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Research article

Stability on a boundary problem with RL-Fractional derivative in the sense of Atangana-Baleanu of variable-order

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Abstract: In this paper, we study the existence and stability of solutions in connection to a non-local multiterm boundary value problem (BVP) with differential equations equipped with the Riemann-Liouville (RL) fractional derivative in the sense of Atangana-Baleanu of variable-order. The results about the existence property are investigated and proved via Krasnoselskii's fixed point theorem. Note that all theorems in the present research are studied based on piece-wise constant functions defined on generalized intervals. We shall convert our main BVP with the RL-fractional derivative of the Atangana-Baleanu type of variable-order to an equivalent BVP of constant order of the RL-Atangana-Baleanu derivative. In the next step, we examine the Ulam-Hyers stability for the supposed variable-order RL-Atangana-Baleanu BVP. Finally, we provide some examples to validate that our results are applicable.

Keywords: Atangana-Baleanu derivative; variable-order; differential equation; fixed point; Ulam-Hyers stability

1. Introduction

The fundamental idea that led to the extension of the constant-order fractional calculus to the variable-order fractional calculus is that we replace the constant τ as a constant order of a given fractional differential equation (FDE) by a function $\tau(\cdot)$ as the variable order. Maybe the mentioned difference seems simple, but from the theoretical point of view, variable-order operators are strong tools for explaining and modeling complex natural phenomena with respect to independent time or space variables.

Recently, Souid et al. [1–5] have contributed in this field with many published papers that are concerned with the study of the existence, uniqueness and stability of solutions to many different problems of FDEs of variable order (implicit, multiterm, resonance, etc.) via different conditions (initial, boundary, impulsive, finite delay, etc.). All the results obtained in these papers are based on different techniques provided in fixed point theory, the theory of measure of non-compactness and the upper-lower solutions method. Also, in these papers, the authors studied the stability of all the proposed problems in the sense of Ulam-Hyers or Ulam-Hyers-Rassias stability. For more details, refer to other studies [6–11].

In [12], Jeelani et al. extended their studies on the existence of solutions by considering nonlinear ML-type integro-differential equation BVPs with variable order

$$\begin{cases} {}^{ML}D_{0^+}^{x(s)}\theta(s) = \varphi(s, \theta(s), {}^{ML}I_{0^+}^{x(s)}\theta(s), {}^{ML}D_{0^+}^{x(s)}\theta(s)), \ \ s \in \mathcal{B} := [0, B], \ \ x(s) \in]1, 2], \\ \theta(0) = 0, \theta(B) = \sum_{l=1}^{m} a_l\theta(s_l), \ \ s_l \in]0, B], \end{cases}$$

where ${}^{ML}D_{0^+}^{x(s)}$, ${}^{ML}I_{0^+}^{x(s)}$ are the derivative and integral operators of the ML-type with variable order $\mu(s)$, respectively, and $\varphi : \mathcal{N} \times \mathbb{R}^2 \to \mathbb{R}$ is a given function.

In view of the above ML-type problem, in this paper, some properties such as the existence and stability of solutions for variable order multiterm BVP with nonlocal conditions

$$\begin{cases} {}^{ABR}\mathbf{D}_{0^+}^{x(s)}\theta(s) = \varphi_\theta(s), \ \ s \in \mathcal{B} := [0, B], \\ \theta(0) = 0, \theta(B) = \sum_{k=1}^{m} a_k\theta(s_k), \ \ s_k \in]0, B], \end{cases} \tag{1.1}$$

are studied, where $\varphi_\theta(s) := \varphi(s, \theta(s), {}^{AB}\mathbf{I}_{0^+}^{x(s)}\theta(s))$, $0 < B < +\infty$, $2 < x(s) \leq 3$, $\varphi : \mathcal{B} \times \mathbb{R}^2 \to \mathbb{R}$ is a given function and ${}^{ABR}\mathbf{D}_{0^+}^{x(s)}$, ${}^{AB}\mathbf{I}_{0^+}^{x(s)}$ are the RL-fractional derivative in the sense of Atangana-Baleanu and the RL-fractional integral in the sense of Atangana-Baleanu of variable-order $x(\cdot)$, respectively. These operators are generalizations of the introduced operators by Atangana-Baleanu [13]. Note that the main difference of our technique is that the generalized intervals and piece-wise constant functions will play a vital role in our study for converting the fractional Atangana-Baleanu problems of variable order into equivalent standard Atangana-Baleanu fractional problems of constant order. In fact, we are studying an abstract variable order boundary value problem, but consider the fact that for studying real phenomena and processes from the mathematical point of view we first have to model them in the framework of non-fractional or fractional initial-boundary value problems.

The paper will be organized as follows. In Section 2, basic definitions and relations which will be applied throughout the next sections are recalled. In Section 3, Krasnoselskii's fixed point theorem (Theorem 10) is used for the first time. Also, the property of Ulam-Hyers stable solutions is analyzed. An example is given in Section 4. Section 5 concludes the paper.

2. Preliminaries

In this section, the basic definitions and relations which will be applied throughout the next sections are recalled.

We denote by $C(\mathcal{B},\mathbb{R})$ the space of $\mathbb{R}$-valued continuous functions defined on $\mathcal{B}$ with the usual supremum norm

$$\|\theta\|_{\mathcal{B}} = \sup\{|\theta(s)| : s \in \mathcal{B}\}$$

for all $\theta \in C(\mathcal{B},\mathbb{R})$.

Definition 1. *([13, 14]) For $-\infty < c < d < +\infty$, we consider the mapping $x : (c,d) \to [0,1]$. Then, the RL-fractional integral in the sense of Atangana-Baleanu of variable-order $x(\cdot)$ for the function $\theta(\cdot) \in L^1(c,d)$ is defined by*

$$^{AB}I_{c^+}^{x(s)}\theta(s) = \frac{1-x(s)}{N(x(s))}\theta(s) + \frac{x(s)}{N(x(s))}\int_c^s \frac{(s-\tau)^{x(\tau)-1}}{\Gamma(x(\tau))}\theta(\tau)d\tau, \quad s > c, \tag{2.1}$$

where $N(x(s))$ is the normalization function:

$$N(x(s)) = 1 - x(s) + \frac{x(s)}{\Gamma(x(s))}.$$

Definition 2. *([13, 14]) For $-\infty < c < d < +\infty$, we consider the mapping $x : [c,d] \to (0,1)$. Then, the RL-fractional derivative in the sense of Atangana-Baleanu of variable-order $x(\cdot)$ for the function $\theta \in L^1(c,d)$ is defined by*

$$^{ABR}D_{c^+}^{x(s)}\theta(s) = \frac{N(x(s))}{1-x(s)}\frac{d}{ds}\int_c^s \mathcal{E}_{x(\tau)}\left(-x(\tau)\frac{(s-\tau)^{x(\tau)}}{(1-x(\tau))}\right)\theta(\tau)d\tau, \quad s > c, \tag{2.2}$$

where $\mathcal{E}_{x(\tau)}(\cdot)$ denotes the Mittag-Leffler function.

Now, we extend Definition 2 to the arbitrary mapping $x : (c,d) \to (n, n+1]$.

Definition 3. *([13, 14]) For the same $c < d$ introduced above, consider $x : (c,d) \to (n, n+1]$ in connection to the function $\theta(\cdot) \in L^1(c,d)$. Set $\nu = x - n : (c,d) \to (0,1]$. Then, the RL-fractional integral in the sense of Atangana-Baleanu of variable-order $x(\cdot)$ is*

$$^{AB}\boldsymbol{I}_{c^+}^{x(s)}\theta(s) = I_{c^+}^{n}\ {}^{AB}I_{c^+}^{\nu(s)}\ \theta(s), \quad s > c. \tag{2.3}$$

Definition 4. *([13, 14]) For the same $c < d$ introduced above, consider $x : (c,d) \to (n, n+1]$ and let $\theta(\cdot)$ be such that $\theta^{(n)}(\cdot) \in L^1(c,d)$. Set $\nu = x - n : (c,d) \to (0,1]$. Then, the RL-fractional derivative in the sense of Atangana-Baleanu of variable-order $x(\cdot)$ is*

$$^{ABR}\boldsymbol{D}_{c^+}^{x(s)}\theta(s) = {}^{ABR}D_{c^+}^{\nu(s)}\theta^{(n)}(s), \quad s > c. \tag{2.4}$$

Remark 5. *For the case $0 < x(s) \leq 1$, we have*

- $\nu = x$
- $^{AB}\boldsymbol{I}_{c^+}^{x(s)}\theta(s) = {}^{AB}I_{c^+}^{x(s)}\theta(s)$

- ${}^{ABR}\boldsymbol{D}^{x(s)}_{c^+}\theta(s) ={}^{ABR} D^{x(s)}_{c^+}\theta(s)$

Lemma 6. *([14]) For $\theta^{(n)}(\cdot) \in L^1(c,d)$ and $\varrho \in (n, n+1]$, we have*

- ${}^{ABR}\boldsymbol{D}^{\varrho}_{c^+}\, {}^{AB}\boldsymbol{I}^{\varrho}_{c^+}\theta(s) = \theta(s),$
- ${}^{AB}\boldsymbol{I}^{\varrho}_{c^+}\, {}^{ABR}\boldsymbol{D}^{\varrho}_{c^+}\theta(s) = \theta(s) - \sum_{k=0}^{n-1} \frac{\theta^{(k)}(c)}{k!}(s-c)^k.$

Remark 7. *[15] For $\theta(\cdot) \in L^1(c,d)$ and $\varrho \in (0,1]$, we have*

$$ {}^{AB}\boldsymbol{I}^{\varrho}_{c^+}\, {}^{ABR}\boldsymbol{D}^{\varrho}_{c^+}\theta(s) = \theta(s). $$

Remark 8. *Generally, for functions $x_1(s)$ and $x_2(s)$, the semi-group property, i.e.,*

$$ {}^{AB}\boldsymbol{I}^{x_1(s)}_{c^+}\, {}^{AB}\boldsymbol{I}^{x_2(s)}_{c^+}\theta(s) \neq {}^{AB}\, \boldsymbol{I}^{x_1(s)+x_2(s)}_{c^+}\theta(s) $$

does not hold.

