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Research article

An implicit fully discrete compact finite difference scheme for time fractional diffusion-wave equation

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Abstract: In this paper, an implicit compact finite difference (CFD) scheme was constructed to get the numerical solution for time fractional diffusion-wave equation (TFDWE), in which the time fractional derivative was denoted by Caputo-Fabrizio (C-F) sense. We proved that the full discrete scheme is unconditionally stable. We also proved that the rate of convergence in time is near to $O(\tau^2)$ and the rate of convergence in space is near to $O(h^4)$. Test problem was considered for regular domain with uniform points to validate the efficiency and accuracy of the method. The numerical results can support the theoretical claims.

Keywords: time fractional diffusion-wave equation (TFDWE); compact finite difference; caputo-fabrizio derivative; unconditional stability; convergence

1. Introduction

Fractional calculus has gained considerable popularity and importance during the past decades, due mainly to its demonstrated applications in many fields, such as physics, engineering etc [1–3]. Due to this fact, many authors have proposed a mass of numerical methods to solve fractional partial differential equations (FPDEs) [4, 5], for instance, finite difference methods (FDMs) [6, 7], spectral methods [8, 9], finite element methods [10, 11], radial basis functions (RBF) methods [12, 13] and so on.

Next, we will introduce some research results on FPDEs. Du et al. considered a numerical scheme with high accuracy for the fractional diffusion-wave equation [14]. Ren and Sun used the finite difference method to obtain numerical schemes of time fractional diffusion-wave equation [15]. Huang et al. considered the linearized numerical schemes for nonlinear time fractional wave equations [16]. Liang et al. studied a high order difference scheme for time fractional telegraph equation in the sense of Caputo [17]. Hosseini et al. applied the RBFs to solve a time fractional telegraph equation defined by Caputo sense [18]. Modanli and Akgul constructed a difference scheme for the fractional telegraph equation [19]. Abdi et al. proposed a compact finite difference (CFD) and rotated point method for 2D time fractional telegraph equation [20]. Nikan investigated the approximate solution of the nonlinear time fractional telegraph equation [21].

There are many results for the fractional diffusion-wave equation, in which fractional derivatives are used with singular kernel. For example, Ali et al. proposed a new numerical approach method for the fractional diffusion-wave equation with fractional derivative in the sense of Riemann-Liouville [22]. Yu constructed a high-order compact finite difference scheme for time fractional mixed diffusion and diffusion-wave equation [23]. Li et al. discussed a fast element-free Galerkin method for the fractional diffusion-wave equation, in which the time fractional derivatives are defined in the Caputo sense [24]. Bhardwaj and Kumar proposed a meshless method for time fractional nonlinear mixed diffusion and diffusion-wave equation [25]. Jiang and Wu studied a time-space fractional diffusion wave equation by fractional Landweber method [26]. Ates and Yıldırım obtained the approximate analytical solution for time-fractional diffusion-wave equations [27].

In order to eliminate the singular kernel in the fractional derivative, Caputo and Fabrizio proposed a new fractional derivative called the Caputo-Fabrizio (C-F) derivative [28]. C-F derivative is a promising differentiation operator and has been widely used to model several problems arising in different fields of science and engineering such as biology, physics, fluid dynamics and control systems [29–31]. The results of fractional diffusion and the diffusion-wave equation with C-F derivative can be found in [32–35].

The traditional finite difference method cannot obtain high-order numerical approximation. In order to obtain higher precision numerical approximation, the compact finite difference method has been studied by many scholars. For example, Gao and Sun considered a compact finite difference scheme with the purpose of solving the fractional sub-diffusion equations for the heat equation in the condition of the Neumann boundary [36, 37]. In order to solve groundwater pollution phenomenon, Li et al. constructed a 2D mathematical model, which has the fourth order accurate [38]. Liao et al. discussed a compact algorithm to analyze nonlinear reaction-diffusion equations [39]. Liao and Sun proposed an implicit scheme to solve the multidimensional parabolic equations [40].

In this paper, the time fractional diffusion-wave equation is discussed in the sense of the C-F derivative. The main purpose of this article is to verify the effectiveness of the compact finite difference method for the time fractional diffusion-wave equation with C-F derivative. An implicit compact finite difference scheme is constructed to obtain the numerical solution for the following equation:

$$\begin{cases} {}_0^{CF}D_t^{\beta}u(x,t) + \dfrac{\partial^2 u(x,t)}{\partial t^2} = a^2\dfrac{\partial^2 u(x,t)}{\partial x^2} + F(x,t), & 0 < x < L,\ \ 0 < t \le T, \\ u(x,0) = \Psi(x),\ \ u_t(x,0) = \psi(x), & 0 \le x \le L, \\ u(0,t) = g_1(x),\ \ u(L,t) = g_2(x), & 0 \le t \le T, \end{cases} \tag{1.1}$$

where $0 < \beta < 1$, $a > 0$, $F(x,t)$ is a known function, $\Psi(x)$, $\psi(x)$, $g_1(x)$ and $g_2(x)$ are given continuous functions and ${}_0^{CF}D_t^{\beta}u(x,t)$ is the C-F fractional derivative, whose definition is as follows

$${}_0^{CF}D_t^{\beta}u(x,t) = \frac{1}{1-\beta}\int_0^t u'(s)e^{-\frac{\beta}{1-\beta}(t-s)}ds = \frac{1}{1-\beta}\int_0^t u'(s)e^{-\eta(t-s)}ds, \quad \eta = \frac{\beta}{1-\beta}.$$

Theoretical analysis and numerical results show that the compact finite difference method is effective for solving the time fractional diffusion-wave equation with C-F derivative.

