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Research article

Note on control for hybrid stochastic systems by intermittent feedback rooted in discrete observations of state and mode with delays

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Abstract: For a hybrid stochastic system, most existing feedback controllers need to observe modes at continuous times, which is feasible when the system's mode is observable and does not incur any cost. However, in most cases, the mode is not readily apparent, and identifying it always incurs a certain expense. Therefore, in order to reduce control costs, when designing a feedback controller, both the state and the mode should be observed at discrete moments. This paper introduces an intermittent feedback controller for stabilizing an unstable hybrid stochastic system through discrete delayed observations of state and mode. By utilizing M-matrix theory, intermittent control approach, and the comparison principle, we propose sufficient conditions for the stabilization theory of hybrid stochastic systems. An illustrative example is taken to validate the proposed theory.

Keywords: hybrid stochastic system; discrete observations; delay; intermittent feedback control

1. Introduction

Stochastic systems have occupied significant positions in diverse fields. Hybrid stochastic differential equations (SDEs) are an important class of stochastic systems, which can effectively describe sudden changes in structures and parameters. Therefore, many scholars have conducted research on hybrid SDEs.

In the study of hybrid SDEs, stability analysis is one of the important research topics [1–8]. Feedback control is referred to as an important approach to ensure the stability of stochastic systems. However, conventional feedback controllers are rooted in the continuous observations of the system's states. Due to practical limitations, data can only be observed at discrete moments, even if the underlying system is continuous. To address this issue and reduce control costs, Mao [9] proposed a feedback control approach for stabilizing a hybrid SDE through discrete-time state observations. Unlike continuous-time controllers, feedback controllers rooted in discrete-time state observations have significant advantages in terms of accuracy and cost. Therefore, discrete-time control strategies [10–17] have been widely studied. Moreover, a delay is frequently present between the state observed moment and its actual time. Thus, delayed feedback control strategy [18–27] has also received extensive attention. Therein, Zhu et al. [22] considered both discrete and time-delay issues in designing the controller. They studied exponential stabilization for hybrid SDEs through feedback control rooted in delayed and discrete observations of the state. In addition, Li et al. [26] developed a delayed feedback controller for stabilizing the switching diffusion system by discrete observations of state and mode.

Moreover, intermittent control strategy has attracted extensive attention from scholars [28–35], particularly been applied in multiagent systems [32,33], complex networks [34] and other fields. Intermittent control splits time into work and rest periods. The controller switches on during the work period and turns off during the rest period, effectively switching the controlled system between closed-loop and open-loop modes. Compared to classical continuous control strategies, intermittent control strategy is more easily acceptable, which can reduce controller wear, extend the controller's lifespan, and lower costs.

To enhance control performance, a growing number of scholars have used hybrid control strategies, which involves the simultaneous use of multiple control strategies. In particular, Jiang et al. [35] considered discrete delayed observations of state and intermittent control method to design the controller. Inspired by these aforementioned works, this paper employs a hybrid control strategy to achieve stabilization, which involves discrete, time-delayed and intermittent components in controller design. Particularly, in this paper, not only the state but also the mode is observed at discrete times. When designing a controller, if the mode is readily apparent (meaning it can be observed without any cost), it can be observed in continuous time. For instance, within a financial system where the mode is referred to as interest rate, this is entirely feasible. However, in most cases, the mode is not evident, and identifying it incurs costs. To lower control expenses, observations pertaining to the state and mode should occur at discrete times. Therefore, our aim is to develop an intermittent feedback controller, rooted in discrete delayed observations related to state and mode, to stabilize an unstable hybrid SDE.

2. Model description and main results

2.1. Symbol explanation

Consider a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_\mu\}_{\mu_{\geq 0}}, P)$ satisfying the usual conditions. Let $w(\mu) = (w_1(\mu) \cdots w_I(\mu))^T$ be an I dimensional Brownian motion defined on the aforementioned probability space. In this probability space, consider a continuous-time Markov chain $\Gamma: \mathbb{R}_+ \to S = \{1,2, \cdots N\}$, whose generator $\Lambda = (\gamma_{uv})_{N\times N}$ is provided by

$$P(\Gamma(\mu+\Delta) = v | \Gamma(\mu) = u) = \begin{cases} \gamma_{uv}\Delta + o(\Delta) & u \neq v, \\ 1 + \gamma_{uv}\Delta + o(\Delta) & u = v, \end{cases}$$

in which $\Delta > 0$, γ_{uv} represents the rate of transition from state u to state v, and $\gamma_{uu} = -\sum_{v\neq u} \gamma_{uv}$.

Let $w(\mu)$ be independent of $\Gamma(\mu)$.

Let v_0 be a positive number, $C([-v_0,0],\mathbb{R}^n)$ represent the family of continuous real-valued functions $\xi:[-v_0,0]\to\mathbb{R}^n$ with $\|\xi\|=\sup_{-v_0\le\theta\le 0}|\xi(\theta)|$. $L^p_{\mathscr{F}_\mu}([-v_0,0];\mathbb{R}^n)$ denotes the family of all $\mathscr{F}_\mu$ measurable $C([-v_0,0],\mathbb{R}^n)$ valued random variables $\gamma=\{\gamma(\theta):-v_0\le\theta\le 0\}$ with $E\|\gamma\|^p<\infty$, where E represents the expectation for probability P.

2.2. Model description

Consider an unstable hybrid SDE

$$d\eta(\mu)=h(\eta(\mu),\Gamma(\mu),\mu)d\mu+k(\eta(\mu),\Gamma(\mu),\mu)dw(\mu), \tag{2.1}$$

on $\mu\ge 0$ along with initial data $\eta(0)=\eta_0\ne 0$ and $\Gamma(0)=\Gamma_0$, where $h:\mathbb{R}^n\times S\times\mathbb{R}_+\to\mathbb{R}^n$ and $k:\mathbb{R}^n\times S\times\mathbb{R}_+\to\mathbb{R}^{n\times I}$. For system (2.1), our objective is to develop a discrete controller $\varpi:\mathbb{R}^n\times S\times\mathbb{R}\to\mathbb{R}^n$ with a time delay v_0 in the drift term to achieve stabilization. Additionally, an intermittent control strategy is incorporated. Thus, the controlled system is as follows:

$$d\lambda(\mu)=\Big(h(\lambda(\mu),\Gamma(\mu),\mu)+\varpi\big(\lambda(\delta_\mu),\Gamma(\delta_\mu),\mu\big)\mathrm{I}(\mu)\Big)d\mu+k(\lambda(\mu),\Gamma(\mu),\mu)dw(\mu), \tag{2.2}$$

in which $\delta_\mu=[\mu/v]v-v_0$, $v>0$ is the duration separating two consecutive discrete observations, $[\mu/v]$ represents the integer part of μ/v, and $\mathrm{I}(\mu)=\sum_{l=0}^{\infty}\mathrm{I}_{[\mu_l,\mu_l+\varphi\Delta]}(\mu)$, $\mu_l=l\Delta$, $l=0,1,2,\cdots$, $\Delta>0$ represents the control period, $\varphi\in[0,1]$ represents the control width, and $\mathrm{I}_{[\mu_l,\mu_{l+1}]}(\mu)=\begin{cases}1, & \mu\in[\mu_l,\mu_l+\varphi\Delta)\\ 0, & \mu\in[\mu_l+\varphi\Delta,\mu_{l+1})\end{cases}$. Namely, the controller will switch on during $[0,\varphi\Delta)$, $[\Delta,(1+\varphi)\Delta)$, $[2\Delta,(2+\varphi)\Delta),\cdots$, switch off during $[\varphi\Delta,\Delta)$, $[(1+\varphi)\Delta,2\Delta),[(2+\varphi)\Delta,3\Delta),\cdots$.

The original system (2.1) has initial data only at $\mu=0$, while the controlled system (2.2) requires initial data

$$\{\lambda(\theta):-v_0\le\theta\le 0\}=\xi\in C([-v_0,0];\mathbb{R}^n),\ \{\Gamma(\theta):-v_0\le\theta\le 0\}=\varsigma\in S. \tag{2.3}$$

To address this issue, set $\lambda(\theta)=\eta(0)$, $\Gamma(\theta)=\Gamma(0)$, $\theta\in[-v_0,0]$.

To study the properties of system (2.2), we introduce the auxiliary system as follows:

$$d\sigma(\mu)=\big(h(\sigma(\mu),\Gamma(\mu),\mu)+\varpi(\sigma(\mu),\Gamma(\mu),\mu)\mathrm{I}(\mu)\big)d\mu+k(\sigma(\mu),\Gamma(\mu),\mu)dw(\mu) \tag{2.4}$$

The system (2.4) is continuous and has no delay term with initial values $\sigma(0)=\sigma_0=\eta_0\ne 0$ and $\Gamma(0)=\Gamma_0$.

2.3. Preliminary knowledge and main result

Assumption 1. There are three positive constants K_1, K_2 and K_3 such that

$$|h(\lambda,u,\mu)-h(\sigma,u,\mu)|\le K_1|\lambda-\sigma|,$$

$$|\varpi(\lambda,u,\mu)-\varpi(\sigma,u,\mu)|\le K_2|\lambda-\sigma|,$$

$$|k(\lambda,u,\mu)-k(\sigma,u,\mu)|\le K_3|\lambda-\sigma|,$$

for $\forall(\lambda,\sigma,u,\mu)\in\mathbb{R}^n\times\mathbb{R}^n\times S\times\mathbb{R}_+$. Additionally,

$$h(0,u,\mu)=0,\ \varpi(0,u,\mu)=0,\ k(0,u,\mu)=0,$$

for $\forall(u,\mu)\in S\times\mathbb{R}_+$.

