Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(9): 5286–5312.
DOI: 10.3934/era.2023269
Received: 25 April 2023
Revised: 16 June 2023
Accepted: 07 July 2023
Published: 24 July 2023

Theory article

Existence of solutions for Kirchhoff-type systems with critical Sobolev exponents in $\mathbb{R}^3$

Xing Yi and Shuhou Ye*

College of Mathematics and Statistics, Hunan Normal University, Hunan, China

* **Correspondence:** Email: 2601320584@qq.com.

Abstract: In this paper, we study the following Kirchhoff-type system:

$$\begin{cases} -(a_1 + b_1 \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \varepsilon f(x), \\ -(a_2 + b_2 \int_{\mathbb{R}^3} |\nabla v|^2 dx)\Delta v = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \varepsilon g(x), \\ (u, v) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \tag{0.1}$$

where $a_1, a_2 \geq 0$, $b_1, b_2 > 0$, $\alpha, \beta > 1$, $\alpha + \beta = 6$ and $f(x), g(x) \geq 0$, $f(x), g(x) \in L^{\frac{6}{5}}(\mathbb{R}^3)$. The aim of this paper is to demonstrate the existence of at least two solutions for system (0.1), utilizing the variational method. To achieve this, we construct an energy functional and analyze its critical points by applying the Ekeland variational principle, the mountain pass lemma and the concentration compactness principle.

Keywords: positive solutions; Kirchhoff-type systems; critical Sobolev exponent; concentration compactness principle; mountain pass lemma

1. Introduction

In this paper, we mainly study the following Kirchhoff-type system

$$\begin{cases} -(a_1 + b_1 \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \varepsilon f(x), \\ -(a_2 + b_2 \int_{\mathbb{R}^3} |\nabla v|^2 dx)\Delta v = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \varepsilon g(x), \\ (u, v) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \tag{1.1}$$

under the condition of $(C.1), (C.2)$ where $(C.1)$: $a_1, a_2 \geq 0$, $b_1, b_2 > 0$, $\alpha, \beta > 1$, $\alpha + \beta = 2^* = 2 \times 3/(3-2) = 6$, $\varepsilon > 0$; $(C.2)$: $f(x), g(x) \geq 0$, $f(x), g(x) \not\equiv 0$, $f(x), g(x) \in L^{\frac{6}{5}}(\mathbb{R}^3)$. $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent where $N = 3$.

It is well known that the critical problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

does not have a minimizing energy solution when Ω is a domain different from $\mathbb{R}^N$. This result is derived from the sharp Sobolev inequality on $\mathbb{R}^N$ [1–3]. However, the situation is different if there is a nonhomogeneous term $f(x)$:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \mu f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Tarantello [4] showed that problem (1.3) has at least two solutions in bounded domains. The main idea is to use the Ekeland variational principle to get one solution u_0 which is a local minimum solution and use the mountain pass theorem to obtain the second solution u_1 with the energy

$$I(u_1) < I(u_0) + \frac{1}{N}S^{\frac{N}{2}}$$

where S is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, namely

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx)^{\frac{N-2}{N}}}.$$

Let U be a solution for the following problem:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u, x \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.4}$$

Then, we know

$$U_{\varepsilon,y}(x) = \frac{(N(N-2))^{\frac{N-2}{4}}\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x-y|^2)^{\frac{N-2}{2}}},$$

which are all positive solutions of (1.4) (see [5, 6]) for any $\varepsilon > 0$ and $y \in \mathbb{R}^N$. Moreover, we know that U satisfies

$$||U||^2 = |U|_{2^*}^{2^*} = S^{\frac{N}{2}} \tag{1.5}$$

where $||U|| = (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{\frac{1}{2}}$ and $|U|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{\frac{1}{s}}$ are the norms of the Sobolev space $D^{1,2}\left(\mathbb{R}^N\right)$ and Lebesgue space $L^s\left(\mathbb{R}^N\right)$, $s \in [2, 2^*]$, respectively. Han [7] extended (1.3) to the following elliptic system

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}u|u|^{\alpha-2}|v|^{\beta} + \varepsilon f(x), & in \quad \Omega, \\ -\Delta v = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}v|v|^{\beta-2} + \varepsilon g(x), & in \quad \Omega, \\ u = v = 0, & on \quad \partial\Omega \end{cases} \tag{1.6}$$

in bounded domains. The author used upper and lower solution methods and variational methods to prove that the problem (1.6) has at least two solutions for both subcritical and critical cases. The first solution $(\tilde{u}_0, \tilde{v}_0)$ of (1.6) is also a local minimizer of the associated functional I and they obtained the second solution $(\tilde{u}_1, \tilde{v}_1)$ with

$$I(\tilde{u}_1, \tilde{v}_1) < I(\tilde{u}_0, \tilde{v}_0) + \frac{2}{N}(\frac{S_{\alpha,\beta}}{2})^{\frac{N}{2}}$$

where

$$S_{\alpha,\beta} = \inf_{(u,v)\in D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)\setminus\{(0,0)\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx}{(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx)^{\frac{2}{\alpha+\beta}}}.$$

Indeed, according to [7] we know that

$$u_0 = kU, \quad v_0 = lU$$

is a solution of the following system

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, \\ -\Delta v = \frac{2\beta}{\alpha+\beta}|u|^\alpha |v|^{\beta-2}v, \\ (u,v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.7}$$

where U is a solution of (1.4),

$$k = [(\frac{2\alpha}{\alpha+\beta})^\beta (\frac{\alpha+\beta}{2\alpha})^{\beta-2}]^{\frac{-1}{2(\alpha+\beta-2)}}, \quad l = [(\frac{2\alpha}{\alpha+\beta})^\alpha (\frac{\alpha+\beta}{2\alpha})^{\alpha-2}]^{\frac{-1}{2(\alpha+\beta-2)}}$$

and

$$S_{\alpha,\beta} = [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}]S.$$

We are certain that the existence of solutions will be affected by the nonhomogeneous term $f(x)$ or $g(x)$ and the existence of a second solution for (1.3) or (1.6) will also be affected by Eq (1.2) or system (1.7).

When $a_1 = a_2$, $b_1 = b_2$, $f = g$ and $u = v$, system (1.1) transforms into the following individual equation.

$$\begin{cases} -(a_1 + b_1 \int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = |u|^{2^*-2}u + \varepsilon f(x), \quad in \quad \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N) \end{cases} \tag{1.8}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - (\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial t}|^2 \, dx)\frac{\partial^2 u}{\partial t^2} = 0$$

presented by Kirchhoff in [8]. Here, the parameters in (1.8) carry the following interpretations: ρ represents the mass density, P_0 the initial tension, h the cross-sectional area, E the Young's modulus of the material and L the length of the string. It was underscored in [9] that the Kirchhoff-type problem models a variety of physical and biological systems where u characterizes a process that relies on its own average (e.g., population density). Early investigations into the Kirchhoff-type problem can be traced back to the work of Bernstein [10] and Pohoẑaev [11]. Nevertheless, Eq (1.8) only attracted significant attention after Lions [12] introduced an abstract framework for such problems.

Liu et al. [5] discovered that problem (1.8) has at least two positive solutions when $N = 3, 4$ under certain assumptions for $f(x)$. Specifically, to consider the existence of a second solution for problem (1.8), they needed to establish the existence of a unique positive solution for (1.8) when $\varepsilon = 0$. In fact, they found that the positive solutions of

$$\begin{cases} -(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = |u|^{2^*-2}u, \quad in \quad \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N) \end{cases}$$

can be expressed as

$$V_{\varepsilon,\lambda,y}(x) = \lambda^{\frac{N-2}{4}} U_{\varepsilon,y}(x)$$

with

$$\lambda = a + bS^{\frac{N}{2}} \lambda^{\frac{N-2}{2}}.$$

The solvability or multiplicity of the Kirchhoff type equation with critical exponent has been extensively studied in recent years; see, for instance, [5, 13–25] and references therein.

Inspired by the ideas presented in [5] and [4], we discuss system (1.1) with $\varepsilon > 0$ small enough. First, we establish the existence of solutions for problem (1.1) when $\varepsilon = 0$:

Theorem 1.1. *Assume that $\varepsilon = 0$ and (u_0, v_0) is a positive solution of (1.7) and*

$$S_1 = \int_{\mathbb{R}^3} |\nabla u_0|^2 dx = \frac{\alpha}{3}(\frac{S_{\alpha,\beta}}{2})^{\frac{3}{2}}, \quad S_2 = \int_{\mathbb{R}^3} |\nabla v_0|^2 dx = \frac{\beta}{3}(\frac{S_{\alpha,\beta}}{2})^{\frac{3}{2}}.$$

Then, we have
(i) If $a_1 = a_2 = 0$, problem (1.1) has a unique positive solution $z = (u', v')$ where

$$u' = (b_1 S_1)^{\frac{\alpha-2}{8}} (b_2 S_2)^{\frac{\beta}{8}} u_0, \quad v' = (b_1 S_1)^{\frac{\alpha}{8}} (b_2 S_2)^{\frac{\beta-2}{8}} v_0.$$

(ii) If a_1 and a_2 are not equal to 0 at the same time, problem (1.1) has a unique positive solution

$$z = (u', v') = (\lambda_1 u_0, \lambda_2 v_0).$$

Indeed, if $a_1 = 0, a_2 \neq 0$,

$$\lambda_1^{\alpha-2}\lambda_2^{\beta} = (\lambda_1^{\alpha}\lambda_2^{\beta-2})^{\frac{\beta}{2+\beta}}(b_1 S_1)^{\frac{4}{\beta+2}}, \quad \lambda_1^{\alpha}\lambda_2^{\beta-2} = a_2 + (b_2 S_2)(b_1 S_1)^{\frac{\alpha}{2+\beta}}(\lambda_1^{\alpha}\lambda_2^{\beta-2})^{\frac{\beta-2}{2+\beta}}.$$