The following theorems will be applied in the next section.

Theorem 9. *([16]) Assume that C is a set in the Banach space X such that it is non-empty and closed. Then, each contraction T on C admits a unique fixed point.*

Theorem 10. *([17]). Let E be a Banach space and let the set M be nonempty, bounded, convex and closed in E. If $\exists\, \mathcal{P}_1, \mathcal{P}_2$ on M such that*

1) $\mathcal{P}_1\xi + \mathcal{P}_2\acute{\xi} \in M$ for all $\xi,\ \acute{\xi} \in M$,

2) $\mathcal{P}_2$ is a contraction,

3) $\mathcal{P}_1$ is completely continuous,

then $\mathcal{P}_1 + \mathcal{P}_2$ admits a fixed point in M.

3. Main results

This section includes some subsections along with main results.

3.1. Existence and uniqueness

Let us prepare some required hypotheses:

(HY1) Let $n \in \mathbb{N}$ be an integer and the finite sequence of points $\{B_k\}_{k=0}^n$ be given such that $0 = B_0 < B_{k-1} < B_k < B_n = N$, $k = 2, ..., n-1$.

Denote $\mathcal{B}_k := (B_{k-1}, B_k]$, $k = 1, ..., n$. Then $\mathcal{P} = \{\mathcal{B}_k : 1 = 1, 2, ..., n\}$ is a partition of the interval $\mathcal{B}$.

Let $x : \mathcal{B} \to (2,3]$ be a piecewise constant function, w.r.t., $\mathcal{P}$ as follows:

$$ x(s) = \sum_{k=1}^{n} x_k I_k(s) = \begin{cases} x_1, & if\ s \in \mathcal{B}_1, \\ x_2, & if\ s \in \mathcal{B}_2, \\ . \\ . \\ . \\ x_n, & if\ s \in \mathcal{B}_n, \end{cases} $$

where the constants x_k are such that $2 < x_k \leq 3$. Moreover, I_k is an indicator of $\mathcal{B}_k$, for $k = 1, 2, ..., n$; that is,

$$I_k(s) = \begin{cases} 1, & for\ s \in \mathcal{B}_k, \\ 0, & for\ elsewhere. \end{cases}$$

(HY2) Let $\varphi : \mathcal{B} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. There is $K_\varphi > 0$ such that

$$|\varphi(s, y_1, z_1) - \varphi(s, y_2, z_2)| \leq K_\varphi \left(|y_1 - y_2| + |z_1 - z_2|\right),$$

$\forall\, y_1, y_2,\ z_1, z_2 \in \mathbb{R}$ and $s \in \mathcal{B}$.

Then, for any $s \in \mathcal{B}_k,\ k = 1, 2, ..., n$ and by Definition 4, the RL-fractional derivative in the sense of Atangana-Baleanu of variable order $x(\cdot)$ for $\theta \in C(\mathcal{B}, \mathbb{R})$ can be considered as a sum of RL-fractional derivatives in the sense of Atangana-Baleanu of constant-orders x_k :

$$\begin{aligned} {}^{ABR}\mathbf{D}_{0^+}^{x(s)}\theta(s) &= {}^{ABR}D_{0^+}^{\nu(s)}\theta^{(2)}(s) \\ &= \frac{N(\nu(s))}{1-\nu(s)}\frac{d}{ds}\int_0^s \mathcal{E}_{\nu(\tau)}\left(-\nu(\tau)\frac{(s-\tau)^{\nu(\tau)}}{(1-\nu(\tau))}\right)\theta^{(2)}(\tau)d\tau \\ &= \frac{N(\nu(s))}{1-\nu(s)}\frac{d}{ds}\Big(\int_0^{N_1} \mathcal{E}_{\nu_1}\left(-\nu_1\frac{(s-\tau)^{\nu_1}}{(1-\nu_1)}\right)\theta^{(2)}(\tau)d\tau \\ &+ \cdots + \int_{B_{k-1}}^s \mathcal{E}_{\nu_k}\left(-\nu_k\frac{(s-\tau)^{\nu_k}}{(1-\nu_k)}\right)\theta^{(2)}(\tau)d\tau\Big). \end{aligned} \tag{3.1}$$

Thus, according to (3.1), the equation of the RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu is rewritten in the form

$$\begin{aligned} &\frac{N(\nu(s))}{1-\nu(s)}\frac{d}{ds}\Big(\int_0^{B_1} \mathcal{E}_{\nu_1}\left(-\nu_1\frac{(s-\tau)^{\nu_1}}{(1-\nu_1)}\right)\theta^{(2)}(\tau)d\tau \\ &\qquad + \cdots + \int_{B_{k-1}}^s \mathcal{E}_{\nu_k}\left(-\nu_k\frac{(s-\tau)^{\nu_k}}{(1-\nu_k)}\right)\theta^{(2)}(\tau)d\tau\Big) = \varphi_\theta(s), \end{aligned} \tag{3.2}$$

for $s \in \mathcal{B}_k$.

Now, we define the solution of the RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu, which is needed in this paper.

Definition 11. *The RL-fractional variable order multiterm BVP* (1.1) *in the sense of Atangana-Baleanu has a solution whenever there exist functions* $\theta_k \in C(0, B_k], \mathbb{R})$ *fulfilling Eq (3.2) so that* $\theta_k(0) = 0$ *and* $\theta(B_k) = \sum_{s_l \in \mathcal{B}_k} a_l\theta(s_l)$*,*

From our previous analysis above, for $k \in \{1, 2, ..., n\}$, the RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu can be expressed with the help of Eq (3.2) on the intervals $\mathcal{B}_k$.

For $0 \leq s \leq B_{k-1}$, by taking $\theta(s) \equiv 0$, and by Eq (2.4), Eq (3.1) becomes

$$
\begin{aligned}
{}^{ABR}\mathbf{D}_{0^+}^{x(s)}\theta(s) &= \frac{N(\nu(s))}{1-\nu(s)}\frac{d}{ds}\Big(\int_{B_{k-1}}^{s} \mathcal{E}_{\nu_k}\left(-\nu_k \frac{(s-\tau)^{\nu_k}}{(1-\nu_k)}\right)\theta^{(2)}(\tau)d\tau\Big) \\
&= {}^{ABR}D_{B_{k-1}^+}^{\nu_k}\theta^{(2)}(s) \\
&= {}^{ABR}\mathbf{D}_{B_{k-1}^+}^{\nu_k+2}\theta(s).
\end{aligned}
$$

So,

$$
{}^{ABR}\mathbf{D}_{0^+}^{x(s)}\theta(s) = {}^{ABR}\,\mathbf{D}_{B_{k-1}^+}^{x_k}\theta(s).
$$

Then, (3.2) is written as follows:

$$
{}^{ABR}\mathbf{D}_{B_{k-1}^+}^{x_k}\theta(s) = \varphi(s,\theta(s), {}^{AB}\,\mathbf{I}_{B_{k-1}^+}^{x_k}\theta(s)), \quad s \in \mathcal{B}_k.
$$

In this step, let us consider the RL-fractional constant order multiterm BVP in the sense of Atangana-Baleanu

$$
\begin{cases} {}^{ABR}\mathbf{D}_{B_{k-1}^+}^{x_k}\theta(s) = \varphi(s,\theta(s), {}^{AB}\,\mathbf{I}_{B_{k-1}^+}^{x_k}\theta(s)), \quad s \in \mathcal{B}_k, \\ \theta(B_{k-1}) = 0, \quad \theta(B_k) = \sum_{s_l\in\mathcal{B}_k} a_l\theta(s_l). \end{cases} \tag{3.3}
$$

In what follows, we assume that $\sum_{s_l\in\mathcal{B}_k} a_l\theta(s_l) \neq B_k - B_{k-1}$. Set

$$
A = \frac{1}{(B_k - B_{k-1}) - \sum_{s_l\in\mathcal{B}_k} a_l(s_l - B_{k-1})}.
$$

An auxiliary lemma is presented for beginning the main results in connection to the existence property for the RL-fractional constant order multiterm BVP (3.3) in the sense of Atangana-Baleanu.

Lemma 12. *The function $\theta \in C([B_{k-1}, B_k], \mathbb{R})$ is a solution of the RL-fractional constant order multiterm BVP* (3.3) *in the sense of Atangana-Baleanu if and only if θ satisfies*

$$
\theta(s) = A\left[\sum_{s_l\in\mathcal{B}_k} a_l\, {}^{AB}\boldsymbol{I}_{B_{k-1}^+}^{x_k}\varphi_\theta(s_l) - {}^{AB}\,\boldsymbol{I}_{B_{k-1}^+}^{x_k}\varphi_\theta(B_k)\right](s - B_{k-1}) + {}^{AB}\,\boldsymbol{I}_{B_{k-1}^+}^{x_k}\varphi_\theta(s). \tag{3.4}
$$

Proof. Let $\theta \in C([B_{k-1}, B_k], \mathbb{R})$ be a solution of the RL-fractional constant order multiterm BVP (3.3) in the sense of Atangana-Baleanu. Let us employ the operator ${}^{AB}\mathbf{I}_{B_{k-1}^+}^{x_k}$ on both sides (3.3), and using Lemma 6 we get

$$
\theta(s) = \eta_1 + \eta_2(s - B_{k-1}) + {}^{AB}\,\mathbf{I}_{B_{k-1}^+}^{x_k}\varphi_\theta(s), \quad s \in \mathcal{B}_k,\ k \in \{1, 2, ..., n\}. \tag{3.5}
$$

By $\theta(B_{k-1}) = 0$, we get $\eta_1 = 0$.

