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Research article

Global existence and long-time behavior of solutions for fully nonlocal Boussinesq equations

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Abstract: In this paper, we study initial boundary value problems for the following fully nonlocal Boussinesq equation

$$ {}_0^CD_t^\beta u + (-\Delta)^\sigma u + (-\Delta)^\sigma {}_0^CD_t^\beta u = -(-\Delta)^\sigma f(u) $$

with spectral fractional Laplacian operators and Caputo fractional derivatives. To our knowledge, there are few results on fully nonlocal Boussinesq equations. The main difficulty is that each term of this equation has nonlocal effect. First, we obtain explicit expressions and some rigorous estimates of the Green operators for the corresponding linear equation. Further, we get global existence and some decay estimates of weak solutions. Second, we establish new chain and Leibnitz rules concerning $(-\Delta)^\sigma$. Based on these results and small initial conditions, we obtain global existence and long-time behavior of weak solutions under different dimensions N by Banach fixed point theorem.

Keywords: Boussinesq equation; fractional operator; global existence; long-time behavior

1. Introduction

Let $Q_T = \Omega \times (0, T)$, $B_T = \partial\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $T > 0$. In this paper, we research initial boundary value problems for the following fully nonlocal Boussinesq equation

$$\begin{cases} {}_0^C D_t^{\beta} u + (-\Delta)^{\sigma} u + (-\Delta)^{\sigma} {}_0^C D_t^{\beta} u = -(-\Delta)^{\sigma} f(u), & \text{in } Q_T, \\ u(x,t) = 0, & \text{on } B_T, \\ u(x,0) = \phi(x), u_t(x,0) = \varphi(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $1 < \beta < 2, 0 < \sigma < 1$ and $N < 4\sigma$. Moreover, $f(u)$ is a given nonlinear function, and $\phi(x), \varphi(x)$ are initial value data. The Caputo fractional operator ${}_0^C D_t^{\beta}$ is defined by

$${}_0^C D_t^{\beta} u(t) = \frac{1}{\sigma(2-\beta)} \int_0^t \frac{1}{(t-s)^{\beta-1}} \frac{d^2}{ds^2} u(s) ds,$$

where σ is the gamma function. It is worth noting that ${}_0^C D_t^{\beta} u$ may turn into the usual derivative u_{tt} when $\beta \to 2$, see [1] for details. The fractional Laplacian operator $(-\Delta)^{\sigma}$ can be defined via spectral decomposition

$$(-\Delta)^{\sigma} u = \sum_{k=1}^{\infty} \mu_k^{\sigma} u_k w_k,$$

where μ_k and $w_k, k \in \mathbb{N}$ are eigenpairs of the following eigenvalue problem

$$\begin{cases} -\Delta w_k = \mu_k w_k, & \text{in } \Omega, \\ w_k = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$v_k = \int_{\Omega} v(x) w_k dx, \text{ with } \|w_k\|_{L^2(\Omega)} = 1.$$

Therefore, it's called the spectral fraction Laplace operator, see [2, 3] for details. Equation (1.1) is nonlocal both in space and time, so we call such a Boussinesq equation a fully nonlocal Boussinesq equation.

Problem (1.1)'s widespread use as a model for anomalous diffusion in physical field serves as a significant incentive for study. Time fractional derivatives are generally exploited to model the omnipresent memory effects such as anomalous diffusion, wave propagations and neuronal transmission in Purkinje cells, etc. For example, in [4], the authors demonstrated how Caputo time fractional derivatives can be used to analyze turbulent eddies' trapping effects. In fact, β order time fractional derivatives have been used for "superdiffusion"-in which particles spread quickly against the laws of Brownian motion. Nevertheless, time fractional derivatives and "anomalous subdiffusion" are frequently linked when $\beta \in (0, 1)$, see [5–7]. Furthermore, space fractional derivatives can be used to describe nonlocal effects, such as anomalous diffusion and Lévy processes. Recently, time or space fractional wave equations have drawn a lot of interest, see [8–16] for examples.

In 1872, J. Boussinesq [17] presented the Boussinesq equation

$$u_{tt} - u_{xx} + \sigma u_{xxxx} = (u^2)_{xx},$$

which can illustrate how small amplitude long waves propagate on the surface of shallow water. The improved Boussinesq equation (IBq equation) may be written as

$$u_{tt} - u_{xx} - u_{xxtt} = (u^2)_{xx},$$

which can describe the continuum limit of shallow water waves in a one-dimensional nonlinear lattice and other modes supporting linear waves with a negative dispersion. They can also explain the lowest-order nonlinear effects in the evolution of perturbations using a dispersion relation similar to that for sound waves (in terms of wave amplitudes). In [18], it was indicated that the IBq equation

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^2)$$

may be deduced from starting with the accurate hydrodynamical set of equations in plasma and modifying the IBq equation in a manner similar to modifying the Korteweg-de Vries equation to derive

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^3),$$

which is called IMBq (modified IBq) equation. During these years, the theory of Boussinesq equations has been developed significantly, see [19–25]. In [19, 20], Wang and Chen studied Cauchy problems for the following generalized Boussinesq equation

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta f(u). \tag{1.2}$$

They discussed whether or not global solutions exist. Moreover, by Banach fixed point theorm, they obtained that small-amplitude solutions exist globally. In [21], the authors researched Cauchy problems for the following Boussinesq equation

$$u_{tt} - \Delta u + \Delta^2 u + \Delta^2 u_{tt} = \Delta f(u).$$

Using the Banach fixed point theorem, they proved that the solution exists locally. Under different dimensions, they obtained global existence of smooth solutions using potential well method. In addition, they acquired the blow-up of solutions. In general, in [23] and [24], the authors studied Cauchy problems for the generalized Boussinesq equation with damping terms, respectively. Using the Banach fixed point theorem, they constructed a class of time-weighted Sobolev spaces, and obtained global existence and long-time behavior of small amplitude solutions. In [26], Li, Yan and Xie studied an extended (3 + 1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq equation, and obtained a family of rational solution through its bilinear form and symbolic computation. In addition, there are many results on nonlocal nonlinear problems, see [27, 28] for examples.

Recently, fully nonlocal evolution equations have received a lot of attention. In [29], Kemppainen, Siljander, Zacher studied classical solutions and large-time behavior for fully nonlocal diffusion equations. In [30], Li, Liu and Wang researched Cauchy problems for Keller-Segel type fully nonlocal diffusion equation. Therefore, the study of fully nonlocal Boussinesq equations has certain theoretical significance. Comparing the Eqs (1.1) and (1.2), we just replace u_{tt} with ${}_0^C D_t^\beta u$ and $-\Delta$ with $(-\Delta)^\sigma$, as it comes to nonlocality and memory effect. In light of these works mentioned above, we aim to investigate Cauchy problems for the fully nonlocal Boussinesq equation in (1.1) and generalize their results in [19, 20] by Wang and Chen. Nevertheless, the spectral fractional Laplacian operator $(-\Delta)^\sigma$ makes no sense in $\mathbb{R}^N$, because the Laplacian operator's spectrum in $\mathbb{R}^N$ is continuous purely. As a result, we research the initial boundary value problem (1.1).