(iii) If $a_1 \neq 0(> 0), a_2 \neq 0(> 0)$, problem (1.1) has at least a positive solution $z = (u', v') = (\lambda_1 u_0, \lambda_2 v_0)$ where

$$\lambda_1 = C_1^{\frac{2-\beta}{8}} C_2^{\frac{\beta}{8}}, \quad \lambda_2 = C_1^{\frac{\alpha}{8}} C_2^{\frac{2-\alpha}{8}}$$

and

$$C_1 = a_1 + C_1^{\frac{2-\beta}{4}} C_2^{\frac{\beta}{4}} b_1 S_1, \quad C_2 = a_2 + C_1^{\frac{\alpha}{4}} C_2^{\frac{2-\alpha}{4}} b_2 S_2.$$

Remark 1.1. *In fact, from the following equations*

$$\lambda_1 = C_1^{\frac{2-\beta}{8}} C_2^{\frac{\beta}{8}}, \quad \lambda_2 = C_1^{\frac{\alpha}{8}} C_2^{\frac{2-\alpha}{8}}$$

and

$$C_1 = a_1 + C_1^{\frac{2-\beta}{4}} C_2^{\frac{\beta}{4}} b_1 S_1, \quad C_2 = a_2 + C_1^{\frac{\alpha}{4}} C_2^{\frac{2-\alpha}{4}} b_2 S_2,$$

we find that the unique or explicit form of (λ_1, λ_2) *is affected by* a_1, a_2, b_1, b_2. *Thus, we can only obtain the uniqueness of* (λ_1, λ_2) *in cases* $(i)-(ii)$ *and the explicit form of* (λ_1, λ_2) *in case* (i). *Furthermore, we believe that if* a_1, a_2, b_1, b_2 *satisfy certain assumptions,* (λ_1, λ_2) *will be unique, and we can also obtain the explicit form of* (λ_1, λ_2).

We define $\varepsilon^* = \frac{g(\sqrt{\frac{9bS^3_{\alpha,\beta}}{20}})\sqrt{S}}{|f|_{\frac{6}{5}}+|g|_{\frac{6}{5}}}$ where $g(\sqrt{\frac{9bS^3_{\alpha,\beta}}{20}}) = \frac{b}{4}(\sqrt{\frac{9bS^3_{\alpha,\beta}}{20}})^3 - \frac{1}{3S^3}(\sqrt{\frac{9bS^3_{\alpha,\beta}}{20}})^5$, $b = \frac{1}{2}\min(b_1, b_2)$. Next, we consider the existence of a local minimum solution for problem (1.1) by applying the Ekeland variational principle.

Theorem 1.2. *Assuming conditions* $(C.1)$ *and* $(C.2)$ *hold, system* (1.1) *has a local minimum solution for any* $\varepsilon \in (0, \varepsilon^*)$.

Remark 1.2. *First, we demonstrate that the minimal value of the set minimization problem can be attained by* (u, v) *and then we prove that* (u, v) *is a solution of* (1.1). *Unlike with single equations, due to the mutual interaction of* (u, v), *it is challenging to obtain* $\|u\| \geq \|u_n\| + o(1), \|v\| \geq \|v_n\| + o(1)$ *where* (u_n, v_n) *represents the minimizing sequence of* c_0. *The definitions of* u_n, v_n, u, v, c_0 *can be found in the proof of Theorem 1.2 in Section 4. By drawing upon the proof method from Theorem 1.3 in [5] and employing meticulous estimates, we can overcome this difficulty.*

Finally, we investigate the existence of a second solution for problem (1.1) by applying the mountain pass lemma and the concentration compactness principle. To obtain the energy estimation of the associated functional Φ_ε for problem (1.1), we will need the explicit form of (λ_1, λ_2). Therefore, when $a_1 = a_2 = 0$ we have:

Theorem 1.3. *Assume* $\alpha = \beta = 3$ *and* $b_1 = b_2$, *there exists* $\varepsilon^{**} \in (0, \varepsilon^*]$ *such that for any* $\varepsilon \in (0, \varepsilon^{**})$, *problem* (1.1) *has another solution. The value of* ε^* *is defined in Theorem 1.2.*

Remark 1.3. *First, we prove that the associated functional* Φ_ε *for problem* (1.1) *satisfies the mountain pass structure, from which we obtain a* (PS) *sequence. Then, we establish the* (PS) *condition by using the concentration compactness principle. From this, we obtain another solution for* (1.1). *Owing to the lack of compactness* $(\alpha + \beta = 2^*)$, *the mutual action of* (u, v) *and the influence of the nonlocal term, there arises a new challenge in employing the concentration compactness principle. Moreover, it is difficult to derive an explicit expression when the values of* a_1 *and* a_2 *are non-zero. The reason for only considering the case where* $\alpha = \beta = 3$ *is that after extensive estimation, it is only when* $\alpha = \beta = 3$ *that* Λ *reaches its minimum value* Λ_{min}. *Therefore, the estimate satisfied by* m *contradicts Lemma 4.4(i). The definitions of* Λ, Λ_{min}, *and* m *as well as the details of their related proofs can be found in Remark 4.2 and the Proof of Theorem 1.3.*

Remark 1.4. *The innovation of this paper lies in overcoming the lack of compactness* $(\alpha + \beta = 2^*)$ *and the mutual interaction of* (u, v) *to demonstrate the existence of at least two solutions for systems*

(1.1). This extends the results from single equations in [5] to a system of equations. We accomplished this by applying the Ekeland variational principle, the mountain pass lemma and the concentration compactness principle as well as through some precise estimates.

The structure of this paper is as follows: Section 2 provides some preliminary background knowledge. Section 3 is dedicated to the proof of Theorem 1.1. Finally, we present the proofs for Theorem 1.2 and Theorem 1.3.

2. Notations and preliminary results

First, we introduce the following notations, which will be useful for proving the upcoming theorems in this section.
• The function space corresponding to problem (1.1) is $E = D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ with the space norm defined as $\|(u,v)\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$. E^* is the dual space of E.
•$(u,v) \in E,\ B_\rho = \{u \in E : \|(u,v)\| < \rho\}$.
•$\overline{B_\rho} = \{(u,v) \in E : \|(u,v)\| \leq \rho\},\ \partial B_\rho = \{(u,v) \in E : \|(u,v)\| = \rho\}$.
• The following elliptic system

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, \\ -\Delta v = \frac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, \\ (u,v) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \end{cases} \tag{2.1}$$

has a positive radial vector solution $z_0 = (u_0, v_0)$ under the condition (C.1) (see [7]).
• $u^+ = \max\{0,u\},\ u^- = \max\{0,-u\}$.

Let us denote the energy functional $\Phi_\varepsilon : E \to \mathbb{R}$ corresponding to (1.1) by

$$\Phi_\varepsilon(u,v) = \frac{1}{2}(a_1\|u\|^2 + a_2\|v\|^2) + \frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4) - \frac{1}{3}\int_{\mathbb{R}^3}(u^+)^\alpha(v^+)^\beta dx - \varepsilon\int_{\mathbb{R}^3}(fu+gv)dx.$$

Obviously, Φ_ε is of C^1 and has the derivative given by

$$\begin{aligned}\langle\Phi_\varepsilon'(u,v),(\varphi,\psi)\rangle =&(a_1 + b_1\|u\|^2)\int_{\mathbb{R}^3}\nabla u\nabla\varphi dx + (a_2 + b_2\|v\|^2)\int_{\mathbb{R}^3}\nabla v\nabla\psi dx \\ &- \frac{\alpha}{3}\int_{\mathbb{R}^3}(v^+)^\beta(u^+)^{\alpha-1}\varphi dx - \frac{\beta}{3}\int_{\mathbb{R}^3}(u^+)^\alpha(v^+)^{\beta-1}\psi dx - \varepsilon\int_{\mathbb{R}^3}(f\varphi + g\psi)dx.\end{aligned}$$

Next, we present the following lemma which can be utilized in the proof of Theorem 1.3.

Lemma 2.1. *Assume $\alpha,\beta > 1, \alpha+\beta = 6$ and define*

$$S = \inf_{u\in D^{1,2}(R^3)\backslash\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2dx}{(\int_{\mathbb{R}^3}|u|^{\alpha+\beta}dx)^{\frac{2}{\alpha+\beta}}},\quad S_{\alpha,\beta} = \inf_{(u,v)\in E\backslash\{(0,0)\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2+|\nabla v|^2dx}{(\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx)^{\frac{2}{\alpha+\beta}}},$$
$$\tilde{S}_{\alpha,\beta} = \inf_{(u,v)\in E\backslash\{(0,0)\}} \frac{\int_{\mathbb{R}^3}a|\nabla u|^2+b|\nabla v|^2dx}{(\int_{\mathbb{R}^3}|u|^\alpha|v|^\beta dx)^{\frac{2}{\alpha+\beta}}}. \tag{2.2}$$

Then,

$$S_{\alpha,\beta} = [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}]S,$$

$$\tilde{S}_{\alpha,\beta} = a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}} [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}]S$$

where a, b is any real number.