Let $s = s_l$ in (3.5). Then, we obtain

$$
\theta(s_l) = \eta_2(s_l - B_{k-1}) + {}^{AB}\,\mathbf{I}_{B_{k-1}^+}^{x_k}\varphi_\theta(s_l).
$$

Thus, we have

$$\begin{aligned}\sum_{s_l\in\mathcal{B}_k} a_l\theta(s_l) &= \sum_{s_l\in\mathcal{B}_k} a_l\Big(\eta_2(s_l-B_{k-1}) +^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l)\Big)\\ &= \eta_2\sum_{s_l\in\mathcal{B}_k} a_l(s_l-B_{k-1}) + \sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l). \end{aligned}\tag{3.6}$$

On the other hand, we have

$$\begin{aligned}\eta_2(B_k-B_{k-1}) +^{AB}\mathbf{I}^{x_k}_{N^+_{k-1}}\varphi_\theta(B_k) &= \theta(B_k)\\ &= \sum_{s_l\in\mathcal{B}_k} a_l\theta(s_l). \end{aligned}\tag{3.7}$$

Hence,

$$\eta_2(B_k-B_{k-1}) +^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k) = \eta_2\sum_{s_l\in\mathcal{B}_k} a_l(s_l-B_{k-1}) + \sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l),\tag{3.8}$$

which implies

$$\eta_2 = A\left[\sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l) -^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\right].\tag{3.9}$$

Substitute (3.9) into (3.5). We obtain (3.4) immediately.

Now we prove the sufficiency. Let $\theta\in C([B_{k-1},B_k],\mathbb{R})$ satisfies (3.4). Employing the operator ${}^{ABR}\mathbf{D}^{x_k}_{B^+_{k-1}}$ on both sides of (3.4), it follows from Lemma 6 and

$${}^{ABR}\mathbf{D}^{x_k}_{B^+_{k-1}}(s-B_{k-1}) = 0$$

that

$$\begin{aligned}{}^{ABR}D^{x(s)}_{B^+_{k-1}}\theta(s) &= A\left[\sum_{k=1}^{m} a_k^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_k) -^{AB}\mathbf{I}^{x(B_k)}_{B^+_{k-1}}\varphi_\theta(B_k)\right]\\ &\times\ {}^{ABR}\mathbf{D}^{x_k}_{B^+_{k-1}}(s-B_{k-1}) +^{ABR}\mathbf{D}^{x_k}_{B^+_{k-1}}\ {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s)\\ &= \varphi_\theta(s). \end{aligned}$$

Let $s=s_l$ in (3.4). Then, we obtain

$$\theta(s_l) = A\left[\sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l) -^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\right](s_l-B_{k-1}) +^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l).$$

Then, we derive

$$
\begin{aligned}
\sum_{s_l\in\mathcal{B}_k} a_l\theta(s_l) &= A\left[\sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_k) -{}^{AB}\,\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\right]\sum_{s_l\in\mathcal{B}_k} a_l(s_l-B_{k-1})\\
&+ \sum_{s_l\in\mathcal{B}_k} a_l^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l)\\
&= A\left[\sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l) -{}^{AB}\,\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\right]\left((B_k-B_{k-1})-\frac{1}{A}\right)\\
&+ \sum_{s_l\in\mathcal{B}_k} a_l^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l)\\
&= A\left[\sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l) -{}^{AB}\,\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\right](B_k-B_{k-1})\\
&- \sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l) +{}^{AB}\,\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\\
&+ \sum_{s_l\in\mathcal{B}_k} a_l^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l)\\
&= A\left[\sum_{s_l\in\mathcal{B}_k} a_l\,{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l) -{}^{AB}\,\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\right](B_k-B_{k-1})\\
&+ {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\\
&= \xi(B_k).
\end{aligned}
$$

The proof is completed.

Lemma 13. *If α is constant such that $\alpha\in(n,n+1],\ n\in\mathbb{B}$ and $\beta=\alpha-n$, then*

1) ${}^{AB}\boldsymbol{I}^{\alpha}_{B^+_{k-1}}\theta=\frac{1-\beta}{N(\beta)}I^n_{B_{k-1}}\theta+\frac{\beta}{N(\beta)}I^{\alpha}_{B_{k-1}}\theta.$

2) For $n=2$, we obtain

$$
{}^{AB}\boldsymbol{I}^{\alpha}_{B^+_{k-1}}\theta(s)=\frac{3-\alpha}{N(\alpha-2)}\int_{B_{k-1}}^{s}(s-\tau)\xi(\tau)d\tau+\frac{\alpha-2}{N(\alpha-2)\Gamma(\alpha)}\int_{B_{k-1}}^{s}(s-\tau)^{\alpha-1}\theta(\tau)d\tau.
$$

Proof. By (2.3) and (2.1), we obtain

$$
\begin{aligned}
{}^{AB}\mathbf{I}^{\alpha}_{B^+_{k-1}}\theta(s) &= I^n_{B^+_{k-1}}\,{}^{AB}I^{\beta}_{B^+_{k-1}}\theta(s)\\
&= I^n_{B^+_{k-1}}\left(\frac{1-\beta}{N(\beta)}\theta(s)+\frac{\beta}{N(\beta)}\int_{B_{k-1}}^{s}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\theta(\tau)d\tau\right)\\
&= I^n_{B^+_{k-1}}\left(\frac{1-\beta}{N(\beta)}\xi(s)+\frac{\beta}{N(\beta)}I^{\beta}_{B^+_{k-1}}\theta(s)\right)\\
&= \frac{1-\beta}{N(\beta)}I^n_{B^+_{k-1}}\theta(s)+\frac{\beta}{N(\beta)}I^{n+\beta}_{B^+_{k-1}}\theta(s)\\
&= \frac{1-\beta}{N(\beta)}I^n_{B^+_{k-1}}\theta(s)+\frac{\beta}{N(\beta)}I^{\alpha}_{B^+_{k-1}}\theta(s).
\end{aligned}
$$

This completes the proof of the first part. Now, we continue the following computations:

$$
\begin{aligned}
{}^{AB}\mathbf{I}^{\alpha}_{B^+_{k-1}}\theta(s) &= \frac{1-\beta}{N(\beta)} I^2_{B_{k-1}}\theta(s) + \frac{\beta}{N(\beta)} I^{\alpha}_{B_{k-1}}\theta(s) \\
&= \frac{3-\alpha}{N(\alpha-2)} \int_{B_{k-1}}^{s} (s-\tau)\theta(\tau)d\tau + \frac{\alpha-2}{N(\alpha-2)\Gamma(\alpha)} \int_{B_{k-1}}^{s} (s-\tau)^{\alpha-1}\theta(\tau)d\tau,
\end{aligned} \tag{3.10}
$$

and the proof is completed.

Theorem 14. *The function $\theta \in C([B_{k-1}, B_k], \mathbb{R})$ is a solution of the RL-fractional constant order multiterm BVP* (3.3) *in the sense of Atangana-Baleanu. Then, it is given by*

$$
\begin{aligned}
\theta(s) &= q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)\varphi_\theta(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k-\tau)\varphi_\theta(\tau)d\tau\right) \\
&+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau\right) \\
&+ q_3 \int_{B_{k-1}}^{s} (s-\tau)\varphi_\theta(\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
q_1 &:= \frac{A(3-x_k)}{N(x_k-2)},\ q_2 := \frac{A(x_k-2)}{N(x_k-2)}, \\
q_3 &:= \frac{3-x_k}{N(x_k-2)},\ q_4 := \frac{x_k-2}{N(x_k-2)}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
\theta(s) &= A\left[\sum_{s_l\in\mathcal{B}_k} a_l\, {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s_l) - {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(B_k)\right](s-B_{k-1}) + {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\varphi_\theta(s) \\
&= A\left[\sum_{s_l\in\mathcal{B}_k} a_l\left(\frac{3-x_k}{N(x_k-2)} \int_{B_{k-1}}^{s_l} (s_l-\tau)\theta(\tau)d\tau + \frac{x_k-2}{N(x_k-2)\Gamma(x_k)} \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1}\theta(\tau)d\tau\right)\right. \\
&\left. -\left(\frac{3-x_k}{N(x_k-2)} \int_{B_{k-1}}^{B_k} (B_k-\tau)\theta(\tau)d\tau + \frac{x_k-2}{N(x_k-2)\Gamma(x_k)} \int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1}\theta(\tau)d\tau\right)\right](s-B_{k-1}) \\
&+ \left(\frac{3-x_k}{N(x_k-2)} \int_{B_{k-1}}^{s} (s-\tau)\theta(\tau)d\tau + \frac{x_k-2}{N(x_k-2)\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\theta(\tau)d\tau\right) \\
&= \frac{A(3-x_k)}{N(x_k-2)}(s-B_{k-1})\left(\sum_{s_l\in\mathcal{N}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)\varphi_\theta(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k-\tau)\varphi_\theta(\tau)d\tau\right) \\
&+ \frac{A(x_k-2)(s-B_{k-1})}{N(x_k-2)\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau\right) \\
&+ \frac{3-x_k}{N(x_k-2)} \int_{B_{k-1}}^{s} (s-\tau)\varphi_\theta(\tau)d\tau + \frac{x_k-2}{N(x_k-2)\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)\varphi_\xi(\tau)d\tau\right)\\
&+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau\right)\\
&+ q_3\int_{B_{k-1}}^{s}(s-\tau)\varphi_\theta(\tau)d\tau+\frac{q_4}{\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau.
\end{aligned}
$$

This completes the proof.

We shall prove the existence results for the RL-fractional constant order multiterm BVP (3.3) in the sense of Atangana-Baleanu. The following result is derived due to Krasnoselskii's fixed point theorem.

Theorem 15. *Assume that both hypotheses* ***(HY1)*** *and* ***(HY2)*** *are satisfied and*

$$L_k K_\varphi (1+\mathcal{G}) < 1. \tag{3.11}$$

Then, the RL-fractional constant order multiterm BVP (3.3) *in the sense of Atangana-Baleanu possesses a solution* $\overline{\theta_k}$ *in* $C([B_{k-1}, B_k], \mathbb{R})$.

