabla v\chi^3|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + \frac{1}{r}\left(\int_{\mathbb{R}^3} |v\chi^2|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\right)^{\frac{6}{\sigma}-1} \left(\int_{\mathbb{R}^3} |v\chi^3|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \\
&\le C\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{3}{\sigma}-\frac{1}{2}} \left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \\
&\quad + Cr^{1-\frac{6}{\sigma}}\left(\int_{\mathbb{R}^3} |v\chi|^2\,\mathrm{d}x\right)^{\frac{3}{\sigma}-\frac{1}{2}} \left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}}.
\end{aligned}
\tag{4.5}
$$

Substituting (4.5) and (4.1) into (4.4) and by using Young's inequality, we observe

$$
\begin{aligned}
\mathcal{J}_2 &\le Cr^{\frac{1}{3}+\frac{2}{\sigma}}\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{3}{\sigma}} \left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \\
&\quad + Cr^{\frac{4}{3}-\frac{4}{\sigma}}\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \left(r^{\frac{11}{6}-\frac{1}{\sigma}}\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + r^{\frac{5}{3}-\frac{2}{\sigma}}\right)^{\frac{3}{\sigma}-\frac{1}{2}} \\
&\le \frac{1}{8}\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x + Cr\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} \\
&\quad + Cr^{\frac{1}{3}+\frac{2}{\sigma}}\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x\right)^{\frac{2}{3}-\frac{2}{\sigma}} \left(r^{-\frac{1}{6}-\frac{1}{\sigma}}\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + r^{-\frac{1}{3}-\frac{2}{\sigma}}\right)^{\frac{3}{\sigma}-\frac{1}{2}} \\
&\le \frac{1}{4}\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x + Cr\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} \\
&\quad + Cr\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{3\sigma}{2(\sigma+6)}} \left(r^{-\frac{1}{6}-\frac{1}{\sigma}}\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + r^{-\frac{1}{3}-\frac{2}{\sigma}}\right)^{\frac{3(6-\sigma)}{2(\sigma+6)}},
\end{aligned}
$$

which follows from Young's inequality that

$$
\begin{aligned}
\mathcal{J}_2 &\le \frac{1}{4}\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x + Cr\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + Cr\left(r^{-\frac{1}{6}-\frac{1}{\sigma}}\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + r^{-\frac{1}{3}-\frac{2}{\sigma}}\right)^{\frac{9}{\sigma+6}} \\
&\le \frac{1}{4}\int_{B_{2r}\backslash B_{\frac{r}{2}}} |v\chi^3|^3\,\mathrm{d}x + Cr\left(\int_{B_{2r}\backslash B_{\frac{r}{2}}} |\nabla v|^2\,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + Cr^{1-\frac{3}{\sigma}}.
\end{aligned}
\tag{4.6}
$$

Plugging (4.3) and (4.6) into (4.2) yields

$$\int_{B_{2r}\setminus B_{\frac{r}{2}}} |v\chi^3|^3 \,\mathrm{d}x \le Cr\left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + Cr^{1-\frac{3}{\sigma}},$$

which implies

$$\int_{B_{\frac{3r}{2}}\setminus B_r} |v|^3 \,\mathrm{d}x \le Cr\left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + Cr^{1-\frac{3}{\sigma}}.$$

Since $\nabla v \in L^2(\mathbb{R}^3)$, we see

$$\frac{1}{r}\int_{B_{\frac{3r}{2}}\setminus B_r} |v|^3 \,\mathrm{d}x \le C\left(\int_{B_{2r}\setminus B_{\frac{r}{2}}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{9}{\sigma+6}} + Cr^{-\frac{3}{\sigma}} \to 0 \quad \text{as } r \to +\infty.$$

The proof of Proposition 4.1 is completed.

With Propositions 3.1 and 4.1 in hand, we are now ready to prove Theorems 1.1 and 1.2. For simplicity, we adopt the following definition:

$$\mathcal{M}_{\gamma,p}v(r) = \left(r^{-\gamma}\int_{B_{\frac{3r}{2}}\setminus B_r} |v(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{4.7}$$

Proof of Theorem 1.1. Let $r > 1$. We first show that $\nabla v \in L^2(\mathbb{R}^3)$ by Proposition 3.1. By virtue of Lemma 2.1, we have

$$\frac{1}{r}\int_{B_{\frac{3r}{2}}\setminus B_r} |v| \,\mathrm{d}x \le \frac{C}{r}\|v\|_{L^{p,q}(B_{\frac{3r}{2}}\setminus B_r)}\|1\|_{L^{\frac{p}{p-1},\frac{q}{q-1}}(B_{\frac{3r}{2}})} \le Cr^{2-\frac{3}{p}}\|v\|_{L^{p,q}(B_{\frac{3r}{2}}\setminus B_r)}. \tag{4.8}$$