Proof. Refer to [26] for the proof of $S_{\alpha,\beta} = [(\frac{\alpha}{\beta})^{\frac{\beta}{6}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{6}}]S$; we will provide the proof of $\tilde{S}_{\alpha,\beta}$ later. Assume that ω_n is a minimizing sequence for S, let $s, t > 0$ to be chosen. Taking $u_n = \frac{s}{\sqrt{a}}\omega_n$, $v_n = \frac{t}{\sqrt{b}}\omega_n$ in (2.2), we have that

$$a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}} \frac{s^2 + t^2}{(s^\alpha t^\beta)^{\frac{2}{\alpha+\beta}}} \frac{\int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx}{(\int_{\mathbb{R}^3} |\omega_n|^{\alpha+\beta} dx)^{\frac{2}{\alpha+\beta}}} \geq \tilde{S}_{\alpha,\beta}. \tag{2.3}$$

Noting that

$$\frac{s^2 + t^2}{(s^\alpha t^\beta)^{\frac{2}{\alpha+\beta}}} = (\frac{s}{t})^{\frac{2\beta}{\alpha+\beta}} + (\frac{t}{s})^{\frac{2\alpha}{\alpha+\beta}}, \tag{2.4}$$

we can proceed to define the function as follows.

$$g(x) = x^{\frac{2\beta}{\alpha+\beta}} + x^{\frac{-2\alpha}{\alpha+\beta}},\ x > 0.$$

When $x = \sqrt{\frac{\alpha}{\beta}}$, there exists the minimum value

$$g(x)_{min} = g(\sqrt{\frac{\alpha}{\beta}}) = (\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{\frac{-\alpha}{\alpha+\beta}}. \tag{2.5}$$

Considering (2.3)–(2.5), we get

$$a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}} [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{\frac{-\alpha}{\alpha+\beta}}]S \geq \tilde{S}_{\alpha,\beta}. \tag{2.6}$$

Then, we need to prove that

$$a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}} [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{\frac{-\alpha}{\alpha+\beta}}]S \leq \tilde{S}_{\alpha,\beta}. \tag{2.7}$$

Let (u_n, v_n) be a minimizing sequence for $\tilde{S}_{\alpha,\beta}$. Define $z_n = s_n v_n$, for some $s_n > 0$ such that

$$\int_{\mathbb{R}^3} |u_n|^{\alpha+\beta} dx = \int_{\mathbb{R}^3} |z_n|^{\alpha+\beta} dx. \tag{2.8}$$

By Young's inequality

$$\int_{\mathbb{R}^3} |u_n|^\alpha |z_n|^\beta dx \leq \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^3} |u_n|^{\alpha+\beta} dx + \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^3} |z_n|^{\alpha+\beta} dx. \tag{2.9}$$

By (2.8), we have

$$(\int_{\mathbb{R}^3} |u_n|^{\alpha}|z_n|^{\beta}dx)^{\frac{2}{\alpha+\beta}} \leq (\int_{\mathbb{R}^3} |u_n|^{\alpha+\beta}dx)^{\frac{2}{\alpha+\beta}} = (\int_{\mathbb{R}^3} |z_n|^{\alpha+\beta}dx)^{\frac{2}{\alpha+\beta}}. \tag{2.10}$$

Using (2.10), we have

$$\frac{\int_{\mathbb{R}^3} a|\nabla u_n|^2 + b|\nabla v_n|^2 dx}{(\int_{\mathbb{R}^3} |u_n|^{\alpha}|v_n|^{\beta}dx)^{\frac{2}{\alpha+\beta}}} = \frac{s_n^{\frac{2\beta}{\alpha+\beta}} \int_{\mathbb{R}^3} a|\nabla u_n|^2 + b|\nabla v_n|^2 dx}{(\int_{\mathbb{R}^3} |u_n|^{\alpha}|z_n|^{\beta}dx)^{\frac{2}{\alpha+\beta}}} \geq$$
$$as_n^{\frac{2\beta}{\alpha+\beta}} \frac{\int_{\mathbb{R}^3} |\nabla u_n|^2 dx}{(\int_{\mathbb{R}^3} |u_n|^{\alpha+\beta}dx)^{\frac{2}{\alpha+\beta}}} + bs_n^{\frac{2\beta}{\alpha+\beta}} s_n^{-2} \frac{\int_{\mathbb{R}^3} |\nabla z_n|^2 dx}{(\int_{\mathbb{R}^3} |z_n|^{\alpha+\beta}dx)^{\frac{2}{\alpha+\beta}}} \geq h(s_n)S,$$

where $h(s_n) = as_n^{\frac{2\beta}{\alpha+\beta}} + bs_n^{\frac{-2\alpha}{\alpha+\beta}}$. Then, we get

$$h(s_n)_{min} = a(\sqrt{\frac{b\alpha}{a\beta}})^{\frac{2\beta}{\alpha+\beta}} + b(\sqrt{\frac{b\alpha}{a\beta}})^{\frac{-2\alpha}{\alpha+\beta}} = a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}} [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{\frac{-\alpha}{\alpha+\beta}}].$$

Therefore, (2.7) is proved. Combining (2.6), we get

$$\tilde{S}_{\alpha,\beta} = a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}} [(\frac{\alpha}{\beta})^{\frac{\beta}{6}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{6}}]S.$$

□

Ekeland's variational principle is a tool used to obtain a local minimum solution. We include it here for the convenience of the readers.

Theorem 2.1. *([27], Theorem 4.1) Let M be a complete metric space with metric d and let $I : M \mapsto (-\infty, +\infty\ ]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\epsilon > 0$ be given and $u \in M$ be such that*

$$I(u) \leq \inf_M I + \epsilon.$$

Then, there exists $v \in M$ such that

$$I(v) \leq I(u), \quad d(u, v) \leq 1.$$

For each $w \in M$, one has

$$I(v) \leq I(w) + \epsilon d(v, w).$$

3. Proof of Theorem 1.1

In this section, we provide the proof of Theorem 1.1. The key idea is to observe that the right-hand side of problem (1.1), namely $a_1 + b_1 \int_{\mathbb{R}^3} |\nabla u|^2 dx$ and $a_2 + b_2 \int_{\mathbb{R}^3} |\nabla v|^2 dx$, can be regarded as two constants. This insight guides us to construct the solution for problem (1.1) by utilizing the solution for problem (2.1) and the method of undetermined coefficients.

Proof of Theorem 1.1. The proof of Theorem 1.1 is inspired by the idea presented in [5]. For any $C_1, C_2 > 0$, let $(u', v') = (\lambda_1 u_0, \lambda_2 v_0)$ where (u', v') is a vector solution of (3.1), $\lambda_1 > 0$, $\lambda_2 > 0$.

$$\begin{cases} -C_1 \Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \\ -C_2 \Delta v = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v. \end{cases} \tag{3.1}$$

By (3.1) and the fact that (u_0, v_0) satisfies (2.1), we can obtain

$$\begin{cases} C_1 = \lambda_1^{\alpha-2}\lambda_2^{\beta}, \\ C_2 = \lambda_1^{\alpha}\lambda_2^{\beta-2}, \end{cases} \tag{3.2}$$

which implies

$$\begin{cases} \lambda_1 = C_1^{\frac{2-\beta}{8}} C_2^{\frac{\beta}{8}}, \\ \lambda_2 = C_1^{\frac{\alpha}{8}} C_2^{\frac{2-\alpha}{8}}. \end{cases} \tag{3.3}$$

Now, we consider the equations

$$C_1 = a_1 + b_1 \int_{\mathbb{R}^3} |\nabla u'|^2 dx, \qquad C_2 = a_2 + b_2 \int_{\mathbb{R}^3} |\nabla v'|^2 dx.$$

Let $\int_{\mathbb{R}^3} |\nabla u_0|^2 dx = S_1$, $\int_{\mathbb{R}^3} |\nabla v_0|^2 dx = S_2$. Thus, C_1, C_2 satisfy

$$\begin{cases} C_1 = a_1 + C_1^{\frac{2-\beta}{4}} C_2^{\frac{\beta}{4}} b_1 S_1, \\ C_2 = a_2 + C_1^{\frac{\alpha}{4}} C_2^{\frac{2-\alpha}{4}} b_2 S_2. \end{cases} \tag{3.4}$$

Next, we consider the existence and uniqueness of the positive solution from (1.1) when $\varepsilon = 0$.
(i) If $a_1 = 0$, $a_2 = 0$, we deduce from (3.4) that

$$\begin{cases} C_1 = (b_1 S_1)^{\frac{2+\alpha}{4}} (b_2 S_2)^{\frac{\beta}{4}}, \\ C_2 = (b_1 S_1)^{\frac{\alpha}{4}} (b_2 S_2)^{\frac{2+\beta}{4}}. \end{cases}$$

Combining with (3.3), we have

$$\begin{cases} \lambda_1 = (b_1 S_1)^{\frac{\alpha-2}{8}} (b_2 S_2)^{\frac{\beta}{8}}, \\ \lambda_2 = (b_1 S_1)^{\frac{\alpha}{8}} (b_2 S_2)^{\frac{\beta-2}{8}}. \end{cases}$$

Hence, we have

$$\begin{cases} u' = (b_1 S_1)^{\frac{\alpha-2}{8}} (b_2 S_2)^{\frac{\beta}{8}} u_0, \\ v' = (b_1 S_1)^{\frac{\alpha}{8}} (b_2 S_2)^{\frac{\beta-2}{8}} v_0. \end{cases} \tag{3.5}$$

(ii) If a_1 and a_2 are not equal to 0 at the same time, we can assume $a_1 = 0$ and $a_2 \neq 0$ which implies that

$$C_1 = C_2^{\frac{\beta}{2+\beta}} (b_1 S_1)^{\frac{4}{\beta+2}}, \qquad C_2 = a_2 + (b_2 S_2)(b_1 S_1)^{\frac{\alpha}{2+\beta}} C_2^{\frac{\beta-2}{2+\beta}}.$$

According to the above, we define
$A = (b_2S_2)(b_1S_1)^{\frac{\alpha}{2+\beta}} > 0, \quad -\frac{1}{3} < k = \frac{\beta-2}{2+\beta} < \frac{3}{7}, B = a_2 > 0$
where $x > B$. We want to determine the number of C_2; we only need to find the solution of

$$f(x) = Ax^k - x + B = 0. \tag{3.6}$$

(1′) When $\beta \in (1, 2)$, $k \in (-\frac{1}{3}, 0)$, we can get

$$f'(x) = Akx^{k-1} - 1 < 0.$$

Considering $f(x) > 0$ as $x \to B$, $f(x) \to -\infty$ as $x \to +\infty$. Thus, there exists a unique $C_2 > B$ such that $f(C_2) = 0$.

(2′) When $\beta = 2,\ k = 0$, from (3.6)

$$\begin{cases} C_1 = (b_2S_2b_1S_1 + a_2)^{\frac{1}{2}}b_1S_1, \\ C_2 = b_2S_2b_1S_1 + a_2. \end{cases}$$

Considering (3.2), (3.3), we have

$$\begin{cases} u' = (b_2S_2b_1S_1 + a_2)^{\frac{1}{4}}u_0, \\ v' = (b_1S_1)^{\frac{1}{2}}v_0. \end{cases}$$

(3′) When $\beta \in (2, 5),\ k \in (0, \frac{3}{7})$, we know $x_0 = (Ak)^{-\frac{1}{k-1}}$ is the only maximum from

$$f'(x) = Akx^{k-1} - 1 = 0.$$

Considering $f(x) > 0$ as $x \to B$, $f(x) \to -\infty$ as $x \to +\infty$. Thus, there exists a unique $C_2 > B$ such that $f(C_2) = 0$. So, we can prove that there exists a unique (u', v') as a result of the only λ_1, λ_2. If $a_1 \neq 0$ and $a_2 = 0$, we can get a unique (u', v') in the same way.

