Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(9): 5559–5572.
DOI: 10.3934/era.2023282
Received: 23 January 2023
Revised: 21 April 2023
Accepted: 25 July 2023
Published: 14 August 2023

Research article

On two-term exponential sums and their mean values

Jianghua Li*, **Zepeng Zhang and Tianyu Pan**

College of Sciences, Xi'an University of Technology, Xi'an, Shaanxi, China

* **Correspondence:** Email: jianghuali@xaut.edu.cn.

Abstract: The mean value problems of exponential sums play a very important role in the research of analytic number theory, and many famous number theory problems are closely related to them. The main purpose of this paper is using some elementary methods and the number of the solutions of the congruence equations to study the calculating problem of some fourth power means of two-term exponential sums, and give exact calculating formulae and asymptotic formula for them.

Keywords: the two-term exponential sums; fourth power mean; elementary method; calculating formula

1. Introduction

As usual, let p be an odd prime. For any integers $h > r \geq 1$ and integers m and n, the two-term exponential sum $S(m, n, h, r; p)$ is defined as

$$S(m, n, h, r; p) = \sum_{a=0}^{p-1} e\left(\frac{ma^h + na^r}{p}\right),$$

where $e(y) = e^{2\pi iy}$ and i is the imaginary unit. That is, $i^2 = -1$.

These sums play very important role in the study of analytic number theory, so many number theorists have studied the various properties of $S(m, n, h, r; p)$ and obtained some meaningful research results. Here we give a few examples. W. P. Zhang and D. Han [1] used analytic methods to study the sixth power mean of $S(1, m, 3, 1; p)$, and proved the following result:

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^3 + ma}{p}\right) \right|^6 = 5p^4 - 8p^3 - p^2, \tag{1.1}$$

where p denotes an odd prime with $3 \nmid (p - 1)$.

Recently, W. P. Zhang and Y. Y. Meng [2] also studied the sixth power mean of $S(m, n, 3, 1; p)$, and obtained some new identities. In fact, they proved that for any odd prime p and integer n with $(n, p) = 1$, one has

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^6 = \begin{cases} 5p^3 \cdot (p-1) & \text{if } p \equiv 5 \bmod 6; \\ p^2 \cdot \left(5p^2 - 23p - d^2\right) & \text{if } p \equiv 1 \bmod 6. \end{cases} \tag{1.2}$$

here, $4p = d^2 + 27 \cdot b^2$, and d is uniquely determined by $d \equiv 1 \bmod 3$.

On the other hand, L. Chen and X. Wang [3] proved

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^4 = \begin{cases} 2p^2(p-2) & \text{if } p \equiv 7 \bmod 12, \\ 2p^3 & \text{if } p \equiv 11 \bmod 12, \\ 2p(p^2 - 10p - 2\alpha^2) & \text{if } p \equiv 1 \bmod 24, \\ 2p(p^2 - 4p - 2\alpha^2) & \text{if } p \equiv 5 \bmod 24, \\ 2p(p^2 - 6p - 2\alpha^2) & \text{if } p \equiv 13 \bmod 24, \\ 2p(p^2 - 8p - 2\alpha^2) & \text{if } p \equiv 17 \bmod 24, \end{cases} \tag{1.3}$$

where the character sums $\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a^3 + a}{p}\right)$ is an integer satisfying the identity (see Theorem 4–11 in [4]):

$$p = \alpha^2 + \beta^2 = \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a^3 + a}{p}\right) \right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a^3 + ra}{p}\right) \right)^2 .$$

$\left(\frac{*}{p}\right)$ denotes the Legendre symbol, and r is a quadratic non-residue modulo p.

In addition, some related papers can also be found in [5-11].

From the formulae (1.1)–(1.3), it is not difficult to see that the content of all these papers only involves $r = 1$ in $S(m, n, h, r; p)$. For the case $r > 1$ in $S(m, n, h, r; p)$, we have not found a corresponding conclusion, at least not so far. Therefore, when $h > r = 2$, the research is difficult, and it is difficult to obtain some ideal results.