Nevertheless, as yet, there are few results on global existence and long-time behavior of solutions for problem (1.1). In reality, the corresponding linear problem has not received much attention. The major difficulty is the nonlocality and nonlinearity of $(-\Delta)^\sigma f(u)$. In addition, from the memory effect of $\partial_t^\beta u$, the definition of weak solutions is difficult to introduce and potential well method may be also ineffective for fully nonlocal Boussinesq equations. Inspired by [16], we first study the corresponding linear Boussinesq equation to obtain explicit expressions of Green operators. Further, we establish some rigorous estimates of the Green operators to acquire global existence and decay properties of weak solutions for linear problems. Second, we establish new chain and Leibniz rules concerning the spectral fractional derivatives. Based on these given results, under different dimensions N and small initial value condition, by Banach fixed point theorm, we obtain global existence and long-time behavior of weak solutions for problem (1.1) in the time-weighted fractional Sobolev spaces. Throughout this paper, we replace $\|\cdot\|_{\mathbb{H}^s(\Omega)}$ with $\|\cdot\|_s$, and the notation $C \lesssim D$ means that there is a constant $M > 0$, such that $C \leq MD$.

The following are major results of this manuscript.

Theorem 1.1. *Suppose that*

$$s = \begin{cases} \sigma, & N < 2\sigma, \\ 2\sigma, & 2\sigma \leq N < 4\sigma, \end{cases}$$

and

$$0 < \alpha < \frac{\beta - 1}{\beta}, \ 2 < q < \frac{1}{\alpha\beta}.$$

If $f \in C^l(\mathbb{R})$ and

$$|f^{(i)}(u)| \leq |u|^{q-i}, \ i = 0, 1, ..., l \leq q,$$

and $\phi, \varphi \in \mathbb{H}^s(\Omega)$ satisfy

$$\|\phi\|_s + \|\varphi\|_s \leq \varepsilon,$$

then problem (1.1) *has a unique global weak solution $u \in C([0, T]; \mathbb{H}^s(\Omega))$ satisfying $u_t \in C([0, T]; \mathbb{H}^s(\Omega))$. Furthermore,*

$$\sup_{0 \leq t \leq T} t^{\alpha\beta} \|u(t)\|_s \leq \zeta, \tag{1.3}$$

where $\varepsilon, \zeta > 0$ are small enough such that $\varepsilon + \zeta^q \leq \zeta$.

Theorem 1.2. *Suppose that all assumptions in Theorem 1.1 hold, and*

$$\frac{\beta - 1}{\beta} < \delta < 1, \ \max\{1, \frac{\delta\beta + 1 - \beta}{\alpha\beta}\} < q < \frac{\delta}{\alpha}.$$

Let u be the global weak solution of problem (1.1)*, then there holds*

$$\sup_{0 \leq t \leq T} t^{\omega} \|u_t(t)\|_s \leq \zeta,$$

where

$$\omega = \beta(\delta - 1) + 1.$$

The paper is organized as follows. In Section 2, we introduce fractional Sobolev space briefly, and give several properties of Mittag-Leffler functions. In Section 3, we study global existence and decay estimates of weak solutions for the corresponding linear Boussinesq equation. In Section 4, for small initial values condition, we establish global existence and long-time behavior of weak solutions for problem (1.1) under different dimensions N.

2. Preliminaries

For simplicity, we use the notation below. Let $L^p(\Omega)$, $1 \leq p \leq \infty$, be Lebesgue spaces endowed with the norm $\|\cdot\|_p$. Specially, we replace $\|\cdot\|_2$ with $\|\cdot\|$. Let $H^\sigma(\Omega)$, $\sigma > 0$, be the usual Sobolev space. Moreover, we introduce the fractional Sobolev space by eigenpairs mentioned above

$$\mathbb{H}^\sigma(\Omega) = \{u | u \in L^2(\Omega), \|u\|^2_{\mathbb{H}^\sigma(\Omega)} = \sum_{k=1}^{\infty} \mu_k^\sigma |(u, w_k)|^2 < \infty\},$$

where $(\cdot,\cdot)$ represents the inner product in $L^2(\Omega)$. Obviously, $(\mathbb{H}^\sigma(\Omega), \|\cdot\|_{\mathbb{H}^\sigma(\Omega)})$ is a Hilbert space, and satisfies $\mathbb{H}^\sigma(\Omega) \subset H^\sigma(\Omega)$. Particularly, $\mathbb{H}^1(\Omega) = H_0^1(\Omega)$. Let $\mathbb{H}^{-\sigma}(\Omega)$ denote the dual space of $\mathbb{H}^\sigma(\Omega)$. Since $\mathbb{H}^\sigma(\Omega) \subset L^2(\Omega)$, we have $\mathbb{H}^\sigma(\Omega) \subset L^2(\Omega) \subset \mathbb{H}^{-\sigma}(\Omega)$.

It is worth noting that $\mathbb{H}^{-\sigma}(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|^2_{\mathbb{H}^{-\sigma}(\Omega)} = \sum_{k=1}^{\infty} \mu_k^{-\sigma} |\langle u, w_k \rangle|^2,$$

where $\langle\cdot,\cdot\rangle$ represents the dual product between $\mathbb{H}^{-\sigma}(\Omega)$ and $\mathbb{H}^\sigma(\Omega)$. Moreover, if $v \in L^2(\Omega)$ and $w \in \mathbb{H}^\sigma(\Omega)$, we have

$$\langle v, w \rangle = (v, w).$$

We may refer to [13, 16, 31] for details on $\mathbb{H}^\sigma(\Omega)$.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ may play an crucial role on existence and decay estimates of solutions. Next, we give the definition and several important properties of $E_{\alpha,\beta}(z)$. For $z \in \mathbb{C}$, the Mittag-Leffler function can be defined by

$$E_{\varpi,\nu}(z) = \sum_{k=0}^{\infty} \frac{1}{\sigma(\varpi k + \nu)} z^k$$

where $\varpi > 0$, $\nu \in \mathbb{R}$ are arbitrary constants, see [1] for details.

Lemma 2.1 ([14, 32]). *If $1 < \varpi < 2$ and $\nu \in \mathbb{R}$, then for all $t \geq 0$,*

$$|E_{\varpi,\nu}(-t)| \leq \frac{C_{\varpi,\nu}}{1+t},$$

where $C_{\varpi,\nu} > 0$ depends only on ϖ, ν.