This assumption implies the linear growth condition

$$|h(\lambda,u,\mu)|\le K_1|\lambda|,\ |\varpi(\lambda,u,\mu)|\le K_2|\lambda|,\ |k(\lambda,u,\mu)|\le K_3|\lambda|, \tag{2.5}$$

for $\forall(\lambda,u,\mu)\in\mathbb{R}^n\times S\times\mathbb{R}_+$.

Remark 1: Under Assumption 1, from the reference [3], it can be inferred that the system (2.2) has a unique solution $\lambda(\mu;\xi,\varsigma,0)$ on $\mu\ge 0$ and

$$E|\lambda(\mu;\xi,\varsigma,0)|^p<\infty,\ \mu\ge 0,\ p>0.$$

Similarly, under Assumption 1, the auxiliary system (2.4) also has a unique solution denoted by $\sigma(\mu;\sigma_0,\Gamma_0,0)$ for $\mu\ge 0$.

Remark 2: Under Assumption 1, here we emphasize the important property from Lemma 2.1 of [1], for $\forall\sigma_0\neq 0$, $P\{\sigma(\mu;\sigma_0,\Gamma_0,0)\neq 0:\mu\ge 0\}=1$. Specifically speaking, when any initial data of system (2.4) is nonzero, almost all trajectories will never reach the origin.

Assumption 2. There exist $m>0$, non-negative numbers τ_u, β_u and c_u, $u\in S$, satisfying

$$\lambda^T\varpi(\lambda,u,\mu)\le-\tau_u|\lambda|^2,$$

and

$$\frac{1}{|\lambda|^2}\left(\lambda^T h(\lambda,u,\mu)+\frac{1}{2}|k(\lambda,u,\mu)|^2\right)-\frac{2-m}{2|\lambda|^4}|\lambda^T k(\lambda,u,\mu)|^2\le\beta_u,$$

for $\forall(\lambda,u,\mu)\in(\mathbb{R}^n-\{0\})\times S\times\mathbb{R}_+$, and $\beta_u-\tau_u\le -c_u$.

Assumption 3. Let m be a positive constant, and $N\times N$ matrix

$$\mathcal{A}(m)=diag\big(\alpha_1(m),\alpha_2(m)\cdots\alpha_N(m)\big)-\Lambda, \tag{2.6}$$

be a non-singular M-matrix, in which $\alpha_u(m)=c_u m$.

Define

$$(b_1,\cdots b_N)^T=\mathcal{A}^{-1}(m)(1,\cdots,1)_N^T, \tag{2.7}$$

and let

$$b_{min}=\min_{u\in S}b_u,\ b_{max}=\max_{u\in S}b_u,\ M=\frac{b_{max}}{b_{min}},\ \chi=\max_{u\in S}(\beta_u+c_u). \tag{2.8}$$

Theorem 1. For a free parameter $\varepsilon\in(0,1)$, let $\Theta=\frac{1}{\chi_1}log\frac{2^{2m_3}b_{max}}{b_{min}\varepsilon}>0$, and $\upsilon^*>0$ be the unique root of Eq (2.9) with respect to $\upsilon_0\ge 0$,

$$\varepsilon(1+K_2v_0)^{\frac{m}{2}}e^{v_0m\left(K_1+0.5(m_1-1)K_3^2+K_2\right)}+2^{m_3}\left(2^{m_3}L_4(m,v_0,\Theta)+L_3(m,v_0,v_0+\Theta)\right)=1, \tag{2.9}$$

where $\chi_1=-\left(m\chi-\frac{1}{b_{max}}-m\chi\varphi\right),\ m>0,\ m_1=2\vee m,\ m_3=0\vee(m-1),$

$$L_3(m,v_0,\Theta)$$
$$=\begin{cases}\left\{3(1+K_2v_0)e^{(\Theta+v_0)2\left(K_1+0.5K_3^2+K_2\right)}\times\left[v_0^2\left(K_1^2+K_2^2\right)+4v_0K_3^2\right]\right\}^{\frac{m}{2}}, m\in(0,2),\\ 3^{m-1}(1+K_2v_0)e^{(\Theta+v_0)m\left(K_1+\frac{1}{2}(m-1)K_3^2+K_2\right)}\times\left[v_0^m\left(K_1^m+K_2^m\right)\right.\\ \left.+\left(\dfrac{m^3}{2(m-1)}\right)^{\frac{m}{2}}\times v_0^{\frac{m}{2}}K_3^m\right], \quad m\in[2,\infty).\end{cases}$$

$$L_4(m,v_0,\Theta)=\begin{cases}\left\{3^2(\Theta+v_0)^2K_2^2\left[L_3(2,v_0,\Theta)+4\left(1-e^{-\hat{\gamma}v}\right)NL_1(2,v_0,\Theta)\right]\right.\\ \left.\times e^{3\left[K_1^2+(\Theta+v_0)^{-1}K_3^2+3K_2^2\right](\Theta+v_0)^2}\right\}^{\frac{m}{2}}, m\in(0,2),\\ 3^{2m-2}(\Theta+v_0)^mK_2^m\left[L_3(m,v_0,\Theta)+2^m\left(1-e^{-\hat{\gamma}v}\right)NL_1(m,v_0,\Theta)\right]\\ \times e^{3^{m-1}\left[K_1^m+\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}(\Theta+v_0)^{-\frac{m}{2}}K_3^m+3^{m-1}K_2^m\right](\Theta+v_0)^m}, \quad m\in[2,\infty).\end{cases}$$

Under Assumptions 1–3, let N^+ be a positive integer, for $\forall v_0\in(0,v^*)$, an intermittent control period $\Delta=(\Theta+2v_0)/N^+v_0$ and a control width $\varphi\in\left(1-\frac{1}{m\chi b_{max}},1\right)$ can be selected to make the controlled system (2.2) almost surely exponentially stable.

Remark 3: It is easy to see that the function

$$h(v_0)\triangleq\varepsilon(1+K_2v_0)^{\frac{m}{2}}e^{v_0m\left(K_1+0.5(m_1-1)K_3^2+K_2\right)}+2^{m_3}\left(2^{m_3}L_4(m,v_0,\Theta)+L_3(m,v_0,v_0+\Theta)\right)$$

on the left side of Eq (2.9) is continuous and increasing. Moreover, $h(0)<1$ and $h(+\infty)=+\infty$. Therefore, Eq (2.9) has a unique positive root.

To prove Theorem 1, a series of lemmas will be introduced in the following section.

3. Lemmas

Lemma 1. For $\forall\mu\geq0,\ j>0$ and $u\in S$, if $s\in[\mu,\mu+j]$, then

$$P(\Gamma(s)\neq u,|\Gamma(\mu)=u)\leq1-e^{-\hat{\gamma}j}, \tag{3.1}$$

where $\hat{\gamma}=\max_{u\in S}(-\gamma_{uu})$.

As for the proof, please refer to the Appendix.

Lemma 2. Under Assumptions 1–3, when $0\leq1-\frac{1}{m\chi b_{max}}<\varphi<1$, the solution of the auxiliary system (2.4) has

$$E|\sigma(\mu)|^m \le ME|\sigma_0|^m e^{-\chi_1\mu}, \tag{3.2}$$

$$\lim_{\mu\to\infty} \sup \frac{1}{\mu} log|\sigma(\mu)| < 0, a.s., \tag{3.3}$$

where $\chi_1 = -\left(m\chi - \frac{1}{b_{max}} - m\chi\varphi\right) > 0.$

As for the proof, please refer to the Appendix.

Lemma 3. Under Assumption 1, for $\forall\Theta > 0$,

$$\sup_{0\le\mu\le\Theta+v_0} E|\lambda(\mu)|^m \le L_1(m, v_0, \Theta)E\|\lambda(0)\|^m, \tag{3.4}$$

where $L_1(m, v_0, \Theta) = \begin{cases} (1+K_2v_0)^{\frac{m}{2}} e^{(\Theta+v_0)m\left(K_1+\frac{1}{2}K_3{}^2+K_2\right)}, m \in (0,2), \\ (1+K_2v_0)e^{(\Theta+v_0)m\left(K_1+\frac{1}{2}(m-1)K_3{}^2+K_2\right)}, m \in [2,\infty). \end{cases}$

Proof. For simplicity, denote $L_1(m, v_0, \Theta) = L_1$. When $m \ge 2$, for initial data $\xi \in C([-v_0, 0], \mathbb{R}^n)$, apply the Itô formula to $|\lambda(\mu)|^m$, and have that

$$E|\lambda(\mu)|^m \le E|\lambda(0)|^m + E\int_0^\mu m\,|\lambda(s)|^{m-2}\left[\lambda(s)^T h(\lambda(s), \Gamma(s), s) + \frac{1}{2}(m-1)\right.$$
$$\times |k(\lambda(s), \Gamma(s), s)|^2]ds + E\int_0^\mu m|\lambda(s)|^{m-1}|\varpi(\lambda(\delta_s), \Gamma(\delta_s), s)\mathrm{I}(s)|\, ds.$$

