Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(9): 5457–5465.
DOI: 10.3934/era.2023277
Received: 10 April 2023
Revised: 11 July 2023
Accepted: 20 July 2023
Published: 03 August 2023

Research article

Local porosity of the free boundary in a minimum problem

Yuwei Hu[1] **and Jun Zheng**[1,2,*]

[1] School of Mathematics, Southwest Jiaotong University, Chengdu 611756, Sichuan, China
[2] Department of Electrical Engineering, Polytechnique Montréal, Montreal H3T 1J4, QC, Canada

* **Correspondence:** Email: zhengjun2014@aliyun.com, zhengjun@swjtu.edu.cn.

Abstract: Given certain set $\mathcal{K}$ and functions q and h, we study geometric properties of the set $\partial\{x \in \Omega : u(x) > 0\}$ for non-negative minimizers of the functional $\mathcal{J}(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p + q(u^+)^\gamma + hu\right) \mathrm{d}x$ over $\mathcal{K}$, where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded domain, $p \in (1, +\infty)$ and $\gamma \in (0, 1]$ are constants, u^+ is the positive part of u and $\partial\{x \in \Omega : u(x) > 0\}$ is the so-called free boundary. Such a minimum problem arises in physics and chemistry for $\gamma = 1$ and $\gamma \in (0, 1)$, respectively. Using the comparison principle of p-Laplacian equations, we establish first the non-degeneracy of non-negative minimizers near the free boundary, then prove the local porosity of the free boundary.

Keywords: free boundary; minimizer; porosity; non-degeneracy; p-Laplacian

1. Introduction

Let Ω be an open bounded domain in $\Omega \subset \mathbb{R}^n (n \geq 2)$, and $p \in (1, +\infty)$ and $\gamma \in (0, 1]$ be constants. Given functions $q \in L^\infty(\Omega)$, $h \in L^\infty(\Omega)$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with

$$q \geq q_0 \text{ and } h \geq 0 \text{ a.e. in } \Omega, \text{ and } \varphi \geq 0 \text{ on } \partial\Omega,$$

where q_0 is a positive constant, we consider the following minimum problem governed by the p-Laplacian

$$\mathcal{J}(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p + q(u^+)^\gamma + hu\right) \mathrm{d}x \to \min \tag{1.1}$$

over the closed and convex set

$$\mathcal{K} = \left\{u \in W^{1,p}(\Omega) : u - \varphi \in W_0^{1,p}(\Omega), u \geq 0 \text{ a.e. in } \Omega\right\},$$

where u^+ is the positive part of u.

The problem (1.1) is known as one-phase free boundary problem, which is often used to model the problems arising in physics or chemistry. For example,

(i) the case of $\gamma = 1$ corresponds to the obstacle problem, describing the problems of equilibrium of elastic membranes, fluid filtration in porous media and control of temperature; see [1–3], etc.
(ii) the case of $\gamma \in (0, 1)$ can be used to model the density of certain chemical species in reaction with a porous catalyst pellet; see [4–6], etc.
(iii) the limit case of $\gamma \to 0$ is reduced to the problem of jets and cavities, which arises in the contexts of combustion theory [7], dams [8] and heat flow [9], etc.

In the past decades, great efforts have been devoted to investigating the existence and regularities of minimizers of $\mathcal{J}(u)$ (see [5, 10–12]), as well as the study of geometric properties of free boundaries with different $\gamma \in (0, 1]$ (see [4, 6, 13–17]), among which the latter brings more difficulties due to the fact that specific estimates of minimizers (or solutions to the corresponding Euler-Lagrange equations) are needed. In general, based on the existence and regularity of minimizers, the local porosity and finite $(n-1)$-dimensional Hausdorff measure of free boundaries can be established only after the optimal growth and non-degeneracy of minimizers are addressed. For example, the porosity of the free boundary in the homogeneous (i.e., $h = 0$) obstacle problem of p-Laplacian types was obtained in [15, 18]; the Hausdorff measure of the free boundary in the homogeneous p-obstacle problem when $p > 2$ and $p \in (1, 2)$ was derived in [16] and [19], respectively; and the Hausdorff measure in the minimum problem (1.1) with $\gamma \in (0, 1)$ and $p = 2$ was established in [4, 6]. It should be mentioned that the authors of [13, 14] proved the porosity and Hausdorff measure of free boundaries in the obstacle problem in the setting of Orlicz-Sobolev spaces. For $\gamma \to 0$, the authors of [20] considered the Hausdorff measure of free boundaries in the homogeneous minimum problem (1.1), while the authors of [21] and [22] studied regularity of free boundaries in the framework of Sobolev spaces with variable exponents and Orlicz-Sobolev spaces, respectively.