Lemma 2.2 ([13, 14]). *If $1 < \varpi < 2$ and $\eta > 0$, then there hold*

$$\partial_t E_{\varpi,1}(-\eta t^\varpi) = -\eta t^{\varpi-1} E_{\varpi,\varpi}(-\eta t^\varpi),$$

and

$$\partial_t (t^{\varpi-1} E_{\varpi,\varpi}(-\eta t^\varpi)) = t^{\varpi-2} E_{\varpi,\varpi-1}(-\eta t^\varpi).$$

Lemma 2.3 ([13, 14]). *If $1 < \varpi < 2$ and $\eta > 0$, then there hold*

$$\partial_t^\varpi E_{\varpi,1}(-\eta t^\varpi) = -\eta E_{\varpi,1}(-\eta t^\varpi),$$

and

$$\partial_t^\varpi (t^{\varpi-1} E_{\varpi,\varpi}(-\eta t^\varpi)) = -\eta t^{\varpi-1} E_{\varpi,\varpi}(-\eta t^\varpi).$$

3. Linear estimates

In this section, we obtain explicit expressions and some rigorous estimates of the Green operators for problem (1.1). First, we study the corresponding linear problem

$$\begin{cases} {}_0^C D_t^\beta u + (-\Delta)^\sigma u + (-\Delta)^\sigma {}_0^C D_t^\beta u = -(-\Delta)^\sigma h(t,x), & \text{in } Q_T, \\ u(x,t) = 0, & \text{on } B_T, \\ u(x,0) = \phi(x), u_t(x,0) = \varphi(x), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where the function $h(t,x)$ is given. Inspired by [12, 13, 16], we try to find the solution of problem (3.1) as follows

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) w_k(x). \tag{3.2}$$

Therefore, it can be inferred that

$$\begin{cases} {}_0^C D_t^\beta u_k + \mu_k^\sigma u_k + \mu_k^\sigma {}_0^C D_t^\beta u_k = -\mu_k^\sigma h_k, \\ u_k(0) = \phi_k, \partial_t u_k(0) = \varphi_k, \end{cases} \tag{3.3}$$

where $h_k = (h, w_k)$, $\phi_k = (\phi, w_k)$ and $\varphi_k = (\varphi, w_k)$. By Laplace transforms, we have

$$\xi^\beta \tilde{u}_k - \xi^{\beta-1}\phi_k - \xi^{\beta-2}\varphi_k + \mu_k^\sigma(\xi^\beta \tilde{u}_k - \xi^{\beta-1}\phi_k - \xi^{\beta-2}\varphi_k) + \mu_k^\sigma \tilde{u}_k = -\mu_k^\sigma \tilde{h}_k,$$

where

$$\tilde{u}_k = \mathcal{L}(u_k(t)) = \int_0^\infty e^{-\xi t} u_k(t) dt.$$

Then, we get

$$\tilde{u}_k = \xi^{\beta-1}\phi_k(\xi^\beta + \eta_k)^{-1} + \xi^{\beta-2}\varphi_k(\xi^\beta + \eta_k)^{-1} - \eta_k \tilde{h}_k(\xi^\beta + \eta_k)^{-1},$$

where

$$\eta_k = \frac{\mu_k^\sigma}{1 + \mu_k^\sigma}$$

satisfies

$$\frac{\mu_1^\sigma}{1 + \mu_1^\sigma} \le \eta_k \le 1.$$

Using the inverse Laplace transform, it is derived from Lemma 2.1 in [33] that

$$\begin{aligned} u_k(t) = {} & E_{\beta,1}(-\eta_k t^\beta)\phi_k + t E_{\beta,2}(-\eta_k t^\beta)\varphi_k \\ & - \eta_k \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(-\eta_k (t-\tau)^\beta) h_k(\tau) d\tau. \end{aligned} \tag{3.4}$$

In terms of (3.4), if (3.2) converges, then we may formally obtain the following weak solution

$$u(t,x) = R_1^\beta(t)\phi(x) + R_2^\beta(t)\varphi(x) + \int_0^t R_3^\beta(t-\tau) h(\tau,x) d\tau, \tag{3.5}$$

where the Green operators are

$$R_1^\beta(t)v = \sum_{k=1}^{\infty} E_{\beta,1}(-\eta_k t^\beta) v_k w_k,$$
$$R_2^\beta(t)v = t\sum_{k=1}^{\infty} E_{\beta,2}(-\eta_k t^\beta) v_k w_k,$$
$$R_3^\beta(t)v = -t^{\beta-1}\sum_{k=1}^{\infty} \eta_k E_{\beta,\beta}(-\eta_k t^\beta) v_k w_k.$$

Then, we get from Lemma 2.2

$$\partial_t R_1^\beta(t)v = -t^{\beta-1}\sum_{k=1}^{\infty} \eta_k E_{\beta,\beta}(-\eta_k t^\beta) v_k w_k,$$
$$\partial_t R_2^\beta(t)v = \sum_{k=1}^{\infty} E_{\beta,1}(-\eta_k t^\beta) v_k w_k,$$
$$\partial_t R_3^\beta(t)v = -t^{\beta-2}\sum_{k=1}^{\infty} \eta_k E_{\beta,\beta-1}(-\eta_k t^\beta) v_k w_k.$$

Definition 3.1. *We say that u is a weak solution of problem* (3.1) *if $u \in L^\infty(0,T;L^2(\Omega))$, $\partial_t u \in L^\infty(0,T;L^2(\Omega))$, $\partial_t^\beta u \in L^2((0,t)\times\Omega)$, $u(0)=\phi$, $\partial_t u(0)=\varphi$ and* (3.5) *holds. Moreover, if $T>0$ can be chosen as any positive number, u is called a global weak solution for problem* (3.1).

We can get the following estimations on the Green operators immediately from Lemmas 2.1 and 2.2.

Lemma 3.2. *If $v \in L^2(\Omega)$, then we get*

$$\|R_1^\beta(t)v\| \lesssim \|v\|,\ \|\partial_t R_1^\beta(t)v\| \lesssim t^{\beta-1}\|v\|,$$
$$\|R_2^\beta(t)v\| \lesssim t^{1-\frac{\beta}{2}}\|v\|,\ \|\partial_t R_2^\beta(t)v\| \lesssim \|v\|,$$
$$\|R_3^\beta(t)v\| \lesssim t^{\frac{\beta}{2}-1}\|v\|,\ \|\partial_t R_3^\beta(t)v\| \lesssim t^{\beta-2}\|v\|.$$

Further, if $v \in \mathbb{H}^s(\Omega)$, then we have

$$\|R_1^\beta(t)v\|_s \lesssim \|v\|_s,\ \|\partial_t R_1^\beta(t)v\|_s \lesssim t^{\beta-1}\|v\|_s,$$
$$\|R_2^\beta(t)v\|_s \lesssim t^{1-\frac{\beta}{2}}\|v\|_s,\ \|\partial_t R_2^\beta(t)v\|_s \lesssim \|v\|_s,$$
$$\|R_3^\beta(t)v\|_s \lesssim t^{\frac{\beta}{2}-1}\|v\|_s,\ \|\partial_t R_3^\beta(t)v\|_s \lesssim t^{\beta-2}\|v\|_s.$$