By Assumption 1, derive that

$$\begin{aligned} E|\lambda(\mu)|^m &\le E|\lambda(0)|^m + \left(mK_1 + \frac{1}{2}m(m-1)K_3^2\right)E\int_0^\mu |\lambda(s)|^m ds + E\int_0^\mu mK_2|\lambda(s)|^{m-1}|\lambda(\delta_s)|ds \\ &\le E|\lambda(0)|^m + \left(mK_1 + \frac{1}{2}m(m-1)K_3^2\right)E\int_0^\mu |\lambda(s)|^m ds + E\int_0^\mu K_2[(m-1)\,|\lambda(s)|^m \\ &\quad +|\lambda(\delta_s)|^m]ds. \end{aligned}$$

Substitute $\int_0^\mu E|\lambda(\delta_s)|^m ds \le \int_0^\mu \sup_{-v_0\le\theta\le s} E|\lambda(\theta)|^m ds \le v_0E\|\lambda(0)\|^m + \int_0^\mu \sup_{0\le\theta\le s} E|\lambda(\theta)|^m ds$ into the above, then

$$\begin{aligned} E|\lambda(\mu)|^m \le E|\lambda(0)|^m &+ \left(mK_1 + \frac{1}{2}m(m-1)K_3{}^2\right)E\int_0^\mu |\lambda(s)|^m\,ds + \int_0^\mu K_2(m-1)E|\lambda(s)|^m ds \\ &+K_2v_0E\|\lambda(0)\|^m + K_2\int_0^\mu \sup_{0\le\theta\le s} E|\lambda(\theta)|^m ds. \end{aligned}$$

Using the Gronwall inequality,

$$\sup_{0\le\theta\le\Theta+v_0} E|\lambda(\theta)|^m \le (1+K_2v_0)e^{(\Theta+v_0)m\left(K_1+\frac{1}{2}(m-1)K_3{}^2+K_2\right)}E\|\lambda(0)\|^m. \tag{3.5}$$

When $m \in (0,2)$,

$$\sup_{0\le\theta\le\Theta+v_0} E|\lambda(\theta)|^m \le \left(\sup_{0\le\theta\le\Theta+v_0} E|\lambda(\theta)|^2\right)^{\frac{m}{2}} \le (1+K_2v_0)^{\frac{m}{2}}e^{(\Theta+v_0)m\left(K_1+\frac{1}{2}K_3{}^2+K_2\right)}E\|\lambda(0)\|^m.$$

Therefore, the inequality (3.4) is proven.
Lemma 4. Under Assumption 1, for $\forall \Theta > 0$,

$$E\left(\sup_{0\le\mu\le\Theta+v_0}|\lambda(\mu)|^m\right)\le L_2(m,v_0,\Theta)E\|\lambda(0)\|^m, \tag{3.6}$$

where

$$L_2(m,v_0,\Theta)=\begin{cases}\Big\{\big(3+\big(6(\Theta+v_0)\big)K_2^2v_0\big)+\big[\big(6(\Theta+v_0)\big)K_1^2+12(\Theta+v_0)K_3^2+\big(6(\Theta+v_0)\big)\\ \times K_2^2\big](1+K_2v_0)\left(e^{(\Theta+v_0)2\left(K_1+\frac{1}{2}K_3^2+K_2\right)}-1\right)\left(2\left(K_1+\frac{1}{2}K_3^2+K_2\right)\right)^{-1}\Big\}^{\frac{m}{2}},\\ m\in(0,2),\\ \Big\{\big[3^{m-1}+\big(6(\Theta+v_0)\big)^{m-1}K_2^mv_0\big]+\Big[\big(6(\Theta+v_0)\big)^{m-1}K_1^m+3^{m-1}\left(\frac{m^3}{2(m-1)}\right)^{\frac{m}{2}}\\ (\Theta+v_0)^{\frac{m-2}{2}}K_3^m+\big(6(\Theta+v_0)\big)^{m-1}K_2^m\Big](1+K_2v_0)\Big(e^{(\Theta+v_0)m\left(K_1+\frac{1}{2}(m-1)K_3^2+K_2\right)}\\ -1)\left(m\left(K_1+\frac{1}{2}(m-1)K_3^2+K_2\right)\right)^{-1}\Big\}, \qquad m\in[2,\infty).\end{cases}$$

Proof. Denote $L_2(m,v_0,\Theta)=L_2$. When $m\ge 2$,

$$\begin{aligned}&E\left(\sup_{0\le s\le\Theta+v_0}|\lambda(s)|^m\right)\\
&\le 3^{m-1}E|\lambda(0)|^m+\big(3(\Theta+v_0)\big)^{m-1}E\int_0^{\Theta+v_0}(|h(\lambda(s),\Gamma(s),s)+\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)\mathrm{I}(s)|)^m ds\\
&\quad+3^{m-1}E\left(\sup_{0\le\mu\le\Theta+v_0}\left|\int_0^\mu k(\lambda(s),\Gamma(s),s)dw(s)\right|^m\right)\\
&\le 3^{m-1}E|\lambda(0)|^m+\big(3(\Theta+v_0)\big)^{m-1}2^{m-1}\int_0^{\Theta+v_0}(E|h(\lambda(s),\Gamma(s),s)|^m\\
&\quad+|\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)\mathrm{I}(s)|^m)ds+3^{m-1}\left(\frac{m^3}{2m-2}\right)^{\frac{m}{2}}(\Theta+v_0)^{\frac{m-2}{2}}\int_0^{\Theta+v_0}E|k(\lambda(s),\Gamma(s),s)|^m ds\\
&\le 3^{m-1}E|\lambda(0)|^m+\left[\big(3(\Theta+v_0)\big)^{m-1}2^{m-1}K_1^m+3^{m-1}\left(\frac{m^3}{2m-2}\right)^{\frac{m}{2}}(\Theta+v_0)^{\frac{m-2}{2}}K_3^m\right]\\
&\quad\times\int_0^{\Theta+v_0}E(|\lambda(s)|^m)ds+\big(3(\Theta+v_0)\big)^{m-1}2^{m-1}K_2^m\int_0^{\Theta+v_0}\sup_{-v_0\le\theta\le s}E|\lambda(\theta)|^m ds\\
&\le\left[3^{m-1}+\big(3(\Theta+v_0)\big)^{m-1}2^{m-1}K_2^mv_0\right]E\|\lambda(0)\|^m+\Big[\big(3(\Theta+v_0)\big)^{m-1}2^{m-1}K_1^m+3^{m-1}\end{aligned}$$

$$\times\left(\frac{m^3}{2m-2}\right)^{\frac{m}{2}}(\Theta+v_0)^{\frac{m-2}{2}}K_3^m+\left(3(\Theta+v_0)\right)^{m-1}2^{m-1}K_2^m\Bigg]\int_0^{\Theta+v_0}\sup_{0\le\theta\le s}E|\lambda(\theta)|^m ds.$$

By (3.5), have that

$$E\left(\sup_{0\le s\le\Theta+v_0}|\lambda(s)|^m\right)$$

$$\le\Bigg\{\left[3^{m-1}+\left(6(\Theta+v_0)\right)^{m-1}K_2^m v_0\right]+\Bigg[\left(6(\Theta+v_0)\right)^{m-1}K_1^m+3^{m-1}\left(\frac{m^3}{2m-2}\right)^{\frac{m}{2}}(\Theta+v_0)^{\frac{m-2}{2}}K_3^m$$

$$+\left(6(\Theta+v_0)\right)^{m-1}K_2^m\Bigg](1+K_2v_0)\left(e^{(\Theta+v_0)m\left(K_1+\frac{1}{2}(m-1)K_3^2+K_2\right)}-1\right)$$

$$\times\left(m\left(K_1+\frac{1}{2}(m-1)K_3^2\times K_2\right)\right)^{-1}\Bigg\}E\|\lambda(0)\|^m.$$

When $m\in(0,2)$,

$$E\left(\sup_{0\le s\le\Theta+v_0}|\lambda(s)|^m\right)$$
$$\le\left(E\sup_{0\le s\le\Theta+v_0}|\lambda(s)|^2\right)^{\frac{m}{2}}$$
$$\le\Big\{\left(3+\left(6(\Theta+v_0)\right)K_2{}^2v_0\right)+\left[\left(6(\Theta+v_0)\right)K_1{}^2+12(\Theta+v_0)K_3{}^2+\left(6(\Theta+v_0)\right)K_2{}^2\right]$$
$$\times(1+K_2v_0)\left(e^{(\Theta+v_0)2\left(K_1+\frac{1}{2}K_3{}^2+K_2\right)}-1\right)\left(2\left(K_1+\frac{1}{2}K_3{}^2+K_2\right)\right)^{-1}\Big\}^{\frac{m}{2}}E\|\lambda(0)\|^m.$$

Inequality (3.6) has been proven.

Lemma 5. Under Assumption 1, for $\forall\Theta>0$,

$$\sup_{0\le\mu\le\Theta}E\left(\sup_{0\le\theta\le v_0}|\lambda(\mu+\theta)-\lambda(\mu)|^m\right)\le L_3(m,v_0,\Theta)E\|\lambda(0)\|^m, \tag{3.7}$$

where $L_3(m,v_0,\Theta)$ can be seen from Theorem 1.