In this paper, we use elementary methods and results on the number of solutions to study the calculating problem of the $2k$-th power mean

$$S_{2k}(p) = \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^{2k},$$

and give an exact calculating formula for $S_4(p)$ with $p \equiv 3 \bmod 4$. That is, we will prove the following two main results:

Theorem 1. Let p be a prime with $p \equiv 3 \bmod 4$. Then, we have the identity

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^4 = p^2 \cdot (7p - 10) .$$

Theorem 2. Let p be a prime with $p \equiv 1 \bmod 4$. Then, we have the identity

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = p^2 + 13p + 4(-1)^{\frac{p-1}{4}} p - 2\alpha(p) - 4(-1)^{\frac{p-1}{4}} \alpha(p) + 4\alpha^2(p),$$

where $\alpha = \alpha(p)$ is defined in (1.3).

Some notes: In Theorem 1, we only discussed a prime p in the case $p \equiv 3 \bmod 4$. If $p \equiv 1 \bmod 4$, then the situation is much more complicated. We cannot calculate the exact value of $S_4(p)$, and we cannot even get a valid asymptotic formula for $S_4(p)$. The reason is that we do not know the exact value of

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+b^2-c^2-1}{p}\right) \tag{1.4}$$

and do not know a valid asymptotic formula for it.

However, if $p \equiv 1 \bmod 4$, we give some ideas and methods of studying $S_4(p)$ (see Lemma 3 and Lemma 4 below). We even have the following:

Conjecture. If prime $p \equiv 1 \bmod 4$, then there is an asymptotic formula

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4+a^2}{p}\right) \right|^4 = 7p^3 + O\left(p^{\frac{5}{2}}\right).$$

To prove this conjecture, we can convert (1.4) to the estimate for character sums

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{ab(a+b-c-1)}{p}\right)$$

or character sums

$$D(p) = \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{ac(a-c-1)(a^2-c^2-1)}{p}\right).$$

If one can give a non-trivial upper bound estimate for $D(p)$, such as $|D(p)| \ll p$, then we can prove that the conjecture is correct.

For any odd prime p with $p \equiv 3 \bmod 4$ and integer $k \geq 3$, whether there exists an exact calculating formula for

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4+a^2}{p}\right) \right|^{2k}$$

is an open problem. Interested readers can continue this research.

2. Several lemmas

To complete the proofs of our main results, we need some simple lemmas. Of course, the proofs of all these lemmas need some knowledge of elementary number theory and analytic number theory, and all these can be found in [12, 13], so we do not have to repeat them here. First, we have the following:

Lemma 1. For any odd prime p, we have the identities

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = 4(2p-3) \ \text{ and } \ \sum_{\substack{a=0 \\ a^4+1\equiv c^4 \bmod p \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = 2.$$

Proof. It is clear that

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ \left(a^2+b^2\right)^2-a^4-b^4\equiv\left(c^2+1\right)^2-c^4-1 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^2b^2\equiv c^2 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1$$

$$= \sum_{\substack{a=0 \\ \left(a^2-1\right)\left(b^2-1\right)\equiv 0 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = 4(p-3)+2+4(p-3)+2+8 = 4(2p-3). \tag{2.1}$$

Similarly, we also have

$$\sum_{\substack{a=0 \\ a^4+1\equiv c^4 \bmod p \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ \left(a^2+1\right)^2-a^4-1\equiv c^4-c^4 \bmod p \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^2\equiv 0 \bmod p \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = 2. \tag{2.2}$$

Now Lemma 1 follows from (2.1) and (2.2). □

Lemma 2. Let p be a prime with $p \equiv 3 \bmod 4$. Then, we have the identities

$$\sum_{\substack{a=0 \\ a^4+1\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = p-1$$

and

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = p^2+p.$$

Proof. First, we prove the second formula in Lemma 2. Let $\overline{b}$ denote the inverse of $b \bmod p$, i.e., $b\overline{b} \equiv 1 (\bmod p)$. From the properties of a complete residue system modulo p, we have

