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Research article

Three identities and a determinantal formula for differences between Bernoulli polynomials and numbers

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Abstract: In the paper, the authors simply review recent results of inequalities, monotonicity, signs of determinants, determinantal formulas, closed-form expressions, and identities of the Bernoulli numbers and polynomials, establish an identity involving the differences between the Bernoulli polynomials and the Bernoulli numbers, present two identities among the differences between the Bernoulli polynomials and the Bernoulli numbers in terms of a determinant and a partial Bell polynomial, and derive a determinantal formula of the differences between the Bernoulli polynomials and the Bernoulli numbers.

Keywords: Bernoulli polynomial; Bernoulli number; difference; identity; inequality; determinant; partial Bell polynomial; determinantal formula

1. A simple review of recent developments

Recall from [1, p. 804, Entry 23.1.1] that the Bernoulli numbers B_n can be generated by

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi. \tag{1.1}$$

Since the function $\frac{x}{\mathrm{e}^x - 1} - 1 + \frac{x}{2}$ is even in $x \in \mathbb{R}$, all the Bernoulli numbers B_{2n+1} for $n \ge 1$ are equal to 0. The first six non-zero Bernoulli numbers B_n are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}.$$

Recall from [2, Chapter 1] that the Bernoulli polynomials $B_n(x)$ can be generated by

$$\frac{z\,\mathrm{e}^{xz}}{\mathrm{e}^z-1}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!},\quad |z|<2\pi. \tag{1.2}$$

It is clear that $B_n(0)=B_n$. The first four Bernoulli polynomials $B_n(x)$ are

$$B_0(x)=1,\qquad B_1(x)=x-\frac{1}{2},\qquad B_2(x)=x^2-x+\frac{1}{6},\qquad B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x.$$

The Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ are classical and fundamental notions in both mathematical sciences and engineering sciences.

We now give a simple review of recent developments of the Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$, including inequalities, monotonicity, determinantal expressions, signs of determinants, and identities related to the Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$.

In [3], Alzer bounded the Bernoulli numbers B_n by the double inequality

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\alpha-2n}}\le|B_{2n}|\le\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\beta-2n}} \tag{1.3}$$

for $n\ge1$, where $\alpha=0$ and $\beta=2+\frac{\ln(1-6/\pi^2)}{\ln 2}=0.6491\dots$ are the best possible in the sense that they can not be replaced by any bigger and smaller constants respectively in the double inequality (1.3).

In [4, 5], Qi bounded the ratio $\frac{B_{2n+2}}{B_{2n}}$ by

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+1)(2n+2)}{\pi^2}<\left|\frac{B_{2n+2}}{B_{2n}}\right|<\frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+1)(2n+2)}{\pi^2}. \tag{1.4}$$

The double inequality (1.4) was generalized and refined in [6, 7]. This double inequality has had a number of non-self citations in over forty-eight articles or preprints published by other mathematicians.

In [8], Y. Shuang et al. proved that the sequence $\left|\frac{B_{2n+2}}{B_{2n}}\right|$ for $n\ge0$ and the sequences

$$\frac{\prod_{k=1}^{\ell}[2(n+1)+k]}{\prod_{k=1}^{\ell}(2n+k)}\left|\frac{B_{2n+2}}{B_{2n}}\right|,\quad n\ge1 \tag{1.5}$$

for fixed $\ell\ge1$ are increasing in n.

In the papers [9, 10], many determinantal expressions of the Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ are reviewed and discovered. For example, the Bernoulli polynomials $B_n(x)$ for $n\ge0$ can be expressed in terms of the determinant of a Hessenberg matrix as

$$B_n(x)=(-1)^n\begin{vmatrix}
1 & 1 & 0 & \cdots & 0 & 0 & 0\\
x & \frac{1}{2}\binom{1}{0} & 1 & \cdots & 0 & 0 & 0\\
x^2 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & \cdots & 0 & 0 & 0\\
x^3 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
x^{n-3} & \frac{1}{n-2}\binom{n-3}{0} & \frac{1}{n-3}\binom{n-3}{1} & \cdots & 1 & 0 & 0\\
x^{n-2} & \frac{1}{n-1}\binom{n-2}{0} & \frac{1}{n-2}\binom{n-2}{1} & \cdots & \frac{1}{2}\binom{n-2}{n-3} & 1 & 0\\
x^{n-1} & \frac{1}{n}\binom{n-1}{0} & \frac{1}{n-1}\binom{n-1}{1} & \cdots & \frac{1}{3}\binom{n-1}{n-3} & \frac{1}{2}\binom{n-1}{n-2} & 1\\
x^{n} & \frac{1}{n+1}\binom{n}{0} & \frac{1}{n}\binom{n}{1} & \cdots & \frac{1}{4}\binom{n}{n-3} & \frac{1}{3}\binom{n}{n-2} & \frac{1}{2}\binom{n}{n-1}
\end{vmatrix} \tag{1.6}$$