Proof. Denote $L_3(m,v_0,\Theta)=L_3$, when $m\ge 2$,

$$
\begin{aligned}
&E\left(\sup_{0\le\theta\le\upsilon_0}|\lambda(\mu+\theta)-\lambda(\mu)|^m\right)\\
&\le(3\upsilon_0)^{m-1}{K_1}^m\int_\mu^{\mu+\upsilon_0}E|\lambda(s)|^m\,ds+(3\upsilon_0)^{m-1}{K_2}^m\int_\mu^{\mu+\upsilon_0}E|\lambda(\delta_s)|^m\,ds\\
&+3^{m-1}\left(\frac{m^3}{2m-2}\right)^{\frac{m}{2}}{\upsilon_0}^{\frac{m-2}{2}}\times{K_3}^m\int_\mu^{\mu+\upsilon_0}E|\lambda(s)|^m\,ds\\
&\le\left[(3\upsilon_0)^{m-1}{K_1}^m+3^{m-1}\left(\frac{m^3}{2m-2}\right)^{\frac{m}{2}}{\upsilon_0}^{\frac{m-2}{2}}{K_3}^m\right]\int_\mu^{\mu+\upsilon_0}E|\lambda(s)|^m\,ds\\
&+(3\upsilon_0)^{m-1}{K_2}^m\int_\mu^{\mu+\upsilon_0}E|\lambda(\delta_s)|^m\,ds.
\end{aligned}
$$

For $\forall\mu\le s\le\mu+\upsilon_0$, by (3.5), obtain that

$$E|\lambda(s)|^m\le(1+K_2\upsilon_0)e^{(\mu+\upsilon_0)m\left(K_1+\frac{1}{2}(m-1){K_3}^2+K_2\right)}E\|\lambda(0)\|^m.$$

$$E|\lambda(\delta_s)|^m\le(1+K_2\upsilon_0)e^{\mu m\left(K_1+\frac{1}{2}(m-1){K_3}^2+K_2\right)}E\|\lambda(0)\|^m.$$

Therefore,

$$
\begin{aligned}
&E\left(\sup_{0\le\theta\le\upsilon_0}|\lambda(\mu+\theta)-\lambda(\mu)|^m\right)\\
&\le\left[(3\upsilon_0)^{m-1}{K_1}^m+3^{m-1}\left(\frac{m^3}{2m-2}\right)^{\frac{m}{2}}{\upsilon_0}^{\frac{m-2}{2}}{K_3}^m\right]\int_\mu^{\mu+\upsilon_0}(1+K_2\upsilon_0)e^{(\mu+\upsilon_0)m\left(K_1+\frac{1}{2}(m-1){K_3}^2+K_2\right)}\\
&\times E\|\lambda(0)\|^m ds+(3\upsilon_0)^{m-1}{K_2}^m\int_\mu^{\mu+\upsilon_0}(1+K_2\upsilon_0)e^{(\mu+\upsilon_0)m\left(K_1+\frac{1}{2}(m-1){K_3}^2+K_2\right)}E\|\lambda(0)\|^m ds.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\sup_{0\le\mu\le\Theta}E\left(\sup_{0\le\theta\le\upsilon_0}|\lambda(\mu+\theta)-\lambda(\mu)|^m\right)\\
&\le\left[3^{m-1}{\upsilon_0}^m{K_1}^m+3^{m-1}\left(\frac{m^3}{2(m-1)}\right)^{\frac{m}{2}}{\upsilon_0}^{\frac{m}{2}}{K_3}^m\right](1\\
&\qquad+K_2\upsilon_0)e^{(\Theta+\upsilon_0)m\left(K_1+\frac{1}{2}(m-1){K_3}^2+K_2\right)}E\|\lambda(0)\|^m\\
&+3^{m-1}{\upsilon_0}^m{K_2}^m(1+K_2\upsilon_0)e^{\Theta m\left(K_1+\frac{1}{2}(m-1){K_3}^2+K_2\right)}E\|\lambda(0)\|^m
\end{aligned}
$$

$$\leq 3^{m-1}(1+K_2\upsilon_0)e^{(\Theta+\upsilon_0)m\left(K_1+\frac{1}{2}(m-1){K_3}^2+K_2\right)}\left[{\upsilon_0}^m({K_1}^m+{K_2}^m)+\left(\frac{m^3}{2(m-1)}\right)^{\frac{m}{2}}{\upsilon_0}^{\frac{m}{2}}{K_3}^m\right]$$

$$\times E\|\lambda(0)\|^m.$$

For $m\in(0,2)$,

$$\sup_{0\leq\mu\leq\Theta}E\left(\sup_{0\leq\theta\leq\upsilon_0}|\lambda(\mu+\theta)-\lambda(\mu)|^m\right)$$
$$\leq\left\{3(1+K_2\upsilon_0)e^{(\Theta+\upsilon_0)2\left(K_1+\frac{1}{2}{K_3}^2+K_2\right)}\times\left[{\upsilon_0}^2\left({K_1}^2+{K_2}^2\right)+4\upsilon_0{K_3}^2\right]\right\}^{\frac{m}{2}}E\|\lambda(0)\|^m.$$

Inequality (3.7) has been proven.

Lemma 6. Under Assumption 1, and $\Theta>0$, for $\mu\in[0,\Theta+\upsilon_0]$,

$$E|\lambda(\mu)-\sigma(\mu)|^m\leq L_4(m,\upsilon_0,\Theta)E\|\lambda(0)\|^m,$$

where $L_4(m,\upsilon_0,\Theta)$ can be seen from Theorem 1.

Proof. Let $L_4(m,\upsilon_0,\Theta)=L_4$, $\sigma(\mu;\sigma_0,\Gamma_0,0)=\sigma(\mu)$. Taking the difference between system (2.2) and system (2.4), obtain that

$$\lambda(\mu)-\sigma(\mu)=\int_0^\mu\big(h(\lambda(s),\Gamma(s),s)-h(\sigma(s),\Gamma(s),s)\big)\,ds+\int_0^\mu(\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)$$
$$-\varpi(\sigma(s),\Gamma(s),s)\big)\mathrm{I}(s)ds+\int_0^\mu\big(k(\lambda(s),\Gamma(s),s)-k(\sigma(s),\Gamma(s),s)\big)\,dw(s),$$

when $m\geq2$. Taking the expectation, yield that

$$E|\lambda(\mu)-\sigma(\mu)|^m$$
$$\leq(3\mu)^{m-1}\int_0^\mu E|h(\lambda(s),\Gamma(s),s)-h(\sigma(s),\Gamma(s),s)|^m ds$$
$$+(3\mu)^{m-1}\int_0^\mu E\left|\big(\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)-\varpi(\sigma(s),\Gamma(s),s)\big)\mathrm{I}(s)\right|^m ds$$
$$+3^{m-1}\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}\mu^{\frac{m-2}{2}}\int_0^\mu E|k(\lambda(s),\Gamma(s),s)-k(\sigma(s),\Gamma(s),s)|^m ds$$
$$\leq\left[(3\mu)^{m-1}K_1^m+3^{m-1}\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}\mu^{\frac{m-2}{2}}K_3^m\right]\int_0^\mu E|\lambda(\mu)-\sigma(\mu)|^m ds+(3\mu)^{m-1}T_1,$$

where

$$
\begin{aligned}
T_1 &= E\int_0^{\mu}\left|\big(\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)-\varpi(\sigma(s),\Gamma(s),s)\big)\mathrm{I}(s)\right|^m ds \\
&\le 3^{m-1}\left[E\int_0^{\mu}(|\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)-\varpi(\lambda(\delta_s),\Gamma(s),s)|^m+|\varpi(\lambda(\delta_s),\Gamma(s),s)\right. \\
&\quad \left.-\varpi(\lambda(s),\Gamma(s),s)|^m+\left|\varpi(\lambda(s),\Gamma(s),s)-\varpi(\sigma(s),\Gamma(s),s)\big)\right|^m ds\right] \\
&\le 3^{m-1}\left[K_2^m(\Theta+v_0)L_3E\|\lambda(0)\|^m+K_2^mE\int_0^{\mu}|\lambda(s)-\sigma(s)|^m ds+T_2\right],
\end{aligned}
$$

and $T_2=\int_0^{\mu}E|\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)-\varpi(\lambda(\delta_s),\Gamma(s),s)|^m ds$.

By Assumption 1, for $lv\le s\le\mu\wedge(l+1)v$, derive that

$$
\begin{aligned}
&E|\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)-\varpi(\lambda(\delta_s),\Gamma(s),s)|^m \\
&=E|\varpi(\lambda(lv-v_0),\Gamma(lv-v_0),s)-\varpi(\lambda(lv-v_0),\Gamma(s),s)|^m \\
&=E\left[E\left(|\varpi(\lambda(lv-v_0),\Gamma(lv-v_0),s)-\varpi(\lambda(lv-v_0),\Gamma(s),s)|^m\,\middle|\mathscr{F}_{\delta_s}\right)\right] \\
&\le E\left[2^mK_2{}^m|\lambda(lv-v_0)|^mE\left(I_{\{\Gamma(s)\neq\Gamma(lv-v_0)\}}\,\middle|\mathscr{F}_{\delta_s}\right)\right] \\
&=E\left[2^mK_2{}^m|\lambda(lv-v_0)|^mE\left(\sum_{u\in S}I_{\{\Gamma(lv-v_0)=u\}}I_{\{\Gamma(s)\neq u\}}\,\middle|\mathscr{F}_{\delta_s}\right)\right] \\
&=E\left[2^mK_2{}^m|\lambda(lv-v_0)|^m\sum_{u\in S}I_{\{\Gamma(lv-v_0)=u\}}P(\Gamma(s)\neq u|\Gamma(lv-v_0)=u)\right].
\end{aligned}
$$