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ (a+c)^2+(b+1)^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^2+2ac+b^2+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1$$

$$= \sum_{\substack{a=0 \\ a(a+2c)+1+2\bar{b}\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1} 1 + \sum_{\substack{a=0 \\ a(a+2c)\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1$$

$$= \sum_{\substack{a=0 \\ ac+b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 - \sum_{\substack{a=0 \\ ac+1\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 + (2p-1) = p^2 - (p-1) + (2p-1)$$

$$= p^2 + p. \tag{2.3}$$

Since $p \equiv 3 \bmod 4$, from the properties of the Legendre symbol modulo p, we have $\left(\frac{-1}{p}\right) = -1$ and

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right) = \sum_{\substack{a=0 \\ (-a)^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{-a}{p}\right) = -\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right)$$

or

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right) = 0. \tag{2.4}$$

Similarly, we also have

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{ab}{p}\right) = \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{ac}{p}\right) = \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{abc}{p}\right) = 0. \tag{2.5}$$

It is clear that we have the identity

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{b}{p}\right)\right)\left(1+\left(\frac{c}{p}\right)\right)$$

$$= \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 + 2\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right) + \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{c}{p}\right)$$

$$+2\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{ac}{p}\right) + \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{ab}{p}\right)$$

$$+\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{abc}{p}\right). \tag{2.6}$$

Combining (2.3)–(2.6), we have the identity

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = p^2 + p.$$

Similarly, we can also deduce the first formula in Lemma 2, which is

$$\sum_{\substack{a=0 \\ a^4+1\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} \left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{c}{p}\right)\right)$$

$$= \sum_{\substack{a=0 \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 + 2\sum_{\substack{a=0 \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right) + \sum_{\substack{a=0 \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} \left(\frac{ac}{p}\right).$$

Using the method (2.3)–(2.6) and removing element b, we immediately get

$$\sum_{\substack{a=0 \\ a^4+1\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = p-1.$$

This proves Lemma 2. □

Lemma 3. Let p be a prime with $p \equiv 1 \bmod 4$. Then, we have the identity

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = p^2 + 13p + 4(-1)^{\frac{p-1}{4}}p - 2\alpha(p) - 4(-1)^{\frac{p-1}{4}}\alpha(p) + 4\alpha^2(p).$$

Proof. The proof of the lemma is mainly divided into three parts:

(i) Summing Legendre symbols containing one variable, we have

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{b}{p}\right) = \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right) = \sum_{\substack{a=0 \\ a^2+b^2+2bc\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right)$$

$$= \sum_{\substack{a=0 \\ a^2+b(b+2c)\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right) = \sum_{\substack{a=0 \\ a^2+bc\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right)$$

$$= 2p + \sum_{\substack{a=0 \\ a^2+c\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1} \left(\frac{a}{p}\right) = 2p + (p-1)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right) = 2p. \tag{2.7}$$

Note the identities

$$\sum_{b=0}^{p-1}\left(\frac{b^2+n}{p}\right) = \begin{cases} p-1 & \text{if } p \mid n, \\ -1 & \text{if } p \nmid n. \end{cases}$$

We have

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{c}{p}\right) = \sum_{\substack{a=0 \\ a+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \left(\frac{c}{p}\right)\left(1+\left(\frac{a}{p}\right)\right)$$

$$= \sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{c}{p}\right) + \sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{c}{p}\right)\left(\frac{c^2+1-b^2}{p}\right)$$

$$= \sum_{c=0}^{p-1}\left(\frac{c}{p}\right)\sum_{b=0}^{p-1}\left(\frac{b^2-c^2-1}{p}\right) = (p-1)\sum_{\substack{c=0\\ p\mid(c^2+1)}}^{p-1}\left(\frac{c}{p}\right) - \sum_{\substack{c=0\\ p\nmid(c^2+1)}}^{p-1}\left(\frac{c}{p}\right)$$

$$= p\sum_{\substack{c=0\\ p\mid(c^2+1)}}^{p-1}\left(\frac{c}{p}\right) - \sum_{c=0}^{p-1}\left(\frac{c}{p}\right) = 2(-1)^{\frac{p-1}{4}}\cdot p. \tag{2.8}$$