Proof. Transform the RL-fractional constant order multiterm BVP (3.3) in the sense of Atangana-Baleanu into a fixed point problem. Define

$$\mathcal{P} : C([B_{k-1}, B_k], \mathbb{R}) \to C([B_{k-1}, B_k], \mathbb{R}),$$

by

$$
\begin{aligned}
(\mathcal{P}\theta)(s) &= q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)\varphi_\theta(\tau)d\tau\right)\\
&+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l\int_{B_{k-1}}^{s_k}(s_k-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau\right)\\
&+ q_3\int_{B_{k-1}}^{s}(s-\tau)\varphi_\theta(\tau)d\tau+\frac{q_4}{\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau.
\end{aligned}
\tag{3.12}
$$

Let

$$R_k \geq \frac{L_k\varphi^\star}{1-L_kK_\varphi(1+\mathcal{G})}, \tag{3.13}$$

where $\mathcal{G} := \dfrac{(3-x_k)(B_k-B_{k-1})^2}{2N(x_k-2)} + \dfrac{(x_k-2)(B_k-B_{k-1})^{x_k}}{N(x_k-2)\Gamma(x_k+1)}$ and $\varphi^\star := \sup_{s\in\mathcal{B}} |\varphi(s,0,0)|$. Let us consider the following set:

$$\Phi_{R_k} = \{\theta \in C([B_{k-1}, B_k], \mathbb{R}),\ \|\theta\|_\infty \leq R_k\}.$$

Clearly, $\Phi_{R_k} \neq \emptyset$ is a closed convex set with the boundedness property.

The properties of fractional integrals and **(HY2)** implies that the operator $\mathcal{P}$ defined in (3.12) is well-defined.

To show the fact that $\mathcal{P}$ satisfies $\mathcal{P}(\Phi_{R_k}) \subseteq (\Phi_{R_k})$, indeed, for $\theta \in \Phi_{R_k}$, the condition **(HY2)** gives (for $s \in \mathcal{B}_k$) that

$$\begin{aligned} |\mathcal{P}\theta(s)| &\leq q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)|\varphi_\theta(\tau)|d\tau + \int_{B_{k-1}}^{B_k}(B_k-\tau)|\varphi_\theta(\tau)|d\tau\right) \\ &+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}|\varphi_\theta(\tau)|d\tau + \int_{B_{k-1}}^{B_k}(B_k-\tau)^{x_k-1}|\varphi_\theta(\tau)|d\tau\right) \\ &+ q_3\int_{B_{k-1}}^{s}(s-\tau)|\varphi_\theta(\tau)|d\tau + \frac{q_4}{\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}|\varphi_\theta(\tau)|d\tau. \end{aligned} \tag{3.14}$$

Moreover, we have for every $\tau \in \mathcal{B}_k$

$$\begin{aligned} |\varphi_\theta(\tau)| &= |\varphi(\tau,\theta(\tau),{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\theta(\tau))| \\ &\leq |\varphi(\tau,\theta(\tau),{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\theta(\tau)) - \varphi(\tau,0,0)| + |\varphi(\tau,0,0)| \\ &= K_\varphi\left(|\theta(\tau)| + |{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\theta(\tau)|\right) + \varphi^\star, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} |{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\theta(s)| &= \left|\frac{3-x_k}{N(x_k-2)}\int_{B_{k-1}}^{s}(s-\tau)\theta(\tau)d\tau + \frac{x_k-2}{N(x_k-2)\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}\theta(\tau)d\tau\right| \\ &\leq \frac{3-x_k}{N(x_k-2)}\int_{N_{k-1}}^{s}(s-\tau)|\xi(\tau)|d\tau + \frac{x_k-2}{N(x_k-2)\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}|\theta(\tau)|d\tau \\ &\leq \frac{3-x_k}{N(x_k-2)}\frac{(B_k-B_{k-1})^2}{2}\|\theta\| + \frac{(x_k-2)(B_k-B_{k-1})^{x_k}}{N(x_k-2)\Gamma(x_k+1)}\|\theta\| \\ &= \left(\frac{3-x_k}{N(x_k-2)}\frac{(B_k-B_{k-1})^2}{2} + \frac{(x_k-2)(B_k-B_{k-1})^{x_k}}{N(x_k-2)\Gamma(x_k+1)}\right)\|\theta\| \\ &= \left(\frac{(3-x_k)(B_k-B_{k-1})^2}{2N(x_k-2)} + \frac{(x_k-2)(B_k-B_{k-1})^{x_k}}{N(x_k-2)\Gamma(x_k+1)}\right)\|\theta\| \\ &= \mathcal{G}\|\theta\|. \end{aligned}$$

Hence,

$$\|\varphi_\theta\| \leq K_\varphi(1+\mathcal{G})\|\theta\| + \varphi^\star, \tag{3.16}$$

and

$$\begin{aligned} |\mathcal{P}\theta(s)| &\leq q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)|\varphi_\theta(\tau)|d\tau + \int_{B_{k-1}}^{B_k}(B_k-\tau)|\varphi_\theta(\tau)|d\tau\right) \\ &+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}|\varphi_\theta(\tau)|d\tau + \int_{B_{k-1}}^{B_k}(B_k-\tau)^{x_k-1}|\varphi_\theta(\tau)|d\tau\right) \\ &+ q_3\int_{B_{k-1}}^{s}(s-\tau)\left(K_\varphi(1+\mathcal{G})\|\theta\| + \varphi^\star\right)d\tau \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1} \left(K_\varphi (1+\mathcal{G}) \|\theta\| + \varphi^\star\right) d\tau. \\
&= \left(K_\varphi (1+\mathcal{G}) \|\theta\| + \varphi^\star\right) \Bigg[q_1(s-B_{k-1}) \left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau) d\tau + \int_{B_{k-1}}^{B_k} (B_k-\tau) d\tau \right) \\
&\quad + \frac{q_2(s-B_{k-1})}{\Gamma(x_k)} \left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1} + \int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1} d\tau \right) \\
&\quad + q_3 \int_{B_{k-1}}^{s} (s-\tau) d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1} d\tau \Bigg] \\
&\le \left(K_\varphi (1+\mathcal{G}) \|\theta\| + \varphi^\star\right) \Bigg[\frac{q_1}{2} (B_k-B_{k-1}) \left(\sum_{s_l \in \mathcal{B}_k} a_l (s_l-B_{k-1})^2 + (B_k-B_{k-1})^2 \right) \\
&\quad + \frac{q_2(B_k-B_{k-1})}{\Gamma(\mu_k+1)} \left(\sum_{s_l \in \mathcal{B}_k} a_l (s_l-B_{k-1})^{x_k} + (B_k-B_{k-1})^{x_k} \right) \\
&\quad + \frac{q_3}{2} (B_k-B_{k-1})^2 + \frac{q_4}{\Gamma(x_k+1)} (B_k-B_{k-1})^{x_k} \Bigg] \\
&= L_k \left(K_\varphi (1+\mathcal{G}) \|\theta\| + \varphi^\star\right) \\
&= L_k K_\varphi (1+\mathcal{G}) \|\theta\| + L_k \varphi^\star, \qquad (3.17)
\end{aligned}
$$

where

$$
\begin{aligned}
L_k &= \frac{q_1}{2} (B_k-B_{k-1}) \left(\sum_{s_l \in \mathcal{B}_k} a_l (s_l-B_{k-1})^2 + (B_k-B_{k-1})^2 \right) \\
&+ \frac{q_2(B_k-B_{k-1})}{\Gamma(x_k+1)} \left(\sum_{s_l \in \mathcal{B}_k} a_l (s_l-B_{k-1})^{x_k} + (B_k-B_{k-1})^{x_k} \right) \\
&+ \frac{q_3}{2} (B_k-B_{k-1})^2 + \frac{q_4}{\Gamma(x_k+1)} (B_k-B_{k-1})^{x_k}. \qquad (3.18)
\end{aligned}
$$

Thus, (3.13) and (3.17) imply that

$$
\begin{aligned}
\|\mathcal{P}\theta\|_\infty &\le L_k K_\varphi (1+\mathcal{G}) \|\theta\| + L_k \varphi^\star \\
&\le L_k K_\varphi (1+\mathcal{G}) R_k + (1 - L_k K_\varphi (1+\mathcal{G})) R_k \\
&\le R_k,
\end{aligned}
$$

which means that $\mathcal{P}(\Phi_{R_k}) \subseteq \Phi_{R_k}$.

We define the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ on Φ_{R_k} by

$$
\begin{aligned}
(\mathcal{P}_1\theta)(s) &= q_1(s-B_{k-1}) \left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau) \varphi_\theta(\tau) d\tau - \int_{B_{k-1}}^{B_k} (B_k-\tau) \varphi_\theta(\tau) d\tau \right) \qquad (3.19) \\
&+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)} \left(\sum_{l=1}^{m} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1} \varphi_\theta(\tau) d\tau - \int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1} \varphi_\theta(\tau) d\tau \right),
\end{aligned}
$$

and

$$(\mathcal{P}_2\theta)(s) = q_3 \int_{B_{k-1}}^{s} (s-\tau)\varphi_\theta(\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau. \quad (3.20)$$

It follows that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$.

We shall investigate that $\mathcal{P}_1$ and $\mathcal{P}_2$ include the conditions of Theorem 10 in several steps:

Step 1: $\mathcal{P}_1\theta + \mathcal{P}_2\acute{\theta} \in \Phi_{R_k}$ for all $\theta,\ \acute{\theta} \in \Phi_{R_k}$.

For $\theta,\ \acute{\theta} \in \Phi_{R_k}$, **(HY2)** and $s \in \mathcal{B}_k$, we have

$$\begin{aligned}
|(\mathcal{P}\theta + \mathcal{P}_2\acute{\theta})(s)| &\leq q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)|\varphi_\theta(\tau)|d\tau + \int_{B_{k-1}}^{B_k} (B_k-\tau)|\varphi_\theta(\tau)|d\tau\right) \\
&+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1}|\varphi_\theta(\tau)|d\tau\right. \\
&+ \left.\int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1}|\varphi_\theta(\tau)|d\tau\right) \\
&+ q_3 \int_{B_{k-1}}^{s} (s-\tau)|\varphi_{\acute{\theta}}(\tau)|d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}|\varphi_{\acute{\theta}}(\tau)|d\tau. \quad (3.21)
\end{aligned}$$

By Eqs (3.16), (3.13) and $\theta,\ \acute{\theta} \in \Phi_{R_k}$, we have

$$\begin{aligned}
|(\mathcal{P}_1\theta + \mathcal{P}_2\acute{\theta})(s)| &\leq \|\varphi_\theta\|\left[q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)d\tau + \int_{B_{k-1}}^{B_k} (B_k-\tau)d\tau\right)\right. \\
&+ \left.\frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1}d\tau + \int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1}d\tau\right)\right] \\
&+ \|\varphi_{\acute{\theta}}\|\left[q_3 \int_{B_{k-1}}^{s} (s-\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}d\tau.\right] \\
&= \left(K_\varphi\,(1+\mathcal{G})\,R_k + \varphi^\star\right)\left[q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)d\tau\right.\right. \\
&+ \left.\int_{B_{k-1}}^{B_k} (B_k-\tau)d\tau\right) \\
&+ \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1}d\tau + \int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1}d\tau\right) \\
&+ \left.q_3 \int_{B_{k-1}}^{s} (s-\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}d\tau.\right] \\
&\leq \left(K_\varphi\,(1+\mathcal{G})\,R_k + \varphi^\star\right)L_k \\
&\leq R_k.
\end{aligned}$$

Therefore,

$$\|\mathcal{P}_1\theta + \mathcal{P}_2\acute{\theta}\|_\infty \leq R_k.$$

Thus, $\mathcal{P}_1\theta + \mathcal{P}_2\acute{\theta} \in \Phi_{R_k}$.