By Lemma 1 and inequality (3.4), have that

$$
\begin{aligned}
&E|\varpi(\lambda(\delta_s),\Gamma(\delta_s),s)-\varpi(\lambda(\delta_s),\Gamma(s),s)|^m \\
&\le E\left[2^mK_2{}^m|\lambda(lv-v_0)|^mN\left(1-e^{-\hat{\gamma}v}\right)\right] \\
&\le 2^mK_2{}^m\left(1-e^{-\hat{\gamma}v}\right)NL_1E\|\lambda(0)\|^m.
\end{aligned}
$$

Hence, for $\mu\in[0,\Theta+v_0]$,

$$
T_2\le\int_0^{\mu}2^mK_2{}^m\left(1-e^{-\hat{\gamma}v}\right)L_1E\|\lambda(0)\|^m ds\le(\Theta+v_0)\,2^mK_2{}^m\left(1-e^{-\hat{\gamma}v}\right)NL_1E\|\lambda(0)\|^m.
$$

$$
\begin{aligned}
T_1\le\ & 3^{m-1}\Big[K_2{}^m(\Theta+v_0)L_3E\|\lambda(0)\|^m+K_2{}^mE\int_0^{\mu}|\lambda(s)-\sigma(s)|^m ds+2^mK_2{}^m\left(1-e^{-\hat{\gamma}v}\right) \\
&\times(\Theta+v_0)NL_1E\|\lambda(0)\|^m\Big].
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&E|\lambda(\mu)-\sigma(\mu)|^m \\
&\leq \left[3^{m-1}(\Theta+\upsilon_0)^{m-1}{K_1}^m+3^{m-1}\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}(\Theta+\upsilon_0)^{\frac{m-2}{2}}{K_3}^m\right]\int_0^\mu E\,|\lambda(s)-\sigma(s)|^m ds \\
&\qquad +3^{m-1}(\Theta+\upsilon_0)^{m-1}T_1 \\
&\leq \left[3^{m-1}(\Theta+\upsilon_0)^{m-1}{K_1}^m+3^{m-1}\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}(\Theta+\upsilon_0)^{\frac{m-2}{2}}{K_3}^m\right]\int_0^\mu E\,|\lambda(s)-\sigma(s)|^m ds \\
&\quad +3^{2m-2}(\Theta+\upsilon_0)^{m-1}\Bigg[{K_2}^m(\Theta+\upsilon_0)L_3E\|\lambda(0)\|^m+{K_2}^mE\int_0^\mu|\lambda(s)-\sigma(s)|^m ds \\
&\quad +2^m{K_2}^m\left(1-e^{-\hat{\gamma}\upsilon}\right)\times(\Theta+\upsilon_0)NL_1E\|\lambda(0)\|^m\Bigg] \\
&\leq \left[3^{m-1}(\Theta+\upsilon_0)^{m-1}{K_1}^m+3^{m-1}\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}(\Theta+\upsilon_0)^{\frac{m-2}{2}}{K_3}^m+3^{2m-2}(\Theta+\upsilon_0)^{m-1}{K_2}^m\right] \\
&\quad \times\int_0^\mu E\,|\lambda(s)-\sigma(s)|^m ds+3^{2m-2}(\Theta+\upsilon_0)^m{K_2}^m\left[L_3+2^m\left(1-e^{-\hat{\gamma}\upsilon}\right)NL_1\right]E\|\lambda(0)\|^m.
\end{aligned}
$$

By Gronwall inequality, we have that

$$
\begin{aligned}
&E|\lambda(\mu)-\sigma(\mu)|^m \\
&\leq 3^{2m-2}(\Theta+\upsilon_0)^m{K_2}^m\left[L_3+2^m\left(1-e^{-\hat{\gamma}\upsilon}\right)NL_1\right] \\
&\quad \times e^{3^{m-1}\left[(\Theta+\upsilon_0)^{m-1}{K_1}^m+\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}(\Theta+\upsilon_0)^{\frac{m-2}{2}}{K_3}^m+3^{m-1}(\Theta+\upsilon_0)^{m-1}{K_2}^m\right](\Theta+\upsilon_0)}E\|\lambda(0)\|^m \\
&\leq 3^{2m-2}(\Theta+\upsilon_0)^m{K_2}^m\left[L_3+2^m\left(1-e^{-\hat{\gamma}\upsilon}\right)NL_1\right]\times e^{3^{m-1}\left[{K_1}^m+\left(\frac{m(m-1)}{2}\right)^{\frac{m}{2}}(\Theta+\upsilon_0)^{-\frac{m}{2}}{K_3}^m+3^{m-1}{K_2}^m\right](\Theta+\upsilon_0)^m} \\
&\quad \times E\|\lambda(0)\|^m.
\end{aligned}
$$

When $m\in(0,2)$, by Hölder inequality, we have that

$$
\begin{aligned}
&E|\lambda(\mu)-\sigma(\mu)|^m \\
&\leq \Big[3^2(\Theta+\upsilon_0)^2{K_2}^2\left[L_3(2,\upsilon_0,\Theta)+4\left(1-e^{-\hat{\gamma}\upsilon}\right)NL_1(2,\upsilon_0,\Theta)\right] \\
&\quad \times e^{3\left[{K_1}^2+(\Theta+\upsilon_0)^{-1}{K_3}^2+3{K_2}^2\right](\Theta+\upsilon_0)^2}\Big]^{\frac{m}{2}}E\|\lambda(0)\|^m.
\end{aligned}
$$

4. Proof of Theorem 1

Proof. Write $\lambda(\mu;\xi,\varsigma,0)=\lambda(\mu)$, $\Gamma(\mu;\varsigma,0)=\Gamma(\mu)$ for $\mu\geq 0$. Similarly, let

$$\sigma\big(\upsilon_0+\Theta;\upsilon_0,\lambda(\upsilon_0),\Gamma(\upsilon_0)\big)=\sigma(\upsilon_0+\Theta).$$

By Lemmas 2 and 3, have that

$$
\begin{aligned}
&E|\sigma(v_0+\Theta)|^m\\
&\le ME|\lambda(v_0)|^m e^{-\chi_1\Theta}\\
&\le M(1+K_2v_0)^{\frac{m_2}{2}}e^{v_0m(K_1+0.5(m_1-1)K_3{}^2+K_2)}E\|\lambda(0)\|^m e^{-\chi_1\Theta},
\end{aligned}
\tag{4.1}
$$

where $m_2=2\wedge m$. By the elementary inequality $(a+b)^m\le 2^{m_3}(a^m+b^m)$ for any $a,b\ge 0$ and Lemma 6,

$$
\begin{aligned}
&E|\lambda(v_0+\Theta)|^m\\
&=E|\sigma(v_0+\Theta)+\lambda(v_0+\Theta)-\sigma(v_0+\Theta)|^m\\
&\le 2^{m_3}(E|\sigma(v_0+\Theta)|^m+E|\lambda(v_0+\Theta)-\sigma(v_0+\Theta)|^m)\\
&\le 2^{m_3}\left(M(1+K_2v_0)^{\frac{m_2}{2}}e^{v_0m(K_1+0.5(m_1-1)K_3{}^2+K_2)}e^{-\chi_1\Theta}+L_4\right)E\|\lambda(0)\|^m
\end{aligned}
\tag{4.2}
$$

Using Lemma 5, obtain that

$$
\begin{aligned}
&E\|\lambda(2v_0+\Theta)\|^m\\
&\le 2^{m_3}\left(E|\lambda(v_0+\Theta)|^m+E\left(\sup_{0\le\theta\le v_0}|\lambda(\theta+v_0+\Theta)-\lambda(v_0+\Theta)|^m\right)\right)\\
&\le 2^{m_3}(E|\lambda(v_0+\Theta)|^m+L_3(m,v_0,v_0+\Theta)E\|\lambda(0)\|^m)\\
&\le 2^{m_3}\left(2^{m_3}\left(M(1+K_2v_0)^{\frac{m_2}{2}}e^{v_0m(K_1+0.5(m_1-1)K_3{}^2+K_2)}e^{-\chi_1\Theta}+L_4\right)+L_3(m,v_0,v_0+\Theta)\right)\\
&\quad\times E\|\lambda(0)\|^m\\
&\le\left[2^{2m_3}Me^{-\chi_1\Theta}(1+K_2v_0)^{\frac{m_2}{2}}e^{v_0m(K_1+0.5(m_1-1)K_3{}^2+K_2)}+2^{m_3}\left(2^{m_3}L_4+L_3(m,v_0,v_0+\Theta)\right)\right]\\
&\quad\times E\|\lambda(0)\|^m\\
&\le\left[\varepsilon(1+K_2v_0)^{\frac{m_2}{2}}e^{v_0m(K_1+0.5(m_1-1)K_3{}^2+K_2)}+2^{m_3}\left(2^{m_3}L_4+L_3(m,v_0,v_0+\Theta)\right)\right]E\|\lambda(0)\|^m,
\end{aligned}
\tag{4.3}
$$

where $\varepsilon=2^{2m_3}Me^{-\chi_1\Theta}$. Since $v_0<v^*$, it is obtained from the definition of v^* that

$$
\varepsilon(1+K_2v_0)^{\frac{m}{2}}e^{v_0m(K_1+0.5(m_1-1)K_3{}^2+K_2)}+2^{m_3}\left(2^{m_3}L_4+L_3(m,v_0,v_0+\Theta)\right)<1.
$$