(ii) Summing Legendre symbols containing two variables, we can get

$$\sum_{\substack{a=0\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{ab}{p}\right) = \sum_{\substack{a=0\\ (ab)^2+b^2\equiv (cb)^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\left(\frac{ab^2}{p}\right) = \sum_{\substack{a=0\\ a^2+1\equiv c^2+\overline{b}^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a}{p}\right)$$

$$= \sum_{\substack{a=0\\ b^2+c^2\equiv a^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a}{p}\right) - \sum_{\substack{a=0\\ c^2\equiv a^2+1 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a}{p}\right)$$

$$= 2(-1)^{\frac{p-1}{4}}\cdot p - \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(1+\left(\frac{1+a^2}{p}\right)\right) = 2(-1)^{\frac{p-1}{4}}\cdot p - \sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right)$$

$$= 2(-1)^{\frac{p-1}{4}}\cdot p - 2\alpha(p). \tag{2.9}$$

and

$$\sum_{\substack{a=0\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{bc}{p}\right) = \sum_{\substack{a=0\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{ac}{p}\right) = \sum_{\substack{a=0\\ (ac)^2+(bc)^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac^2}{p}\right)$$

$$= \sum_{\substack{a=0\\ a^2+b^2\equiv 1+\overline{c}^2 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a}{p}\right) = \sum_{\substack{a=0\\ a^2+b^2\equiv 1+c^2 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a}{p}\right) - \sum_{\substack{a=0\\ a^2+b^2\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a}{p}\right)$$

$$= 2p - \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(1+\left(\frac{1-a^2}{p}\right)\right) = 2p - 2(-1)^{\frac{p-1}{4}}\alpha(p). \tag{2.10}$$

(iii) Summing Legendre symbols containing three variables, we can obtain

$$\sum_{\substack{a=0\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{abc}{p}\right) = \sum_{\substack{a=1\\ b^2a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ab^2c}{p}\right) = \sum_{\substack{a=1\\ b^2(a^2+1)\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac}{p}\right)$$

$$= \sum_{\substack{a=1\\ a^2+1\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac}{p}\right) + \sum_{\substack{a=1\\ (b(a^2+1))^2\equiv (a^2+1)(c^2+1) \bmod p\\ (a^2+1,p)=1}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac}{p}\right)$$

$$
\begin{aligned}
&= (p-1)\left(\sum_{\substack{a=1\\ a^2+1\equiv 0 \bmod p}}^{p-1}\left(\frac{a}{p}\right)\right)^2 + \sum_{\substack{a=1\\ b^2\equiv(a^2+1)(c^2+1) \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac}{p}\right)\\
&= 4(p-1) + \sum_{\substack{a=1\\ b^2\equiv(a^2+1)(c^2+1) \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac}{p}\right) - \sum_{\substack{a=1\\ (a^2+1)(c^2+1)\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac}{p}\right)\\
&= 4p + \sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac}{p}\right)\left(1+\left(\frac{(a^2+1)(c^2+1)}{p}\right)\right) - 2(-1)^{\frac{p-1}{4}}\sum_{c=1}^{p-1}\left(\frac{c}{p}\right)\\
&= 4p + \left(\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right)\right)^2 = 4p + 4\alpha^2(p).
\end{aligned}
\tag{2.11}
$$

Combining (2.3), (2.6)–(2.11) we have the identity

$$
\sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = p^2 + 13p + 4(-1)^{\frac{p-1}{4}}p - 2\alpha(p) - 4(-1)^{\frac{p-1}{4}}\alpha(p) + 4\alpha^2(p).
$$

This proves Lemma 3. □

Lemma 4. Let p be a prime with $p \equiv 1 \bmod 4$. Then, we have the identities

$$
\sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right) = 4\sum_{\substack{a=0\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{ab(a+b-c-1)}{p}\right) + O(p).
$$

Proof. Let

$$
S = \sum_{\substack{a=0\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right).
$$

The proof of the lemma is divided into the following four parts:

(i) There is no Legendre symbol after S. From a complete residue system modulo p, we have

$$
\begin{aligned}
&\sum_{\substack{a=0\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right) = \sum_{\substack{a=0\\ (a+c)^2+(b+1)^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b}{p}\right)\\
&= \sum_{\substack{a=0\\ a(a+2c)+b^2+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b}{p}\right) = \sum_{\substack{a=0\\ ac+b^2+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b}{p}\right)\\
&= \sum_{\substack{c=0\\ b^2+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\left(\frac{b}{p}\right) + \sum_{\substack{a=1\\ c+b^2+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b}{p}\right)
\end{aligned}
$$

$$= \left(\frac{2}{p}\right)p + \sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a+b}{p}\right) = \left(\frac{2}{p}\right)p. \tag{2.12}$$

(ii) Legendre symbol contains one variable after S. Then, we can get

$$\begin{aligned}
&\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{b}{p}\right) \\
= &\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{a}{p}\right) \\
= &\sum_{\substack{a=0 \\ a^2+b(b+2c)\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-1}{p}\right)\left(\frac{a}{p}\right) = \sum_{\substack{a=0 \\ a^2+bc\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-1}{p}\right)\left(\frac{a}{p}\right) \\
= &\sum_{\substack{a=0 \\ a^2\equiv 1 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right) + \sum_{\substack{a=0 \\ a^2+c\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-1}{p}\right)\left(\frac{a}{p}\right) \\
= &\left(\frac{2}{p}\right)p + \sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b-1}{p}\right)\left(\frac{a}{p}\right) = \left(\frac{2}{p}\right)p + 1.
\end{aligned} \tag{2.13}$$

From (2.11) we have

$$\begin{aligned}
&\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{c}{p}\right) \\
= &\sum_{\substack{a=0 \\ a^2c^2+b^2c^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac+bc-c-1}{p}\right)\left(\frac{c}{p}\right) = \sum_{\substack{a=0 \\ a^2+b^2\equiv 1+\bar{c}^2 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a+b-1-\bar{c}}{p}\right) \\
= &\left(\frac{2}{p}\right)p - \sum_{\substack{a=0 \\ a^2+b^2\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a+b-1}{p}\right) = \left(\frac{2}{p}\right)p - \sum_{\substack{a=0 \\ a^2+b^2+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\left(\frac{a+b}{p}\right) \\
= &\left(\frac{2}{p}\right)p - \sum_{\substack{a=0 \\ a^2+1+2\bar{b}\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+1}{p}\right)\left(\frac{b}{p}\right) \\
= &\left(\frac{2}{p}\right)p - \left(\frac{2}{p}\right)\sum_{\substack{a=0 \\ a^2+1+b\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+1}{p}\right)\left(\frac{b}{p}\right) \\
= &\left(\frac{2}{p}\right)p - \left(\frac{2}{p}\right)\sum_{a=0}^{p-1}\left(\frac{a+1}{p}\right)\left(\frac{a^2+1}{p}\right)
\end{aligned}$$

$$= \left(\frac{2}{p}\right)p - \left(\frac{2}{p}\right)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-2a+2}{p}\right)$$
$$= \left(\frac{2}{p}\right)p - \left(\frac{2}{p}\right)\sum_{a=1}^{p-1}\left(\frac{2a-2+\overline{a}}{p}\right). \tag{2.14}$$

(iii) Legendre symbol contains one variable after S. Then, we can get

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{ab}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2b^2+b^2\equiv c^2b^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\left(\frac{ab+b-cb-1}{p}\right)\left(\frac{a}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2+1\equiv c^2+\overline{b}^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+1-c-\overline{b}}{p}\right)\left(\frac{ab}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{ac}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{bc}{p}\right). \tag{2.15}$$

(iv) Legendre symbol contains one variable after S. Then, we can get

$$\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{abc}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2c^2+b^2c^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{ac+bc-c-1}{p}\right)\left(\frac{abc^3}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2+b^2\equiv 1+\overline{c}^2 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a+b-1-\overline{c}}{p}\right)\left(\frac{ab}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{ab}{p}\right) - \sum_{\substack{a=1 \\ a^2+b^2\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b-1}{p}\right)\left(\frac{ab}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(\frac{ab}{p}\right) + O(p). \tag{2.16}$$

From (2.12)–(2.16) and the properties of the Legendre symbol modulo p, we have

$$\begin{aligned}
&\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)\\
=&\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a+b-c-1}{p}\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{b}{p}\right)\right)\left(1+\left(\frac{c}{p}\right)\right)\\
=&\ 4\sum_{\substack{a=0 \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{ab(a+b-c-1)}{p}\right)+O(p).
\end{aligned}$$