Substituting (4.8) into (3.1) leads to

$$\int_{B_r} |\nabla v|^2 \,\mathrm{d}x \le C\left(1 + r^{-\frac{1}{3}-\frac{2}{\sigma}} + r^{2-\frac{3}{p}}\|v\|_{L^{p,q}(B_{\frac{3r}{2}}\setminus B_r)}\right). \tag{4.9}$$

Since $v \in L^{p,q}(\mathbb{R}^3)$ with $(p, q) \in [1, \frac{3}{2})\times[1, +\infty]$ or $p = q = \frac{3}{2}$, letting $r \to +\infty$ in (4.9) and making use of Fatou's lemma, we see

$$\int_{\mathbb{R}^3} |\nabla v|^2 \,\mathrm{d}x \le \liminf_{r\to\infty}\int_{B_r} |\nabla v|^2 \,\mathrm{d}x \le C. \tag{4.10}$$

We next prove the vanishing property of $\|\nabla v\|_{L^2(\mathbb{R}^3)}$. To this end, by the standard iteration argument to (3.8) , we observe

$$\int_{B_r} |\nabla v|^2 \,\mathrm{d}x \le \frac{C}{r^2}\int_{B_{\frac{3r}{2}}\setminus B_r} |v|^2 \,\mathrm{d}x + \frac{C}{r}\int_{B_{\frac{3r}{2}}\setminus B_r} |v|^3 \,\mathrm{d}x + \frac{C}{r}\int_{B_{\frac{3r}{2}}\setminus B_r} |v| \,\mathrm{d}x. \tag{4.11}$$

Inserting (4.8) into (4.11), we arrive at

$$\int_{B_r} |\nabla v|^2 \,\mathrm{d}x \le \frac{C}{r^2} \int_{B_{\frac{3r}{2}} \setminus B_r} |v|^2 \,\mathrm{d}x + \frac{C}{r} \int_{B_{\frac{3r}{2}} \setminus B_r} |v|^3 \,\mathrm{d}x + C r^{2-\frac{3}{p}} \|v\|_{L^{p,q}(B_{\frac{3r}{2}} \setminus B_r)}. \tag{4.12}$$

Since $v \in L^{p,q}(\mathbb{R}^3)$ with $(p, q) \in [1, \frac{3}{2}) \times [1, +\infty]$ or $p = q = \frac{3}{2}$, we have

$$r^{2-\frac{3}{p}} \|v\|_{L^{p,q}(B_{\frac{3r}{2}} \setminus B_r)} \to 0 \quad \text{as } r \to +\infty,$$

which together with Proposition 4.1 and (4.12) yields

$$\lim_{r\to\infty} \int_{B_r} |\nabla v|^2 \,\mathrm{d}x = 0.$$

By virtue of (4.10) and the Lebesgue dominated convergence theorem, one can see

$$\int_{\mathbb{R}^3} |\nabla v|^2 \,\mathrm{d}x = 0.$$

It follows from the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ that

$$\|v\|_{L^6(\mathbb{R}^3)} \le C \|\nabla v\|_{L^2(\mathbb{R}^3)} = 0.$$

Hence, $v = 0$ in $\mathbb{R}^3$.

Furthermore, combining (1.2) and $(1.1)_2$, we conclude that ρ is a constant in $\mathbb{R}^3$. The proof of Theorem 1.1 is then finished.

We proceed to give the proof of Theorem 1.2.

Proof of Theorem 1.2. (a) Since $v \in \dot{M}^{p,\gamma}(\mathbb{R}^3)$ for $1 \le p < \gamma < \frac{3}{2}$, by virtue of Hölder's inequality, we derive that

$$\frac{1}{r} \int_{B_{\frac{3r}{2}} \setminus B_r} |v| \,\mathrm{d}x \le C r^{2-\frac{3}{\gamma}} r^{\frac{3}{\gamma}} \left(r^{-3} \int_{B_{\frac{3r}{2}} \setminus B_r} |v|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \le C r^{2-\frac{3}{\gamma}} \|v\|_{\dot{M}^{p,\gamma}}. \tag{4.13}$$

Substituting (4.13) into (3.1), we find

$$\int_{B_r} |\nabla v|^2 \,\mathrm{d}x \le C \left(1 + r^{-\frac{1}{3} - \frac{2}{\sigma}} + r^{2-\frac{3}{\gamma}} \|v\|_{\dot{M}^{p,\gamma}(\mathbb{R}^3)} \right).$$