(iii) If $a_1 \neq 0(> 0), a_2 \neq 0(> 0)$, we deduce from (3.4) that

$$\frac{C_1 - a_1}{C_2 - a_2} = \frac{b_1S_1}{b_2S_2} \times \frac{C_2}{C_1}. \tag{3.7}$$

Let $C_1 = \lambda C_2$. By the above equality, we have

$$(b_2S_2C_2)\lambda^2 - (a_1b_2S_2)\lambda - (C_2 - a_2)b_1S_1 = 0.$$

Because $\lambda = \frac{C_1}{C_2} > 0$, we deduce from (3.7) that

$$\lambda = \frac{a_1b_2S_2 + \sqrt{(a_1b_2S_2)^2 + 4(C_2 - a_2)b_1S_1b_2S_2C_2}}{2b_2S_2C_2}.$$

Then, we have

$$C_2 - a_2 = (b_2S_2)^{\frac{4-\alpha}{4}}C_2^{\frac{2-\alpha}{4}}(\frac{a_1b_2S_2 + \sqrt{(a_1b_2S_2)^2 + 4(C_2 - a_2)b_1S_1b_2S_2C_2}}{2})^{\frac{\alpha}{4}}.$$

Let $A^{'} = (\frac{1}{2})^{\frac{\alpha}{4}}(b_2S_2)^{\frac{4-\alpha}{4}}$, $\quad B^{'} = a_1b_2S_2$, $\quad C^{'} = a_2$, $\quad D^{'} = 4b_1b_2S_1S_2$ and

$$h(x) = A^{'}x^{\frac{2-\alpha}{4}}(B^{'} + \sqrt{(B')^2 + D'x(x-C')})^{\frac{\alpha}{4}} - x + C^{'}.$$

As a result of

$$x \to +\infty,\ h(x) \to -\infty; \quad x \to C^{'},\ h(x) > 0.$$

Then, there exists a $C_2 > 0$ such that $h(C_2) = 0$. Because the uniqueness is not clear, we have some difficulties in considering the existence of the second solution of problem (1.1). Hence, the Theorem 1.1 is proved.

4. Proofs of Theorem 1.2 and Theorem 1.3

In this section, we establish the existence of two solutions for (1.1) by using some variational methods. First, we will present the proofs of Theorem 1.2 and Theorem 1.3 utilizing various Lemmas for each proof, respectively. We consider the existence of a local minimum solution for problem (1.1) by applying the Ekeland variational principle.

Lemma 4.1. *Assume that* $(C.1), (C.2)$ *hold. Then, there exists* $\rho > 0$ *such that for any* $\varepsilon \in (0, \varepsilon^*)$, *one has* $\Phi_\varepsilon|_{\partial B\rho} \geq \widetilde{\alpha}$ *for some* $\widetilde{\alpha} > 0$. *For the definition of* ε^*, *please refer to Page 4, line 26.*

Proof. By the Hölder inequalities, $b_1 > 0$, $b_2 > 0$, the complete square formula and Sobolev inequalities, one has

$$\begin{aligned}\Phi_\varepsilon(u,v) =& \frac{1}{2}(a_1\|u\|^2 + a_2\|v\|^2) + \frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4) - \frac{1}{3}\int_{\mathbb{R}^3}(u^+)^\alpha(v^+)^\beta dx \\ &- \varepsilon\int_{\mathbb{R}^3}(fu + gv)dx \\ \geq& \frac{1}{2}(a_1\|u\|^2 + a_2\|v\|^2) + \frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4) - \frac{1}{3} \times \frac{\|(u,v)\|^6}{S^3_{\alpha,\beta}} \\ &- \frac{\varepsilon}{\sqrt{S}}(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})\|(u,v)\| \\ \geq& [\frac{b}{4}\|(u,v)\|^3 - \frac{1}{3}\frac{\|(u,v)\|^5}{S^3_{\alpha,\beta}} - \frac{\varepsilon}{\sqrt{S}}(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})]\|(u,v)\|\end{aligned}$$

where $b = \frac{1}{2}\min(b_1, b_2)$. Let $g(t) = \frac{b}{4}t^3 - \frac{1}{3S^3}t^5$, then $\max_{t\geq 0} g(t) = g(\rho) > 0$ with $\rho = \sqrt{\frac{9bS^3_{\alpha,\beta}}{20}}$. Owing to $\varepsilon^* = \frac{g(\rho)\sqrt{S}}{|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}}}$, for any $\varepsilon \in (0, \varepsilon^*)$, we have

$$\Phi_\varepsilon(u,v) \geq [g(\rho) - \frac{\varepsilon}{\sqrt{S}}(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})]\rho = \tilde{\alpha} > 0(\forall (u,v) \in \partial B_\rho).$$

Hence, we conclude the proof. □

Lemma 4.2. *Suppose that* $(C.1), (C.2)$ *hold. Then, for any* $\varepsilon \in (0, \varepsilon^*)$ *one has* $c_0 = \inf\limits_{u \in \overline{B_\rho}} \Phi_\varepsilon(u, v) \in (-\infty, 0)$ *where* ρ, ε^* *is given by Lemma 4.1.*

Proof. Firstly, we choose a $(u, v) \in E$ such that $\int_{\mathbb{R}^3}(fu + gv)dx > 0$. Then, for any $t > 0$ we have

$$\begin{aligned}\Phi_\varepsilon(tu, tv) =&\frac{1}{2}(a_1\|u\|^2 + a_2\|v\|^2)t^2 + \frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4)t^4 - \frac{1}{3}t^6 \int_{\mathbb{R}^3}(u^+)^\alpha(v^+)^\beta dx\\ &- \varepsilon t \int_{\mathbb{R}^3}(fu + gv)dx.\end{aligned}$$

Hence, there exists a sufficiently small $t > 0$ such that $\|t(u, v)\| \le \rho$ and $\Phi_\varepsilon(tu, tv) < 0$ which leads to

$$c_0 \le \Phi_\varepsilon(tu, tv) < 0.$$

As $\|(u, v)\| \le \rho$, we obtain

$$\begin{aligned}\Phi_\varepsilon(u, v) >& -\frac{1}{3}\int_{\mathbb{R}^3}(u^+)^\alpha(v^+)^\beta dx - \varepsilon \int_{\mathbb{R}^3}(fu + gv)dx\\ \ge& -\frac{1}{3}\frac{\|(u, v)\|^6}{S^3_{\alpha,\beta}} - \frac{\varepsilon}{\sqrt{S}}(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})\|(u, v)\| > -\infty.\end{aligned}$$

Consequently, we can establish that $c_0 \in (-\infty, 0)$, thereby completing the proof.

Proof of Theorem 1.2. By applying Lemma 4.2, we find that $c_0 = \inf\limits_{u \in \overline{B_\rho}} \Phi_\varepsilon(u, v) \in (-\infty, 0)$. Moreover, we know that $\Phi_\varepsilon|_{\partial B\rho} > 0$ from Lemma 4.1. Therefore, we deduce that the minimum cannot be attained on $\partial B\rho$. According to Lemma 4.1, Lemma 4.2 and Theorem 2.1 (Ekeland variational principle), there exists $(u_n, v_n) \in B_\rho$ such that $\Phi_\varepsilon(u_n, v_n) \to c_0$ and $\Phi'_\varepsilon(u_n, v_n) \to 0$ in E^*. The above proof can be referred to [[28], pp. 534-535]. Consequently, there exists $(u, v) \in E$ satisfying:

$$\begin{aligned}&u_n \rightharpoonup u, v_n \rightharpoonup v \qquad in\ \ E,\\ &u_n \to u, v_n \to v, \qquad in\ \ L^s_{loc}(\mathbb{R}^3) \times L^s_{loc}(\mathbb{R}^3), (1 \le s < 2^*)\\ &u_n \to u, v_n \to v, \qquad a.e.\ \ on\ \ \mathbb{R}^3 \times \mathbb{R}^3.\end{aligned}$$

First, we prove that (u, v) is a minimizer for c_0. Noting that $\overline{B_\rho}$ is closed and convex,

$$(u, v) \in \overline{B_\rho}, \quad \Phi_\varepsilon(u, v) \ge c_0.$$

Therefore, what we need to prove next is that $\Phi_\varepsilon(u, v) \le c_0$. The key idea here is that since $\Phi_\varepsilon(u_n, v_n)$ converges to c_0, we need to establish the inequality relationship between $\Phi_\varepsilon(u, v)$ and $\Phi_\varepsilon(u_n, v_n)$.