The remaining part of this paper is organized as follows: In Section 2, we introduce some basic knowledge and present a CFD scheme for Eq (1.1). In Section 3, we use the mathematical induction and energy inequality method to analyze the unconditionally stable and convergence of the CFD scheme. In Section 4, we provide a detailed exposition of the theoretical aspect by the numerical experiment. Some conclusions are given in Section 5.

2. Construction of the CFD scheme

In this section, some basic knowledge is introduced and the CFD scheme for Eq (1.1) is presented. Due to the arbitrariness of C, we allow the value of C to be different at different locations.

For any positive integers M and N, let $x_j = jh$ ($j = 0, 1, 2, \cdots, M$) with $h = L/M$ and $t_n = n\tau$ ($n = 0, 1, 2, \cdots, N$) with $\tau = T/N$, where h and τ are space and time step size, respectively. Define the grid function $u_j^n = u(x_j, t_n)$ and $F_j^n = F(x_j, t_n)$. Some notations, inner products, norms and lemmas are as follows.

$$\delta_x^2 u_j^n = \frac{1}{h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \qquad \langle u^n, v^n\rangle = h\sum_{j=1}^{M-1} u_j^n v_j^n,$$

$$\|u^n\| = \sqrt{\langle u^n, u^n\rangle}, \qquad \|\delta_x^2 u^n\| = \sqrt{\langle \delta_x^2 u^n, \delta_x^2 u^n\rangle},$$

$$\|u\|_{\mathcal{H}} = \sqrt{(u, u)_{\mathcal{H}}}, \qquad (u, v)_{\mathcal{H}} = \langle \delta_x u, \delta_x v\rangle - \frac{h^2}{12}\langle \delta_x^2 u, \delta_x^2 v\rangle.$$

Denote $V_h = \{v | v = (v_0, v_1, \cdots, v_M),\ v_0 = v_M = 0\}$ as the grid function space on $\Omega_h = \{x_j | 0 \le j \le M\}$. Define the compact finite difference operator as

$$\mathcal{H}u_j^n = \begin{cases} \frac{1}{12}(u_{j-1}^n + 10u_j^n + u_{j+1}^n) = (I + \frac{h^2}{12}\delta_x^2)u_j^n, & 1 \le j \le M-1, \\ u_j^n, & j = 0, M. \end{cases}$$

Lemma 2.1. *([41]) Suppose $g(x) \in C^6[0, L]$ and $x_i = ih$, $0 \le i \le M$, then*

$$\frac{1}{12}[g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})] - \frac{1}{h^2}[g(x_{i+1}) - 2g(x_i) + g(x_{i-1})] = \frac{h^4}{240}g^{(6)}(\xi_i),$$

where $\xi_i \in (x_{i-1}, x_{i+1})$, $1 \le i \le M-1$.

Lemma 2.2. *([30]) Assume $0 < \beta < 1$, let $\eta = \frac{\beta}{1-\beta}$. Let $u(t)$ be a continuous differentiable function for $t \ge 0$, then*

$${}_0^{CF}D_t^\beta u(t_n) = \frac{1}{1-\beta}\sum_{i=1}^{n}\frac{u(t_k) - u(t_{k-1})}{\eta\tau}e^{-\eta(n-i)\tau}(1 - e^{-\eta\tau}) + O(\tau^2).$$

For a given discrete grid point (x_j, t_n), by Eq (1.1), we have

$${}_0^{CF}D_t^\beta u(x_j, t_n) + \frac{\partial^2 u(x_j, t_n)}{\partial t^2} = \frac{a^2}{2}\frac{\partial^2 u(x_j, t_{n+1})}{\partial x^2} + \frac{a^2}{2}\frac{\partial^2 u(x_j, t_{n-1})}{\partial x^2} + F(x_j, t_n). \tag{2.1}$$

Applying compact finite difference operator $\mathcal{H}$ to Eq (2.1), we have

$$\mathcal{H}{}_0^{CF}D_t^\beta u(x_j, t_n) + \mathcal{H}\frac{\partial^2 u(x_j, t_n)}{\partial t^2} = \frac{a^2}{2}\mathcal{H}\frac{\partial^2 u(x_j, t_{n+1})}{\partial x^2} + \frac{a^2}{2}\mathcal{H}\frac{\partial^2 u(x_j, t_{n-1})}{\partial x^2} + \mathcal{H}F(x_j, t_n). \tag{2.2}$$

First, we will use Lemma 2.2 to discretize C-F derivative, then we have

$$\begin{aligned}
&\mathcal{H}_0^{CF}D_t^{\beta}u(x_j,t_n)=\frac{(1-e^{-\eta\tau})}{(1-\beta)\eta\tau}\sum_{i=1}^{n}(\mathcal{H}u_j^i-\mathcal{H}u_j^{i-1})e^{-\eta(n-i)\tau}+O(\tau^2),\\
&\mathcal{H}\frac{\partial^2u(x_j,t_n)}{\partial t^2}=\mathcal{H}\Big(\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\tau^2}\Big)=\frac{1}{\tau^2}(\mathcal{H}u_j^{n+1}-2\mathcal{H}u_j^n+\mathcal{H}u_j^{n-1})+O(\tau^2).
\end{aligned}\tag{2.3}$$