and, consequently, the Bernoulli numbers B_n for $n \ge 0$ can be expressed as

$$B_n = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2}\binom{1}{0} & 1 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n-2}\binom{n-3}{0} & \frac{1}{n-3}\binom{n-3}{1} & \cdots & 1 & 0 & 0 \\ 0 & \frac{1}{n-1}\binom{n-2}{0} & \frac{1}{n-2}\binom{n-2}{1} & \cdots & \frac{1}{2}\binom{n-2}{n-3} & 1 & 0 \\ 0 & \frac{1}{n}\binom{n-1}{0} & \frac{1}{n-1}\binom{n-1}{1} & \cdots & \frac{1}{3}\binom{n-1}{n-3} & \frac{1}{2}\binom{n-1}{n-2} & 1 \\ 0 & \frac{1}{n+1}\binom{n}{0} & \frac{1}{n}\binom{n}{1} & \cdots & \frac{1}{4}\binom{n}{n-3} & \frac{1}{3}\binom{n}{n-2} & \frac{1}{2}\binom{n}{n-1} \end{vmatrix}. \tag{1.7}$$

In [11], basing on the increasing property of the sequences in (1.5), among other things, Qi determined signs of certain Toeplitz–Hessenberg determinants whose elements involve the Bernoulli numbers B_{2n}. For example, for $n \ge 1$ and $\alpha > \frac{5}{6}$,

$$(-1)^n \begin{vmatrix} B_2 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ B_4 & B_2 & -\alpha & \cdots & 0 & 0 & 0 \\ B_6 & B_4 & B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{2n-4} & B_{2n-6} & B_{2n-8} & \cdots & B_2 & -\alpha & 0 \\ B_{2n-2} & B_{2n-4} & B_{2n-6} & \cdots & B_4 & B_2 & -\alpha \\ B_{2n} & B_{2n-2} & B_{2n-4} & \cdots & B_6 & B_4 & B_2 \end{vmatrix} < 0 \tag{1.8}$$

and

$$(-1)^n \begin{vmatrix} B_2 & -B_0 & 0 & \cdots & 0 & 0 & 0 \\ B_4 & B_2 & -B_0 & \cdots & 0 & 0 & 0 \\ B_6 & B_4 & B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{2n-4} & B_{2n-6} & B_{2n-8} & \cdots & B_2 & -B_0 & 0 \\ B_{2n-2} & B_{2n-4} & B_{2n-6} & \cdots & B_4 & B_2 & -B_0 \\ B_{2n} & B_{2n-2} & B_{2n-4} & \cdots & B_6 & B_4 & B_2 \end{vmatrix} < 0. \tag{1.9}$$

The rising factorial $(\alpha)_k$ is defined [12] by

$$(\alpha)_k = \prod_{\ell=0}^{k-1}(\alpha+\ell) = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+k-1), & k \ge 1; \\ 1, & k = 0. \end{cases}$$

The central factorial numbers of the second kind $T(n,k)$ for $n \ge k \ge 0$ can be generated [13, 14] by

$$\frac{1}{k!}\left(2\sinh\frac{x}{2}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{x^n}{n!}.$$

In [15], considering the power series expansion

$$\left(\frac{\sin x}{x}\right)^{\alpha} = 1 + \sum_{m=1}^{\infty}(-1)^m\left[\sum_{k=1}^{2m}\frac{(-\alpha)_k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}}\right]\frac{(2x)^{2m}}{(2m)!}$$

for $\alpha < 0$, which was established in [16, Theorem 4.1], X.-Y. Chen et al. derived the closed-form expression

$$B_{2n} = \frac{2^{2n-1}}{2^{2n-1}-1}\sum_{k=1}^{2n}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\frac{T(2n+j,j)}{\binom{2n+j}{j}}, \quad n \ge 1 \tag{1.10}$$

and two identities

$$\sum_{j=1}^{2n}(-1)^j\binom{4n+2}{2j}(2^{2j-1}-1)(2^{4n-2j+1}-1)B_{2j}B_{4n-2j+2} = 0, \quad n \ge 1 \tag{1.11}$$

and

$$\sum_{j=1}^{n-1}\binom{2n}{2j}(1-2^{2j-1}-2^{2n-2j-1})B_{2j}B_{2n-2j} = (2^{2n}-1)B_{2n}, \quad n \ge 2. \tag{1.12}$$

There have been a simple review about closed-form formulas for the Bernoulli numbers and polynomials at the web sites `https://math.stackexchange.com/a/4256911` (accessed on 5 February 2023), `https://math.stackexchange.com/a/4256914` (accessed on 5 February 2023), and `https://math.stackexchange.com/a/4656534` (accessed on 11 March 2023). For more recent developments of the Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$, please refer to the monograph [17], to the papers [18–24], and to the articles [25–33].