This proves Lemma 4. □

3. Proofs of the Theorems

Now, we apply Lemmas 1–4 in Section 2 to complete the proofs of our theorems. For any positive integer $q>1$ and integer n, note the trigonometric identities

$$\sum_{a=0}^{q} e\left(\frac{na}{q}\right)=\begin{cases} q & \text{if } q \mid n; \\ 0 & \text{if } q \nmid n. \end{cases} \tag{3.1}$$

If $(n,p)=1$, then we have

$$\sum_{a=0}^{p-1} e\left(\frac{na^2}{p}\right)=1+\sum_{a=1}^{p-1}(1+\chi_2(a))\,e\left(\frac{na}{p}\right)=\chi_2(n)\cdot\tau(\chi_2), \tag{3.2}$$

where $\chi_2=\left(\frac{*}{p}\right)$ denotes the Legendre symbol modulo p, and $\tau(\chi)=\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.

From (3.1), (3.2), and the properties of a reduced residue system modulo p, we have the identities

$$\begin{aligned}
S_4(p)=&\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a^2}{p}\right)\right|^4\\
=&\sum_{m=0}^{p-1}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}e\left(\frac{m(a^4+b^4-c^4-d^4)+a^2+b^2-c^2-d^2}{p}\right)\\
=&\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{m(a^4+b^4-c^4-d^4)}{p}\right)e\left(\frac{a^2+b^2-c^2-d^2}{p}\right)\\
=&\ p\cdot\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+d^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}e\left(\frac{a^2+b^2-c^2-d^2}{p}\right)
\end{aligned}$$

$$
\begin{aligned}
&= p \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{d^2\left(a^2+b^2-c^2-1\right)}{p}\right) + p \sum_{\substack{a=0\\ a^4+b^4\equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) \\
&= p \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^2\left(a^2+b^2-c^2-1\right)}{p}\right) \\
&\quad + p \sum_{\substack{a=0\\ a^4+b^4\equiv c^4 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) - p \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 \\
&= p^2 \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + p \cdot \tau(\chi_2) \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+b^2-c^2-1}{p}\right) \\
&\quad + p \sum_{\substack{a=0\\ a^4+1\equiv c^4 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{b^2(a^2+1-c^2)}{p}\right) - p \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 \\
&\quad + p \sum_{\substack{a=0\\ a^4\equiv c^4 \bmod p}}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right) \\
&= p^2 \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + p \cdot \tau(\chi_2) \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+b^2-c^2-1}{p}\right) \\
&\quad + p^2 \cdot \sum_{\substack{a=0\\ a^4+1\equiv c^4 \bmod p\\ a^2+1\equiv c^2 \bmod p}}^{p-1} \sum_{c=0}^{p-1} 1 + p \cdot \tau(\chi_2) \sum_{\substack{a=0\\ a^4+1\equiv c^4 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+1-c^2}{p}\right) \\
&\quad - p \sum_{\substack{a=0\\ a^4+1\equiv c^4 \bmod p}}^{p-1} \sum_{c=0}^{p-1} 1 - p \sum_{\substack{a=0\\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + p \sum_{\substack{a=0\\ a^4\equiv c^4 \bmod p}}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right). \qquad (3.3)
\end{aligned}
$$

Now, we prove Theorem 1. If $p \equiv 3 \bmod 4$, then $S_4(p)$ is a real number, $\tau(\chi_2) = i \cdot \sqrt{p}$ (see Theorem 9.13 in [1]) is a pure imaginary number, and

$$
\begin{aligned}
\sum_{\substack{a=0\\ a^4\equiv c^4 \bmod p}}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right) &= 1 + \sum_{\substack{a=1\\ a^4\equiv 1 \bmod p}}^{p-1} \sum_{c=1}^{p-1} e\left(\frac{c^2\left(a^2-1\right)}{p}\right) \\
&= 1 + 2(p-1) = 2p-1. \qquad (3.4)
\end{aligned}
$$