Therefore, there exists $\zeta>0$ such that

$$
\varepsilon(1+K_2v_0)^{\frac{m}{2}}e^{v_0m(K_1+0.5(m_1-1)K_3{}^2+K_2)}+2^{m_3}\left(2^{m_3}L_4+L_3(m,v_0,v_0+\Theta)\right)=e^{-\zeta(2v_0+\Theta)}.
$$

It is concluded from (4.3) that

$$
E\|\lambda(2v_0+\Theta)\|^m\le e^{-\zeta(2v_0+\Theta)}E\|\lambda(0)\|^m.
$$

Further considering the solution $\lambda(\mu)$ on $\mu\ge\Theta+2v_0$, there is a N^+ such that $\Theta+2v_0=N^+\Delta v_0$. Meanwhile, $\lambda(\mu)$ can be referred to as the solution of the Eq (2.2) with the initial value $\lambda(N^+\Delta v_0)$, $\Gamma(N^+\Delta v_0)$. By following the same procedure as mentioned above, show that

$$
\begin{aligned}
E\|\lambda(2N^+\Delta v_0)\|^m&\le e^{-\zeta N^+\Delta v_0}E\|\lambda(N^+\Delta v_0)\|^m\\
&\le e^{-\zeta N^+\Delta v_0}e^{-\zeta(2v_0+\Theta)}E\|\lambda(0)\|^m\\
&\le e^{-2\zeta N^+\Delta v_0}E\|\lambda(0)\|^m.
\end{aligned}
$$

Repeating the above procedure, have that

$$E\|\lambda(zN^{+}\Delta v_0)\|^m \leq e^{-z\zeta N^{+}\Delta v_0} E\|\lambda(0)\|^m . z = 1,2,\cdots$$

By Lemma 4, have that

$$E\left(\sup_{zN^{+}\Delta v_0 \leq \mu \leq (z+1)N^{+}\Delta v_0} E|\lambda(\mu)|^m\right) \leq L_2(m, v_0, N^{+}\Delta v_0 - v_0) e^{-z\zeta N^{+}\Delta v_0} E\|\lambda(0)\|^m. \quad (4.4)$$

From the Markov inequality and inequality (4.4), it can be concluded that, for all $z \geq 0$,

$$P\left(\sup_{zN^{+}\Delta v_0 \leq \mu \leq (z+1)N^{+}\Delta v_0} |\lambda(\mu)|^m \geq e^{-0.5z\zeta N^{+}\Delta v_0}\right)$$

$$\leq e^{0.5z\zeta N^{+}\Delta v_0} E\left(\sup_{zN^{+}\Delta v_0 \leq \mu \leq (z+1)N^{+}\Delta v_0} |\lambda(\mu)|^m\right)$$

$$\leq e^{-0.5z\zeta N^{+}\Delta v_0} L_2(m, v_0, N^{+}\Delta v_0 - v_0) E\|\lambda(0)\|^m.$$

According to the $\mathrm{Borel-Cantelli}$ lemma, there is a set $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$, then for almost all $\omega \in \Omega_0$, there is an integer $z_0 = z_0(\omega)$ such that for $\forall z > z_0(\omega)$,

$$\sup_{zN^{+}\Delta v_0 \leq \mu \leq (z+1)N^{+}\Delta v_0} |\lambda(\mu)|^m < e^{-0.5z\zeta N^{+}\Delta v_0} \forall z > z_0(\omega). a.s.$$

Hence,

$$\lim_{\mu\to\infty} \sup \frac{1}{\mu} log(|\lambda(\mu,\omega)|) \leq -\frac{0.5z\zeta N^{+}\Delta v_0}{mzN^{+}\Delta v_0} = -\frac{\zeta}{2m}.$$

The proof is completed.

5. Numerical example

Consider a stochastic differential equation

$$d\eta(\mu) = h(\eta(\mu), \Gamma(\mu), \mu)d\mu + k(\eta(\mu), \Gamma(\mu), \mu)dw(\mu), \quad (5.1)$$

in which $\Gamma(\mu)$ represents a Markov chain that takes values in $S = \{1,2\}$ with the generator $\Lambda = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$, and

$$h(\eta, 1, \mu) = 0.2\eta, \ k(\eta, 1, \mu) = 0.4\eta, \quad (5.2)$$

$$h(\eta, 2, \mu) = 0.4\eta, \ k(\eta, 2, \mu) = 0.5\eta. \quad (5.3)$$

Using the Euler-Maruyama numerical method, with $\eta(0) = 1$, $\Gamma(0) = 1$ and the step size 10^{-5}, it is seen that the system (5.1) is unstable, as shown in Figure 1.

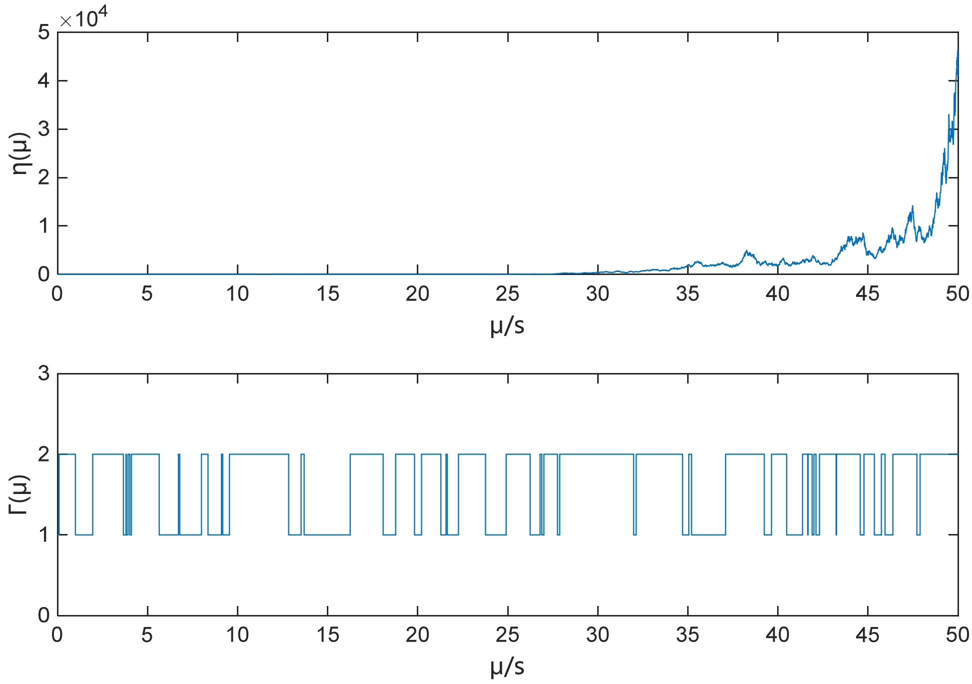

Figure 1. Trajectory simulation of $\eta(\mu)$ and $\Gamma(\mu)$ for system (5.1).

In order to stabilize the system (5.1), the control function is designed as follows:

$$\varpi(\eta, 1, \mu) = -0.4\eta, \ \varpi(\eta, 2, \mu) = -0.5\eta.$$

By simple calculations, Assumption 1 is satisfied with $K_1 = 0.4$, $K_2 = 0.5$ and $K_3 = 0.5$. Choosing $m = 1$, we can infer from Assumption 2 that

$$\beta_1 = 0.2, \ \beta_2 = 0.4, \ c_1 = 0.2, \ c_2 = 0.1.$$

From (2.6), obtain a non-singular M-matrix.

$$\mathcal{A} = \begin{pmatrix} c_1 m & 0 \\ 0 & c_2 m \end{pmatrix} - \Lambda = \begin{pmatrix} 2.2 & -2 \\ -1 & 1.1 \end{pmatrix}.$$

By (2.7) and (2.8), we have

$$b_{max} = 7.6191 \ \ b_{min} = 7.3809 \ \ \chi = max(\beta_u + c_u) = 0.5.$$

By Lemma 2, it is known that if $\varphi \in (0.7375,1)$, then the auxiliary controlled stochastic system $d\sigma(\mu) = \big(h(\sigma(\mu), \Gamma(\mu), \mu) + \varpi(\sigma(\mu), \Gamma(\mu), \mu)\mathrm{I}(\mu)\big)d\mu + k(\sigma(\mu), \Gamma(\mu), t)dw(\mu)$ is almost surely exponentially stable.