2. A motivation of this paper

Let

$$Q_n(x) = B_n(x) - B_n, \quad n \ge 0$$

denote the differences between the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n. Subtracting (1.1) from (1.2) on both sides yields

$$\frac{z(\mathrm{e}^{xz}-1)}{\mathrm{e}^{z}-1} = \sum_{n=0}^{\infty}[B_n(x)-B_n]\frac{z^n}{n!} = \sum_{n=0}^{\infty}Q_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi. \tag{2.1}$$

It is easy to see that

$$Q_0(x) = B_0(x) - B_0 = 0 \tag{2.2}$$

and $Q_n(0) = 0$ for $n \ge 0$. Accordingly, Eq (2.1) can be reformulated as

$$\frac{\mathrm{e}^{xz}-1}{\mathrm{e}^{z}-1} = \sum_{n=0}^{\infty}\frac{Q_{n+1}(x)}{(n+1)!}z^n, \quad |z| < 2\pi. \tag{2.3}$$

The values of $Q_n(x)$ for $1 \le n \le 4$ are

$$Q_1(x) = x, \qquad Q_2(x) = x^2 - x, \qquad Q_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \qquad Q_4(x) = x^4 - 2x^3 + x^2.$$

For $\alpha,\beta \in \mathbb{R}$ such that $\alpha \neq \beta$, $(\alpha,\beta) \neq (0,1)$, and $(\alpha,\beta) \neq (1,0)$, let

$$Q_{\alpha,\beta}(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{1 - e^{-t}}, & t \neq 0; \\ \beta - \alpha, & t = 0. \end{cases}$$

In the papers [34–36], the monotonicity and logarithmic convexity of $Q_{\alpha,\beta}(t)$ were discussed and the following conclusions were acquired:

1. the function $Q_{\alpha,\beta}(t)$ is increasing on $(0,\infty)$ if and only if $(\beta-\alpha)(1-\alpha-\beta) \geq 0$ and $(\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \geq 0$,
2. the function $Q_{\alpha,\beta}(t)$ is decreasing on $(0,\infty)$ if and only if $(\beta-\alpha)(1-\alpha-\beta) \leq 0$ and $(\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \leq 0$,
3. the function $Q_{\alpha,\beta}(t)$ is increasing on $(-\infty,0)$ if and only if $(\beta-\alpha)(1-\alpha-\beta) \geq 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \geq 0$,
4. the function $Q_{\alpha,\beta}(t)$ is decreasing on $(-\infty,0)$ if and only if $(\beta-\alpha)(1-\alpha-\beta) \leq 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \leq 0$,
5. the function $Q_{\alpha,\beta}(t)$ is increasing on $(-\infty,\infty)$ if and only if $(\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \geq 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \geq 0$,
6. the function $Q_{\alpha,\beta}(t)$ is decreasing on $(-\infty,\infty)$ if and only if $(\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \leq 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \leq 0$,
7. the function $Q_{\alpha,\beta}(t)$ on $(-\infty,\infty)$ is logarithmically convex if $\beta-\alpha > 1$ and logarithmically concave if $0 < \beta-\alpha < 1$,
8. if $1 > \beta-\alpha > 0$, then $Q_{\alpha,\beta}(t)$ is 3-log-convex on $(0,\infty)$ and 3-log-concave on $(-\infty,0)$,
9. if $\beta-\alpha > 1$, then $Q_{\alpha,\beta}(t)$ is 3-log-concave on $(0,\infty)$ and 3-log-convex on $(-\infty,0)$.

The monotonicity of $Q_{\alpha,\beta}(t)$ on $(0,\infty)$ was used in [34, 37, 38] to present necessary and sufficient conditions for some functions involving ratios of the gamma and q-gamma functions to be logarithmically completely monotonic. The logarithmic convexity of $Q_{\alpha,\beta}(t)$ on $(0,\infty)$ was employed in [36, 39] to provide alternative proofs for Elezović-Giordano-Pečarić's theorem. For more detailed information, please refer to [40, 41] and related references therein. The above texts are extracted and modified from [42, pp. 486–487].

The generating function $\frac{e^{xz}-1}{e^z-1}$ in (2.3) can be reformulated as

$$\frac{e^{xz}-1}{e^z-1} = \frac{e^{-(1-x)z} - e^{-z}}{1-e^{-z}} = Q_{1-x,1}(z).$$

Consequently, we deduce properties of the generating function $Q_{1-x,1}(t) = \frac{e^{xt}-1}{e^t-1}$ in (2.3) as follows:

1. the function $Q_{1-x,1}(t)$ is increasing on $(0,\infty)$ if and only if $x(x-1) \geq 0$ and $x(|x|+x-2) \geq 0$,
2. the function $Q_{1-x,1}(t)$ is decreasing on $(0,\infty)$ if and only if $x(x-1) \leq 0$ and $x(|x|+x-2) \leq 0$,
3. the function $Q_{1-x,1}(t)$ is increasing on $(-\infty,0)$ if and only if $x(x-1) \geq 0$ and $x(x-|x|) \geq 0$,
4. the function $Q_{1-x,1}(t)$ is decreasing on $(-\infty,0)$ if and only if $x(x-1) \leq 0$ and $x(x-|x|) \leq 0$,
5. the function $Q_{1-x,1}(t)$ is increasing on $(-\infty,\infty)$ if and only if $x(|x|+x-2) \geq 0$ and $x(x-|x|) \geq 0$,
6. the function $Q_{1-x,1}(t)$ is decreasing on $(-\infty,\infty)$ if and only if $x(|x|+x-2) \leq 0$ and $x(x-|x|) \leq 0$,
7. the function $Q_{1-x,1}(t)$ on $(-\infty,\infty)$ is logarithmically convex if $x > 1$ and logarithmically concave if $0 < x < 1$,

8. if $0 < x < 1$, then the function $Q_{1-x,1}(t)$ is 3-log-convex on $(0, \infty)$ and 3-log-concave on $(-\infty, 0)$,
9. if $x > 1$, then $Q_{1-x,1}(t)$ is 3-log-concave on $(0, \infty)$ and 3-log-convex on $(-\infty, 0)$.

What properties do the polynomials $Q_n(x) = B_n(x) - B_n$ for $n \ge 0$, the differences between the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n, possess?

3. An identity involving differences between the Bernoulli polynomials and numbers

In this section, we establish an identity involving the polynomials $Q_n(x) = B_n(x) - B_n$ for $n \ge 0$, the differences between the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n.

Theorem 3.1. *For $n \ge 1$, we have*

$$\sum_{k=0}^{n} \binom{n+2}{k+1} Q_{k+1}\left(\frac{1}{x}\right) Q_{n-k+1}(x) x^k = 0. \tag{3.1}$$

Proof. The identity (3.1) can be reformulated as

$$\sum_{k=0}^{n} \binom{n+2}{k+1} Q_{k+1}\left(\frac{1}{x}\right) Q_{n+2-(k+1)}(x) x^{k+1} = 0, \quad \sum_{k=1}^{n+1} \binom{n+2}{k} Q_{k}\left(\frac{1}{x}\right) Q_{n+2-k}(x) x^{k} = 0,$$

and

$$\sum_{k=0}^{n+2} \binom{n+2}{k} Q_{k}\left(\frac{1}{x}\right) Q_{n+2-k}(x) x^{k} = Q_0\left(\frac{1}{x}\right) Q_{n+2}(x) + Q_{n+2}\left(\frac{1}{x}\right) Q_0(x) x^{n+2} = 0,$$

where we used the identity (2.2). The last equation means that the identity (3.1) is equivalent to

$$A_n(x) = 0, \quad n \ge 3, \tag{3.2}$$

where

$$A_n(x) = \sum_{k=1}^{n-1} \frac{Q_k(\frac{1}{x}) x^k}{k!} \frac{Q_{n-k}(x)}{(n-k)!}, \quad n \ge 2.$$

On both sides of the identity

$$B_n(x+h) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) h^{n-k}, \quad n \ge 0,$$

which is listed in [1, p. 804, Entry 23.1.7], taking $x = 0$ yields

$$B_n(h) = \sum_{k=0}^{n} \binom{n}{k} B_k h^{n-k}, \quad n \ge 0.$$

This implies that

$$\frac{Q_n(x)}{n!} = \sum_{k=0}^{n-1} \frac{B_k}{k!} \frac{x^{n-k}}{(n-k)!}, \quad n \ge 0.$$

Let

$$P_k(x) = \frac{Q_k(\frac{1}{x})x^k}{k!} = \sum_{j=0}^{k-1} \frac{B_j x^j}{j!(k-j)!} \quad \text{and} \quad R_{n,k}(x) = \frac{Q_{n-k}(x)}{(n-k)!} = \sum_{j=1}^{n-k} \frac{B_{n-k-j} x^j}{j!(n-k-j)!}$$

for $1 \le k \le n-1$ and $n \ge 3$. Therefore, we obtain

$$A_n(x) = \sum_{k=1}^{n-1} P_k(x) R_{n,k}(x)$$

with $A_n(0) = 0$. This means that $A_n(x)$ is a polynomial in x of degree $n-1$. Hence, in order to verify the equality (3.2), it is sufficient to show

$$A_n^{(q)}(0) = 0, \quad 0 \le q \le n-1, \quad n \ge 3.$$

It is immediate that

$$R_{n,k}(0) = 0, \quad R_{n,k}^{(m)}(0) = \begin{cases} \dfrac{B_{n-k-m}}{(n-k-m)!}, & 1 \le m \le n-k; \\ 0, & m \ge n-k+1, \end{cases} \tag{3.3}$$

and

$$P_k^{(m)}(0) = \begin{cases} \dfrac{B_m}{(k-m)!}, & 0 \le m \le k-1; \\ 0, & m \ge k. \end{cases} \tag{3.4}$$