Proof. We obtain directly from Lemma 2.1

$$\|R_1^\beta(t)v\|^2 = \sum_{k=1}^{\infty}[E_{\beta,1}(-\eta_k t^\beta)v_k]^2 \lesssim \sum_{k=1}^{\infty} v_k^2 = \|v\|^2,$$
$$\|\partial_t R_1^\beta(t)v\| = t^{\beta-1}(\sum_{k=1}^{\infty}\eta_k^2[E_{\beta,\beta}(-\eta_k t^\beta)v_k]^2)^{\frac{1}{2}} \lesssim t^{\beta-1}\|v\|,$$

$$\|R_2^{\beta}(t)v\| \lesssim t(\sum_{k=1}^{\infty}\frac{t^{\beta}}{(1+\eta_k t^{\beta})^2}t^{-\beta}v_k^2)^{\frac{1}{2}} \lesssim t^{1-\frac{\beta}{2}}\|v\|,$$
$$\|\partial_t R_2^{\beta}(t)v\| = (\sum_{k=1}^{\infty}[E_{\beta,1}(-\eta_k t^{\beta})v_k]^2)^{\frac{1}{2}} \lesssim \|v\|,$$

and

$$\begin{aligned}
\|R_3^{\beta}(t)v\| &= t^{\beta-1}(\sum_{k=1}^{\infty}\eta_k^2[E_{\beta,\beta}(-\eta_k t^{\beta})v_k]^2)^{\frac{1}{2}} \\
&\lesssim t^{\beta-1}(\sum_{k=1}^{\infty}\frac{\mu_k^{2\sigma}}{(1+\mu_k^{\sigma})^2}\frac{t^{\beta}}{(1+\eta_k t^{\beta})^2}t^{-\beta}v_k^2)^{\frac{1}{2}} \\
&\lesssim t^{\frac{\beta}{2}-1}\|v\|, \\
\|\partial_t R_3^{\beta}(t)v\| &= t^{\beta-2}(\sum_{k=1}^{\infty}\eta_k^2[E_{\beta,\beta-1}(-\eta_k t^{\beta})v_k]^2)^{\frac{1}{2}} \\
&\lesssim t^{\beta-2}(\sum_{k=1}^{\infty}\frac{\mu_k^{2\sigma}}{(1+\mu_k^{\sigma})^2}\frac{1}{(1+\eta_k t^{\beta})^2}v_k^2)^{\frac{1}{2}} \\
&\lesssim t^{\beta-2}\|v\|.
\end{aligned}$$

Thus, $R_1^{\beta}(t)v$ is uniformly convergent with regard to t, and $\partial_t R_1^{\beta}(t)v$ is convergent in $L^2(\Omega)$. Therefore, $\partial_t R_1^{\beta}(t)v$ exists. In a similar way, other conclusions of Lemma 3.2 are easily obtained from Lemma 2.1.

Next, more rigorous estimations are acquired for the Green operators.

Lemma 3.3. *Suppose that*

$$0 < \alpha < \frac{\beta-1}{\beta},\ 1 < q < 1+\frac{1}{\alpha}.$$

If $v \in \mathbb{H}^s(\Omega)$*, then we get*

$$\begin{aligned}
&\|R_1^{\beta}(t)v\|_s \lesssim t^{-\alpha\beta}\|v\|_s,\ \|R_2^{\beta}(t)v\|_s \lesssim t^{-\alpha\beta}\|v\|_s, \\
&\|R_3^{\beta}(t)v\|_s \lesssim t^{\alpha\beta(q-1)-1}\|v\|_s.
\end{aligned}$$

Proof. By Young inequality, we obtain from Lemma 2.1

$$\begin{aligned}
\|R_1^{\beta}(t)v\|_s = (\sum_{k=1}^{\infty}\mu_k^s[E_{\beta,1}(-\eta_k t^{\beta})v_k]^2)^{\frac{1}{2}} &\lesssim t^{-\alpha\beta}(\sum_{k=1}^{\infty}\mu_k^s\frac{t^{2\alpha\beta}}{(1+\eta_k t^{\beta})^2}v_k^2)^{\frac{1}{2}} \\
&\lesssim t^{-\alpha\beta}\|v\|_s,
\end{aligned}$$

and

$$\|R_2^{\beta}(t)v\|_s = t(\sum_{k=1}^{\infty}\mu_k^s[E_{\beta,2}(-\eta_k t^{\beta})v_k]^2)^{\frac{1}{2}} \lesssim t(\sum_{k=1}^{\infty}\mu_k^s\frac{t^{2\alpha\beta+2}}{(1+\eta_k t^{\beta})^2}t^{-2\alpha\beta-2}v_k^2)^{\frac{1}{2}}$$

$$\lesssim t^{-\alpha\beta}\|v\|_s,$$

and

$$\begin{aligned}
\|R_3^\beta(t)v\|_s &= t^{\beta-1}\Big(\sum_{k=1}^{\infty}\mu_k^s[E_{\beta,\beta}(-\eta_k t^\beta)v_k]^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\beta-1}\Big(\sum_{k=1}^{\infty}\mu_k^s\frac{t^{2\beta-2\alpha\beta(q-1)}}{(1+\eta_k t^\beta)^2}t^{-2\beta+2\alpha\beta(q-1)}v_k^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\alpha\beta(q-1)-1}\|v\|_s.
\end{aligned}$$

Lemma 3.4. *Suppose that*

$$\frac{\beta-1}{\beta}<\delta<1,\ \max\{1,\frac{\delta\beta+1-\beta}{\alpha\beta}\}<q<\frac{\delta}{\alpha},$$

where α is already determined in Lemma 3.3. If $v\in\mathbb{H}^s(\Omega)$, then we get

$$\begin{aligned}
&\|\partial_t R_1^\beta(t)v\|_s\lesssim t^{\beta(1-\delta)-1}\|v\|_s,\ \|\partial_t R_2^\beta(t)v\|_s\lesssim t^{\beta(1-\delta)-1}\|v\|_s,\\
&\|\partial_t R_3^\beta(t)v\|_s\lesssim t^{\beta(1-\delta+\alpha q)-2}\|v\|_s.
\end{aligned}$$

Proof. By Young inequality, we obtain from Lemma 2.1

$$\begin{aligned}
\|\partial_t R_1^\beta(t)v\|_s &= t^{\beta-1}\Big(\sum_{k=1}^{\infty}\mu_k^s[E_{\beta,\beta}(-\eta_k t^\beta)v_k]^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\beta-1}\Big(\sum_{k=1}^{\infty}\mu_k^s\frac{t^{2\beta\delta}}{(1+\eta_k t^\beta)^2}t^{-2\beta\delta}v_k^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\beta(1-\delta)-1}\|v\|_s,
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_t R_2^\beta(t)v\|_s &= \Big(\sum_{k=1}^{\infty}\mu_k^s[E_{\beta,1}(-\eta_k t^\beta)v_k]^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\beta(1-\delta)-1}\Big(\sum_{k=1}^{\infty}\mu_k^s\frac{t^{2\beta\delta+2-2\beta}}{(1+\eta_k t^\beta)^2}v_k^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\beta(1-\delta)-1}\|v\|_s,
\end{aligned}$$

and

$$\begin{aligned}
\|R_3^\beta(t)v\|_s &= t^{\beta-2}\Big(\sum_{k=1}^{\infty}\mu_k^s[E_{\beta,\beta-1}(-\eta_k t^\beta)v_k]^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\beta-2}\Big(\sum_{k=1}^{\infty}\mu_k^s\frac{t^{2\beta(\delta-\alpha q)}}{(1+\eta_k t^\beta)^2}t^{-2\beta(\delta-\alpha q)}v_k^2\Big)^{\frac{1}{2}}\\
&\lesssim t^{\beta(1-\delta+\alpha q)-2}\|v\|_s.
\end{aligned}$$

Proposition 3.5.