Step 2: $\mathcal{P}_2$ is a contraction.

Let $\theta, \tilde{\theta} \in C([B_{k-1}, B_k], \mathbb{R})$ and $s \in \mathcal{B}_k$. We have

$$\begin{aligned}
|\mathcal{P}_2\theta(s) - \mathcal{P}_2\tilde{\theta}(s)| &\leq q_3 \int_{B_{k-1}}^{s} (s-\tau)|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau \\
&+ \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau. \\
&\leq q_1(s - B_{k-1})\left(\sum_{l=1}^{m} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau\right. \\
&+ \left.\int_{B_{k-1}}^{B_k} (B_k-\tau)|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau\right) \\
&+ \frac{q_2(s - B_{k-1})}{\Gamma(x_k)}\left(\sum_{l=1}^{m} a_l \int_{B_{k-1}}^{s_l} (s_l-\tau)^{x_k-1}|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau\right. \\
&+ \left.\int_{B_{k-1}}^{B_k} (B_k-\tau)^{x_k-1}|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau\right) \\
&+ q_3 \int_{B_{k-1}}^{s} (s-\tau)|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau \\
&+ \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)|d\tau. \qquad (3.22)
\end{aligned}$$

On the other side, for every $\tau \in \mathcal{B}_k$, we write

$$\begin{aligned}
|\varphi_\theta(\tau) - \varphi_{\tilde{\theta}}(\tau)| &= |\varphi(\tau, \theta(\tau), {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\theta(\tau)) - \varphi(\tau, \tilde{\theta}(\tau), {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\tilde{\theta}(\tau))| \\
&\leq K_\varphi\left(|\theta(\tau) - \tilde{\theta}(\tau)| + |{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\theta(\tau) - {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}\tilde{\theta}(\tau)|\right) \\
&= K_\varphi\left(|(\theta - \tilde{\theta})(\tau)| + |{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}(\theta - \tilde{\theta})(\tau)|\right),
\end{aligned}$$

and

$$\begin{aligned}
|{}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}(\theta - \tilde{\theta})(s)| &= |\frac{3 - x_k}{N(x_k - 2)} \int_{B_{k-1}}^{s} (s-\tau)(\theta - \tilde{\theta})(\tau)d\tau \\
&+ \frac{x_k - 2}{N(x_k - 2)\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}(\theta - \tilde{\theta})(\tau)d\tau| \\
&\leq \frac{3 - x_k}{N(x_k - 2)} \int_{B_{k-1}}^{s} (s-\tau)|(\theta - \tilde{\theta})(\tau)|d\tau \\
&+ \frac{x_k - 2}{N(x_k - 2)\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}|(\theta - \tilde{\theta})(\tau)|d\tau \\
&\leq \frac{3 - x_k}{N(x_k - 2)} \frac{(B_k - B_{k-1})^2}{2}||\theta - \tilde{\theta}|| \\
&+ \frac{(x_k - 2)(B_k - B_{k-1})^{x_k}}{N(x_k - 2)\Gamma(x_k + 1)}||\theta - \tilde{\theta}||
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{3-x_k}{N(x_k-2)}\frac{(B_k-B_{k-1})^2}{2}+\frac{(x_k-2)(B_k-B_{k-1})^{x_k}}{N(x_k-2)\Gamma(x_k+1)}\right)\|\theta-\tilde{\theta}\| \\
&= \left(\frac{(3-x_k)(B_k-B_{k-1})^2}{2N(x_k-2)}+\frac{(x_k-2)(B_k-B_{k-1})^{x_k}}{N(x_k-2)\Gamma(x_k+1)}\right)\|\theta-\tilde{\theta}\| \\
&= \mathcal{G}\|\theta-\tilde{\theta}\|.
\end{aligned}
$$

Hence,

$$\|\varphi_\theta-\varphi_{\tilde{\theta}}\| \leq K_\varphi\,(1+\mathcal{G})\,\|\theta-\tilde{\theta}\|. \tag{3.23}$$

By replacing (3.23) in the inequality (3.22), we obtain

$$
\begin{aligned}
|\mathcal{P}_2\theta(s)-\mathcal{P}_2\tilde{\theta}(s)| \leq\ & K_\varphi\,(1+\mathcal{G})\,\|\theta-\tilde{\theta}\|\Bigg[\frac{q_1}{2}(B_k-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k}a_l(s_l-B_{k-1})^2+(B_k-B_{k-1})^2\right) \\
+\ & \frac{q_2(B_k-B_{k-1})}{\Gamma(x_k+1)}\left(\sum_{s_l\in\mathcal{B}_k}a_l(s_l-B_{k-1})^{x_k}+(B_k-B_{k-1})^{x_k}\right) \\
+\ & \frac{q_3}{2}(B_k-B_{k-1})^2+\frac{q_4}{\Gamma(x_k+1)}(B_k-B_{k-1})^{x_k}\Bigg] \\
\leq\ & L_kK_\varphi\,(1+\mathcal{G})\,\|\theta-\tilde{\theta}\|.
\end{aligned} \tag{3.24}
$$

Consequently by (3.11), $\mathcal{P}_2$ is a contraction.

Step 3: $\mathcal{P}_1$ is continuous.

The Continuity of φ implies that $\mathcal{P}_1$ is continuous.

Step 4: $\mathcal{P}_1(\Phi_{R_k})$ is bounded in Φ_{R_k}.

Similar to Step 1, we know that $\mathcal{P}_1(\Phi_{R_k})\subset\Phi_{R_k}$. It implies that $\mathcal{P}_1(\Phi_{R_i})$ is a bounded set in Φ_{R_k}.

Step 5: $\mathcal{P}_1(\Phi_{R_k})$ is equicontinuous .

For arbitrary $t_1,t_2\in\mathcal{B}_k$, with $t_1<t_2$, let $\xi\in\Phi_{R_k}$. We write

$$
\begin{aligned}
|\mathcal{P}_1(\theta)(t_2)\ -\ & \mathcal{P}_1(\theta)(t_1)| = \Bigg|q_1(t_2-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k}a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)\varphi_\theta(\tau)d\tau\right) \\
+\ & \frac{q_2(t_2-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k}a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau\right) \\
-\ & q_1(t_1-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k}a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)\varphi_\theta(\tau)d\tau\right) \\
-\ & \frac{q_2(t_1-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k}a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau-\int_{N_{k-1}}^{N_k}(N_k-\tau)^{x_k-1}\varphi_\xi(\tau)d\tau\right).\Bigg| \\
\leq\ & q_1(t_2-t_1)|\left(\sum_{s_l\in\mathcal{B}_k}a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)\varphi_\theta(\tau)d\tau\right)| \\
+\ & \frac{q_2(t_2-t_1)}{\Gamma(x_k)}|\left(\sum_{s_l\in\mathcal{B}_k}a_l\int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau-\int_{B_{k-1}}^{B_k}(B_k-\tau)^{x_k-1}\varphi_\theta(\tau)d\tau\right)\Bigg|.
\end{aligned}
$$

Hence, $|\mathcal{P}_1(\theta)(t_2) - \mathcal{P}_1(\theta)(t_1)| \to 0$ as $t_2 - t_1 \to 0$. It implies that $\mathcal{P}_1(\Phi_{R_k})$ is equicontinuous. Therefore, all conditions of Theorem 10 hold and we deduce that $\mathcal{P}$ has a fixed point $\overline{\theta}_k \in \Phi_{R_k}$. Then the RL-fractional constant order multiterm BVP (3.3) in the sense of Atangana-Baleanu has a solution $\overline{\theta}_k \in C([B_{k-1}, B_k], \mathbb{R})$.

Now, we can present the final theorem about the existence result in connection to the RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu.

Theorem 16. *Assume that conditions **(HY1)**, **(HY2)** and inequality (3.11) hold for all $k \in \{1, \dots, n\}$. Then, the RL-fractional variable order multiterm BVP* (1.1) *in the sense of Atangana-Baleanu possesses a solution in $C([0, B], \mathbb{R})$.*

Proof. For each $k \in \{1, \dots, n\}$ and based on Theorem 15, we know that the RL-fractional constant order multiterm BVP (3.3) in the sense of Atangana-Baleanu possesses a solution $\overline{\theta}_k \in C([B_{k-1}, B_k], \mathbb{R})$. We define

$$\theta_1(s) = \overline{\theta}_1(s), \quad s \in \mathcal{B}_1,$$

and for any $k \in \{2, ..., n\}$,

$$\theta_k(s) = \begin{cases} 0, & s \in [0, B_{k-1}], \\ \overline{\theta}_k(s), & s \in \mathcal{B}_k. \end{cases}$$

Thus, $\theta_k \in C([0, B_k], \mathbb{R})$ solves the integral Eq (3.2) for $s \in \mathcal{B}_k$ with $\theta_k(0) = 0$ and $\theta_k(B_k) = \sum_{s_l \in \mathcal{B}_k} a_l \theta_k(s_l) = \sum_{s_l \in \mathcal{B}_k} a_l \overline{\theta}_k(s_l)$.