Differentiating $q \ge 2$ times the polynomial $A_n(x)$, taking the limit $x \to 0$, and interchanging the order of repeated sums give

$$\begin{aligned} A_n^{(q)}(0) &= \sum_{k=1}^{n-1} \sum_{j=0}^{q-1} \binom{q}{j} P_k^{(j)}(0) R_{n,k}^{(q-j)}(0) \\ &= \sum_{j=0}^{q-1} \binom{q}{j} \sum_{k=1}^{n-1} P_k^{(j)}(0) R_{n,k}^{(q-j)}(0) \\ &= \begin{cases} 0, & n-q < 1 \\ \displaystyle\sum_{j=0}^{q-1} \binom{q}{j} \sum_{k=j+1}^{j+n-q} \frac{B_j}{(k-j)!} \frac{B_{n-k-(q-j)}}{[n-k-(q-j)]!}, & n-q \ge 1 \end{cases} \\ &= \begin{cases} 0, & n-q < 1 \\ \displaystyle\left[\sum_{j=0}^{q-1} \binom{q}{j} B_j\right] \left[\sum_{\ell=1}^{n-q} \frac{B_{n-q-\ell}}{\ell!(n-q-\ell)!}\right], & n-q \ge 1 \end{cases} \\ &= 0, \end{aligned}$$

where we used the derivatives in (3.3) and (3.4) and utilized the identity

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad n = 2, 3, \dots, \tag{3.5}$$

which is collected in [43, p. 591, Entry 24.5.3] and [44, p. 206, (15.14)].

Moreover, by the identity (3.5) again, it is easy to see that

$$A_n'(0) = \sum_{j=0}^{n-2} \frac{B_j}{j!(n-1-j)!} = \frac{1}{(n-1)!} \sum_{j=0}^{n-2} \binom{n-1}{j} B_j = 0, \quad n \ge 3.$$

The proof of the identity (3.1) is thus complete. □

4. Two identities among differences between the Bernoulli polynomials and numbers

In this section, we demonstrate two identities among $Q_n(x)$ and $Q_n(\frac{1}{x})$.

The partial Bell polynomials $B_{n,k}$ for $n \ge k \ge 0$ are defined in [45, Definition 11.2] and [46, p. 134, Theorem A] by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{\ell_i \ge 0 \text{ for } 1 \le i \le n-k+1, \\ \sum_{i=1}^{n-k+1} i\ell_i = n, \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

This kind of polynomials $B_{n,k}$ are important in analytic combinatorics, analytic number theory, analysis, and other areas in mathematical sciences. In recent years, some novel conclusions and applications of partial Bell polynomials $B_{n,k}$ have been discovered, carried out, reviewed, and surveyed in the papers [12, 16, 47–54], for example.

Theorem 4.1. *For $n \ge 1$, we have*

$$Q_n\left(\frac{1}{x}\right) = (-1)^{n-1} \frac{n!}{x^{2n-1}} \begin{vmatrix} \frac{Q_2(x)}{2!} & \frac{Q_1(x)}{1!} & 0 & \cdots & 0 & 0 & 0 \\ \frac{Q_3(x)}{3!} & \frac{Q_2(x)}{2!} & \frac{Q_1(x)}{1!} & \cdots & 0 & 0 & 0 \\ \frac{Q_4(x)}{4!} & \frac{Q_3(x)}{3!} & \frac{Q_2(x)}{2!} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{Q_{n-2}(x)}{(n-2)!} & \frac{Q_{n-3}(x)}{(n-3)!} & \frac{Q_{n-4}(x)}{(n-4)!} & \cdots & \frac{Q_2(x)}{2!} & \frac{Q_1(x)}{1!} & 0 \\ \frac{Q_{n-1}(x)}{(n-1)!} & \frac{Q_{n-2}(x)}{(n-2)!} & \frac{Q_{n-3}(x)}{(n-3)!} & \cdots & \frac{Q_3(x)}{3!} & \frac{Q_2(x)}{2!} & \frac{Q_1(x)}{1!} \\ \frac{Q_n(x)}{n!} & \frac{Q_{n-1}(x)}{(n-1)!} & \frac{Q_{n-2}(x)}{(n-2)!} & \cdots & \frac{Q_4(x)}{4!} & \frac{Q_3(x)}{3!} & \frac{Q_2(x)}{2!} \end{vmatrix}, \tag{4.1}$$

where the determinant of order 0 *is regarded as* 1 *by convention.*

For $n \ge 0$, we have

$$Q_{n+1}\left(\frac{1}{x}\right) = \frac{n+1}{x^{n+1}} \sum_{k=0}^{n} (-1)^k \frac{k!}{x^k} B_{n,k}\left(\frac{Q_2(x)}{2}, \frac{Q_3(x)}{3}, \dots, \frac{Q_{n-k+2}(x)}{n-k+2}\right). \tag{4.2}$$