Although geometric properties of free boundaries in the minimum problem (1.1) have been well studied in different cases, however, there is no result reported on the porosity of the free boundary for the problem (1.1) with a general $\gamma \in (0, 1]$ and in inhomogeneous case, i.e., the case of $h \neq 0$. Indeed, it is very meaningful to study porosity of free boundaries, either based on the practical needs of engineering or problems in physics or from mathematical perspectives. For example, the porosity of a free boundary has a significant impact on the dynamics of a free boundary, and has many applications in context of fluid flow in porous media [23–25]. From a mathematical point of view, the study of porosity of a set arises naturally in some problems in real analysis, especially in the differentiation theory; see [26, 27] for comprehensive surveys. As indicated in [27], the notion of a porous set $E \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ characterize the size of "hole" in the set E near to x. In particular, for a porous set E, it is not only nowhere dense but is "small" due to the fact that "holes" near to each point $x \in E$ are "big" in a certain sense; see Definition 3.1 in Section 3. Thus, the notion of porosity provides a kind of geometric characterization of a set.

In this paper, based on the results of existence, regularity and optimal growth of minimizers obtained in [11], we establish first the non-degeneracy of minimizers near the free boundary for the inhomogeneous minimum problem (1.1) and then establish a local porosity property of the free boundary whenever $\gamma \in (0, 1]$. The main technical tool used in this paper is the comparison principle of p-Laplacian equations.

In the rest of the paper, we introduce first basic notations. In Section 2, we introduce some technical lemmas that are needed in the proof of the main result. In Section 3, we state the definition of porosity and present the main result obtained in this paper, as well as its proof. Concluding remarks are given in Section 4.

Notation In this paper, for a positive constant r, $B_r(x)$ denotes a ball in $\mathbb{R}^n$ with center $x \in \Omega$ and radius r. For simplicity, we write $B_r := B_r(0)$. Let $\Omega^+ := \{x \in \Omega : u(x) > 0\}$ and $\partial\Omega^+$ denote the boundary of Ω^+, i.e., $\partial\Omega^+ := \partial\{x \in \Omega : u(x) > 0\}$, which is known as the free boundary of the minimum problem (1.1).

2. Auxiliary results

We present first the results on the existence of a non-negative minimizer of the functional $\mathcal{J}(u)$ over the set $\mathcal{K}$.

Lemma 2.1. *[11, Lemma 4.2] The function $\mathcal{J}(u)$ admits at least one non-negative minimizer u over the set $\mathcal{K}$. Moreover, u is a weak solution of the equation*

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = q\gamma u^{\gamma-1} + h \text{ in } \Omega^+. \tag{2.1}$$

The following lemma, which was proved in [11] by using the De Giorgi iteration, is concerned with the local $C^{1,\alpha}$-regularity of minimizers and will be applied to the proof of main result obtained in this paper.

Lemma 2.2. *[11, Theorem 3.1] If u is a non-negative minimizer of $\mathcal{J}(u)$ over the set $\mathcal{K}$, then, $u \in C^{1,\alpha}_{loc}(\Omega)$ with some $\alpha \in (0,1)$. More precisely, for any $\Omega' \subsetneq \Omega$, there exists a positive constant C_1 depending only on n, p, γ, $\|q\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,p}(\Omega)}$ and the diameter of Ω such that $\|u\|_{C^{1,\alpha}(\Omega')} \leq C_1$.*

As mentioned in Section 1 that geometric properties are established based on specific estimates of minimizers, we need the following optimal growth result to prove the main result obtained in this paper. It indicates that non-negative minimizers can not grow too fast near the free boundary.

Lemma 2.3. *[11, Theorem 4.1] Let $x_0 \in \partial\Omega^+$ and $B_{r_0}(x_0) \subsetneq \Omega$ with some $r_0 > 0$. For any non-negative minimizer u of $\mathcal{J}(u)$ over the set $\mathcal{K}$, there exists a positive constant C_2 depending only on n, p, γ, $\|q\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,p}(\Omega)}$ and the diameter of Ω such that*

$$|u(x)| \leq C_2|x - x_0|^{\frac{p}{p-\gamma}}, \forall x \in B_r(x_0)$$

holds true for all $r \in (0, r_0)$.