This paper aims to develop an intermittent feedback controller with discrete observations of both state and mode with delays for stabilizing system (5.1). The controlled system becomes

$$d\lambda(\mu) = \Big(h(\lambda(\mu), \Gamma(\mu), \mu) + \varpi\big(\lambda(\delta_\mu), \Gamma(\delta_\mu), \mu\big)\mathrm{I}(\mu)\Big)\, d\mu + k(\lambda(\mu), \Gamma(\mu), \mu)dw(\mu). \quad (5.4)$$

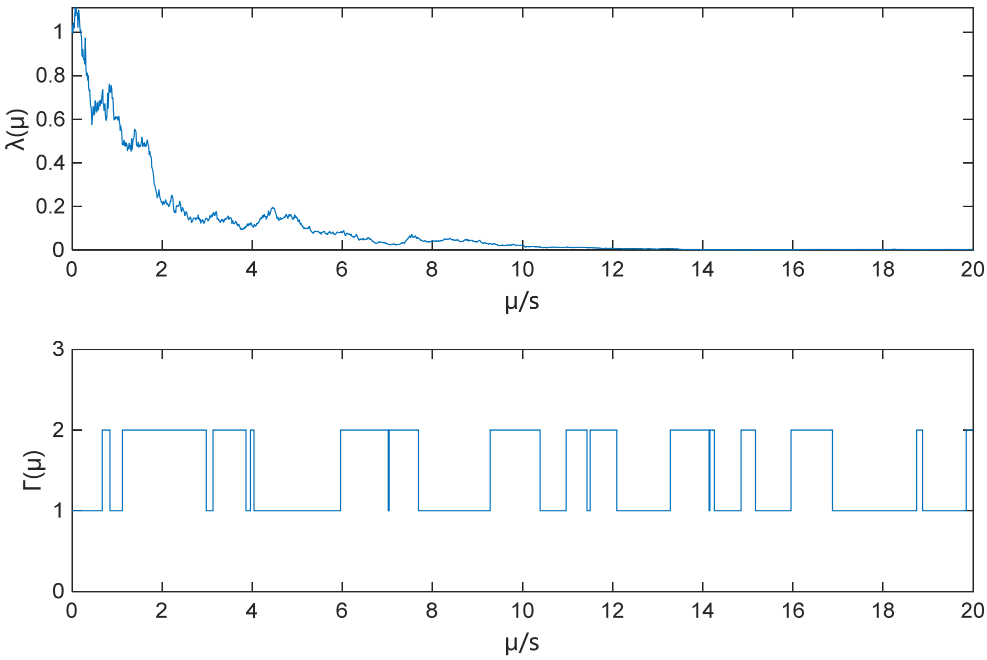

Figure 2: Trajectory simulation of $\lambda(\mu)$ and $\Gamma(\mu)$ for system (5.4).

We choose $\varphi = 0.95$, $\varepsilon = 0.9$, and it can be obtained that $\chi_1 = 0.1062$, $\Theta = 1.0126$. Then (2.9) becomes

$$0.9(1 + 0.5\upsilon_0)^{\frac{1}{2}} e^{1.025\upsilon_0} + L_4(1, \upsilon_0, 1.0126) + L_3(1, \upsilon_0, 1.0126 + \upsilon_0) = 1,$$

and its unique root is $\upsilon^* = 4.6436 \times 10^{-5}$. Taking $\upsilon = 10^{-4}$, $\upsilon_0 = 10^{-5}$, $\Delta = 10^{-5}$, from Theorem 1, we can get that the system (5.4) is almost surely exponentially stable. The simulated trajectory is shown in Figure 2 with the step size 10^{-5}.

6. Conclusions

This paper delves into the issue of exponential stabilization for hybrid stochastic systems by employing an intermittent feedback control with discrete delayed observations related to both state and mode. Compared to state discrete observations of the feedback controller, model observations are also performed at discrete times, which is more practical and cost-saving. Using M-matrix theory and intermittent control approach, the feedback stabilization theory of hybrid stochastic systems is established. However, this study focuses on the stabilization for hybrid stochastic systems and the underlying systems do not consider some important practical factors, such as time delay, which will be addressed in future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix

Proof of Lemma 1.

Given $\Gamma(\mu)=u$, define stopping time $\bar{\kappa}_u=inf\{s\geq\mu:\Gamma(s)\neq u\}$. Let $inf\,\Phi=\infty$. Since $\bar{\kappa}_u-\mu$ conforms to an exponential distribution with the parameter $-\gamma_{uu}$, one has that, for $s\in[\mu,\mu+j]$,

$$\begin{aligned}&P(\Gamma(s)\neq u|\Gamma(\mu)=u)\\ \leq\ &P(\bar{\kappa}_u-\mu\leq j|\Gamma(\mu)=u)\\ =\ &\int_0^j-\gamma_{uu}e^{\gamma_{uu}s}ds\\ =\ &1-e^{\gamma_{uu}j}\\ \leq\ &1-e^{-\hat{\gamma}j}.\end{aligned}\tag{A1}$$

Thus, assertion (3.1) is obtained.

Proof of Lemma 2.

Define $H(\sigma,u,\mu)=b_u|\sigma|^m,\ (\sigma,u,\mu)\in(\mathbb{R}^n-\{0\})\times S\times\mathbb{R}_+$, then

$$EH(\sigma(\mu),\Gamma(\mu),\mu) = EH(\sigma(0),\Gamma(0),0) + E\int_0^\mu LH\,(\sigma(s),\Gamma(s),s)ds, \tag{A2}$$

where generalized Itô operator $LH\colon (\mathbb{R}^n - \{0\}) \times \mathbb{R}_+ \times S \to \mathbb{R}$ is given by

$$\begin{aligned} LH(\sigma,u,\mu) = b_u m|\sigma|^m \Big[\tfrac{1}{|\sigma|^2}\Big(\sigma^T\big(h(\sigma,u,\mu) + \varpi(\sigma,u,\mu)\mathrm{I}(\mu)\big) + \tfrac{1}{2}|k(\sigma,u,\mu)|^2\Big) \\ -\tfrac{(2-m)}{2|\sigma|^4}|\sigma^T k(\sigma,u,\mu)|^2\Big] + \textstyle\sum_{v=1}^N \gamma_{uv}\, b_v|\sigma|^m. \end{aligned} \tag{A3}$$

As $\mu \in [\mu_l,\mu_l + \varphi\Delta)$, $\mathrm{I}(\mu) = 1$. By Assumptions 2 and 3, conclude that

$$\begin{aligned} LH(\sigma,u,\mu) &\le b_u m|\sigma|^m\big(\beta_\mu - \tau_\mu\big) + \sum_{v=1}^N \gamma_{uv}\, b_v|\sigma|^m \\ &\le -b_u m c_u|\sigma|^m + \sum_{v=1}^N \gamma_{uv}\, b_v|\sigma|^m = -|\sigma|^m\left(b_u\alpha_u(m) - \sum_{v=1}^N \gamma_{uv}\, b_v\right). \end{aligned}$$

By (2.6) and (2.7), have that

$$b_u\alpha_u(m) - \textstyle\sum_{v=1}^N \gamma_{uv}\, b_v = 1,$$

hence,

$$LH(\sigma,u,\mu) \le -|\sigma|^m = -b_u|\sigma|^m\tfrac{1}{b_u} \le -\tfrac{1}{b_{max}}H(\sigma,u,\mu). \tag{A4}$$

As $\mu \in [\mu_l + \varphi\Delta, \mu_{l+1})$, $\mathrm{I}(\mu) = 0$, then

$$\begin{aligned} &LH(\sigma,u,\mu) \\ &\le b_u m|\sigma|^m\beta_\mu + \sum_{v=1}^N \gamma_{uv}\, b_v|\sigma|^m \\ &\le b_u m|\sigma|^m(\chi - c_u) + \sum_{v=1}^N \gamma_{uv}\, b_v|\sigma|^m \\ &\le m\chi|\sigma|^m b_u - |\sigma|^m\left(m b_u c_u - \sum_{v=1}^N \gamma_{uv}\, b_v\right) \\ &= m\chi|\sigma|^m b_u - |\sigma|^m\left(b_u\alpha_u(m) - \sum_{v=1}^N \gamma_{uv}\, b_v\right). \end{aligned}$$

Since $b_u\alpha_u(m) - \sum_{v=1}^N \gamma_{uv}\, b_v = 1$,

$$LH(\sigma,u,\mu) \le m\chi|\sigma|^m b_u - |\sigma|^m = b_u|\sigma|^m\left(m\chi - \tfrac{1}{b_u}\right) \le \left(m\chi - \tfrac{1}{b_{max}}\right)H(\sigma,u,\mu). \tag{A5}$$

From (A4) and (A5),

$$LH(\sigma,u,\mu) \le \left[-\tfrac{1}{b_{max}}\mathrm{I}(\mu) + \left(m\chi - \tfrac{1}{b_{max}}\right)\big(1 - \mathrm{I}(\mu)\big)\right]H(\sigma,u,\mu).$$

For any integer $\iota \geq 1$, define stopping time $\kappa_\iota = inf\{\mu \geq \mu_0 : |\sigma(\mu)| \geq \iota\}$. It is evident that when $\iota \to \infty$, $\kappa_\iota \to \infty$. When $\mu \geq 0$, by utilizing the Itô formula, yield that

$$E[H(\sigma(\mu \wedge \kappa_\iota), \Gamma(\mu \wedge \kappa_\iota), \mu \wedge \kappa_\iota)]e^{-\int_0^{\mu\wedge\kappa_\iota}\left[\frac{1}{-b_{max}}\mathrm{I}(s)+\left(m\chi-\frac{1}{b_{max}}\right)(1-\mathrm{I}(s))ds\right]}$$

$$= EH(\sigma_0, \Gamma_0, 0) + E\int_0^{\mu\wedge\kappa_\iota} e^{-\int_0^s\left[-\frac{1}{b_{max}}\mathrm{I}(\theta)+\left(m\chi-\frac{1}{b_{max}}\right)(1-\mathrm{I}(\theta))d\theta\right]}$$

$$\times \left[LH(\sigma(s), \Gamma(s), s) - \left(-\frac{1}{b_{max}}\mathrm{I}(s) + \left(m\chi - \frac{1}{b_{max}}\right)\left(1 - \mathrm{I}(s)\right)\right)H(\sigma(s), \Gamma(s), s)\right] ds$$

$$\leq EH(\sigma_0, \Gamma_0, 0).$$

When $\iota \to \infty$, get that

$$EH(\sigma(\mu), \Gamma(\mu), \mu)e^{-\int_0^{\mu}\left[-\frac{1}{b_{max}}\mathrm{I}(s)+\left(m\chi-\frac{1}{b_{max}}\right)(1-\mathrm{I}(s))ds\right]} \leq EH(\sigma_0, \Gamma_0, 0),$$

which means that

$$b_{min}E|\sigma(\mu)|^m \leq b_{max}E|\sigma_0|^m e^{\int_0^{\mu}\left[-\frac{1}{b_{max}}\mathrm{I}(s)+\left(m\chi-\frac{1}{b_{max}}\right)(1-\mathrm{I}(s))\right]ds} \tag{A6}$$