Proof. The Wronski formula reads that, if $a_0 \ne 0$ and

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{4.3}$$

is a formal series, then the coefficients of the reciprocal series

$$\frac{1}{P(x)} = b_0 + b_1 x + b_2 x^2 + \cdots \tag{4.4}$$

are given by

$$b_n = \frac{(-1)^n}{a_0^{n+1}} \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-3} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_0 & 0 & 0 \\ a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & \cdots & a_1 & a_0 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_2 & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \quad n \ge 0. \tag{4.5}$$

This can be found in [55, p. 17, Theorem 1.3], [11, Lemma 2.1 and Section 5], and [9, Lemma 2.4]. It is easy to see that the equalities (4.3) and (4.4) are equivalent to the identities $a_0b_0 = 1$ and $\sum_{k=0}^{n} a_k b_{n-k} = 0$ for $n \ge 1$. See [47, 56–58].

Let β be a fixed real number and let

$$a_n = \frac{x^{n+\beta}}{(n+1)!} Q_{n+1}\left(\frac{1}{x}\right) \quad \text{and} \quad b_n = \frac{1}{(n+1)!x^{\beta}} Q_{n+1}(x) \tag{4.6}$$

for $n \ge 0$. It is easy to verify that $a_0b_0 = 1$. The identity (3.1) in Theorem 3.1 is equivalent to the equality $\sum_{k=0}^{n} a_k b_{n-k} = 0$ for $n \ge 1$. Therefore, the sequences a_n and b_n defined in (4.6) satisfy the relation (4.5). Interchanging the roles of a_n and b_n and simplifying yield (4.1).

On the other hand, if the sequences a_n and b_n satisfy $a_0 = b_0 = 1$ and meet the equalities (4.3) and (4.4), then

$$b_n = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k k! B_{n,k}(1!a_1, 2!a_2, \dots, (n-k+1)!a_{n-k+1}). \tag{4.7}$$

See the papers [47, 48, 54, 59]. When $\beta = 1$ in (4.6), it follows that $a_0 = b_0 = 1$ and $\sum_{k=0}^{n} a_k b_{n-k} = 0$ for $n \ge 1$. Interchanging the roles of a_n and b_n in (4.7) and applying the sequences a_n and b_n in (4.6) result in (4.2). The proof of Theorem 4.1 is complete. □

5. A determinantal formula of differences between Bernoulli polynomials and numbers

In this section, we derive a determinantal formula of the difference $Q_n(x)$ as follows.

Theorem 5.1. *For $n \ge 1$, the difference $Q_n(x)$ can be computed by*

$$Q_n(x) = (-1)^{n-1} n x \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ \frac{x}{2} & \frac{1}{2}\binom{1}{0} & 1 & \cdots & 0 & 0 \\ \frac{x^2}{3} & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & \cdots & 0 & 0 \\ \frac{x^3}{4} & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{x^{n-3}}{n-2} & \frac{1}{n-2}\binom{n-3}{0} & \frac{1}{n-3}\binom{n-3}{1} & \cdots & 1 & 0 \\ \frac{x^{n-2}}{n-1} & \frac{1}{n-1}\binom{n-2}{0} & \frac{1}{n-2}\binom{n-2}{1} & \cdots & \frac{1}{2}\binom{n-2}{n-3} & 1 \\ \frac{x^{n-1}}{n} & \frac{1}{n}\binom{n-1}{0} & \frac{1}{n-1}\binom{n-1}{1} & \cdots & \frac{1}{3}\binom{n-1}{n-3} & \frac{1}{2}\binom{n-1}{n-2} \end{vmatrix}. \tag{5.1}$$

Proof. The power series expansion (2.3) implies that

$$\frac{Q_{n+1}(x)}{n+1}=\lim_{z\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,z^n}\left(\frac{\mathrm{e}^{xz}-1}{\mathrm{e}^z-1}\right)=\lim_{z\to0}\frac{\mathrm{d}^n\,Q_{1-x,1}(z)}{\mathrm{d}\,z^n},\quad n\ge0.$$

The generating function $Q_{1-x,1}(z)$ can be rewritten as

$$Q_{1-x,1}(z)=x\frac{(\mathrm{e}^{xz}-1)/(xz)}{(\mathrm{e}^z-1)/z}=x\frac{\int_1^e s^{xz-1}\,\mathrm{d}\,s}{\int_1^e s^{z-1}\,\mathrm{d}\,s}$$

with

$$\lim_{z\to0}\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\int_1^e s^{z-1}\,\mathrm{d}\,s=\lim_{z\to0}\int_1^e s^{z-1}\ln^k s\,\mathrm{d}\,s=\int_1^e\frac{\ln^k s}{s}\,\mathrm{d}\,s=\frac{1}{k+1}$$

and

$$\lim_{z\to0}\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\int_1^e s^{xz-1}\,\mathrm{d}\,s=x^k\lim_{z\to0}\int_1^e s^{xz-1}\ln^k s\,\mathrm{d}\,s=x^k\int_1^e\frac{\ln^k s}{s}\,\mathrm{d}\,s=\frac{x^k}{k+1}$$

for $k\ge0$.