In addition to the aforementioned lemmas, we need the following weak comparison principle of p-Laplacian equations.

Lemma 2.4. *[28, Lemma 4.1] If u, $v \in W^{1,p}(\Omega)$ satisfy $-\Delta_p u \leq -\Delta_p v$ in Ω and $u \leq v$ on $\partial\Omega$, then, $u \leq v$ in Ω.*

3. Local porosity of the free boundary

In this section, we state and prove the main result obtained in this paper, namely, the local porosity of the free boundary. We introduce first the notion of *porosity*.

Definition 3.1. A set E is said to be porous with porosity constant δ, if there exist constants $r_1 > 0$ and $\delta > 0$ such that

$$\forall x \in E, \forall r \in (0, r_1) \quad \Rightarrow \quad \exists y \in \mathbb{R}^n \text{ s.t. } B_{\delta r}(y) \subset B_r(x) \setminus E.$$

It is well-known that a porous set has Hausdorff dimension not exceeding $n - C\delta^n$, where $C = C(n)$ is a constant depending only on n. In particular, the n-dimensional Lebesgue measure of a porous set is zero; see [15, 29, 30].

The following theorem is main result obtained in this paper.

Theorem 3.1. *Let u be a non-negative minimizer of $\mathcal{J}(u)$ over the set $\mathcal{K}$, then for every compact set $D \subsetneq \Omega$, the intersection $D \cap \partial\Omega^+$ is porous with porosity constant δ depending only on q_0, n, p, γ, $\|q\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\varphi\|_{W^{1,p}(\Omega)}$ and the diameter of Ω.*

In order to prove Theorem 3.1, we need to prove a non-degeneracy property of non-negative minimizers, which indicates that non-negative minimizers can not grow too slowly near the free boundary.

Lemma 3.1. *Let u be a non-negative minimizer of $\mathcal{J}(u)$ over the set $\mathcal{K}$, then for every $y \in \partial\Omega^+$ satisfying $B_{r_2}(y) \subsetneq \Omega$ with some $r_2 > 0$, there exists a positive constant C_3 depending only on q_0, n, p and γ such that*

$$\sup_{\Omega^+ \cap B_r(y)} u \geq C_3 r^{\frac{p}{p-\gamma}} \tag{3.1}$$

holds true for all $r \in (0, r_2)$.

Proof. Let $y \in \Omega^+$ and $B_{r_2}(y) \subsetneq \Omega$ with some $r_2 > 0$. Define

$$w(x) := |u(x)|^{\frac{p-\gamma}{p-1}} \qquad \text{and} \qquad v(x) := c|x - y|^{\frac{p}{p-1}},$$

where $c := \frac{p-\gamma}{p}\left(\frac{q_0\gamma}{n}\right)^{\frac{1}{p-1}}$.

It follows that

$$\nabla w = \frac{p-\gamma}{p-1}|u|^{\frac{2-p-\gamma}{p-1}} u \nabla u \qquad \text{and} \qquad \nabla v = c\frac{p}{p-1}|x-y|^{\frac{2-p}{p-1}}(x-y).$$

Furthermore, it holds that

$$\begin{aligned} \Delta_p v =& \operatorname{div}\left(c^{p-2}\left(\frac{p}{p-1}\right)^{p-2}|x-y|^{\frac{p-2}{p-1}} c\frac{p}{p-1}|x-y|^{\frac{2-p}{p-1}}(x-y)\right) \\ =& c^{p-1}\left(\frac{p}{p-1}\right)^{p-1} \operatorname{div}\,(x-y) \\ =& nc^{p-1}\left(\frac{p}{p-1}\right)^{p-1}. \end{aligned}$$