Second, applying Lemma 2.1 to discretize the first term and the second term on the right of Eq (2.2), we have

$$\frac{a^2}{2}\mathcal{H}\frac{\partial^2u(x_j,t_{n+1})}{\partial x^2}+\frac{a^2}{2}\mathcal{H}\frac{\partial^2u(x_j,t_{n-1})}{\partial x^2}=\frac{a^2}{2}\delta_x^2u_j^{n+1}+\frac{a^2}{2}\delta_x^2u_j^{n-1}+O(h^4).\tag{2.4}$$

Substituting Eqs (2.3) and (2.4) into Eq (2.2), we can obtain

$$\begin{aligned}
&\lambda\sum_{i=1}^{n}(\mathcal{H}u_j^i-\mathcal{H}u_j^{i-1})e^{-\eta(n-i)\tau}+\frac{1}{\tau^2}(\mathcal{H}u_j^{n+1}-2\mathcal{H}u_j^n+\mathcal{H}u_j^{n-1})\\
&=\frac{a^2}{2}\delta_x^2u_j^{n+1}+\frac{a^2}{2}\delta_x^2u_j^{n-1}+\mathcal{H}F_j^n+R_j^n,
\end{aligned}\tag{2.5}$$

where $\lambda=\frac{(1-e^{-\eta\tau})}{\beta\tau}$ and $|R_j^n|\le C(\tau^2+h^4)$.

We denote the exact solution and numerical solution with u_j^n and U_j^n, respectively. Now, omitting the error term R_j^n from Eq (2.5), the exact value u_j^n is approximated by U_j^n, which is the numerical approximation, and the resulted CFD scheme is as follows

$$\begin{aligned}
&\lambda\sum_{i=1}^{n}(\mathcal{H}U_j^i-\mathcal{H}U_j^{i-1})e^{-\eta(n-i)\tau}+\frac{1}{\tau^2}(\mathcal{H}U_j^{n+1}-2\mathcal{H}U_j^n+\mathcal{H}U_j^{n-1})\\
&=\frac{a^2}{2}\delta_x^2U_j^{n+1}+\frac{a^2}{2}\delta_x^2U_j^{n-1}+\mathcal{H}F_j^n.
\end{aligned}\tag{2.6}$$

Additionally, we can obtain the discrete initial and boundary condition as follows

$$\begin{aligned}
&U_j^0=\varphi(x_j), && 0\le j\le M,\\
&U_0^n=0,\ \ U_M^n=0, && 1\le n\le N.
\end{aligned}$$

3. Stability analysis and error estimation of the CFD scheme

Before starting the stability and convergence analysis of CFD scheme Eq (2.6), we will introduce some useful lemmas. In the following analysis, omit the subscript j.

Lemma 3.1. *([42]) For arbitrary $u,v\in V_h$, it holds that $\langle\delta_x^2u^n,v^n\rangle=-\langle\delta_xu^n,\delta_xv^n\rangle$.*

By the definition of $\mathcal{H}$ and Lemma 3.1, the following lemma can be obtained

Lemma 3.2. *Suppose $u,v\in V_h$, then $-\langle\delta_x^2u,\mathcal{H}v\rangle=(u,v)_{\mathcal{H}}$.*

Lemma 3.3. *([43]) Let $v \in V_h$, then the following inequality holds*

$$\frac{1}{3}\|v\|^2 \le \|\mathcal{H}v\|^2 \le \|v\|^2.$$

Lemma 3.4. *([44]) Let $u^{-1} = u^0 - \tau\psi$ and $\varepsilon^{-1} = u(x, t_{-1}) - u^{-1}$, then*

$$|\varepsilon^{-1}| \le C\tau^2, \quad \|u^{-1}\| \le \frac{3}{2}\|u^0\| + \frac{1}{2}\|u^1\|.$$

For the full discrete scheme Eq (2.6), we have the following stability result as Theorem 3.5, which shows that the full discrete scheme is unconditionally stable.

Theorem 3.5. *Let U^n be the numerical solution of Eq (2.6). The full discrete scheme Eq (2.6) is unconditionally stable in the sense that for all $\tau > 0$, it holds that*

$$\|U^n\| \le C(\|U^0\| + \max_{0\le s\le n-1} \|F^s\|),$$

where C is a constant.

Proof. By Eq (2.6), we have

$$\begin{aligned} &\mathcal{H}U^{n+1} - \frac{1}{2}a^2\tau^2\delta_x^2 U^{n+1} \\ &= 2\mathcal{H}U^n + \frac{1}{2}a^2\tau^2\delta_x^2 U^{n-1} - \mathcal{H}U^{n-1} - \lambda\tau^2\sum_{i=1}^{n}(\mathcal{H}U^i - \mathcal{H}U^{i-1})e^{-\eta(n-i)\tau} + \tau^2\mathcal{H}F^n. \end{aligned} \tag{3.1}$$