Then, the function

$$\theta(s) = \begin{cases} \theta_1(s), & s \in \mathcal{B}_1, \\ \theta_2(s) = \begin{cases} 0, & s \in \mathcal{B}_1, \\ \overline{\theta}_2(s), & s \in \mathcal{B}_2, \end{cases} \\ . \\ . \\ . \\ \theta_n(s) = \begin{cases} 0, & s \in [0, B_{n-1}], \\ \overline{\theta}_n(s), & s \in \mathcal{B}_n, \end{cases} \end{cases}$$

gives the solution for the RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu.

3.2. Ulam-Hyers stability with respect to partition $\mathcal{P}$

For the implicit RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu, we adopt a definition in [18] in connection to the Ulam-Hyers stability.

Let $\mathcal{P} = \{\mathcal{B}_k : 1 = 1, 2, ..., n\}$ be a partition of the interval $\mathcal{B}$.

Definition 17. *([18, 19]) The RL-fractional variable order multiterm BVP* (1.1) *in the sense of Atangana-Baleanu is Ulam-Hyers stable if for all $k \in \{1, 2, ..., n\}$, $\exists\, c_\varphi > 0$, s.t., $\forall\, \epsilon > 0$ and for each solution $y \in C([B_{k-1}, B_k], \mathbb{R})$ of the inequality*

$$|\boldsymbol{D}^{x(s)}_{B^+_{k-1}} y(s) - \varphi_y(s)| \le \epsilon, \quad s \in \mathcal{B}_k := (B_{k-1}, B_k], \tag{3.25}$$

$\exists$ *a solution* $\theta \in C([B_{k-1}, B_k], \mathbb{R})$ *of* (3.3) *with*

$$|y(s) - \theta(s)| \leq c_{\varphi}\epsilon, \; s \in [B_{k-1}, B_k].$$

Remark 18. $y \in C([B_{k-1}, B_k], \mathbb{R})$ *is a solution of* (3.25) *iff there exists* $h \in C([B_{k-1}, B_k], \mathbb{R})$ *(depending on y), s.t.,*

(i) $|h(s)| \leq \epsilon, \; \forall \, s \in [B_{k-1}, B_k]$,

(ii) $D^{x(s)}_{N^+_{k-1}} y(s) = \varphi_y(s) + h(s), \; \forall \, s \in \mathcal{B}_k$.

Lemma 19. *If* $y \in C([B_{k-1}, B_k], \mathbb{R})$ *is a solution of* (3.25), *then y satisfies*

$$|y(s) - A_y - q_3 \int_{B_{k-1}}^{s} (s-\tau)\varphi_y(\tau)d\tau - \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k - 1}\varphi_y(\tau)d\tau| \leq \epsilon L_k, \tag{3.26}$$

where

$$\begin{aligned} A_y &= q_1(s - B_{k-1})\left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l - \tau)\varphi_y(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k - \tau)\varphi_y(\tau)d\tau\right) \\ &+ \frac{q_2(s - B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l - \tau)^{x_k-1}\varphi_y(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k - \tau)^{x_k-1}\varphi_y(\tau)d\tau\right). \end{aligned} \tag{3.27}$$

Proof. By Remark 18, we have that

$$^{ABR}\mathbf{D}^{x_k}_{B^+_{k-1}} y(s) = \varphi(s, y(s), ^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}} y(s)) + h(s) = \varphi_y(s) + h(s),$$

and

$$\begin{cases} ^{ABR}\mathbf{D}^{x_k}_{B^+_{k-1}} y(s) = \varphi_y(s) + h(s), \quad s \in \mathcal{B}_k, \\ y(B_{k-1}) = 0, \quad\; y(B_k) = \sum_{s_l \in \mathcal{B}_k} a_l y(s_l). \end{cases} \tag{3.28}$$

Then, by Theorem 14, we get

$$\begin{aligned} y(s) &= q_1(s - B_{k-1})\left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l - \tau)(\varphi_y(\tau) + h(\tau))d\tau - \int_{B_{k-1}}^{B_k} (B_k - \tau)(\varphi_y(\tau) + h(\tau))d\tau\right) \\ &+ \frac{q_2(s - B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l - \tau)^{x_k-1}(\varphi_y(\tau) + h(\tau))d\tau\right. \\ &- \left.\int_{B_{k-1}}^{B_k} (B_k - \tau)^{x_k-1}(\varphi_y(\tau) + h(\tau))d\tau\right) \\ &+ q_3 \int_{N_{k-1}}^{s} (s-\tau)(\varphi_y(\tau) + h(\tau))d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}(\varphi_y(\tau) + h(\tau))d\tau. \\ &= q_1(s - B_{k-1})\left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l - \tau)\varphi_y(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k - \tau)\varphi_y(\tau)d\tau\right) \\ &+ \frac{q_2(s - B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l \in \mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l} (s_l - \tau)^{x_k-1}\varphi_y(\tau)d\tau - \int_{B_{k-1}}^{B_k} (B_k - \tau)^{x_k-1}\varphi_y(\tau)d\tau\right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)h(\tau)d\tau - \int_{B_{k-1}}^{B_k}(B_k-\tau)h(\tau)d\tau\right)\\
&+ \quad \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}h(\tau)d\tau - \int_{B_{k-1}}^{B_k} B_k-\tau)^{x_k-1}h(\tau)d\tau\right)\\
&+ \quad q_3\int_{B_{k-1}}^{s}(s-\tau)(\varphi_y(\tau)+h(\tau))d\tau + \frac{q_4}{\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}(\varphi_y(\tau)+h(\tau))d\tau.\\
&= \quad A_y + q_3\int_{B_{k-1}}^{s}(s-\tau)\varphi_y(\tau)d\tau + \frac{q_4}{\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}\varphi_y(\tau)d\tau\\
&+ \quad q_1(s-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)h(\tau)d\tau - \int_{B_{k-1}}^{B_k}(B_k-\tau)h(\tau)d\tau\right)\\
&+ \quad \frac{q_2(s-B_{k-1})}{\Gamma(x_k)}\left(\sum_{s_l\in\mathcal{B}_k} a_l \int_{B_{k-1}}^{s_l}(s_l-\tau)^{x_k-1}h(\tau)d\tau - \int_{B_{k-1}}^{B_k}(B_k-\tau)^{x_k-1}h(\tau)d\tau\right)\\
&+ \quad q_3\int_{B_{k-1}}^{s}(s-\tau)h(\tau)d\tau + \frac{q_4}{\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}h(\tau)d\tau. \qquad (3.29)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& |y(s) - A_y - q_3\int_{B_{k-1}}^{s}(s-\tau)\varphi_y(\tau)d\tau - \frac{q_4}{\Gamma(x_k)}\int_{B_{k-1}}^{s}(s-\tau)^{x_k-1}\varphi_y(\tau)d\tau|\\
\le \quad & \epsilon\left[\frac{q_1}{2}(B_k-B_{k-1})\left(\sum_{s_l\in\mathcal{B}_k} a_l(s_l-B_{k-1})^2 + (B_k-B_{k-1})^2\right)\right.\\
+ \quad & \frac{q_2(B_k-B_{k-1})}{\Gamma(x_k+1)}\left(\sum_{s_l\in\mathcal{B}_k} a_l(s_l-B_{k-1})^{x_k} + (B_k-B_{k-1})^{x_k}\right)\\
+ \quad & \left.\frac{q_3}{2}(B_k-B_{k-1})^2 + \frac{q_4}{\Gamma(x_k+1)}(B_k-B_{k-1})^{x_k}\right]\\
\le \quad & \epsilon L_k.
\end{aligned}
$$

Now, the proof is complete.

Theorem 20. *Suppose that **(HY1)**, **(HY2)** and* (3.11) *hold. Then, the RL-fractional variable order multiterm BVP* (1.1) *in the sense of Atangana-Baleanu is Ulam-Hyers stable.*

Proof. For each $\epsilon > 0$, let $y \in C([B_{k-1}, B_k], \mathbb{R})$ be a solution of (3.25). According to Theorem 16, the RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu admits a solution $\theta \in C([0, B], \mathbb{R})$ as $\theta(s) = \theta_k(s)$ for $s \in [0, B_k],\ k = 1, 2, ..., n$, in which

$$\theta_1(s) = \overline{\theta}_1(s), \quad s \in \mathcal{B}_1, \qquad (3.30)$$

and for any $k \in \{2, ..., n\}$

$$\theta_k(s) = \begin{cases} 0, & s \in [0, B_{k-1}],\\ \overline{\theta}_k(s), & s \in \mathcal{B}_k, \end{cases} \qquad (3.31)$$

and $\overline{\theta}_k \in C([B_{k-1}, B_k], \mathbb{R})$ is a solution of the RL-fractional constant order multiterm BVP in the sense of Atangana-Baleanu

$$\begin{cases} {}^{ABR}\mathbf{D}^{x_k}_{B^+_{k-1}}\theta(s) = \varphi(s, \theta(s), {}^{AB}\mathbf{I}^{x_k}_{B^+_{k-1}}(s)), \quad s \in \mathcal{B}_k, \\ \theta(B_{k-1}) = y(B_{k-1}) = 0, \quad \theta(B_k) = y(B_k) = \sum_{s_l \in \mathcal{B}_k} a_l \theta(s_l) = \sum_{s_l \in \mathcal{B}_k} a_l y(s_l). \end{cases} \tag{3.32}$$

Then, by Theorem 14, the solution of constant order BVP (3.32) takes the form

$$\overline{\theta}_k(s) = A_{\overline{\theta}} + q_3 \int_{B_{k-1}}^{s} (s-\tau)\varphi_{\overline{\theta}}(\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\varphi_{\overline{\theta}}(\tau)d\tau.$$

On the other hand, since $\theta(B_{k-1}) = y(B_{k-1}) = 0$ and $\sum_{s_l \in \mathcal{B}_k} a_l \theta(s_l) = \sum_{s_l \in \mathcal{B}_k} a_l y(s_l)$, then $A_y = A_{\overline{\theta}}$. Then we have

$$\overline{\theta}_k(s) = A_y + q_3 \int_{B_{k-1}}^{s} (s-\tau)\varphi_{\overline{\theta}}(\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\varphi_{\overline{\theta}}(\tau)d\tau. \tag{3.33}$$