For any $r \in (0, r_2)$, we deduce by (2.1) that in $\Omega^+ \cap B_r(y)$:

$$\Delta_p w = \operatorname{div}\left(\left(\frac{p-\gamma}{p-1}\right)^{p-1}|u|^{-\gamma}u|\nabla u|^{p-2}\nabla u\right)$$

$$
\begin{aligned}
&=\left(\frac{p-\gamma}{p-1}\right)^{p-1}\left(|\nabla u|^{p-2}\nabla u\nabla\left(|u|^{-\gamma}u\right)+|u|^{-\gamma}u\cdot\operatorname{div}\ \left(|\nabla u|^{p-2}\nabla u\right)\right)\\
&=\left(\frac{p-\gamma}{p-1}\right)^{p-1}\left(\frac{\nabla u|u|^{\gamma}-\gamma\nabla u|u|^{\gamma}}{|u|^{2\gamma}}|\nabla u|^{p-2}\nabla u+|u|^{-\gamma}u\Delta_p u\right)\\
&=\left(\frac{p-\gamma}{p-1}\right)^{p-1}\left((1-\gamma)\frac{|\nabla u|^{p}}{|u|^{\gamma}}+|u|^{-\gamma}u\Delta_p u\right)\\
&=\left(\frac{p-\gamma}{p-1}\right)^{p-1}\left((1-\gamma)\frac{|\nabla u|^{p}}{|u|^{\gamma}}+q\gamma+hu|u|^{-\gamma}\right)\\
&\geq\left(\frac{p-\gamma}{p-1}\right)^{p-1}q_0\gamma\\
&=nc^{p-1}\left(\frac{p}{p-1}\right)^{p-1}\\
&=\Delta_p v,
\end{aligned}
$$

namely, $-\Delta_p w\leq-\Delta_p v$ in $\Omega^+\cap B_r(y)$.

By the definitions of $w(x)$ and $v(x)$, and the fact that $u(x)=0$ on $\partial\Omega^+\cap B_r(y)$, we have

$$w(x)=0\leq v(x)\text{ on }\partial\Omega^+\cap B_r(y).$$

If $w(x)\leq v(x)$ on $\Omega^+\cap\partial B_r(y)$, then by the comparison principle (Lemma 2.4), we have

$$w(x)\leq v(x)\text{ in }\Omega^+\cap B_r(y).$$

However, $w(y)>0=v(y)$, which is a contradiction. Therefore, there exists $y_0\in\Omega^+\cap\partial B_r(y)$ such that $w(y_0)\geq v(y_0)$. Then we have

$$|u(y_0)|^{\frac{p-\gamma}{p-1}}=w(y_0)\geq v(y_0)=cr^{\frac{p}{p-1}},$$

or, equivalently,

$$|u(y_0)|\geq c^{\frac{p-1}{p-\gamma}}r^{\frac{p}{p-\gamma}},$$

which implies that

$$\sup_{\Omega^+\cap B_r(y)}u\geq c^{\frac{p-1}{p-\gamma}}r^{\frac{p}{p-\gamma}}:=C_3r^{\frac{p}{p-\gamma}}.$$

Finally, for $y\in\partial\Omega^+$, due to the continuity of minimizers (Lemma 2.2), we can take first $y_j\in\partial\Omega^+$ such that $y_j\to y$, and then conclude (3.1) by taking limits. The proof is complete.

Now we are ready for the proof of Theorem 3.1, which is guaranteed by the optimal growth (Lemma 2.3) and non-degeneracy (Lemma 3.1) of non-negative minimizers near the free boundary.

Proof. [Proof of Theorem 3.1] Since local properties of the free boundary are considered in this paper, without loss of generality, we assume that $\overline{B_3}\subsetneq\Omega$, and that the compact set D is the closed unit ball $\overline{B_1}$. We apply the techniques of [14, 15] to the proof.

For any $z\in\partial\Omega^+\cap B_1$, let $r<\min\left\{\frac{2}{3},r_0,r_2\right\}$ be a positive constant, where r_0 and r_2 are determined by Lemma 2.3 and Lemma 3.1, respectively.

By Lemma 3.1, there exists a point $Y \in \Omega^+ \cap B_r(z)$ such that

$$u(Y) \geq C_3 r^{\frac{p}{p-\gamma}}. \tag{3.2}$$

For $Y \in \Omega^+ \cap \overline{B_r}(z)$, define $d_Y := \text{dist}(Y, \overline{B_r}(z) \setminus \Omega^+)$ and take $z_Y \in \partial\Omega^+ \cap \overline{B_r}(z)$ with $d_Y = |Y - z_Y|$. We claim that $B_r(z_Y) \subsetneqq \Omega$. Indeed, for any $x \in B_r(z_Y)$, it holds that

$$|x| \leq |x - z_Y| + |z_Y - Y| + |Y - z| + |z| \leq r + d_Y + r + 1 \leq 3r + 1 < 3.$$

Therefore, $B_r(z_Y) \subset \overline{B_3} \subsetneqq \Omega$.