Multiply both sides of Eq (3.1) by $\mathcal{H}U^{n+1}$ simultaneously. Do the inner product and we have

$$\begin{aligned} &\langle\mathcal{H}U^{n+1}, \mathcal{H}U^{n+1}\rangle - \frac{1}{2}a^2\tau^2\langle\delta_x^2 U^{n+1}, \mathcal{H}U^{n+1}\rangle \\ &= 2\langle\mathcal{H}U^n, \mathcal{H}U^{n+1}\rangle + \frac{1}{2}a^2\tau^2\langle\delta_x^2 U^{n-1}, \mathcal{H}U^{n+1}\rangle - \langle\mathcal{H}U^{n-1}, \mathcal{H}U^{n+1}\rangle \\ &- \lambda\tau^2\sum_{i=1}^{n}\Big(\langle\mathcal{H}U^i, \mathcal{H}U^{n+1}\rangle - \langle\mathcal{H}U^{i-1}, \mathcal{H}U^{n+1}\rangle\Big)e^{-\eta(n-i)\tau} + \tau^2\langle\mathcal{H}F^n, \mathcal{H}U^{n+1}\rangle. \end{aligned} \tag{3.2}$$

According to the Cauchy-Schwarz inequality, norm-equivalence theorem and Lemmas 3.2 and 3.3, we have

$$
\begin{aligned}
\|\mathcal{H}U^{n+1}\| \leq & 2\|\mathcal{H}U^{n}\| + \frac{2}{3}Ca^2\tau^2\|\mathcal{H}U^{n-1}\| - \|\mathcal{H}U^{n-1}\| - \lambda\tau^2\sum_{i=1}^{n}\|\mathcal{H}U^{i}\|e^{-\eta(n-i)\tau} \\
& + \lambda\tau^2\sum_{i=1}^{n}\|\mathcal{H}U^{i-1}\|e^{-\eta(n-i)\tau} + \tau^2\|\mathcal{H}F^{n}\| \\
\leq & 2\|\mathcal{H}U^{n}\| + \frac{2}{3}Ca^2\tau^2\|\mathcal{H}U^{n-1}\| + \|\mathcal{H}U^{n-1}\| + \lambda\tau^2\sum_{i=1}^{n}\|\mathcal{H}U^{i}\|e^{-\eta(n-i)\tau} \\
& + \lambda\tau^2\sum_{i=1}^{n}\|\mathcal{H}U^{i-1}\|e^{-\eta(n-i)\tau} + \tau^2\|\mathcal{H}F^{n}\| \\
\leq & C\|\mathcal{H}U^{n}\| + C\|\mathcal{H}U^{n-1}\| + \lambda\tau^2\sum_{i=1}^{n-1}\|\mathcal{H}U^{i}\|e^{-\eta(n-i)\tau} \\
& + \lambda\tau^2\sum_{i=1}^{n-1}\|\mathcal{H}U^{i-1}\|e^{-\eta(n-i)\tau} + C\|\mathcal{H}F^{n}\|.
\end{aligned}
\tag{3.3}
$$

For the Eq (3.3), when $n = 0$, using Lemma 3.4 we have

$$
\begin{aligned}
\|\mathcal{H}U^{1}\| \leq & C\|\mathcal{H}U^{0}\| + \|\mathcal{H}U^{-1}\| + C\|\mathcal{H}F^{1}\| \\
\leq & C\|\mathcal{H}U^{0}\| + \frac{3}{2}\|\mathcal{H}U^{0}\| + \frac{1}{2}\|\mathcal{H}U^{1}\| + C\|\mathcal{H}F^{0}\| \\
\leq & C(\|\mathcal{H}U^{0}\| + \|\mathcal{H}F^{0}\|).
\end{aligned}
\tag{3.4}
$$

Assume that Eq (3.3) holds for $m = 1, 2, \cdots, n-1$, which means that

$$
\|\mathcal{H}U^{m}\| \leq C(\|\mathcal{H}U^{0}\| + \max_{1\leq s\leq m-1}\|\mathcal{H}F^{s}\|). \tag{3.5}
$$

Now, we will prove it holds for $m = n$. Let $A_n = C(\|\mathcal{H}U^{0}\| + \max_{1\leq s\leq n-1}\|\mathcal{H}F^{s}\|)$. According to Eqs (3.3)–(3.5), we can obtain

$$
\begin{aligned}
& \|\mathcal{H}U^{n}\| \\
& \leq C\|\mathcal{H}U^{n-1}\| + \|\mathcal{H}U^{n-2}\| + \lambda\tau^2\sum_{i=1}^{n-1}(\|\mathcal{H}U^{i} + \|\mathcal{H}U^{i-1}\|)\|e^{-\eta(n-i)\tau} + C\|\mathcal{H}f^{n-1}\| \\
& \leq \lambda\tau^2\sum_{i=1}^{n-1}\|\mathcal{H}U^{i-1}\|e^{-\eta(n-i)\tau} + \lambda\tau^2\sum_{i=1}^{n-1}\|\mathcal{H}U^{i}\|e^{-\eta(n-i)\tau} + C(\|\mathcal{H}U^{0}\| + \max_{1\leq s\leq n-1}\|\mathcal{H}F^{s}\|) \\
& \leq \lambda\tau^2\sum_{i=1}^{n-1}e^{-\eta(n-i)\tau}A_{k-1} + \lambda\tau^2\sum_{i=1}^{n-1}e^{-\eta(n-i)\tau}A_{k} + C(\|\mathcal{H}U^{0}\| + \max_{1\leq s\leq n-1}\|\mathcal{H}F^{s}\|) \\
& \leq C(\|\mathcal{H}U^{0}\| + \max_{1\leq s\leq n-2}\|\mathcal{H}F^{s}\|) + C(\|\mathcal{H}U^{0}\| + \max_{1\leq s\leq n-1}\|\mathcal{H}F^{s}\|) \\
& \leq C(\|\mathcal{H}U^{0}\| + \max_{1\leq s\leq n-1}\|\mathcal{H}F^{s}\|).
\end{aligned}
\tag{3.6}
$$