In order to eliminate $\frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4)$, we have the following estimates. From $\Phi'_\varepsilon(u_n, v_n) \to 0$ in E^*, it holds that

$$\begin{aligned}&\langle \Phi_\varepsilon^{'}(u_n, v_n), (u, v)\rangle\\ &= (a_1 + b_1\|u_n\|^2)\int_{\mathbb{R}^3} \nabla u_n \nabla u dx + (a_2 + b_2\|v_n\|^2)\int_{\mathbb{R}^3} \nabla v_n \nabla v dx\end{aligned}$$

$$- \frac{\alpha}{3}\int_{\mathbb{R}^3}(u_n^+)^{\alpha-1}(v_n^+)^{\beta}udx - \frac{\beta}{3}\int_{\mathbb{R}^3}(u_n^+)^{\alpha}(v_n^+)^{\beta-1}vdx - \varepsilon\int_{\mathbb{R}^3}(fu+gv)dx + o(1)$$
$$= (a_1 + b_1\|u_n\|^2)\|u\|^2 + (a_2 + b_2\|v_n\|^2)\|v\|^2 - 2\int_{\mathbb{R}^3}(u^+)^{\alpha}(v^+)^{\beta}dx - \varepsilon\int_{\mathbb{R}^3}(fu+gv)dx + o(1)$$
$$= o(1).$$

Combining above equality and weakly lower semi-continuity of norm, we have

$$\begin{aligned}
\Phi_\varepsilon(u,v) =& \frac{1}{2}(a_1\|u\|^2 + a_2\|v\|^2) + \frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4) - \frac{1}{3}\int_{\mathbb{R}^3}(u^+)^{\alpha}(v^+)^{\beta}dx \\
&- \varepsilon\int_{\mathbb{R}^3}(fu+gv)dx \\
\leq& \frac{1}{2}(a_1\|u\|^2 + a_2\|v\|^2) + \frac{1}{4}(b_1\|u_n\|^2\|u\|^2 + b_2\|v_n\|^2\|v\|^2) - \frac{1}{3}\int_{\mathbb{R}^3}(u^+)^{\alpha}(v^+)^{\beta}dx \\
&- \varepsilon\int_{\mathbb{R}^3}(fu+gv)dx - \frac{1}{4}(a_1 + b_1\|u_n\|^2)\|u\|^2 - \frac{1}{4}(a_2 + b_2\|v_n\|^2)\|v\|^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^3}(u^+)^{\alpha}(v^+)^{\beta}dx + \frac{\varepsilon}{4}\int_{\mathbb{R}^3}(fu+gv)dx + o(1) \\
\leq& \frac{a_1}{4}\|u\|^2 + \frac{a_2}{4}\|v\|^2 + \frac{1}{6}\int_{\mathbb{R}^3}(u^+)^{\alpha}(v^+)^{\beta}dx - \frac{3\varepsilon}{4}\int_{\mathbb{R}^3}(fu+gv)dx + o(1) \\
\leq& \frac{a_1}{4}\|u_n\|^2 + \frac{a_2}{4}\|v_n\|^2 + \frac{1}{6}\int_{\mathbb{R}^3}(u_n^+)^{\alpha}(v_n^+)^{\beta}dx - \frac{3\varepsilon}{4}\int_{\mathbb{R}^3}(fu_n+gv_n)dx + o(1) \\
\leq& \Phi_\varepsilon(u_n,v_n) - \frac{1}{4}\langle\Phi_\varepsilon'(u_n,v_n),(u_n,v_n)\rangle + o(1) = c_0.
\end{aligned}$$

Hence, we get (u,v) is a minimizer for c_0.

Now, we need to prove (u,v) is a solution of (1.1) with $\Phi_\varepsilon(u,v) = c_0$. On one hand, we have

$$\begin{aligned}
c_0 =& \Phi_\varepsilon(u,v) - \frac{1}{6}\langle\Phi_\varepsilon'(u_n,v_n),(u,v)\rangle + o(1) \\
=& \frac{1}{2}(a_1\|u\|^2 + a_2\|v\|^2) + \frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4) - \frac{1}{3}\int_{\mathbb{R}^3}(u^+)^{\alpha}(v^+)^{\beta}dx \\
&- \varepsilon\int_{\mathbb{R}^3}(fu+gv)dx - \frac{a_1}{6}\int_{\mathbb{R}^3}\nabla u_n\nabla udx - \frac{a_2}{6}\int_{\mathbb{R}^3}\nabla v_n\nabla vdx \\
&- \frac{b_1}{6}\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}\nabla u_n\nabla udx - \frac{b_2}{6}\int_{\mathbb{R}^3}|\nabla v_n|^2dx\int_{\mathbb{R}^3}\nabla v_n\nabla vdx \\
&+ \frac{\alpha}{18}\int_{\mathbb{R}^3}(v_n^+)^{\beta}(u_n^+)^{\alpha-1}udx + \frac{\beta}{18}\int_{\mathbb{R}^3}(u_n^+)^{\alpha}(v_n^+)^{\beta-1}vdx + \frac{\varepsilon}{6}\int_{\mathbb{R}^3}(fu+gv)dx + o(1) \\
=& \frac{1}{3}(a_1\|u\|^2 + a_2\|v\|^2) + \frac{1}{4}(b_1\|u\|^4 + b_2\|v\|^4) - \frac{b_1}{6}\|u_n\|^2\|u\|^2 - \frac{b_2}{6}\|v_n\|^2\|v\|^2 \\
&- \frac{5\varepsilon}{6}\int_{\mathbb{R}^3}(fu+gv)dx + o(1).
\end{aligned}$$

On the other hand, we have

$$c_0 = \Phi_\varepsilon(u_n,v_n) - \frac{1}{6}\langle\Phi_\varepsilon'(u_n,v_n),(u_n,v_n)\rangle + o(1)$$

$$
\begin{aligned}
&=\frac{1}{2}(a_1\|u_n\|^2 + a_2\|v_n\|^2) + \frac{1}{4}(b_1\|u_n\|^4 + b_2\|v_n\|^4) - \frac{1}{3}\int_{\mathbb{R}^3}(u_n^+)^\alpha(v_n^+)^\beta dx \\
&\quad - \varepsilon\int_{\mathbb{R}^3}(fu_n + gv_n)dx - \frac{a_1}{6}\|u_n\|^2 - \frac{a_2}{6}\|v_n\|^2 - \frac{b_1}{6}\|u_n\|^4 - \frac{b_2}{6}\|v_n\|^4 \\
&\quad + \frac{1}{3}\int_{\mathbb{R}^3}(v_n^+)^\beta(u_n^+)^\alpha dx + \frac{\varepsilon}{6}\int_{\mathbb{R}^3}(fu_n + gv_n)dx + o(1) \\
&=\frac{a_1}{3}\|u_n\|^2 + \frac{a_2}{3}\|v_n\|^2 + \frac{b_1}{12}\|u_n\|^4 + \frac{b_2}{12}\|v_n\|^4 - \frac{5\varepsilon}{6}\int_{\mathbb{R}^3}(fu + gv)dx.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\frac{1}{3}a_1\|u\| + \frac{1}{3}a_2\|v\|^2 + \frac{1}{4}b_1\|u\|^4 + \frac{1}{4}b_2\|v\|^4 - \frac{b_1}{6}\|u_n\|^2\|u\|^2 - \frac{b_2}{6}\|v_n\|^2\|v\|^2 + o(1) \\
=&\frac{1}{3}a_1\|u_n\| + \frac{1}{3}a_2\|v_n\|^2 + \frac{1}{12}b_1\|u_n\|^4 + \frac{1}{12}b_2\|v_n\|^4 + o(1) \\
\geq&\frac{1}{3}a_1\|u\| + \frac{1}{3}a_2\|v\|^2 + (\frac{1}{4} - \frac{1}{6})b_1\|u\|^4 + (\frac{1}{4} - \frac{1}{6})b_2\|v\|^4 + o(1) \\
\geq&\frac{1}{3}a_1\|u\| + \frac{1}{3}a_2\|v\|^2 + (\frac{1}{4} - \frac{1}{6})b_1\|u\|^4 + \frac{1}{4}b_2\|v\|^4 - \frac{1}{6}b_2\|v_n\|^2\|v\|^2 + o(1).
\end{aligned}
$$

So, we obtain

$$
\|u\|^2 \geq \|u_n\|^2 + o(1).
$$

Following the above steps, we have

$$
\begin{aligned}
&\frac{1}{3}a_1\|u\| + \frac{1}{3}a_2\|v\|^2 + \frac{1}{4}b_1\|u\|^4 + \frac{1}{4}b_2\|v\|^4 - \frac{b_1}{6}\|u_n\|^2\|u\|^2 - \frac{b_2}{6}\|v_n\|^2\|v\|^2 + o(1) \\
=&\frac{1}{3}a_1\|u_n\| + \frac{1}{3}a_2\|v_n\|^2 + \frac{1}{12}b_1\|u_n\|^4 + \frac{1}{12}b_2\|v_n\|^4 + o(1) \\
\geq&\frac{1}{3}a_1\|u\| + \frac{1}{3}a_2\|v\|^2 + (\frac{1}{4} - \frac{1}{6})b_1\|u\|^4 + (\frac{1}{4} - \frac{1}{6})b_2\|v\|^4 + o(1) \\
\geq&\frac{1}{3}a_1\|u\| + \frac{1}{3}a_2\|v\|^2 + \frac{1}{4}b_2\|u\|^4 - \frac{1}{6}b_2\|u_n\|^2\|u\|^2 + (\frac{1}{4} - \frac{1}{6})b_1\|v\|^4 + o(1).
\end{aligned}
$$

Thus, we obtain

$$
\|v\|^2 \geq \|v_n\|^2 + o(1).
$$

Using again the weakly lower semi-continuity of norm, we get $\|(u,v)\| = \|(u_n,v_n)\| + o(1)$. Combining $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in E, we have $u_n \to u, v_n \to v$ in E and then $\Phi_\varepsilon(u_n,v_n) \to c_0 = \Phi_\varepsilon(u,v), \Phi'_\varepsilon(u_n,v_n) \to \Phi'_\varepsilon(u,v) = 0$ in E^*. We complete the proof.

Now, we give proof of the second positive solution by the mountain pass lemma and the concentration compactness principle.