By Lemma 2.3, we obtain

$$u(x) \leq C_2 |x - z_Y|^{\frac{p}{p-\gamma}}, \forall x \in B_r(z_Y). \tag{3.3}$$

We infer from (3.2) and (3.3) that

$$C_3 r^{\frac{p}{p-\gamma}} \leq u(Y) \leq C_2 d_Y^{\frac{p}{p-\gamma}}.$$

Setting $\delta = \left(\frac{C_3}{C_2}\right)^{\frac{p-\gamma}{p}}$, we have $\delta r \leq d_Y$. Therefore, $B_{\delta r}(Y) \cap B_r(z) \subset \Omega^+$.

Selecting a point y on the line with z and Y as endpoints such that $|y - Y| = \frac{\delta r}{2}$, we claim that

$$B_{\frac{\delta r}{2}}(y) \subset B_{\delta r}(Y) \cap B_r(z) \subset B_r(z) \setminus \partial\Omega^+ \subset B_r(z) \setminus \left(\partial\Omega^+ \cap \overline{B_1}\right), \tag{3.4}$$

which shows that $\partial\Omega^+ \cap \overline{B_1}$ is porous with the porosity constant $\frac{\delta}{2}$. Indeed, for every point $y_0 \in B_{\frac{\delta r}{2}}(y)$, on one hand, it holds that

$$|y_0 - Y| \leq |y_0 - y| + |y - Y| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r.$$

On the other hand, since $|y - z| = |z - Y| - |y - Y|$, it holds that

$$|y_0 - z| \leq |y_0 - y| + |y - z| \leq |y_0 - y| + (|z - Y| - |y - Y|) < \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r.$$

Therefore, (3.4) holds true and the proof is complete.

4. Conclusions

As stated in the introduction, the regularity of free boundaries is based on the regularity of minimizers. In this paper, we first proved non-degeneracy of non-negative minimizers near the free boundary by using the comparison principle. Then, based on this result and the optimal growth of non-negative minimizers, we proved that the free boundary is locally porous, which indicates that the n-dimensional Lebesgue measure of the free boundary is zero. In future work, we will study the finite dimensional Hausdorff measure of the free boundary, so as to provide more geometrical characterizations for the free boundary.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable suggestions and comments that contribute to improving the quality of the paper. This work was supported in part by the National Natural Science Foundation of China under Grant 11901482.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. C. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
2. L. A. Caffarelli, The obstacle problem revisited, *J. Fourier Anal. Appl.*, **4** (1998), 383–402. https://doi.org/10.1007/BF02498216
3. C. Lederman, A free boundary problem with a volume penalization, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **23** (1996), 249–300.
4. H. W. Alt, D. Phillips, A free boundary problem for semi-linear elliptic equations, *J. Reine Angew. Math.*, **368** (1986), 63–107. https://doi.org/10.1515/crll.1986.368.63
5. D. Phillips, A minimization problem and the regularity of solutions in the presence of a free boundary, *Indiana Univ. Math. J.*, **32** (1983), 1–17.
6. D. Phillips, Hausdorff measure estimates of a free boundary for a minimum problem, *Commun. Partial Differ. Equations*, **8** (1983), 1409–1454. https://doi.org/10.1080/03605308308820309
7. H. Berestycki, L. A. Caffarelli, L. Nirenberg, Uniform estimates for regularization of free boundary problems, *Anal. Partial Differ. Equations*, **122** (1990), 567–619.
8. H. W. Alt, G. Gilardi, The behavior of the free boundary for the dam problem, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **9** (1982), 571–626.
9. A. Acker, Heat flow inequalities with applications to heat flow optimization problems, *SIAM J. Math. Anal.*, **8** (1977), 604–618. https://doi.org/10.1137/0508048
10. H. W. Alt, L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.*, **1981** (1981), 105–144. https://doi.org/10.1515/crll.1981.325.105
11. J. Zheng, L. S. Tavares, A free boundary problem with subcritical exponents in Orlicz spaces, *Ann. Mat. Pura Appl.*, **201** (2022), 695–731. https://doi.org/10.1007/s10231-021-01134-1
12. J. D. Rossi, P. Y. Wang, The limit as $p \to \infty$ in a two-phase free boundary problem for the p-Laplacian, *Interfaces Free Boundaries*, **18** (2016), 115–135. https://doi.org/10.4171/IFB/359
13. S. Challal, A. Lyaghfouri, J. F. Rodrigues, On the A-obstacle problem and the Hausdorff measure of its free boundary, *Ann. Mat. Pura Appl.*, **191** (2012), 113–165. https://doi.org/10.1007/s10231-010-0177-7
14. S. Challal, A. Lyaghfouri, Porosity of free boundaries in A-obstacle problems, *Nonlinear Anal. Theory Methods Appl.*, **70** (2009), 2772–2778. https://doi.org/10.1016/j.na.2008.04.002