Using Lemma 3.3, we can obtain $\|U\| \le \sqrt{3}\|\mathcal{H}U\|$, $\|\mathcal{H}U\| \le \|U\|$, then

$$\|U^n\| \le C(\|U^0\| + \max_{1\le s\le n-1} \|F^s\|),$$

which completes the proof. □

We denote the exact solution and numerical solution with u^n and U^n, respectively. Let $e^n = u^n - U^n$. Now, we aim at deriving the error estimates for $\|e^n\|$, which is given in the following theorem.

Theorem 3.6. *Assume $u(x,t) \in C^{6,3}_{x,t}$. Let $e^n = u^n - U^n$, and it holds that*

$$\|e^n\| \le C(\tau^2 + h^4),$$

where C is a constant.

Proof. Subtracting Eq (2.6) from Eq (2.5), we have

$$\begin{aligned} &\lambda \sum_{i=1}^{n} (\mathcal{H}e^i - \mathcal{H}e^{i-1}) e^{-\eta(n-i)\tau} + \frac{1}{\tau^2}(\mathcal{H}e^{n+1} - 2\mathcal{H}e^n + \mathcal{H}e^{n-1}) \\ &= \frac{a^2}{2}\delta_x^2 e^{n+1} + \frac{a^2}{2}\delta_x^2 e^{n-1} + R^n. \end{aligned} \tag{3.7}$$

By the similar deduction as Eq (3.1), we can obtain

$$\begin{aligned} &\mathcal{H}e^{n+1} - \frac{1}{2}a^2\tau^2\delta_x^2 e^{n+1} \\ &= 2\mathcal{H}e^n + \frac{1}{2}a^2\tau^2\delta_x^2 e^{n-1} - \mathcal{H}e^{n-1} - \lambda\tau^2 \sum_{i=1}^{n} (\mathcal{H}e^i - \mathcal{H}e^{i-1}) e^{-\eta(n-i)\tau} + \tau^2 R^n. \end{aligned} \tag{3.8}$$

Multiply both sides of Eq (3.8) by $\mathcal{H}e^{n+1}$ simultaneously. Do the inner product and we have

$$\begin{aligned} &\langle \mathcal{H}e^{n+1}, \mathcal{H}e^{n+1}\rangle - \frac{1}{2}a^2\tau^2\langle \delta_x^2 e^{n+1}, \mathcal{H}e^{n+1}\rangle \\ &= 2\langle \mathcal{H}e^n, \mathcal{H}e^{n+1}\rangle + \frac{1}{2}a^2\tau^2\langle \delta_x^2 e^{n-1}, \mathcal{H}e^{n+1}\rangle - \langle \mathcal{H}e^{n-1}, \mathcal{H}e^{n+1}\rangle \\ &- \lambda\tau^2 \sum_{i=1}^{n} \left(\langle \mathcal{H}e^i, \mathcal{H}e^{n+1}\rangle - \langle \mathcal{H}e^{i-1}, \mathcal{H}e^{n+1}\rangle\right) e^{-\eta(n-i)\tau} + \tau^2\langle R^n, \mathcal{H}e^{n+1}\rangle. \end{aligned} \tag{3.9}$$

According to the Cauchy-Schwarz inequality, norm-equivalence theorem and Lemmas 3.2 and 3.3, we can get

$$\begin{aligned} \|\mathcal{H}e^{n+1}\| \le& 2\|\mathcal{H}e^n\| + \frac{2}{3}Ca^2\tau^2\|\mathcal{H}e^{n-1}\| - \|\mathcal{H}e^{n-1}\| - \lambda\tau^2 \sum_{i=1}^{n} \|\mathcal{H}e^i\| e^{-\eta(n-i)\tau} \\ &+ \lambda\tau^2 \sum_{i=1}^{n} \|\mathcal{H}e^{i-1}\| e^{-\eta(n-i)\tau} + \tau^2\|R^n\| \\ \le& C\|\mathcal{H}e^n\| + C\|\mathcal{H}e^{n-1}\| + \lambda\tau^2 \sum_{i=1}^{n-1} \|\mathcal{H}e^i\| e^{-\eta(n-i)\tau} \\ &+ \lambda\tau^2 \sum_{i=1}^{n-1} \|\mathcal{H}e^{i-1}\| e^{-\eta(n-i)\tau} + C\|R^n\|. \end{aligned} \tag{3.10}$$

For the Eq (3.10), when $n = 0$, we have

$$\|\mathcal{H}e^1\| \leq C\|\mathcal{H}e^0\| + C\|\mathcal{H}e^{-1}\| + C\|R^0\|. \tag{3.11}$$