Let $s \in \mathcal{B}_k$, where $k \in \{1, 2, ..., n\}$. According to Lemma 19 and (3.23), (3.30), (3.31) and (3.33), we get

$$\begin{aligned}
& |y(s) - \theta(s)| = |y(s) - \theta_k(s)| = |y(s) - \overline{\theta}_k(s)| \\
= & \left| y(s) - A_y - q_3 \int_{B_{k-1}}^{s} (s-\tau)\varphi_{\overline{\theta}}(\tau)d\tau - \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\varphi_{\overline{\theta}}(\tau)d\tau \right| \\
\leq & \left| y(s) - A_y - q_3 \int_{B_{k-1}}^{s} (s-\tau)\varphi_y(\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}\varphi_y(\tau)d\tau \right| \\
+ & q_3 \int_{B_{k-1}}^{s} (s-\tau)|\varphi_y(\tau) - \varphi_{\overline{\theta}}(\tau)|d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}|(\varphi_y\tau) - \varphi_{\overline{\theta}}(\tau)|d\tau \\
\leq & \epsilon L_k + K_\varphi (1 + \mathcal{G}) \|y - \theta\| \left(q_3 \int_{B_{k-1}}^{s} (s-\tau)d\tau + \frac{q_4}{\Gamma(x_k)} \int_{B_{k-1}}^{s} (s-\tau)^{x_k-1}d\tau \right) \\
\leq & \epsilon L_k + K_\varphi (1 + \mathcal{G}) \left(\frac{q_3}{2}(B_k - B_{k-1})^2 + \frac{q_4}{\Gamma(x_k+1)}(B_k - B_{k-1})^{x_k} \right) \|y - \theta\| \\
\leq & \epsilon L_k + K_\varphi (1 + \mathcal{G}) L_k \|y - \theta\| \\
\leq & \epsilon L_k + \nu \|y - \theta\|,
\end{aligned}$$

where

$$\nu = L_k K_\varphi (1 + \mathcal{G}).$$

Then

$$\|y - \theta\|(1 - \nu) \leq \epsilon L_k,$$

and so by assuming $c_\varphi := \frac{L_k}{(1-\nu)}$,

$$\|y - \theta\| \leq c_\varphi \epsilon,$$

i.e.,

$$|y(s) - \theta(s)| \leq c_\varphi \epsilon, \quad s \in [B_{k-1}, B_k].$$

Finally, Definition 17 implies that the RL-fractional variable order multiterm BVP (1.1) in the sense of Atangana-Baleanu is Ulam-Hyers stable.

4. Example

In this section, we investigate some examples.

Example 21. *Consider the RL-fractional variable order multiterm BVP in the sense of Atangana-Baleanu as*

$$\begin{cases} {}^{ABR}\boldsymbol{D}_{0^+}^{x(s)}\theta(s) = \frac{1}{100(1+|\theta(s)|+|{}^{AB}\boldsymbol{I}_{0^+}^{x(s)}\theta(s))|}, & s \in \mathcal{B} := [0,2], \\ \theta(0) = 0, \quad \theta(2) = \frac{1}{2}\theta(\frac{1}{2}) + \frac{2}{3}\theta(\frac{3}{2}), \end{cases} \tag{4.1}$$

where

$$x(s) = \begin{cases} \frac{7}{3}, & s \in \mathcal{B}_1 := [0,1], \\ \frac{8}{3}, & s \in \mathcal{B}_2 := [1,2]. \end{cases} \tag{4.2}$$

Let

$$\varphi(s,y,z) = \frac{1}{100(1+|y|+|z|)}, \ (s,y,z) \in [0,2] \times \mathbb{R} \times \mathbb{R}.$$

Let $y_1, y_2, z_1, z_2 \in \mathbb{R}$ *and* $s \in \mathcal{B}$. *Then, we have*

$$\begin{aligned} |\varphi(s,y_1,z_1) - \varphi(s,y_2,z_2)| &= \left|\left(\frac{1}{100(1+|y_1|+|z_1|)} - \frac{1}{100(1+|y_2|+|z_2|)}\right)\right| \\ &\le \frac{||y_2|+|z_2|-|y_1|-|z_1||}{100(1+|y_1|+|z_1|)(1+|y_2|+|z_2|)} \\ &\le \frac{1}{100}(|y_1-y_2|+|z_1-z_2|). \end{aligned}$$

Hence, condition ***(HY2)*** *holds with* $K_\varphi = \frac{1}{100}$.

By (4.2)*, according to (3.3) we design two auxiliary RL-fractional constant order multiterm BVPs in the sense of Atangana-Baleanu as follows:*

$$\begin{cases} {}^{ABR}\boldsymbol{D}_{0^+}^{\frac{7}{3}}\theta(s) = \frac{1}{100(1+|\xi(s)|+|{}^{AB}\boldsymbol{I}_{0^+}^{\frac{7}{3}}\xi(s)|)}, & s \in \mathcal{B}_1, \\ \theta(0) = 0, \ \xi(1) = \frac{1}{2}\theta(\frac{1}{2}), \end{cases} \tag{4.3}$$

and

$$\begin{cases} {}^{ABR}\boldsymbol{D}_{1^+}^{\frac{8}{3}}\theta(s) = \frac{1}{100(1+|\xi(s)|+|{}^{AB}\boldsymbol{I}_{1^+}^{\frac{8}{3}}\xi(s)|)}, & s \in \mathcal{B}_2, \\ \theta(1) = 0, \ \theta(2) = \frac{2}{3}\xi(\frac{3}{2}). \end{cases} \tag{4.4}$$

Next, we prove that condition (3.11) is fulfilled for $k = 1$. *Indeed, we calculate the following values:*

$$A = \frac{1}{(1-0) - \frac{1}{2}(\frac{1}{2}-0)} = \frac{4}{3}, \quad N(x_1 - 2) = 1 - (\frac{7}{3} - 2) + \frac{\frac{7}{3}-2}{\Gamma(\frac{7}{3}-2)} = 0.79109,$$

$$q_1 := \frac{4}{3}\frac{(3-\frac{7}{3})}{N(x_1-2)} = 1.12361, \ q_2 := \frac{\frac{4}{3}(\frac{7}{3}-2)}{N(x_1-2)} = 0.56180,$$

$$q_3 := \frac{3-\frac{7}{3}}{N(x_1-2)} = 0.84271, \ q_4 := \frac{\frac{7}{3}-2}{N(x_1-2)} = 0.42135,$$

$$\mathcal{G} := \frac{(3-\frac{7}{3})(1-0)^2}{2N(x_1-2)} + \frac{(\frac{7}{3}-2)(1-0)^{\frac{7}{3}}}{N(x_1-2)\Gamma(\frac{7}{3}+1)} = 0.29353,$$

and

$$\begin{aligned} L_1 &= \frac{q_1}{2}(1-0)\left(\frac{1}{2}(\frac{1}{2}-0)^2+(1-0)^2\right) \\ &+ \frac{q_2(1-0)}{\Gamma(\frac{7}{3}+1)}\left(\frac{1}{2}(\frac{1}{2}-0)^{\frac{7}{3}}+(1-0)^{\frac{7}{3}}\right) \\ &+ \frac{q_3}{2}(1-0)^2+\frac{q_4}{\Gamma(\frac{7}{3}+1)}(1-0)^{\frac{7}{3}} = 1.42733. \end{aligned} \tag{4.5}$$

Hence,

$$L_1 K_\varphi (1+\mathcal{G}) \simeq 0.018462 < 1.$$

Condition (3.11) is achieved. From Theorem 15, the problem (4.3) admits a solution $\theta_1 \in C([0,1],\mathbb{R})$, *where*

$$\theta_1(s) = \overline{\theta}_1(s), \quad s \in \mathcal{B}_1.$$

In the next step, we investigate the satisfication of condition (3.11) *for* $k = 2$*. First, we calculate the following values:*

$$A = \frac{1}{(2-1)-\frac{2}{3}(\frac{3}{2}-1)} = \frac{3}{2}, \quad N(x_2-2) = 1-(\frac{8}{3}-2)+\frac{\frac{8}{3}-2}{\Gamma(\frac{8}{3}-2)} = 0.8256587410811,$$

$$q_1 := \frac{3}{2}\frac{(3-\frac{8}{3})}{N(x_2-2)}0.302788534246790, \; q_2 := \frac{\frac{3}{2}(\frac{8}{3}-2)}{N(x_2-2)} = 0.60557706849359,$$

$$q_3 := \frac{3-\frac{8}{3}}{N(x_2-2)} = 0.40371804566239, \; q_4 := \frac{\frac{8}{3}-2}{N(x_2-2)} = 0.80743609132479,$$

$$\mathcal{G} := \frac{(3-\frac{8}{3})(2-1)^2}{2N(x_2-2)} + \frac{(\frac{8}{3}-2)(2-1)^{\frac{8}{3}}}{N(x_2-2)\Gamma(\frac{8}{3}+1)} = 0.33885494976913,$$

and

$$\begin{aligned} L_2 &= \frac{q_1}{2}(2-1)\left(\frac{2}{3}(\frac{3}{2}-1)^2+(2-1)^2\right) \\ &+ \frac{q_2(2-1)}{\Gamma(\frac{8}{3}+1)}\left(\frac{2}{3}(\frac{3}{2}-1)^{\frac{8}{3}}+(2-1)^{\frac{8}{3}}\right) \\ &+ \frac{q_3}{2}(2-1)^2+\frac{q_4}{\Gamma(\frac{8}{3}+1)}(2-1)^{\frac{8}{3}} = 0.44418362357126. \end{aligned} \tag{4.6}$$

Hence,

$$L_2 K_\varphi (1+\mathcal{G}) \simeq 0.00594697443025 < 1.$$

Thus, condition (3.11) is satisfied.