So, the sum of the coefficients of $\tau(\chi_2)$ in (23) must be zero. That is,

$$\sum_{\substack{a=0\\a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right)+\sum_{\substack{a=0\\a^4+1\equiv c^4 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+1-c^2}{p}\right)=0. \tag{3.5}$$

Combining (3.3)–(3.5), Lemma 1 and Lemma 2, we have

$$\begin{aligned}S_4(p) &= \sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a^2}{p}\right)\right|^4\\ &= 4p^2(2p-3)+2p^2-p(p-1)-p\left(p^2+p\right)+p(2p-1)\\ &= p^2(7p-10).\end{aligned}$$

This proves Theorem 1. □

Theorem 2 and some notes follows from (3.3), Lemma 3 and Lemma 4.

This completes the proofs of all our results.

4. Conclusions

The main result of this paper is to give an exact calculating formula for the fourth power mean of one special two-term exponential sum. That is,

$$S_4(p)=\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a^2}{p}\right)\right|^4=p^2\cdot(7p-10)$$

with the case $p\equiv 3 \bmod 4$. If $p\equiv 1 \bmod 4$, then we cannot get any non-trivial results for $S_4(p)$ yet. We also point out the key to the problem, which is we do not know the exact value of

$$\sum_{\substack{a=0\\a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right) \tag{4.1}$$

and do not know a valid asymptotic formula for it. However, we get some periodic results. At the same time, some problems to be further studied are also proposed.

In any case, our work provides a new method for the study of relevant problems. We have a reason to believe that this work will play a positive role in promoting the study of relevant problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the reviewers for their very helpful and detailed comments, which have significantly improved the presentation of this paper. This work is supported by the National Natural Science Foundation of China (12126357), and the Natural Science Foundation of Shannxi Province (2022JM-017, 2023-JC-YB-013).

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. W. P. Zhang, D. Han, On the sixth power mean of the two-term exponential sums, *J. Number Theo.*, **136** (2014), 403–413. https://doi.org/10.1016/j.jnt.2013.10.022
2. W. P. Zhang, Y. Y. Meng, On the sixth power mean of the two-term exponential sums, *Acta Math. Sin.*, **38** (2022), 510–518. https://doi.org/10.1007/s10114-022-0541-8
3. L. Chen, X. Wang, A new fourth power mean of two-term exponential sums, *Open Math.*, **17** (2019), 407–414. https://doi.org/10.1515/math-2019-0034
4. W. P. Zhang, H. L. Li, Elementary Number Theory, Shaanxi Normal University Press, Xi'an, 2013.
5. H. Zhang, W. P. Zhang, The fourth power mean of two-term exponential sums and its application, *Math. Rep.*, **19** (2017), 75–81.
6. T. T. Wang, W. P. Zhang, On the fourth and sixth power mean of mixed exponential sums, *Sci. Sin. Math.*, **38** (2011), 265–270. https://doi.org/10.1360/012010-753
7. W. P. Zhang, J. Y. Hu, The number of solutions of the diagonal cubic congruence equation $\bmod\ p$, *Math. Rep.*, **20** (2018), 70–76.
8. B. C. Berndt, R. J. Evans, The determination of Gauss sums, *Bull. Am. Math. Soc.*, **5** (1981), 107–128. https://doi.org/10.1090/S0273-0979-1981-14930-2
9. Z. Y. Chen, W. P. Zhang, On the fourth-order linear recurrence formula related to classical Gauss sums, *Open Math.*, **15** (2017), 1251–1255. https://doi.org/10.1515/math-2017-0104
10. L. Chen, On the classical Gauss sums and their some properties, *Symmetry*, **10** (2018), 625. https://doi.org/10.3390/sym10110625
11. W. P. Zhang, X. D. Yuan, On the classical Gauss sums and their some new identities, *AIMS Math.*, **2022** (2022), 5860–5870. https://doi.org/10.3934/math.2022325
12. T. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976. https://doi.org/10.1007/978-1-4757-5579-4
13. K. Ireland, M. Rosen, A classical introduction to modern number theory, Springer-Verlag, New York, 1982. https://doi.org/10.1007/978-1-4757-1779-2

AIMS Press