Now, we turn to analyze $\|\mathcal{H}e^{-1}\|$. Using Lemma 3.4, we can obtain

$$\begin{aligned}\|e^{-1}\| =&\|u(x, t_{-1}) - U^{-1}\|\\ =&\|u(x, t_{-1}) - u^{-1} + u^{-1} - U^{-1}\|\\ =&\|\varepsilon^{-1} + u^0 - \tau\psi - (U^0 - \tau\psi)\|\\ \leq& C(\tau^2).\end{aligned}$$

Thus, $\|\mathcal{H}e^{-1}\| \leq \|e^{-1}\| \leq C(\tau^2)$. According to Eq (3.11), we have $\|\mathcal{H}e^1\| \leq C(\tau^2 + h^4)$.
Assume when $m = 1, 2, \cdots, n-1$, it holds that

$$\|\mathcal{H}e^m\| \leq C(\tau^2 + h^4). \tag{3.12}$$

Now, we will prove it holds for $m = n$. According to Eqs (3.10)–(3.12) and by the similar deduction as Eq (3.6), we can derive that

$$\begin{aligned}\|\mathcal{H}e^n\| \leq& C\|\mathcal{H}e^{n-1}\| + C\|\mathcal{H}e^{n-2}\| + \lambda\tau^2\sum_{i=1}^{n-2}\|\mathcal{H}e^i\|e^{-\eta(n-i)\tau}\\ &+ \lambda\tau^2\sum_{i=1}^{n-2}\|\mathcal{H}e^{i-1}\|e^{-\eta(n-i)\tau} + C\|R^{n-1}\|\\ \leq&\lambda\tau^2\sum_{i=1}^{n-2}\|\mathcal{H}e^i\|e^{-\eta(n-i)\tau} + \lambda\tau^2\sum_{i=1}^{n-2}\|\mathcal{H}e^{i-1}\|e^{-\eta(n-i)\tau} + C(\tau^2 + h^4)\\ \leq& C(\tau^2 + h^4)\lambda\tau^2\sum_{i=1}^{n-2}e^{-\eta(n-i)\tau} + C(\tau^2 + h^4)\\ \leq& C(\tau^2 + h^4).\end{aligned} \tag{3.13}$$

According to Lemma 3.3, it holds that $\|e^n\| \leq \sqrt{3}\|\mathcal{H}e^n\|$, then the following estimate is obtained

$$\|e^n\| \leq C(\tau^2 + h^4).$$

The proof of the theorem is completed. □

4. Numerical results

This section is devoted to do some numerical simulation, which will show that the proposed numerical method is accurate and convergent. In the process of experiment, we applyied the L_∞ norm to compute the numerical results. The numerical experiment was carried out using MATLAB2017a under the environment of Inter Core i5–8265U computer with 4GB internal storage and 1.60GHZ. The L_∞ norm error can be obtained by the following formula

$$e_\infty(\tau, h) = \|u^n - U^n\|_\infty.$$

Example 1. *We consider the time fractional diffusion-wave equation (TFDWE) with $a = 1$*

$$\begin{cases} {}_0^{CF}D_t^{\beta}u(x,t) + \dfrac{\partial^2 u(x,t)}{\partial t^2} = a^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} + F(x,t), & 0 < x < 1,\ \ 0 < t \leq 1, \\ u(x,0) = \sin(2\pi x),\ \ u_t(x,0) = 0, & 0 \leq x \leq 1, \\ u(0,t) = 0, \quad u(1,t) = 0, & 0 \leq t \leq 1, \end{cases}$$

where $0 < \beta < 1$, $u(x,t) = (t^3 + 1)\sin(2\pi x)$ *and*

$$F(x,t) = \frac{1}{1-\beta}\Big(\frac{3t^2}{\eta} - \frac{6t}{\eta^2} + \frac{6}{\eta^3}(1 - e^{-\eta t})\Big)\sin(2\pi x) + 6t\sin(2\pi x) + 4\pi^2(t^3+1)\sin(2\pi x).$$

The result of Example 1 will be shown in Tables 1 and 2 and Figures 1–3. Tables 1 and 2 show the maximum error, convergence order and Central Processing Unit (CPU) time of the CFD scheme Eq (2.6) in time and in space, respectively. We first verify the time convergence accuracy by using the following formula

$$Order = \log_2\Big(\frac{e_\infty(\tau_1, h)}{e_\infty(\tau, h)}\Big),$$

where $\tau_1 = 2\tau$.

We first investigate the temporal convergence rate. To this end, M is chosen big enough such that the errors stemming from the spatial approximation are negligible. The numerical results for a fixed value of β (where $\beta = 0.2, 0.4, 0.6, 0.8$, respectively) and $h = 1/100$, with different values of τ at time $T = 1$ are reported in Table 1. From the data in Table 1, it can be seen that the rate of convergence in time is near to $O(\tau^2)$, which has a nice agreement with theoretical one in Theorem 3.6.

Second, we verify the spatial accuracy of convergence by using the following formula

$$Order = \log_2\Big(\frac{e_\infty(\tau_1, 2h)}{e_\infty(\tau_2, h)}\Big),$$

where $\tau_1 = 4h^2$ and $\tau_2 = h^2$.