(i) If $h \in L^\infty(0,T;L^2(\Omega))$, $\phi \in L^2(\Omega)$ and $\varphi \in L^2(\Omega)$, then problem (3.1) *has a unique global weak solution $u \in C([0,T];L^2(\Omega))$ satisfying $u_t \in C([0,T];L^2(\Omega))$. Moreover, for all $t>0$, there hold*

$$\|u(t)\| \lesssim \|\phi\| + t^{1-\frac{\beta}{2}}\|\varphi\| + t^{\frac{\beta-1}{2}}\|h\|_{L^2((0,t)\times\Omega)},$$

$$\|u_t(t)\| \lesssim t^{\beta-1}\|\phi\| + \|\varphi\| + t^{\beta-1}\|h\|_{L^\infty(0,t;L^2(\Omega))},$$

and

$$\|\partial_t^\beta u\|_{L^2((0,t)\times\Omega)} \lesssim t^{\frac{1}{2}}\|\phi\| + t^{\frac{3-\beta}{2}}\|\varphi\| + (t^{\frac{\beta}{2}}+1)\|h\|_{L^2((0,t)\times\Omega)}.$$

(ii) If $h \in L^\infty(0,T;\mathbb{H}^s(\Omega))$, $\phi \in \mathbb{H}^s(\Omega)$, and $\varphi \in \mathbb{H}^s(\Omega)$, then problem (3.1) *has a unique global weak solution $u \in C([0,T];\mathbb{H}^s(\Omega))$ satisfying $u_t \in C([0,T];\mathbb{H}^s(\Omega))$. Moreover, for all $t>0$, there hold*

$$\|u(t)\|_s \lesssim \|\phi\|_s + t^{1-\frac{\beta}{2}}\|\varphi\|_s + t^{\frac{\beta-1}{2}}\|h\|_{L^2(0,t;\mathbb{H}^s(\Omega))},$$

and

$$\|u_t(t)\|_s \lesssim t^{\beta-1}\|\phi\|_s + \|\varphi\|_s + t^{\beta-1}\|h\|_{L^\infty(0,t;\mathbb{H}^s(\Omega))}.$$

Proof. First, we prove Proposition 3.5.(i). By Hölder inequality, we get from Lemma 3.2

$$\begin{aligned}
\|u(t)\| &\le \|R_1^\beta(t)\phi\| + \|R_2^\beta(t)\varphi\| + \int_0^t \|R_3^\beta(t-\tau)h(\tau,\cdot)\|d\tau \\
&\lesssim \|\phi\| + t^{1-\frac{\beta}{2}}\|\varphi\| + \int_0^t (t-\tau)^{\frac{\beta}{2}-1}\|h(\tau,\cdot)\|d\tau \\
&\lesssim \|\phi\| + t^{1-\frac{\beta}{2}}\|\varphi\| + (\int_0^t (t-\tau)^{\beta-2}d\tau)^{\frac{1}{2}}(\int_0^t \|h(\tau,\cdot)\|^2 d\tau)^{\frac{1}{2}} \\
&\lesssim \|\phi\| + t^{1-\frac{\beta}{2}}\|\varphi\| + t^{\frac{\beta-1}{2}}\|h\|_{L^2((0,t)\times\Omega)}.
\end{aligned}$$

Therefore, we have $u \in C([0,T];L^2(\Omega))$. Furthermore, we obtain that u is continuous absolutely with regard to t from (3.5). Then, it is deduced from Lemma 3.2 that $\partial_t u$ exists and

$$u_t(t,x) = \partial_t R_1^\beta(t)\phi + \partial_t R_2^\beta(t)\varphi + \int_0^t \partial_t R_3^\beta(t-\tau)h(\tau,x)d\tau.$$

Furthermore, we get from Lemma 2.3

$$\begin{aligned}
\partial_t^\beta u_k(t) =& -\eta_k E_{\beta,1}(-\eta_k t^\beta)\phi_k - t\eta_k E_{\beta,2}(-\eta_k t^\beta)\varphi_k \\
&+ \eta_k^2 \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(-\eta_k(t-\tau)^\beta)h_k(\tau)d\tau - \eta_k h_k(t).
\end{aligned}$$

Hence, by Young inequality, we have from Lemma 2.1

$$\|\partial_t^\beta u\|_{L^2((0,t)\times\Omega)} \lesssim t^{\frac{1}{2}}\|\phi\| + t^{\frac{3-\beta}{2}}\|\varphi\| + (t^{\frac{\beta}{2}}+1)\|h\|_{L^2((0,t)\times\Omega)}.$$

Further, we also obtain

$$\begin{aligned}\|u_t(t)\| &\le \|\partial_t R_1^\beta(t)\phi\| + \|\partial_t R_2^\beta(t)\varphi\| + \int_0^t \|\partial_t R_3^\beta(t-\tau)h(\tau,\cdot)\|d\tau \\ &\lesssim t^{\beta-1}\|\phi\| + \|\varphi\| + \int_0^t (t-\tau)^{\beta-2}\|h(\tau,\cdot)\|d\tau \\ &\lesssim t^{\beta-1}\|\phi\| + \|\varphi\| + t^{\beta-1}\|h\|_{L^\infty(0,t;L^2(\Omega))}.\end{aligned}$$

Therefore, $u \in C([0,T];L^2(\Omega))$ and $u_t \in C([0,T];L^2(\Omega))$.