By $v \in \dot{M}^{p,\gamma}(\mathbb{R}^3)$, letting $r \to \infty$ then yields that

$$\int_{\mathbb{R}^3} |\nabla v|^2 \,\mathrm{d}x \le C. \tag{4.14}$$

Conducting the standard iteration argument on (3.8) and utilizing (4.13), we can obtain that

$$\begin{aligned} \int_{B_r} |\nabla v|^2 \,\mathrm{d}x &\le \frac{C}{r^2} \int_{B_{\frac{3r}{2}} \setminus B_r} |v|^2 \,\mathrm{d}x + \frac{C}{r} \int_{B_{\frac{3r}{2}} \setminus B_r} |v|^3 \,\mathrm{d}x + \frac{C}{r} \int_{B_{\frac{3r}{2}} \setminus B_r} |v| \,\mathrm{d}x \\ &\le \frac{C}{r^2} \int_{B_{\frac{3r}{2}} \setminus B_r} |v|^2 \,\mathrm{d}x + \frac{C}{r} \int_{B_{\frac{3r}{2}} \setminus B_r} |v|^3 \,\mathrm{d}x + C r^{2-\frac{3}{\gamma}} \|v\|_{\dot{M}^{p,\gamma}(\mathbb{R}^3)}. \end{aligned} \tag{4.15}$$

By (4.14) and the assumption (1.7), we can deduce from Proposition 4.1 that

$$\lim_{r\to\infty}\frac{1}{r^2}\int_{B_{\frac{3r}{2}}\setminus B_r}|v|^2\,\mathrm{d}x=0 \tag{4.16}$$

and

$$\lim_{r\to\infty}\frac{1}{r}\int_{B_{\frac{3r}{2}}\setminus B_r}|v|^3\,\mathrm{d}x=0. \tag{4.17}$$

Since $v\in\dot{M}^{p,\gamma}(\mathbb{R}^3)$ with $1\le p<\gamma<\frac{3}{2}$, we have that

$$\lim_{r\to\infty}r^{2-\frac{3}{\gamma}}\|v\|_{\dot{M}^{p,\gamma}(\mathbb{R}^3)}=0. \tag{4.18}$$

Plugging (4.16)–(4.18) into (4.15) and using the Lebesgue dominated convergence theorem lead to

$$\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x=0,$$

which together with the Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ implies that v vanishes and ρ is a constant in $\mathbb{R}^3$.

(b) and (c). In the case of $0\leqslant\gamma<1\leqslant p<\frac{3}{2}$, by Hölder's inequality and the definition $\mathcal{M}_{\gamma,p}v(r)$ given in (4.7), we can readily see that

$$\frac{1}{r}\int_{B_{\frac{3r}{2}}\setminus B_r}|v|\,\mathrm{d}x\le Cr^{2-\frac{3}{p}+\frac{\gamma}{p}}\mathcal{M}_{\gamma,p}v(r).$$

It follows from (3.1) that

$$\int_{B_r}|\nabla v|^2\,\mathrm{d}x\le C\left(1+r^{-\frac{1}{3}-\frac{2}{\sigma}}+r^{2-\frac{3}{p}+\frac{\gamma}{p}}\mathcal{M}_{\gamma,p}v(r)\right). \tag{4.19}$$

In the case (b), i.e., when $v\in M^p_\gamma(\mathbb{R}^3)$ with $2p+\gamma<3$, according to the definition of $\|\cdot\|_{M^p_\gamma}$, we obtain

$$r^{2-\frac{3}{p}+\frac{\gamma}{p}}\mathcal{M}_{\gamma,p}v(r)\le r^{2-\frac{3}{p}+\frac{\gamma}{p}}\|v\|_{M^p_\gamma}\to 0\qquad(r\to\infty). \tag{4.20}$$

In the case (c), i.e., when $v\in M^p_{\gamma,0}(\mathbb{R}^3)$ with $2p+\gamma=3$, from the definition of the space $M^p_{\gamma,0}(\mathbb{R}^3)$, we have

$$r^{2-\frac{3}{p}+\frac{\gamma}{p}}\mathcal{M}_{\gamma,p}v(r)\ \to 0\qquad(r\to\infty). \tag{4.21}$$

Substituting (4.20) or (4.21) into (4.19) separately, we see

$$\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x\le C. \tag{4.22}$$

Combining (3.8), (4.20)–(4.22), the assumption (1.7) and Proposition 4.1 and repeating the previous estimation process of (4.15)–(4.18), we can also find

$$\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x=0,$$

which implies the desired conclusion. The proof of Theorem 1.2 is finished.

5. Conclusions

This paper is concerned with the Liouville-type theorem for the stationary barotropic compressible Navier–Stokes equations in $\mathbb{R}^3$. We proved that smooth solutions must be trivial under the L^∞ boundedness of the density and some new assumptions on the velocity field. This work contains two main results. The first one allows the velocity field to be in the appropriate Lorentz space $L^{p,q}(\mathbb{R}^3)$ and gives a delicate condition related to the growth rate of the local mean oscillation of a "potential" Ψ with the velocity $v = \operatorname{div}\Psi$. The subsequent corollary is a weaker result phrased in terms of the BMO^{-1} space. The second main result addresses the case of allowing the velocity being in the (local) Morrey space. Our work improves the result of Li-Niu [14] and also extends the result of Liu [16] to the framework of Morrey spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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