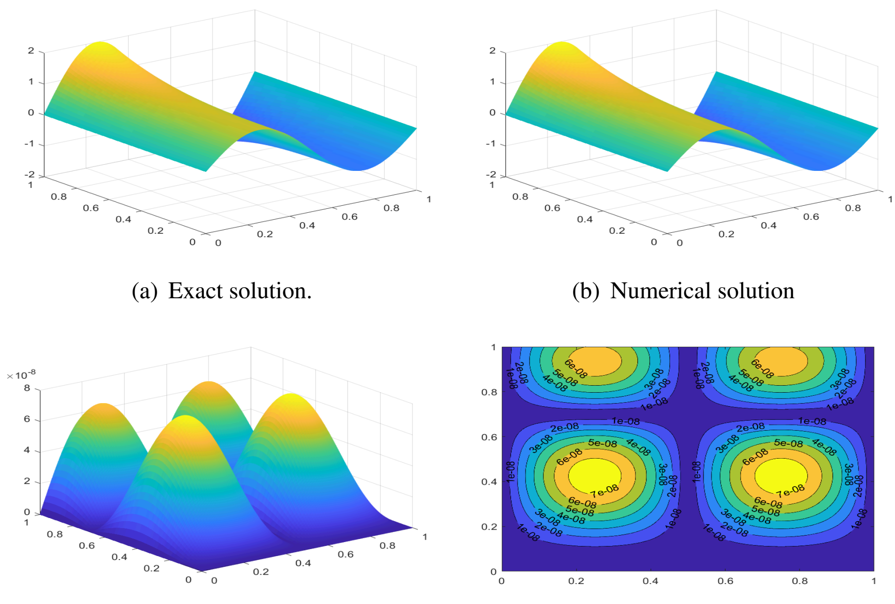
Now, we check the spatial accuracy by fixing the time step sufficiently small to avoid contamination of the temporal error. The numerical results for a fixed value of β (where $\beta = 0.2, 0.4, 0.6, 0.8$, respectively), with different values of h ($\tau = h^2$) at time $T = 1$ are reported in Table 2. From the data in Table 2, it can be seen that the rate of convergence in space is near to $O(h^4)$, which has a nice agreement with theoretical one in Theorem 3.6.

In Figures 1–3, we take $N = 100$ and $M = 5000$. We use the (a), (b), (c) and (d) to denote the exact solution, numerical solution, absolute error and contour plot of error, respectively. The results of $\beta = 0.15, 0.5, 0.95$ are shown in Figures 1–3. By Figures 1–3, we can find that the numerical solution of Eq (1.1) is infinitely close to the exact solution when β takes different values.

It is found that the present method is a reliable approach to deal with the one-dimensional problem in regular domain with uniform points. In the future, we want to extend the considered problem to multi-term time-fractional mixed problems using the proposed method.

Table 1. L_∞ norm errors, convergence orders and CPU time with $\beta = 0.2, 0.4, 0.6, 0.8$.

β	h	τ	$e_\infty(\tau, h)$	$order$	CPU time(s)
		1/10	3.0593×10^{-2}	—	0.0098
		1/20	7.3141×10^{-3}	2.06	0.0237
0.2	1/100	1/40	1.8103×10^{-3}	2.01	0.0417
		1/80	4.5144×10^{-4}	2.00	0.0915
		1/160	1.1275×10^{-4}	2.00	0.2573
		1/10	3.0342×10^{-2}	—	0.0189
		1/20	7.2626×10^{-3}	2.07	0.0230
0.4	1/100	1/40	1.7982×10^{-3}	2.01	0.0424
		1/80	4.4848×10^{-4}	2.00	0.0929
		1/160	1.1201×10^{-4}	2.00	0.2523
		1/10	2.9990×10^{-2}	—	0.0193
		1/20	7.1960×10^{-3}	2.06	0.0321
0.6	1/100	1/40	1.7830×10^{-3}	2.01	0.0399
		1/80	4.4477×10^{-4}	2.00	0.0932
		1/160	1.1109×10^{-4}	2.00	0.2575
		1/10	2.9495×10^{-2}	—	0.0175
		1/20	7.1187×10^{-3}	2.05	0.0226
0.8	1/100	1/40	1.7666×10^{-3}	2.01	0.0399
		1/80	4.4084×10^{-4}	2.00	0.0973
		1/160	1.1011×10^{-4}	2.00	0.2568



(a) Exact solution.

(b) Numerical solution

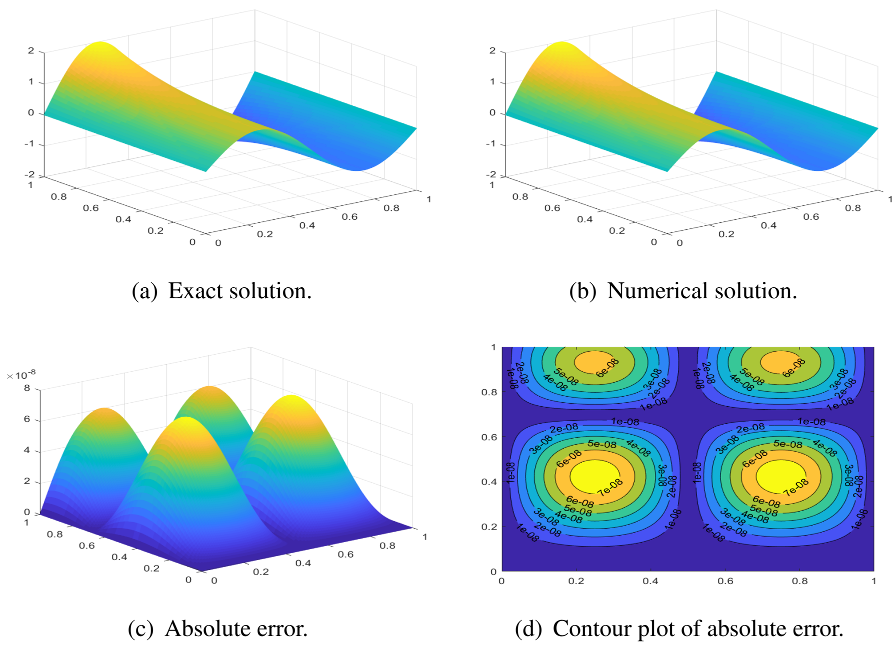
(c) Absolute error.