Lemma 4.3. *Suppose that* $(C.1),(C.2)$ *hold. Then, there exists* $(u^*,v^*) \in E$ *such that* $\|(u^*,v^*)\| > \rho$ *and* $\Phi_\varepsilon(u^*,v^*) < 0$ *where* ρ *is given by Lemma 4.1.*

Proof. Let $u_0 = kU,\ v_0 = lU$, we can obtain $\frac{k}{l} = \sqrt{\frac{\alpha}{\beta}}$ and

$$\begin{cases} k = 3^{\frac{1}{4}}\alpha^{\frac{\beta-2}{8}}\beta^{-\frac{\beta}{8}} = (\frac{\alpha}{\beta})^{\frac{\beta}{8}}(\frac{3}{\alpha})^{\frac{1}{4}}, \\ l = 3^{\frac{1}{4}}\alpha^{-\frac{\alpha}{8}}\beta^{\frac{\alpha-2}{8}} = (\frac{\beta}{\alpha})^{\frac{\alpha}{8}}(\frac{3}{\beta})^{\frac{1}{4}}, \end{cases}$$

where U is a solution of (1.4). Then, for any $t > 0$ it holds that

$$\begin{aligned} \Phi_\varepsilon(tu_0, tv_0) =& \frac{1}{2}(a_1\|tu_0\|^2 + a_2\|tv_0\|^2) + \frac{1}{4}(b_1\|tu_0\|^4 + b_2\|tv_0\|^4) - \frac{t^6}{3}\int_{\mathbb{R}^3}(u_0^+)^\alpha(v_0^+)^\beta \\ &- \varepsilon t\int_{\mathbb{R}^3}(fu_0 + gv_0)dx \\ =& \frac{1}{2}(a_1k^2 + a_2l^2)S^{\frac{3}{2}}t^2 + \frac{1}{4}(b_1k^4 + b_2l^4)S^3t^4 - \frac{1}{3}k^\alpha l^\beta S^{\frac{3}{2}}t^6 - \varepsilon t\int_{\mathbb{R}^3}(fu_0 + gv_0)dx \\ \le& \frac{1}{2}(a_1k^2 + a_2l^2)S^{\frac{3}{2}}t^2 + \frac{1}{4}(b_1k^4 + b_2l^4)S^3t^4 - \frac{1}{3}k^\alpha l^\beta S^{\frac{3}{2}}t^6. \end{aligned}$$

Hence, there exists a sufficiently large $t_0 > 0$ such that

$$\|t_0(u_0, v_0\| > \rho \quad and \quad \Phi_\varepsilon(t_0u_0, t_0v_0) < 0.$$

Let $(u^*, v^*) = (t_0u_0, t_0v_0)$. This completes the proof. □

According to lemma 4.1, lemma 4.3, we can find (u, v) such that

$$\inf_{\partial B_\rho}\Phi_\varepsilon(u, v) \triangleq d > \Phi_\varepsilon(0, 0) = 0,$$
$$(u^*, v^*) \notin \overline{B_\rho}\ \ satisfy\ \ \Phi_\varepsilon(u^*, v^*) < d.$$

Then, we define

$$m \triangleq \inf_{P\in A}\max_{u\in P}\Phi_\varepsilon(u, v) \ge d,$$

where A is the set of all passes which connect 0 and $e = (u^*, v^*), i, e.,$

$$A = \{P \in C([0, 1], X)|P(0) = 0, P(1) = e\}.$$

Remark 4.1. *For any $\varepsilon \in (0, \varepsilon^*)$, we can obtain a nonnegative bounded (PS) sequence.*

Proof. By the mountain pass theorem [6], there exists $(u_n, v_n) \in E$ such that $I(u_n, v_n) \to m$ and $I'(u_n, v_n) \to 0$ in E^*. Thus, we can get

$$\begin{aligned} \frac{5\varepsilon}{6}\int_{\mathbb{R}^3}(fu_n + gv_n)dx + m + o(\|u_n, v_n\|) =& \Phi_\varepsilon(u_n, v_n) - \frac{1}{6}\langle\Phi_\varepsilon'(u_n, v_n), (u_n, v_n)\rangle \\ =& \frac{a_1}{3}\|u_n\|^2 + \frac{a_2}{3}\|v_n\|^2 + \frac{1}{12}\|u_n\|^4 + \frac{1}{12}\|v_n\|^4 \\ \ge& Q_1\|(u_n, v_n)\|^2. \end{aligned} \tag{4.1}$$

where Q_1 is a positive constant with a sufficiently small value.

By the Hölder inequality and the Sobolev inequality, we get

$$\frac{5\varepsilon}{6}\int_{\mathbb{R}^3}(fu_n+gv_n)dx \le \frac{5\varepsilon}{6}|f|_2|u_n|_2+\frac{5\varepsilon}{6}|g|_2|v_n|_2 \le C(\epsilon)\|(u_n,v_n)\| \tag{4.2}$$

where $C(\varepsilon)$ is a a sufficiently small constant that depends only on ε. Combining (4.1) with (4.2), we conclude that (u_n, v_n) is bounded in E.

Since $u = u^+ - u^-, v = v^+ - v^-$, we have

$$\begin{aligned}
o(1) =& -\langle \Phi_\varepsilon'(u_n,v_n),(u_n^-,v_n^-)\rangle \\
=& -a_1\int_{\mathbb{R}^3}\nabla u_n\nabla u_n^- dx - a_2\int_{\mathbb{R}^3}\nabla v_n\nabla v_n^- dx - b_1\|u_n\|^2\int_{\mathbb{R}^3}\nabla u_n\nabla u_n^- dx \\
& - b_2\|v_n\|^2\int_{\mathbb{R}^3}\nabla v_n\nabla v_n^- dx + \varepsilon\int_{\mathbb{R}^3}(fu_n^-+gv_n^-)dx \\
=& a_1\|u_n^-\|^2 + a_2\|v_n^-\|^2 + b_1\|u_n\|^2\|u_n^-\|^2 + b_2\|v_n\|^2\|v_n^-\|^2 + \varepsilon\int_{\mathbb{R}^3}(fu_n^-+gv_n^-)dx \\
\ge& b_1\|u_n\|^2\|u_n^-\|^2 + b_2\|v_n\|^2\|v_n^-\|^2 \\
\ge& b_1\|u_n^-\|^4 + b_2\|v_n^-\|^4
\end{aligned}$$

which implies $\|u_n^-\| = 0,\ \|v_n^-\| = 0, n \to \infty$. According to Hölder and Sobolev inequality, we have

$$0 \le \varepsilon\int_{\mathbb{R}^3}(fu_n^-+gv_n^-)dx \le C_\varepsilon(|f|_{\frac{6}{5}}\|u_n^-\| + |g|_{\frac{6}{5}}\|v_n^-\|).$$

Therefore,

$$\varepsilon\int_{\mathbb{R}^3}(fu_n^-+gv_n^-)dx = 0, n\to\infty.$$

Next, we need to verify that

$$\Phi_\varepsilon(u_n^+,v_n^+)\to m, \qquad \langle\Phi_\varepsilon'(u_n^+,v_n^+),(\varphi,\psi)\rangle\to 0.$$

Given that $\|u_n^-\| = 0,\ \|v_n^-\| = 0, n\to\infty$ and $\varepsilon\int_{\mathbb{R}^3}(fu_n^-+gv_n^-)dx = 0, n\to\infty$, we have

$$\begin{aligned}
\Phi_\varepsilon(u_n,v_n) =& \frac{1}{2}(a_1\|u_n\|^2+a_2\|v_n\|^2) + \frac{1}{4}(b_1\|u_n\|^4+b_2\|v_n\|^4) - \frac{1}{3}\int_{\mathbb{R}^3}(u^+)^\alpha(v^+)^\beta dx \\
& - \varepsilon\int_{\mathbb{R}^3}(fu_n+gv_n)dx \\
=& \frac{1}{2}[a_1(\|u_n^+\|^2+\|u_n^-\|^2) + a_2(\|v_n^+\|^2+\|v_n^-\|^2)] - \frac{1}{3}\int_{\mathbb{R}^3}(u_n^+)^\alpha(v_n^+)^\beta dx \\
& + \frac{1}{4}[b_1(\|u_n^+\|^4+\|v_n^+\|^4) + b_2(\|v_n^+\|^4+\|v_n^-\|^4)] - \varepsilon\int_{\mathbb{R}^3}g(v_n^+-v_n^-)dx \\
& - \varepsilon\int_{\mathbb{R}^3}f(u_n^+-u_n^-)dx
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}(a_1(\|u_n^+\|^2+a_2(\|v_n^+\|^2)+\frac{1}{4}(b_1(\|u_n^+\|^4+b_2(\|v_n^+\|^4)\\
&-\frac{1}{3}\int_{\mathbb{R}^3}(u_n^+)^\alpha(v_n^+)^\beta dx-\varepsilon\int_{\mathbb{R}^3}(fu_n^++gv_n^+)dx+o(1)\\
=&\Phi_\varepsilon(u_n^+,v_n^+)+o(1).
\end{aligned}
$$

Given that $\|u_n^-\|=0,\ \|v_n^-\|=0, n\to\infty$ and $\varepsilon\int_{\mathbb{R}^3}(fu_n^-+gv_n^-)dx=0, n\to\infty$, we have

$$
\begin{aligned}
\langle\Phi_\varepsilon'(u_n,v_n),(\varphi,\psi)\rangle=&a_1\int_{\mathbb{R}^3}\nabla u_n\nabla\varphi dx+a_2\int_{\mathbb{R}^3}\nabla v_n\nabla\psi dx+b_1\|u_n\|^2\int_{\mathbb{R}^3}\nabla u_n\nabla\varphi dx\\
&+b_2\|v_n\|^2\int_{\mathbb{R}^3}\nabla v_n\nabla\psi dx-\frac{\alpha}{3}\int_{\mathbb{R}^3}(v_n^+)^\beta(u_n^+)^{\alpha-1}\varphi dx\\
&-\frac{\beta}{3}\int_{\mathbb{R}^3}(u_n^+)^\alpha(v_n^+)^{\beta-1}\psi dx-\varepsilon\int_{\mathbb{R}^3}(f\varphi+g\psi)dx\\
=&a_1\int_{\mathbb{R}^3}\nabla(u_n^+-u_n^-)\nabla\varphi dx+a_2\int_{\mathbb{R}^3}\nabla(v_n^+-v_n^-)\nabla\psi dx\\
&+b_1(\|u_n^+\|^2+\|u_n^-\|^2)\int_{\mathbb{R}^3}\nabla(u_n^+-u_n^-)\nabla\varphi dx\\
&+b_2(\|v_n^+\|^2+\|v_n^-\|^2)\int_{\mathbb{R}^3}(\nabla v_n^+-v_n^-)\nabla\psi dx\\
&-\frac{\alpha}{3}\int_{\mathbb{R}^3}(v_n^+)^\beta(u_n^+)^{\alpha-1}\varphi dx-\frac{\beta}{3}\int_{\mathbb{R}^3}(u_n^+)^\alpha(v_n^+)^{\beta-1}\psi dx-\varepsilon\int_{\mathbb{R}^3}(f\varphi+g\psi)dx\\
=&a_1\int_{\mathbb{R}^3}\nabla u_n^+\nabla\varphi dx+a_2\int_{\mathbb{R}^3}\nabla v_n^+\nabla\psi dx+b_1\|u_n^+\|^2\int_{\mathbb{R}^3}\nabla u_n^+\nabla\varphi dx\\
&+b_2\|v_n^+\|^2\int_{\mathbb{R}^3}\nabla v_n^+\nabla\psi dx-\frac{\alpha}{3}\int_{\mathbb{R}^3}(v_n^+)^\beta(u_n^+)^{\alpha-1}\varphi dx\\
&-\frac{\beta}{3}\int_{\mathbb{R}^3}(u_n^+)^\alpha(v_n^+)^{\beta-1}\psi dx-\varepsilon\int_{\mathbb{R}^3}(f\varphi+g\psi)dx+o(1)\\
=&\langle\Phi_\varepsilon'(u_n^+,v_n^+),(\varphi,\psi)\rangle+o(1).
\end{aligned}
$$

Then, we can obtain a nonnegative bounded sequence for Φ_ε. We complete the proof.