Let $u(z)$ and $v(z)\ne0$ be two differentiable functions, let $U_{(n+1)\times1}(z)$ be an $(n+1)\times1$ matrix whose elements are $u_{k,1}(z)=u^{(k-1)}(z)$ for $1\le k\le n+1$, let $V_{(n+1)\times n}(z)$ be an $(n+1)\times n$ matrix whose elements are

$$v_{ij}(z)=\begin{cases}\dbinom{i-1}{j-1}v^{(i-j)}(z), & i-j\ge0\\ 0, & i-j<0\end{cases}$$

for $1\le i\le n+1$ and $1\le j\le n$, and let $\left|W_{(n+1)\times(n+1)}(z)\right|$ denote the determinant of the $(n+1)\times(n+1)$ matrix

$$W_{(n+1)\times(n+1)}(z)=\begin{pmatrix}U_{(n+1)\times1}(z) & V_{(n+1)\times n}(z)\end{pmatrix}.$$

Then the nth derivative of the ratio $\frac{u(z)}{v(z)}$ can be computed [60, p. 40, Exercise 5] by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left[\frac{u(z)}{v(z)}\right]=(-1)^n\frac{\left|W_{(n+1)\times(n+1)}(z)\right|}{v^{n+1}(z)}. \tag{5.2}$$

See also [61, Lemma 1], [11, Section 2], [9, p. 94, The first proof of Theorem 1.2], and [10, Lemma 1].

Applying the formula (5.2) to the functions

$$u(z)=\int_1^e s^{xz-1}\,\mathrm{d}\,s\quad\text{and}\quad v(z)=\int_1^e s^{z-1}\,\mathrm{d}\,s$$

yields

$$\begin{aligned}\lim_{z\to0}\frac{\mathrm{d}^n\,Q_{1-x,1}(z)}{\mathrm{d}\,z^n}&=x\lim_{z\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,z^n}\left(\frac{\int_1^e s^{xz-1}\,\mathrm{d}\,s}{\int_1^e s^{z-1}\,\mathrm{d}\,s}\right)\\&=x\lim_{z\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,z^n}\left[\frac{u(z)}{v(z)}\right]\end{aligned}$$

$$
= x\lim_{z\to0}\frac{(-1)^n}{v^{n+1}(z)}\begin{vmatrix}
u(z) & v(z) & 0 & \cdots & 0\\
u'(z) & v'(z) & v(z) & \cdots & 0\\
u''(z) & v''(z) & \binom{2}{1}v'(z) & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
u^{(n-1)}(z) & v^{(n-1)}(z) & \binom{n-1}{1}v^{(n-2)}(z) & \cdots & v(z)\\
u^{(n)}(z) & v^{(n)}(z) & \binom{n}{1}v^{(n-1)}(z) & \cdots & \binom{n}{n-1}v'(z)
\end{vmatrix}
$$

$$
= \frac{(-1)^n x}{v^{n+1}(0)}\begin{vmatrix}
u(0) & v(0) & 0 & \cdots & 0 & 0\\
u'(0) & v'(0) & v(0) & \cdots & 0 & 0\\
u''(0) & v''(0) & \binom{2}{1}v'(0) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
u^{(n-1)}(0) & v^{(n-1)}(0) & \binom{n-1}{1}v^{(n-2)}(0) & \cdots & \binom{n-1}{n-2}v'(0) & v(0)\\
u^{(n)}(0) & v^{(n)}(0) & \binom{n}{1}v^{(n-1)}(0) & \cdots & \binom{n}{n-2}v''(0) & \binom{n}{n-1}v'(0)
\end{vmatrix}
$$

$$
= (-1)^n x\begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0\\
\frac{x}{2} & \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 & 0\\
\frac{x^2}{3} & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 & 0\\
\frac{x^3}{4} & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{2}\binom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{x^{n-2}}{n-1} & \frac{1}{n-1}\binom{n-2}{0} & \frac{1}{n-2}\binom{n-2}{1} & \binom{n-2}{2}\frac{1}{n-3} & \cdots & 1 & 0\\
\frac{x^{n-1}}{n} & \frac{1}{n}\binom{n-1}{0} & \frac{1}{n-1}\binom{n-1}{1} & \frac{1}{n-2}\binom{n-1}{2} & \cdots & \frac{1}{2}\binom{n-1}{n-2} & 1\\
\frac{x^n}{n+1} & \frac{1}{n+1}\binom{n}{0} & \frac{1}{n}\binom{n}{1} & \frac{1}{n-1}\binom{n}{2} & \cdots & \frac{1}{3}\binom{n}{n-2} & \frac{1}{2}\binom{n}{n-1}
\end{vmatrix}.
$$

The determinantal formula (5.1) is thus proved. □

Remark 5.1. The formula (4.5) can also be proved by the formula (5.2). For details, please refer to [11, Section 5].