(d) Contour plot of absolute error.

Figure 1. The results for Example 1 with $\beta = 0.15$.

Table 2. L_∞ norm errors, convergence orders and CPU time with $\beta = 0.3, 0.5, 0.7, 0.9$.

β	τ	h	$e_\infty(\tau, h)$	*order*	*CPU* time(s)
0.3	1/25	1/5	4.3923×10^{-3}	—	0.0212
	1/100	1/10	2.6180×10^{-4}	4.07	0.0250
	1/400	1/20	1.6755×10^{-5}	3.97	0.3728
	1/1600	1/40	1.0394×10^{-6}	4.01	3.9936
	1/6400	1/80	6.5165×10^{-8}	4.00	136.8142
0.5	1/25	1/5	4.7558×10^{-3}	—	0.0133
	1/100	1/10	2.8305×10^{-4}	4.07	0.0228
	1/400	1/20	1.8110×10^{-5}	3.97	0.2663
	1/1600	1/40	1.1234×10^{-6}	4.01	4.0769
	1/6400	1/80	7.0513×10^{-8}	3.99	126.7946
0.7	1/25	1/5	5.6247×10^{-3}	—	0.0140
	1/100	1/10	3.3395×10^{-4}	4.07	0.0264
	1/400	1/20	2.1380×10^{-5}	3.97	0.2563
	1/1600	1/40	1.3266×10^{-6}	4.01	3.9717
	1/6400	1/80	8.3430×10^{-8}	3.99	126.3879
0.9	1/25	1/5	7.6281×10^{-3}	—	0.0130
	1/100	1/10	4.5231×10^{-4}	4.08	0.0228
	1/400	1/20	2.9091×10^{-5}	3.96	0.2127
	1/1600	1/40	1.8076×10^{-6}	4.01	4.0002
	1/6400	1/80	1.1941×10^{-7}	3.92	126.6125



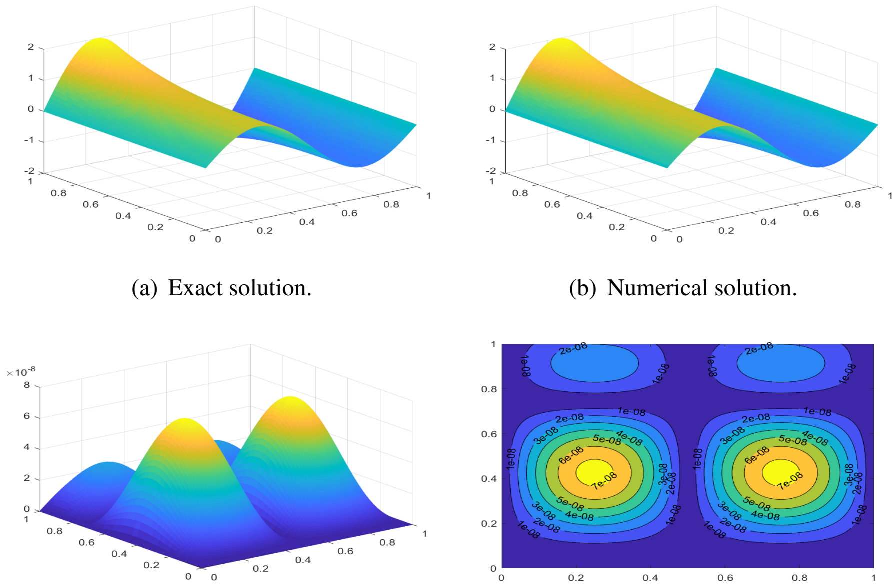
(a) Exact solution.

(b) Numerical solution.

(c) Absolute error.

(d) Contour plot of absolute error.

Figure 2. The results for Example 1 with $\beta = 0.5$.



(a) Exact solution.

(b) Numerical solution.

(c) Absolute error.

(d) Contour plot of absolute error.

Figure 3. The results for Example 1 with $\beta = 0.95$.

5. Conclusions

In this paper, we constructed an implicit numerical scheme for TFDWE. Compared with the traditional DWEs, the C-F derivative was used in our paper. We proved that the implicit numerical scheme was unconditionally stable. We also proved that the rate of convergence in time is near to $O(\tau^2)$ and the rate of convergence in space is near to $O(h^4)$. The numerical experiments verified our theoretical results. In the future, we will work on the numerical solutions of multidimensional FPDEs and irregular region equations. Also in the future, we would like to investigate fractional derivatives in both space and time.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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