Lemma 4.4. *Suppose that $(C.1),(C.2)$ hold. Then, there exists $\varepsilon^{**}\in(0,\varepsilon^*)$ such that for any $\varepsilon\in(0,\varepsilon^{**})$ where Λ is the Maximum value for p(t), the following statements hold:*

$$
\begin{aligned}
&(i)\quad a_1=a_2=0,\quad m\le\sup_{t\ge0}\Phi_\varepsilon(tu',tv')<\Lambda-\varepsilon^2(|f|_{\frac{6}{5}}+|g|_{\frac{6}{5}})S^{\frac{1}{4}};\\
&(ii)\quad a_1=0,a_2\ne0,\quad m\le\sup_{t\ge0}\Phi_\varepsilon(tu',tv')<\Lambda-\frac{9\varepsilon^2|g|^2_{\frac{6}{5}}}{16a_2S}-\varepsilon^2|f|_{\frac{6}{5}}S^{\frac{1}{4}};\\
&(iii)\quad a_1\ne0,a_2\ne0,\quad m\le\sup_{t\ge0}\Phi_\varepsilon(tu',tv')<\Lambda-\frac{9\varepsilon^2|f|^2_{\frac{6}{5}}}{16a_1S}-\frac{9\varepsilon^2|g|^2_{\frac{6}{5}}}{16a_2S}.
\end{aligned}
$$

Proof. Let

$$
h(t)=\Phi_\varepsilon(tu',tv')=\frac{1}{2}(a_1\|u'\|^2+a_2\|v'\|^2)t^2+\frac{1}{4}(b_1\|u'\|^4+b_2\|v'\|^4)t^4-\frac{1}{3}t^6\int_{\mathbb{R}^3}(u')^\alpha(v')^\beta dx
$$

$$- \varepsilon t \int_{\mathbb{R}^3} (fu' + gv')dx,$$

and

$$p(t) = \frac{1}{2}(a_1\|u'\|^2 + a_2\|v'\|^2)t^2 + \frac{1}{4}(b_1\|u'\|^4 + b_2\|v'\|^4)t^4 - \frac{1}{3}t^6 \int_{\mathbb{R}^3} (u')^\alpha (v')^\beta dx.$$

Then, there exists $t_1 > 0$ such that $p'(t_1) = 0$. In this case, we have

$$t_1^2 = \frac{\frac{1}{4}(b_1\|u'\|^4 + b_2\|v'\|^4) + \sqrt{\frac{1}{16}(b_1\|u'\|^4 + b_2\|v'\|^4)^2 + \frac{1}{2}(a_1\|u'\|^2 + a_2\|v'\|^2)\int_{\mathbb{R}^3}(u')^\alpha (v')^\beta dx}}{\int_{\mathbb{R}^3}(u')^\alpha (v')^\beta dx}. \tag{4.3}$$

On the other hand, we know that (u', v') satisfies

$$a_1\|u'\|^2 + a_2\|v'\|^2 + b_1\|u'\|^4 + b_2\|v'\|^4 = 2\int_{\mathbb{R}^3} (u')^\alpha (v')^\beta dx. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain $t_1 = 1$ and

$$\Lambda = \max_{t>0} p(t) = p(t_1) = \frac{1}{3}(a_1\|u'\|^2 + a_2\|v'\|^2) + \frac{1}{12}(b_1\|u'\|^4 + b_2\|v'\|^4).$$

Let $\varepsilon_1 \in (0, \varepsilon^*]$. Then, for $t_2 \in (0, t_1)$ and $\varepsilon \in (0, \varepsilon_1)$, we have:
(i) when $a_1 = a_2 = 0$,

$$\begin{aligned}\max_{0\le t\le t_2} h(t) &\le \max_{0\le t\le t_2}(\frac{1}{2}(a_1\|u'\|^2 + a_2\|v'\|^2)t^2 + \frac{1}{4}(b_1\|u'\|^4 + b_2\|v'\|^4)t^4 \\ &< \Lambda - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}}.\end{aligned}$$

(ii) when $a_1 = 0, a_2 \neq 0$,

$$\begin{aligned}\max_{0\le t\le t_2} h(t) &\le \max_{0\le t\le t_2}(\frac{1}{2}(a_1\|u'\|^2 + a_2\|v'\|^2)t^2 + \frac{1}{4}(b_1\|u'\|^4 + b_2\|v'\|^4)t^4 \\ &< \Lambda - \frac{9\varepsilon^2|g|^2_{\frac{6}{5}}}{16a_2S} - \varepsilon^2|f|_{\frac{6}{5}}S^{\frac{1}{4}}.\end{aligned}$$

(iii) when $a_1 \neq 0, a_2 \neq 0$,

$$\begin{aligned}\max_{0\le t\le t_2} h(t) &\le \max_{0\le t\le t_2}(\frac{1}{2}(a_1\|u'\|^2 + a_2\|v'\|^2)t^2 + \frac{1}{4}(b_1\|u'\|^4 + b_2\|v'\|^4)t^4 \\ &< \Lambda - \frac{9\varepsilon^2|f|^2_{\frac{6}{5}}}{16a_1S} - \frac{9\varepsilon^2|g|^2_{\frac{6}{5}}}{16a_2S}.\end{aligned}$$

Choosing $\varepsilon^{**} \in (0, \varepsilon_1]$ for any $\varepsilon \in (0, \varepsilon^{**})$, we can deduce that for all $t \ge t_2$,

$$\max_{t\ge t_2} h(t) \le \max_{t\ge t_2} p(t) - \varepsilon t_2 \int_{\mathbb{R}^3}(fu' + gv')dx = \Lambda - \varepsilon t_2 \int_{\mathbb{R}^3}(fu' + gv')dx.$$

Furthermore, we obtain the following inequalities:
(i) when $a_1 = a_2 = 0$,

$$\max_{t\geq t_2} h(t) < \Lambda - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}}.$$

(ii) when $a_1 = 0, a_2 \neq 0$,

$$\max_{t\geq t_2} h(t) < \Lambda - \frac{9\varepsilon^2|g|^2_{\frac{6}{5}}}{16a_2S} - \varepsilon^2|f|_{\frac{6}{5}}S^{\frac{1}{4}}.$$

(iii) when $a_1 \neq 0, a_2 \neq 0$,

$$\max_{t\geq t_2} h(t) < \Lambda - \frac{9\varepsilon^2|f|^2_{\frac{6}{5}}}{16a_1S} - \frac{9\varepsilon^2|g|^2_{\frac{6}{5}}}{16a_2S}.$$

Therefore, we complete the proof.

Proof of Theorem 1.3. According to Remark 4.1, we can get that $\{(u_n, v_n)\}$ is bounded and nonnegative. Up to a subsequence, there exists $(u, v) \subset E$ such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in E, $u_n \to u$, $v_n \to v$ in $L^s_{loc}(\mathbb{R}^3) \times L^s_{loc}(\mathbb{R}^3)(1 \leq s < 2^*)$ and $u_n \to u, v_n \to v$ *a.e in* $\mathbb{R}^3$. By applying the concentration compactness principle (see Proposition 2.2 in [29]), we can find non-negative measures μ and ν on $\mathbb{R}^3$, a vector function (u, v) and an at most countable set Γ such that as $n \to \infty$,

$$|\nabla u_n|^2 + |\nabla v_n|^2 \rightharpoonup \mu, \quad |u_n|^\alpha |v_n|^\beta \rightharpoonup \nu \tag{4.5}$$

in the sense of measure and

$$\begin{aligned} &(i) \quad \nu = |u|^\alpha |v|^\beta + \sum_{i\in\Gamma} \nu_i \delta_{x_i}, \quad \mu \geq (|\nabla u|^2 + |\nabla v|^2) + \sum_{i\in\Gamma} \mu_i \delta_{x_i}; \\ &(ii) \quad \mu_i \geq S_{\alpha,\beta}(\nu_i)^{\frac{2}{\alpha+\beta}}, \quad i \in \Gamma. \end{aligned} \tag{4.6}$$

Here, δ_{x_i} is the Dirac delta measure concentrated x_i. We claim that $\Gamma = \emptyset$. Suppose by contradiction that $\Gamma \neq \emptyset$. To obtain a contradiction, we estimate $m = \lim_{n\to\infty} \Phi_\varepsilon(u_n, v_n)$ by utilizing the assumption $\Gamma \neq \emptyset$ and the concentration compactness principle. By comparing this estimation of m with the one provided in Lemma 4.1, we deduce a contradiction. To do this, we first present the following relevant estimates. Fix $k \in \Gamma$. For $\rho > 0$, assume that $\varphi^k_\rho \in C^\infty_0\left(\mathbb{R}^3\right)$ satisfies $\varphi^k_\rho \in [0, 1]$,

$$\varphi^k_\rho(x) = 1, \ \ for \ \ |x - a_k| \leq \frac{\rho}{2}; \ \ \varphi^k_\rho(x) = 0, \ \ for \ \ |x - a_k| \geq \rho$$

and $|\nabla\varphi^k_\rho| \leq \frac{\varrho}{2}$. It follows from $\langle(\Phi'_\rho(u_n, v_n), (\varphi^k_\rho u_n, 0)\rangle \to 0$ that

$$\begin{aligned} &(a_1 + b_1\|u_n\|^2)(\textstyle\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\varphi^k_\rho dx + \int_{\mathbb{R}^3} |\nabla u_n|^2\varphi^k_\rho dx) \\ = &\tfrac{\alpha}{3}\textstyle\int_{\mathbb{R}^3} u^\alpha_n v^\beta_n \varphi^k_\rho dx + \varepsilon\int_{\mathbb{R}^3} f\varphi^k_\rho u_n dx + o(1). \end{aligned} \tag{4.7}$$