Next, we prove Proposition 3.5.(ii). By Hölder inequality, we obtain from Lemma 3.2

$$\begin{aligned}\|u(t)\|_s &\le \|R_1^\beta(t)\phi\|_s + \|R_2^\beta(t)\varphi\|_s + \int_0^t \|R_3^\beta(t-\tau)h(\tau,\cdot)\|_s d\tau \\ &\lesssim \|\phi\|_s + t^{1-\frac{\beta}{2}}\|\varphi\|_s + \int_0^t (t-\tau)^{\frac{\beta}{2}-1}\|h(\tau,\cdot)\|_s d\tau \\ &\lesssim \|\phi\|_s + t^{1-\frac{\beta}{2}}\|\varphi\|_s + (\int_0^t (t-\tau)^{\beta-2}d\tau)^{\frac{1}{2}}(\int_0^t \|h(\tau,\cdot)\|_s^2 d\tau)^{\frac{1}{2}} \\ &\lesssim \|\phi\|_s + t^{1-\frac{\beta}{2}}\|\varphi\|_s + t^{\frac{\beta-1}{2}}\|h\|_{L^2(0,t;\mathbb{H}^s(\Omega))},\end{aligned}$$

and

$$\begin{aligned}\|u_t(t)\|_s &\le \|\partial_t R_1^\beta(t)\phi\|_s + \|\partial_t R_2^\beta(t)\varphi\|_s + \int_0^t \|\partial_t R_3^\beta(t-\tau)h(\tau,\cdot)\|_s d\tau \\ &\lesssim t^{\beta-1}\|\phi\|_s + \|\varphi\|_s + \int_0^t (t-\tau)^{\beta-2}\|h(\tau,\cdot)\|_s d\tau.\end{aligned}$$

In the inequality above, we represent the final term by

$$I = Z * \|h\|_s := \int_0^t (t-\tau)^{\beta-2}\|h(\tau,\cdot)\|_s d\tau.$$

Therefore,

$$\begin{aligned}\|I\|_{L^2(0,t)} = \|Z * \|h\|_s\|_{L^2(0,t)} &\le \|Z\|_{L^1(0,t)}\|\|h\|_s\|_{L^2(0,t)} \\ &\le t^{\beta-1}\|h\|_{L^2(0,t;\mathbb{H}^s(\Omega))}.\end{aligned}$$

Then,

$$\|u_t\|_{L^2(0,t;\mathbb{H}^s(\Omega))} \lesssim t^{\beta-\frac{1}{2}}\|\phi\|_s + t^{\frac{1}{2}}\|\varphi\|_s + t^{\beta-1}\|h\|_{L^2(0,t;\mathbb{H}^s(\Omega))}.$$

Furthermore, we also have

$$\begin{aligned}\|u_t\|_s &\lesssim t^{\beta-1}\|\phi\|_s + \|\varphi\|_s + \int_0^t (t-\tau)^{\beta-2}\|h(\tau,\cdot)\|_s d\tau \\ &\lesssim t^{\beta-1}\|\phi\|_s + \|\varphi\|_s + t^{\beta-1}\|h\|_{L^\infty(0,t;\mathbb{H}^s(\Omega))}.\end{aligned}$$

Therefore, $u \in C([0,T];\mathbb{H}^s(\Omega))$ and $u_t \in C([0,T];\mathbb{H}^s(\Omega))$.

Proposition 3.6. *Suppose that $h \in L^\infty(0,T;\mathbb{H}^s(\Omega))$, $\phi \in \mathbb{H}^s(\Omega)$, and $\varphi \in \mathbb{H}^s(\Omega)$. If u is a global weak solution of problem* (3.1)*, then we have*

$$\|u(t)\|_s \lesssim t^{-\alpha\beta}(\|\phi\|_s + \|\varphi\|_s) + \int_0^t (t-\tau)^{\alpha\beta(q-1)-1}\|h(\tau,\cdot)\|_s d\tau,$$

and

$$\|u_t(t)\|_s \lesssim t^{\beta(1-\delta)-1}(\|\phi\|_s + \|\varphi\|_s) + \int_0^t (t-\tau)^{\beta(1-\delta+\alpha q)-2}\|h(\tau,\cdot)\|_s d\tau.$$

Remark 3.7. *It is simple to observe that Proposition 3.6 is more rigorous than Proposition 3.5, which initially appears to be weak, when it comes to estimates of weak solutions. Proposition 3.6 cannot be ignored because Proposition 3.5 can be used to establish local existence theorems to problem* (1.1)*, but Proposition 3.6 cannot.*

4. Proofs of main results

In this section, by constructing time-weighted fractional Sobolev spaces and Banach fixed point theorem, we get global existence and long-time behavior of weak solutions for problem (1.1). Now, we provide the definition of weak solutions for problem (1.1).

Definition 4.1. *We say that u is a weak solution of problem* (1.1) *if $u \in C([0,T];\mathbb{H}^s(\Omega))$, $u_t \in C([0,T];\mathbb{H}^s(\Omega))$, $u(0) = \phi$, $\partial_t u(0) = \varphi$, and there holds*

$$u(t,x) = R_1^\beta(t)\phi(x) + R_2^\beta(t)\varphi(x) + \int_0^t R_3^\beta(t-\tau) f(u(\tau,x))d\tau.$$

Moreover, if $T > 0$ can be chosen as any positive number, u is called a global weak solution for problem (1.1).

First, we need the following lemmas to establish chain and Leibnitz rules concerning $(-\Delta)^\sigma$.

Lemma 4.2 ([34]). *Let $D^s = (-\Delta)^{\frac{s}{2}}$, for any $s \geq 0$, then we have*

$$\|D^s f(u)\|_{L^r(\mathbb{R}^N)} \lesssim \|u\|^{q-1}_{L^{(q-1)r_1}(\mathbb{R}^N)}\|D^s u\|_{L^{r_2}(\mathbb{R}^N)},$$

where

$$r^{-1} = r_1^{-1} + r_2^{-1},\ r_1 \in (1,\infty],\ r_2 \in (1,\infty),$$

and

$$\|D^s(vw)\|_{L^r(\mathbb{R}^N)} \lesssim \|D^s v\|_{L^{r_1}(\mathbb{R}^N)}\|w\|_{L^{q_2}(\mathbb{R}^N)} + \|v\|_{L^{q_1}(\mathbb{R}^N)}\|D^s w\|_{L^{r_2}(\mathbb{R}^N)},$$

where

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{r_2} + \frac{1}{q_1},\ r_i \in (1,\infty),\ q_i \in (1,\infty], i = 1,2.$$

Since Ω is regular enough, it has the so-called extension property, namely: For any $s \in (0,1)$, there exists an extension $\breve{u}$ of $u \in H^s(\Omega)$ such that $\breve{u} \in H^s(\mathbb{R}^N)$ and $\breve{u}|_\Omega = u$ where

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi < \infty\}.$$

Moreover, $\|\breve{u}\|_{H^s(\mathbb{R}^N)} \le C\|u\|_{H^s(\Omega)}$. In particular, taking such extension is the trivial one, namely the extension by zero outside Ω, there holds

$$\|\breve{u}\|_{H^s(\mathbb{R}^N)} \asymp \|u\|_{H^s(\Omega)},$$

where

$$\|\breve{u}\|_{H^s(\mathbb{R}^N)} := \|D^s\breve{u}\|_{L^2(\mathbb{R}^N)},$$

and $A \asymp B$ represents that there are two constants $c_1, c_2 > 0$ satisfying $c_1 A \le B \le c_2 A$, see [3, 35] for details. Moreover, from [3], the space $\mathbb{H}^s(\Omega)$ is redefined by

$$\mathbb{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : \operatorname{supp}(u) \subset \bar{\Omega}\},$$

and these two norms $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{\mathbb{H}^s(\Omega)}$ on $\mathbb{H}^s(\Omega)$ are equivalent. Therefore, we conclude that

$$\|\breve{u}\|_{H^s(\mathbb{R}^N)} \asymp \|u\|_{\mathbb{H}^s(\Omega)}.$$

Further, let $s \in [1, 2)$ and $u \in \mathbb{H}^s(\Omega)$. Taking $s = 1 + \delta$, we derive

$$\|u\|_{\mathbb{H}^s(\Omega)} = \|\nabla u\|_{\mathbb{H}^\delta(\Omega)} \asymp \|\nabla u\|_{H^\delta(\Omega)} \asymp \|\nabla\breve{u}\|_{H^\delta(\mathbb{R}^N)}.$$