15. L. Karp, T. Kilpeläinen, A. Petrosyan, H. Shahgholian, On the porosity of free boundaries in degenerate variational inequalities, *J. Differ. Equations*, **164** (2000), 110–117. https://doi.org/10.1006/jdeq.1999.3754

16. K. Lee, H. Shahgholian, Hausdorff measure and stability for the p-obstacle problem ($2 < p < \infty$), *J. Differ. Equations*, **195** (2003), 14–24. https://doi.org/10.1016/j.jde.2003.06.002

17. J. Zheng, B. H. Feng, P. H. Zhao, A remark on the two-phase obstacle-type problem for the p-Laplacian, *Adv. Calc. Var.*, **11** (2018), 325–334. https://doi.org/10.1515/acv-2015-0049

18. J. Zheng, Z. H. Zhang, P. H. Zhao, Porosity of free boundaries in the obstacle problem for quasilinear elliptic equations, *Proc. Math. Sci.*, **123** (2013), 373–382. https://doi.org/10.1007/s12044-013-0137-4

19. J. Zheng, P. H. Zhao, A remark on Hausdorff measure in obstacle problems, *J. Anal. Appl.*, **31** (2012), 427–439. https://doi.org/10.4171/ZAA/1467

20. R. Leit*ão*, E. V. Teixeira, Regularity and geometric estimates for minima of discontinuous functionals, *Rev. Mat. Iberoam.*, **31** (2015), 69–108. https://doi.org/10.4171/rmi/827

21. F. Ferrari, C. Lederman, Regularity of flat free boundaries for a $p(x)$-Laplacian problem with right hand side, *Nonlinear Anal.*, **212** (2021). https://doi.org/10.1016/j.na.2021.112444

22. J. E. M. Braga, P. R. P. Regis, On the full regularity of the free boundary for minima of Alt-Caffarelli functionals in Orlicz spaces, *Ann. Fenn. Math.*, **47** (2022), 961–977. https://doi.org/10.54330/afm.120561

23. J. R. Ockendon, W. R. Hodgkings, *Moving Boundary Problems in Heat Flow and Diffusion*, Clarendon Press, Oxford, 1975.

24. C. J. Duijn, I. S. Pop, Crystal dissolution and precipitation in porous media: pore scale analysis, *J. Reine Angew. Math.*, **2004** (2004), 171–211. https://doi.org/10.1515/crll.2004.2004.577.171

25. U. Hornung, W. Jäger, Diffusion, convection, adsorption, and reaction of chemicals in porous media, *J. Differ. Equations*, **92** (1991), 199–225. https://doi.org/10.1016/0022-0396(91)90047-D

26. L. Zajíček, On σ-porous sets in abstract space, *Abstr. Appl. Anal.*, **2005** (2005), 509–534. https://doi.org/10.1155/AAA.2005.509

27. L. Zajíček, Porosity and σ-porosity, *Real Anal. Exch.*, **13** (1987), 314–350. https://doi.org/10.2307/44151885

28. Z. Tan, F. Fang, Orlicz-Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations, *J. Math. Anal. Appl.*, **402** (2013), 348–370. https://doi.org/10.1016/j.jmaa.2013.01.029

29. L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC press, Boca Raton, 1992.

30. P. Koskela, S. Rohde, Hausdorff dimension and mean porosity, *Math. Ann.*, **309** (1997), 593–609. https://doi.org/10.1007/s002080050129