By $0 \leq 1 - \frac{1}{m\chi b_{max}} < \varphi < 1$, $\mu_l = l\Delta$,

$$-\frac{1}{b_{max}} \leq -\frac{1}{b_{max}}\varphi \leq -\frac{1}{b_{max}}\varphi + \left(m\chi - \frac{1}{b_{max}}\right)(1-\varphi) = m\chi - \frac{1}{b_{max}} - m\chi\varphi.$$

Let $l > N^+$, when $\mu \in [\mu_l, \mu_l + \varphi\Delta)$,

$$\int_0^{\mu}\left[-\frac{1}{b_{max}}\mathrm{I}(s) + \left(m\chi - \frac{1}{b_{max}}\right)(1 - \mathrm{I}(s))\right]ds$$
$$= (m\chi - m\chi\varphi)l\Delta + \left(m\chi - \frac{1}{b_{max}} - m\chi\right)\mu$$
$$= \left(m\chi - \frac{1}{b_{max}} - m\chi\varphi\right)l\Delta - \frac{1}{b_{max}}(\mu - l\Delta)$$
$$\leq \left(m\chi - \frac{1}{b_{max}} - m\chi\varphi\right)\mu.$$

When $\mu \in [\mu_l + \varphi\Delta, \mu_{l+1})$,

$$\int_0^{\mu}\left[-\frac{1}{b_{max}}\mathrm{I}(s) + \left(m\chi - \frac{1}{b_{max}}\right)(1 - \mathrm{I}(s))\right]ds$$
$$= \left(m\chi - \frac{1}{b_{max}}\right)\mu - m\chi(l+1)\varphi\Delta$$
$$\leq \left(m\chi - \frac{1}{b_{max}} - m\chi\varphi\right)\mu.$$

Substituting this into (A6), yield that

$$E|\sigma(\mu)|^m \leq \frac{b_{max}}{b_{min}} E|\sigma_0|^m e^{\left(m\chi - \frac{1}{b_{max}} - m\chi\varphi\right)\mu} \leq ME|\sigma_0|^m e^{\left(m\chi - \frac{1}{b_{max}} - m\chi\varphi\right)\mu},$$

hence $E|\sigma(\mu)|^m \leq ME|\sigma_0|^m e^{-\chi_1 \mu}$. Inequality (3.2) has been proven.

Next, we start to prove inequality (3.3). Similarly to (A3), employing the Itô formula to $|\sigma(\mu)|^m$, obtain that

$$\begin{aligned} &E|\sigma(\mu)|^m \\ &\leq E|\sigma(0)|^m + E\int_0^\mu m\,|\sigma(s)|^m \left[\frac{1}{|\sigma(s)|^2}\left(\sigma(s)^T\left(h(\sigma(s),\Gamma(s),s) + \varpi(\sigma(s),\Gamma(s),s)\mathrm{I}(s)\right)\right.\right. \\ &\left.\left. + \frac{1}{2}|k(\sigma(s),\Gamma(s),s)|^2\right) - \frac{(2-m)}{2|\sigma(s)|^4}|\sigma(s)^T k(\sigma(s),\Gamma(s),s)|^2\right] ds. \end{aligned}$$

As $\mu \in [\mu_l, \mu_l + \varphi\Delta)$, $\mathrm{I}(\mu) = 1$. By Assumption 2,

$$E|\sigma(\mu)|^m \leq E|\sigma(0)|^m + mE\int_0^\mu (\beta_u - \tau_u)\,|\sigma(s)|^m ds \leq E|\sigma(0)|^m - c_u m\int_0^\mu E\,|\sigma(s)|^m ds.$$

As $\mu \in [\mu_l + \varphi\Delta, \mu_{l+1})$, $\mathrm{I}(\mu) = 0$,

$$E|\sigma(\mu)|^m \leq E|\sigma(0)|^m + \beta_u m\int_o^\mu E|\sigma(s)|^m\, ds.$$

By (2.8), we have that

$$\sup_{0\leq s\leq \mu} E|\sigma(s)|^m \leq E|\sigma(0)|^m + \chi m\int_0^\mu \sup_{0\leq\theta\leq s} E|\sigma(\theta)|^m\, ds.$$

Gronwall inequality leads to

$$\sup_{0\leq s\leq \mu} E|\sigma(s)|^m \leq E|\sigma(0)|^m e^{\chi m\mu}.$$

Hence,

$$\int_0^\Delta \sup_{0\leq\theta\leq s} E|\sigma(\theta)|^m ds \leq \frac{1}{\chi m}(e^{\chi m\Delta} - 1)E|\sigma(0)|^m.$$

If $m = 2$,

$$\int_0^\Delta \sup_{0\leq\theta\leq s} E|\sigma(\theta)|^2 ds \leq \frac{1}{2\chi}(e^{2\chi\Delta} - 1)E|\sigma(0)|^2. \tag{A7}$$

For a non-negative integer l, we have that

$$\begin{aligned} E\left(\sup_{\mu_l\leq\mu\leq\mu_{l+1}} |\sigma(\mu)|^2\right) &\leq 3E|\sigma(\mu_l)|^2 + 3E\left|\int_{\mu_l}^{\mu_{l+1}} h(\sigma(s),\Gamma(s),s) + \varpi(\sigma(s),\Gamma(s),s)\times\mathrm{I}(s)\,ds\right|^2 \\ &+3E\left(\sup_{\mu_l\leq s\leq\mu_{l+1}}\left|\int_{\mu_l}^{s} k(\sigma(s),\Gamma(s),s)dw(s)\right|^2\right). \quad \text{(A8)} \end{aligned}$$

Under Assumption 1 and utilizing the Burkholder − Davis − Gundy inequality, we yield that

$$E\left(\sup_{0\le\mu\le\Delta}|\sigma(\mu)|^2\right)\le 3E|\sigma(0)|^2+6\Delta\int_0^{\Delta}E(|h(\sigma(s),\Gamma(s),s)|^2$$

$$+|\varpi(\sigma(s),\Gamma(s),s)\times \mathrm{I}(s)|^2)ds+12\int_0^{\Delta}E|k(\sigma(s),\Gamma(s),s)|^2ds$$

$$\le 3E|\sigma(0)|^2+(6\Delta K_1^2+12K_3^2+6\Delta K_2^2)\int_0^{\Delta}\sup_{0\le\mu\le s}|\sigma(\mu)|^2ds.$$

Substituting (A7) into the above inequality, we obtain that

$$E\left(\sup_{0\le\mu\le\Delta}|\sigma(\mu)|^2\right)\le 3E|\sigma(0)|^2+\left(6\Delta {K_1}^2+12{K_3}^2+6\Delta {K_2}^2\right)\frac{1}{2\chi}(e^{2\chi\Delta}-1)E|\sigma(0)|^2.$$

For $m\in(0,2)$, by Hölder inequality, we get that

$$E\left(\sup_{0\le\mu\le\Delta}|\sigma(\mu)|^m\right)\le J^{\frac{m}{2}}E|\sigma(0)|^m,$$

where $J=3+\left(6\Delta {K_1}^2+12{K_3}^2+6\Delta {K_2}^2\right)\frac{1}{2\chi}(e^{2\chi\Delta}-1)$.

Repeating the above process, we have that

$$E\left(\sup_{q\Delta\le\mu\le(q+1)\Delta}|\sigma(\mu)|^m\right)\le J^{\frac{m}{2}}E|\sigma(q\Delta)|^m.q=1,2,\cdots$$

Using Chebyshev's inequality, have we that

$$P\left(\sup_{q\Delta\le\mu\le(q+1)\Delta}|\sigma(\mu)|^m\ge e^{-0.5\chi_1 q\Delta}\right)\le e^{0.5\chi_1 q\Delta}E\left(\sup_{q\Delta\le\mu\le(q+1)\Delta}|\sigma(\mu)|^m\right)$$

$$\le e^{0.5\chi_1 q\Delta}J^{\frac{m}{2}}E|\sigma(q\Delta)|^m$$

$$\le e^{-0.5\chi_1 q\Delta}J^{\frac{m}{2}}ME|\sigma(0)|^m.$$

According to the $\mathrm{Borel-Cantelli}$ lemma, there is a set $\Omega_0\in\mathscr{F}$ with $P(\Omega_0)=1$, then for almost all $\omega\in\Omega_0$, there exists an integer $q_0=q_0(\omega)$ such that for $\forall q\ge q_0$,

$$\sup_{q\Delta\le\mu\le(q+1)\Delta}|\sigma(\mu)|^m\le e^{-0.5\chi_1 q\Delta}.$$

Therefore, for $q\Delta\le\mu\le(q+1)\Delta$,

$$\frac{1}{\mu}\log|\sigma(\mu)|\le-\frac{0.5\chi_1 q\Delta}{(q+1)\Delta m}.$$

Letting $\mu\to\infty$, obtain that

$$\lim_{\mu\to\infty}\sup\frac{1}{\mu}\log|\sigma(\mu)| \le -\frac{\chi_1}{2m}. a.s.$$

This completes the proof.