Moreover, we get

$$\|\nabla\breve{u}\|_{H^\delta(\mathbb{R}^N)} = \|D^\delta\nabla\breve{u}\|_{L^2(\mathbb{R}^N)} = \|D^s\breve{u}\|_{L^2(\mathbb{R}^N)} = \|\breve{u}\|_{H^s(\mathbb{R}^N)}.$$

Therefore, we obtain

$$\|\breve{u}\|_{H^s(\mathbb{R}^N)} \asymp \|u\|_{\mathbb{H}^s(\Omega)}.$$

Therefore, we obtain the following chain and Leibniz rules concerning the spectral fractional derivatives, which plays an fundamental role on existence of weak solutions.

Lemma 4.3. *Suppose that $s \in (0, 2)$ and $u, v \in \mathbb{H}^s(\Omega)$, then there hold*

$$\|f(u)\|_s \lesssim \|u\|_\infty^{q-1}\|u\|_s,$$

$$\|uv\|_s \lesssim \|u\|_s\|v\|_\infty + \|u\|_\infty\|v\|_s.$$

Proof. By Lemma 4.2, we have

$$\begin{aligned}\|f(u)\|_s \lesssim \|f(\breve{u})\|_{H^s(\mathbb{R}^N)} &= \|D^s f(\breve{u})\|_{L^2(\mathbb{R}^N)} \\ &\lesssim \|\breve{u}\|_{L^\infty(\mathbb{R}^N)}^{q-1}\|D^s\breve{u}\|_{L^2(\mathbb{R}^N)} \\ &= \|u\|_{L^\infty(\Omega)}^{q-1}\|\breve{u}\|_{H^s(\mathbb{R}^N)} \lesssim \|u\|_\infty^{q-1}\|u\|_s,\end{aligned}$$

and

$$\begin{aligned}\|uv\|_s \lesssim \|\breve{u}\breve{v}\|_{H^s(\mathbb{R}^N)} &= \|D^s(\breve{u}\breve{v})\|_{L^2(\mathbb{R}^N)} \\ &\lesssim \|D^s\breve{u}\|_{L^2(\mathbb{R}^N)}\|\breve{v}\|_{L^\infty(\mathbb{R}^N)} + \|\breve{u}\|_{L^\infty(\mathbb{R}^N)}\|D^s\breve{v}\|_{L^2(\mathbb{R}^N)} \\ &= \|\breve{u}\|_{H^s(\mathbb{R}^N)}\|\breve{v}\|_{L^\infty(\mathbb{R}^N)} + \|\breve{u}\|_{L^\infty(\mathbb{R}^N)}\|\breve{v}\|_{H^s(\mathbb{R}^N)} \\ &\lesssim \|u\|_{\mathbb{H}^s(\Omega)}\|v\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\|v\|_{\mathbb{H}^s(\Omega)} \\ &= \|u\|_s\|v\|_\infty + \|u\|_\infty\|v\|_s.\end{aligned}$$

Lemma 4.4. *Suppose that $s \in (0, 2)$ and $u, v \in \mathbb{H}^s(\Omega)$, then there holds*

$$\|f(u) - f(v)\|_s \lesssim \|w\|_\infty^{q-2}\|w\|_s\|u - v\|_\infty + \|w\|_\infty^{q-1}\|u - v\|_s,$$

where $w = \delta u + (1 - \delta)v$ for some $\delta \in (0, 1)$.

Proof. By Lemma 4.3, we have

$$\begin{aligned}\|f(u) - f(v)\|_s &= \|f'(w)(u - v)\|_s \\ &\lesssim \|f'(w)\|_s\|u - v\|_\infty + \|f'(w)\|_\infty\|u - v\|_s \\ &\lesssim \|w\|_\infty^{q-2}\|w\|_s\|u - v\|_\infty + \|w\|_\infty^{q-1}\|u - v\|_s.\end{aligned}$$

Now, we prove main results of this manuscript.

Proof of Theorem 1.1. Define

$$V = \{v | v \in L^\infty(0, T; \mathbb{H}^s(\Omega)), \|v\|_V \leq \zeta\},$$

where

$$\|v\|_V = \sup_{0<t\leq T} t^{\alpha\beta}\|v(t)\|_s,$$

and $\rho(v, w) = \|v - w\|_V$ for any $v, w \in V$. Consequently, it is evident that the metric space (V, ρ) is complete. Moreover, The operator P on V is defined as

$$P(u(t)) = R_1^\beta(t)\phi + R_2^\beta(t)\varphi + \int_0^t R_3^\beta(t - \tau)f(u(\tau))d\tau.$$

By Sobolev embedding theorem, it is easy to get $P(u) \in L^\infty(0, T; \mathbb{H}^s(\Omega))$ from Lemma 3.2. For any $u \in V$, using Proposition 3.6, Lemma 4.3 and Sobolev embedding theorem, we obtain

$$\begin{aligned}\|P(u(t))\|_s &\lesssim t^{-\alpha\beta}\|\phi\|_s + t^{-\alpha\beta}\|\varphi\|_s + \int_0^t (t - \tau)^{\alpha\beta(q-1)-1}\|f(u(\tau))\|_s d\tau \\ &\lesssim t^{-\alpha\beta}\|\phi\|_s + t^{-\alpha\beta}\|\varphi\|_s + \int_0^t (t - \tau)^{\alpha\beta(q-1)-1}\|u(\tau)\|_\infty^{q-1}\|u(\tau)\|_s d\tau \\ &\lesssim t^{-\alpha\beta}\|\phi\|_s + t^{-\alpha\beta}\|\varphi\|_s + \zeta^q \int_0^t (t - \tau)^{\alpha\beta(q-1)-1}\tau^{-\alpha\beta q}d\tau \\ &\lesssim t^{-\alpha\beta}(\|\phi\|_s + \|\varphi\|_s + \zeta^q).\end{aligned}$$

Hence, when ε and ζ are small enough, we get

$$\|P(u)\|_V \lesssim \varepsilon + \zeta^q \leq \zeta.$$

Thus, we acquire $P(u) \in V$. Next, we prove that $P : V \to V$ is contractive. Taking any $u, v \in V$, by Sobolev embedding theorem, we obtain from Lemma 4.4

$$\|P(u(t)) - P(v(t))\|_s \leq \int_0^t \|R_3^\beta(t - \tau)(f(u(\tau)) - f(v(\tau)))\|_s d\tau$$

$$
\begin{aligned}
&\lesssim \int_0^t (t-\tau)^{\alpha\beta(q-1)-1} \|(1-\vartheta)u+\vartheta v)\|_s^{q-1}\|u(\tau)-v(\tau)\|_s d\tau \\
&\lesssim \zeta^{q-1}\|u-v\|_V \int_0^t (t-\tau)^{\alpha\beta(q-1)-1}\tau^{-\alpha\beta q} d\tau \\
&\lesssim t^{-\alpha\beta}\zeta^{q-1}\rho(u,v),
\end{aligned}
$$

for some $\vartheta \in (0,1)$. Then,

$$\rho(P(u),P(v)) \lesssim \zeta^{q-1}\rho(u,v).$$

Taking ζ small enough, we conclude that P is contractive. Using Banach fixed point theorem, we derive that P has a unique fixed point $u \in V$.