The conclusion of Theorem 15 follows that the auxiliary RL-fractional constant order multiterm BVP (4.4) in the sense of Atangana-Baleanu possesses a solution $\theta_2 \in C([0,2],\mathbb{R})$*, where*

$$\theta_2(s) = \begin{cases} 0, & s \in [0,1], \\ \overline{\theta}_2(s), & s \in \mathcal{B}_2. \end{cases}$$

Therefore, by Theorem 16, the RL-fractional variable order multiterm BVP (4.1) in the sense of Atangana-Baleanu has a solution

$$\theta(s) = \begin{cases} \theta_1(s), & s \in \mathcal{B}_1, \\ \theta_2(s) = \begin{cases} 0, & s \in \mathcal{B}_1, \\ \overline{\theta}_2(s), & s \in \mathcal{B}_2, \end{cases} \end{cases}$$

Moreover, according to Theorem 20, the RL-fractional variable order multiterm BVP (4.1) in the sense of Atangana-Baleanu is Ulam-Hyers stable.

Example 22. *Consider the RL-fractional variable order multiterm BVP in the sense of Atangana-Baleanu as*

$$\begin{cases} {}^{ABR}\boldsymbol{D}_{0^+}^{x(s)}\theta(s) = \frac{1}{(s+4)(1+|\theta(s)|+|{}^{AB}\boldsymbol{I}_{0^+}^{x(s)}\theta(s)|)}, & s \in \mathcal{B} :=]0,2], \\ \theta(0) = 0, \quad \theta(2) = \frac{1}{4}\theta(\frac{1}{4}) + \frac{1}{2}\theta(\frac{1}{2}) + \frac{2}{3}\theta(\frac{3}{2}), \end{cases} \tag{4.7}$$

where

$$x(s) = \begin{cases} \frac{7}{3}, & s \in \mathcal{B}_1 := [0,1], \\ \frac{8}{3}, & s \in \mathcal{B}_2 :=]1,2]. \end{cases} \tag{4.8}$$

Let

$$\varphi(s,y,z) = \frac{1}{(s+4)(1+|y|+|z|)}, \ (s,y,z) \in [0,2]\times\mathbb{R}\times\mathbb{R}.$$

Let $y_1, y_2, z_1, z_2 \in \mathbb{R}$ *and* $s \in \mathcal{B}$*. Then, we have*

$$\begin{aligned} |\varphi(s,y_1,z_1) - \varphi(s,y_2,z_2)| &= \left|\left(\frac{1}{(s+4)(1+|y_1|+|z_1|)} - \frac{1}{(s+4)(1+|y_2|+|z_2|)}\right)\right| \\ &\le \frac{||y_2|+|z_2|-|y_1|-|z_1||}{(s+4)(1+|y_1|+|z_1|)(1+|y_2|+|z_2|)} \\ &\le \frac{1}{4}(|y_1-y_2|+|z_1-z_2|). \end{aligned}$$

*Hence, condition **(HY2)** holds with* $K_\varphi = \frac{1}{4}$*.*

By (4.8) *and according to (3.3), we obtain two auxiliary RL-fractional constant order multiterm BVPs in the sense of Atangana-Baleanu as follows:*

$$\begin{cases} {}^{ABR}\boldsymbol{D}_{0^+}^{\frac{7}{3}}\theta(s) = \frac{1}{(s+4)(1+|\xi(s)|+|{}^{AB}\boldsymbol{I}_{0^+}^{\frac{7}{3}}\xi(s)|)}, & s \in \mathcal{B}_1, \\ \theta(0) = 0, \ \theta(1) = \frac{1}{4}\theta(\frac{1}{4}) + \frac{1}{2}\theta(\frac{1}{2}), \end{cases} \tag{4.9}$$

and

$$\begin{cases} {}^{ABR}\boldsymbol{D}_{1^+}^{\frac{8}{3}}\theta(s) = \frac{1}{(s+4)(1+|\xi(s)|+|{}^{AB}\boldsymbol{I}_{1^+}^{\frac{8}{3}}\xi(s)|)}, \quad s \in \mathcal{B}_2, \\ \theta(1) = 0,\ \theta(2) = \frac{2}{3}\xi(\frac{3}{2}). \end{cases} \tag{4.10}$$

Next, we prove that condition (3.11) is fulfilled for $k = 1$. Indeed, we calculate the following values:

$$A = \frac{1}{(1-0) - \frac{1}{4}(\frac{1}{4} - 0) - \frac{1}{2}(\frac{1}{2} - 0)} = \frac{16}{11}, \quad N(x_1 - 2) = 1 - (\frac{7}{3} - 2) + \frac{\frac{7}{3} - 2}{\Gamma(\frac{7}{3} - 2)} = 0.79109,$$

$$q_1 := \frac{4}{3}\frac{(3 - \frac{7}{3})}{N(x_1 - 2)} = 1.032982,\ q_2 := \frac{\frac{4}{3}(\frac{7}{3} - 2)}{N(x_1 - 2)} = 0.612886,$$

$$q_3 := \frac{3 - \frac{7}{3}}{N(x_1 - 2)} = 0.84271,\ q_4 := \frac{\frac{7}{3} - 2}{N(x_1 - 2)} = 0.42135,$$

$$\mathcal{G} := \frac{(3 - \frac{7}{3})(1-0)^2}{2N(x_1 - 2)} + \frac{(\frac{7}{3} - 2)(1-0)^{\frac{7}{3}}}{N(x_1 - 2)\Gamma(\frac{7}{3} + 1)} = 0.29353,$$

and

$$\begin{aligned} L_1 &= \frac{q_1}{2}(1-0)\left(\frac{1}{4}(\frac{1}{4} - 0)^2 + \frac{1}{2}(\frac{1}{2} - 0)^2 + (1-0)^2\right) \\ &+ \frac{q_2(1-0)}{\Gamma(\frac{7}{3} + 1)}\left(\frac{1}{4}(\frac{1}{4} - 0)^{\frac{7}{3}} + \frac{1}{2}(\frac{1}{2} - 0)^{\frac{7}{3}} + (1-0)^{\frac{7}{3}}\right) \\ &+ \frac{q_3}{2}(1-0)^2 + \frac{q_4}{\Gamma(\frac{7}{3} + 1)}(1-0)^{\frac{7}{3}} = 1.406810. \end{aligned} \tag{4.11}$$

Hence,

$$L_1 K_\varphi (1 + \mathcal{G}) \simeq 0.49989239 < 1.$$

Condition (3.11) is achieved. From Theorem 15, problem (4.9) admits a solution $\theta_1 \in C([0,1],\mathbb{R})$, where

$$\theta_1(s) = \overline{\theta}_1(s), \quad s \in \mathcal{B}_1.$$

In the next step, we investigate the fullfilment of condition (3.11) *for $k = 2$. First, we calculate the following values:*

$$A = \frac{1}{(2-1) - \frac{2}{3}(\frac{3}{2} - 1)} = \frac{3}{2}, \quad N(x_2 - 2) = 1 - (\frac{8}{3} - 2) + \frac{\frac{8}{3} - 2}{\Gamma(\frac{8}{3} - 2)} = 0.8256587410811,$$

$$q_1 := \frac{3}{2}\frac{(3 - \frac{8}{3})}{N(x_2 - 2)}0.302788534246790,\ q_2 := \frac{\frac{3}{2}(\frac{8}{3} - 2)}{N(x_2 - 2)} = 0.60557706849359,$$

$$q_3 := \frac{3 - \frac{8}{3}}{N(x_2 - 2)} = 0.40371804566239,\ q_4 := \frac{\frac{8}{3} - 2}{N(x_2 - 2)} = 0.80743609132479,$$

$$\mathcal{G} := \frac{(3 - \frac{8}{3})(2-1)^2}{2N(x_2 - 2)} + \frac{(\frac{8}{3} - 2)(2-1)^{\frac{8}{3}}}{N(x_2 - 2)\Gamma(\frac{8}{3} + 1)} = 0.33885494976913,$$

and

$$
\begin{aligned}
L_2 &= \frac{q_1}{2}(2-1)\left(\frac{2}{3}(\frac{3}{2}-1)^2+(2-1)^2\right)\\
&+ \frac{q_2(2-1)}{\Gamma(\frac{8}{3}+1)}\left(\frac{2}{3}(\frac{3}{2}-1)^{\frac{8}{3}}+(2-1)^{\frac{8}{3}}\right)\\
&+ \frac{q_3}{2}(2-1)^2+\frac{q_4}{\Gamma(\frac{8}{3}+1)}(2-1)^{\frac{8}{3}} = 0.44418362357126. \qquad (4.12)
\end{aligned}
$$

Hence,

$$L_2K_\varphi(1+\mathcal{G}) \simeq 0.148674 < 1.$$

Thus, condition (3.11) is satisfied.

The conclusion of Theorem 15 follows that the auxiliary RL-fractional constant order multiterm BVP (4.10) in the sense of Atangana-Baleanu possesses a solution $\theta_2 \in C([0,2],\mathbb{R})$, where

$$\theta_2(s) = \begin{cases} 0, & s \in [0,1], \\ \overline{\theta}_2(s), & s \in \mathcal{B}_2. \end{cases}$$

Therefore, by Theorem 16, the RL-fractional variable order multiterm BVP (4.7) in the sense of Atangana-Baleanu has a solution

$$\theta(s) = \begin{cases} \theta_1(s), & s \in \mathcal{B}_1, \\ \theta_2(s) = \begin{cases} 0, & s \in \mathcal{B}_1, \\ \overline{\theta}_2(s), & s \in \mathcal{B}_2, \end{cases} \end{cases}$$

Moreover, according to Theorem 20, the RL-fractional variable order multiterm BVP (4.7) in the sense of Atangana-Baleanu is Ulam-Hyers stable.

5. Conclusions

This research introduced an RL-fractional variable order multiterm BVP of order $2 < x(s) \leq 3$ in the sense of Atangana-Baleanu. The analytical solutions were successfully studied from three points of view: uniqueness via the Banach's fixed point theorem, existence via Krasnoselskii's fixed point theorem and Ulam-Hyers stability. The newly obtained results generalized some existing theorems for the delayed RL-FDEs of constant order by extending the order of derivatives as the variable order. Finally two examples were given to examine the potential our theorems. Further investigation on this open research subject is warranted. In fact, our proposed BVP may possibly be extended to other fractional models. We can study different types of real mathematical models such as pantograph systems, Langevin equations or hybrid systems in the context of fractional variable order structures, and then, by using some numerical algorithms, one can analyze numerical solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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