Remark 5.2. For $n \ge 1$, the determinantal formula (5.1) can be reformulated as

$$
\frac{Q_n(x)}{x} = (-1)^{n-1}n\begin{vmatrix}
1 & 1 & 0 & \cdots & 0 & 0\\
\frac{x}{2} & \frac{1}{2}\binom{1}{0} & 1 & \cdots & 0 & 0\\
\frac{x^2}{3} & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & \cdots & 0 & 0\\
\frac{x^3}{4} & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{x^{n-3}}{n-2} & \frac{1}{n-2}\binom{n-3}{0} & \frac{1}{n-3}\binom{n-3}{1} & \cdots & 1 & 0\\
\frac{x^{n-2}}{n-1} & \frac{1}{n-1}\binom{n-2}{0} & \frac{1}{n-2}\binom{n-2}{1} & \cdots & \frac{1}{2}\binom{n-2}{n-3} & 1\\
\frac{x^{n-1}}{n} & \frac{1}{n}\binom{n-1}{0} & \frac{1}{n-1}\binom{n-1}{1} & \cdots & \frac{1}{3}\binom{n-1}{n-3} & \frac{1}{2}\binom{n-1}{n-2}
\end{vmatrix}.
$$

Since

$$
\lim_{x\to0}\frac{Q_n(x)}{x} = \lim_{x\to0}\frac{B_n(x)-B_n}{x} = \lim_{x\to0}B_n'(x) = n\lim_{x\to0}B_{n-1}(x) = nB_{n-1},
$$

taking the limit $x \to 0$ on both sides of the above determinantal formula gives

$$nB_{n-1} = (-1)^{n-1} n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2}\binom{1}{0} & 1 & \cdots & 0 & 0 \\ 0 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & \cdots & 0 & 0 \\ 0 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{n-2}\binom{n-3}{0} & \frac{1}{n-3}\binom{n-3}{1} & \cdots & 1 & 0 \\ 0 & \frac{1}{n-1}\binom{n-2}{0} & \frac{1}{n-2}\binom{n-2}{1} & \cdots & \frac{1}{2}\binom{n-2}{n-3} & 1 \\ 0 & \frac{1}{n}\binom{n-1}{0} & \frac{1}{n-1}\binom{n-1}{1} & \cdots & \frac{1}{3}\binom{n-1}{n-3} & \frac{1}{2}\binom{n-1}{n-2} \end{vmatrix}$$

for $n \ge 1$. Consequently, we recover the determinantal formula (1.7).

Remark 5.3. The determinantal formula (5.1) can be rearranged as

$$Q_n(x) = (-1)^{n-1} n \begin{vmatrix} x & 1 & 0 & \cdots & 0 & 0 \\ \frac{x^2}{2} & \frac{1}{2}\binom{1}{0} & 1 & \cdots & 0 & 0 \\ \frac{x^3}{3} & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & \cdots & 0 & 0 \\ \frac{x^4}{4} & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{x^{n-2}}{n-2} & \frac{1}{n-2}\binom{n-3}{0} & \frac{1}{n-3}\binom{n-3}{1} & \cdots & 1 & 0 \\ \frac{x^{n-1}}{n-1} & \frac{1}{n-1}\binom{n-2}{0} & \frac{1}{n-2}\binom{n-2}{1} & \cdots & \frac{1}{2}\binom{n-2}{n-3} & 1 \\ \frac{x^n}{n} & \frac{1}{n}\binom{n-1}{0} & \frac{1}{n-1}\binom{n-1}{1} & \cdots & \frac{1}{3}\binom{n-1}{n-3} & \frac{1}{2}\binom{n-1}{n-2} \end{vmatrix}. \tag{5.3}$$

Differentiating with respect to x on both sides of (5.3) and making use of the relation $B_n'(x) = nB_{n-1}(x)$, we recover the determinantal formula (1.6).

Remark 5.4. In theory, the determinantal formula (5.1) in Theorem 5.1 can be obtained by algebraically subtracting the determinant (1.7) from the determinant (1.6).

6. Conclusions

In this paper, about the Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$, we simply reviewed the inequalities (1.3) and (1.4), the increasing property of the sequence in (1.5), the determinantal formulas (1.6) and (1.7), the negativity of two determinants in (1.8) and (1.9), the closed-form formula (1.10), and the identities (1.11) and (1.12), established the identity (3.1) in Theorem 3.1 in which the differences $Q_n(x)$ between the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n are involved, presented two identities (4.1) and (4.2) among the differences $Q_n(x)$ in terms of a beautiful Hessenberg determinant and the partial Bell polynomials $B_{n,k}$ in Theorem 4.1, and derived a determinantal formula (5.1) for the difference $Q_n(x)$ in Theorem 5.1.

To the best of our authors' knowledge, the difference $Q_n(x)$ has been investigated in this paper for the first time in the mathematical community.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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