Remarkably, we have get the solution $u \in C((0,T];\mathbb{H}^s(\Omega))$ and

$$\sup_{0<t\le T} t^{\alpha\beta}\|u(t)\|_s \le \zeta. \tag{4.1}$$

from the above proof. Next, we show

$$u \in C([0,T];\mathbb{H}^s(\Omega)).$$

In reality, we just have to verify that there exist $T_0 > 0$ small enough such that problem (1.1) admits a weak solution in

$$X = \{u | u \in C([0,T_0];\mathbb{H}^s(\Omega)), \|u\|_X \le \zeta\},$$

where

$$\|u\|_X = \max_{t\in[0,T_0]} \|u(t)\|_s.$$

Using Lemma 3.2 and Sobolev embedding theorem, we get $P(u) \in C([0,T];\mathbb{H}^s(\Omega))$. For any $u \in X$, using Lemma 3.2, Lemma 4.3 and Sobolev embedding theorem, we acquire

$$
\begin{aligned}
\|P(u(t))\|_s &\lesssim \|\phi\|_s + t^{1-\frac{\beta}{2}}\|\varphi\|_s + \int_0^t (t-\tau)^{\frac{\beta}{2}-1}\|f(u(\tau))\|_s d\tau \\
&\lesssim \|\phi\|_s + t^{1-\frac{\beta}{2}}\|\varphi\|_s + \int_0^t (t-\tau)^{\frac{\beta}{2}-1}\|u(\tau)\|_\infty^{q-1}\|u(\tau)\|_s d\tau \\
&\lesssim \varepsilon + T_0^{\frac{\beta}{2}} 2^q \zeta^q \\
&\le \zeta,
\end{aligned}
$$

where $T_0 < 1$ is small enough such that

$$\varepsilon + T_0^{\frac{\beta}{2}} 2^q \zeta^q \le \zeta.$$

Then, when ε and ζ are small enough, we derive

$$\|P(u)\|_X \le 2\zeta.$$

Thus, we get $P(u) \in X$. Next, we verify that $P : X \to X$ is contractive. Taking any $u, v \in X$, using Lemma 3.2, Lemma 4.4 and Sobolev embedding theorem, we obtain

$$\|P(u(t)) - P(v(t))\|_s \le \int_0^t \|R_3^\beta(t-\tau)(f(u(\tau)) - f(v(\tau)))\|_s d\tau$$

$$
\begin{aligned}
&\lesssim \int_0^t (t-\tau)^{\frac{\beta}{2}-1} \|(1-\vartheta)u + \vartheta v\|_s^{q-1} \|u(\tau) - v(\tau)\|_s d\tau \\
&\lesssim 2^{q-1} \zeta^{q-1} T_0^{\frac{\beta}{2}} \|u - v\|_X.
\end{aligned}
$$

Then,

$$
\|P(u) - P(v)\|_X \lesssim \zeta^{q-1} T_0^{\frac{\beta}{2}} \|u - v\|_X.
$$

Taking T_0 small enough, we infer that P is contractive. By Banach fixed point theorem, we know that problem (1.1) admits a unique weak solution $\bar{u}$ in $C([0, T_0]; \mathbb{H}^s(\Omega))$. What's more, we may take T_0 satisfying

$$
\sup_{t \in [0, T_0]} t^{\alpha\beta} \|\bar{u}(t)\|_s \leq \zeta. \tag{4.2}
$$

Therefore, using uniqueness of the solution, we obtain $u = \bar{u}$, i.e., $u \in C([0, T]; \mathbb{H}^s(\Omega))$. Furthermore, we derive from (4.1) and (4.2) that (1.3) holds.

Proof of Theorem 1.2. By Proposition 3.5, we get $u \in C^1([0, T]; \mathbb{H}^s(\Omega))$. Therefore,

$$
u_t(t, x) = \partial_t R_1^\beta(t)\phi(x) + \partial_t R_2^\beta(t)\varphi(x) + \int_0^t \partial_t R_3^\beta(t-\tau) f(u(\tau, x)) d\tau.
$$

Using Lemma 3.4, Lemma 4.3 and Sobolev embedding theorem, we obtain

$$
\begin{aligned}
\|u_t(t)\|_s &\lesssim t^{\beta(1-\delta)-1}(\|\phi\|_s + \|\varphi\|_s) + \int_0^t (t-\tau)^{\beta(1-\delta+\alpha q)-2} \|f(u(\tau))\|_s d\tau \\
&\lesssim t^{\beta(1-\delta)-1}(\|\phi\|_s + \|\varphi\|_s) + \int_0^t (t-\tau)^{\beta(1-\delta+\alpha q)-2} \|u(\tau)\|_\infty^{q-1} \|u(\tau)\|_s d\tau \\
&\lesssim t^{\beta(1-\delta)-1}(\|\phi\|_s + \|\varphi\|_s) + \zeta^q \int_0^t (t-\tau)^{\beta(1-\delta+\alpha q)-2} \tau^{-\alpha\beta q} d\tau \\
&\lesssim t^{\beta(1-\delta)-1}(\varepsilon + \zeta^q).
\end{aligned}
$$

Then, we have

$$
t^\omega \|u_t(t)\|_s \lesssim \varepsilon + \zeta^q \leq \zeta.
$$

Therefore, we conclude that

$$
\sup_{0 \leq t \leq T} t^\omega \|u_t(t)\|_s \leq \zeta,
$$

where

$$
\omega = \beta(\delta - 1) + 1.
$$

5. Conclusions

In this paper, we study initial boundary value problems for fully nonlocal Boussinesq equations. We overcome full nonlocal effects generated by ${}_0^C D_t^\beta$ and $(-\Delta)^\sigma$, and obtain some new results as follows: (a) We obtain explicit expressions and some rigorous estimates of the Green operators for the corresponding linear equation; (b) We establish new chain and Leibnitz rules concerning $(-\Delta)^\sigma$; (c) We establish time-wighted fractional Sobolev spaces and obtain global existence and long-time behavior of weak solutions. Moreover, our work adds some novelty results to the subject of Boussinesq equations, which may provide a certain theoretical support for the study of fully nonlocal wave equations and have certain theoretical significance.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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