In the same way, it follows from $\langle(\Phi'_\rho(u_n, v_n), (0, \varphi^k_\rho v_n)\rangle \to 0$ that

$$\begin{aligned} &(a_2 + b_2\|v_n\|^2)(\textstyle\int_{\mathbb{R}^3} v_n\nabla v_n\nabla\varphi^k_\rho dx + \int_{\mathbb{R}^3} |\nabla v_n|^2\varphi^k_\rho dx) \\ = &\tfrac{\beta}{3}\textstyle\int_{\mathbb{R}^3} u^\alpha_n v^\beta_n \varphi^k_\rho dx + \varepsilon\int_{\mathbb{R}^3} g\varphi_\rho v^k_n dx + o(1). \end{aligned} \tag{4.8}$$

First, we need to solve the lack of compactness problem from the critical Sobolev exponent which causes the invariance of dilation. Combining (4.7), (4.8) and Hölder inequality, we have

$$\begin{aligned} A_1 &= \lim_{\rho\to 0}\limsup_{n\to\infty}\ (a_1+b_1\|u_n\|^2)|\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\varphi_\rho^k dx| \\ &\le \lim_{\rho\to 0}\limsup_{n\to\infty}\ C(\int_{B_{\rho(a_k)}}|\nabla u_n|^2dx)^{\frac{1}{2}}(\int_{B_{\rho(a_k)}}|\nabla\varphi_\rho^k|^2|u_n|^2dx)^{\frac{1}{2}} \\ &\le \lim_{\rho\to 0}\ C(\int_{B_{\rho(a_k)}}|u|^6dx)^{\frac{1}{6}} \\ &= 0 \end{aligned} \tag{4.9}$$

and

$$A_2 = \lim_{\rho\to 0}\limsup_{n\to\infty}\ (a_2+b_2\|v_n\|^2)|\int_{\mathbb{R}^3} v_n\nabla v_n\nabla\varphi_\rho^k dx| = 0 \tag{4.10}$$

where $B_\rho(a_k)=\{x\in R^3 : |x-a_k|<\rho\}$. By (4.5) and (4.6), we have

$$\begin{aligned} &\lim_{\rho\to 0}\limsup_{n\to\infty}(a_1+b_1\|u_n\|^2)\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho^k dx+(a_2+b_2\|v_n\|^2)\int_{\mathbb{R}^3}|\nabla v_n|^2\varphi_\rho^k dx \\ \ge& \lim_{\rho\to 0}\limsup_{n\to\infty}(a_1\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho^k dx+a_2\int_{\mathbb{R}^3}|\nabla v_n|^2\varphi_\rho^k dx)+[b_1(\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho^k dx)^2 \\ &+b_2(\int_{\mathbb{R}^3}|\nabla v_n|^2\varphi_\rho^k dx)^2] \\ \ge& \lim_{\rho\to 0}\limsup_{n\to\infty}(a_1\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho^k dx+a_2\int_{\mathbb{R}^3}|\nabla v_n|^2\varphi_\rho^k dx) \\ &+\tfrac{1}{2}(\sqrt{b_1}\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho^k dx+\sqrt{b_2}\int_{\mathbb{R}^3}|\nabla v_n|^2\varphi_\rho^k dx)^2 \\ \ge& \min(a_1,a_2)S_{\alpha,\beta}\nu_i^{\frac{1}{3}}+\tfrac{1}{2}\min(b_1,b_2)S_{\alpha,\beta}^2\nu_i^{\frac{2}{3}}, \end{aligned} \tag{4.11}$$

$$\lim_{\rho\to 0}\limsup_{n\to\infty}(\frac{\alpha}{3}+\frac{\beta}{3})\int_{\mathbb{R}^3}u_n^\alpha v_n^\beta\varphi_\rho^k dx = 2\lim_{\rho\to 0}\int_{\mathbb{R}^3}u^\alpha v^\beta\varphi_\rho^k+2\nu_i = 2\nu_i, \tag{4.12}$$

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}(f\varphi_\rho^k u_n+g\varphi_\rho^k v_n)dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3}(f\varphi_\rho^k u+g\varphi_\rho^k v)dx = 0. \tag{4.13}$$

We can deduce from (4.7)–(4.13) that

$$\nu_i \ge \frac{1}{2}\min(a_1,a_2)S_{\alpha,\beta}\nu_i^{\frac{1}{3}}+\frac{1}{4}\min(b_1,b_2)S_{\alpha,\beta}^2\nu_i^{\frac{2}{3}}.$$

So, we have

$$\begin{aligned} \nu_i &\ge \Big(\frac{\min(b_1,b_2)S_{\alpha,\beta}^2+\sqrt{[\min(b_1,b_2)]^2S_{\alpha,\beta}^4+32\min(a_1,a_2)S_{\alpha,\beta}}}{8}\Big)^3, \\ \mu_i &\ge \min(a_1,a_2)\frac{\min(b_1,b_2)S_{\alpha,\beta}^3+\sqrt{[\min(b_1,b_2)]^2S_{\alpha,\beta}^6+32\min(a_1,a_2)S_{\alpha,\beta}^3}}{8}. \end{aligned}$$

For $R>0$, assume that $\varphi_R\in C_0^\infty(\mathbb{R}^3)$ satisfies $\varphi_R\in[0,1]$,

$$\varphi_R(x)=1,\ \ for\ \ |x|<R, \qquad \varphi_R(x)=0,\ \ for\ \ |x|>2R,$$

and $|\nabla\varphi_R|<\frac{2}{R}$. By applying the concentration compactness principle, we obtain

$$m = \lim_{n\to\infty}\Phi_\varepsilon(u_n,v_n)-\frac{1}{4}\langle\Phi_\varepsilon'(u_n,v_n),(u_n,v_n)\rangle$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{4}(a_1\|u_n\|^2 + a_2\|v_n\|^2) + \frac{1}{6}\int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx - \frac{3\varepsilon}{4}\int_{\mathbb{R}^3}(fu_n + gv_n)dx \\
&\geq \lim_{R\to\infty}\lim_{n\to\infty} \frac{1}{4}(a_1\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_R dx + a_2\int_{\mathbb{R}^3}|\nabla v_n|^2\varphi_R dx) + \frac{1}{6}\int_{\mathbb{R}^3} u_n^\alpha v_n^\beta \varphi_R dx \\
&\quad - \frac{3\varepsilon}{4}\int_{\mathbb{R}^3}(fu + gv)dx \\
&\geq \frac{a_1}{4}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{a_2}{4}\int_{\mathbb{R}^3}|\nabla v|^2 dx + \frac{1}{4}\mu_i + \frac{1}{6}\nu_i - \frac{3\varepsilon}{4}\int_{\mathbb{R}^3}(fu + gv)dx.
\end{aligned}
$$

Hence, we can infer that

$$
m \geq \frac{a_1}{4}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{a_2}{4}\int_{\mathbb{R}^3}|\nabla v|^2 dx + \frac{1}{4}\mu_i + \frac{1}{6}\nu_i - \frac{3\varepsilon}{4}\int_{\mathbb{R}^3}(fu + gv)dx. \tag{4.14}
$$

(i) If $a_1 = a_2 = 0$, by (4.14), we need to demonstrate that

$$
m \geq \frac{1}{4}\mu_i + \frac{1}{6}\nu_i - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}} \geq \Lambda - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}}. \tag{4.15}
$$

By Lemma 2.1 and the fact that (u_0, v_0) satisfies (2.1), we can obtain:

$$
\int_{\mathbb{R}^3} |u_0|^\alpha |v_0|^\beta dx = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 + |\nabla v_0|^2 dx = (\frac{S_{\alpha,\beta}}{2})^{\frac{3}{2}} \tag{4.16}
$$

and

$$
S_1 = \int_{\mathbb{R}^3} |\nabla u_0|^2 dx = \frac{\alpha}{3}(\frac{S_{\alpha,\beta}}{2})^{\frac{3}{2}}, \ \ S_2 = \int_{\mathbb{R}^3} |\nabla v_0|^2 dx = \frac{\beta}{3}(\frac{S_{\alpha,\beta}}{2})^{\frac{3}{2}}. \tag{4.17}
$$

Combing with (4.16), (4.17) and (3.5), (u', v') satisfies

$$
\begin{gathered}
\int_{\mathbb{R}^3} |\nabla u'|^2 dx = \frac{1}{72} b_1^{\frac{\alpha-2}{4}} b_2^{\frac{\beta}{4}} \alpha^{\frac{\alpha+2}{4}} \beta^{\frac{\beta}{4}} S^3_{\alpha,\beta}, \qquad \int_{\mathbb{R}^3} |\nabla v'|^2 dx = \frac{1}{72} b_1^{\frac{\alpha}{4}} b_2^{\frac{\beta-2}{4}} \alpha^{\frac{\alpha}{4}} \beta^{\frac{\beta+2}{4}} S^3_{\alpha,\beta}, \\
\int_{\mathbb{R}^3} (u')^\alpha (v')^\beta dx = \frac{1}{1728} b_1^{\frac{\alpha}{2}} b_2^{\frac{\beta}{2}} \alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} S^6_{\alpha,\beta}.
\end{gathered} \tag{4.18}
$$

Consequently, we have

$$
\begin{aligned}
p(t) \quad &= \frac{1}{20736} b_1^{\frac{\alpha}{2}} b_2^{\frac{\beta}{2}} \alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} S^6_{\alpha,\beta}(\alpha+\beta)t^4 - \frac{1}{5184} b_1^{\frac{\alpha}{2}} b_2^{\frac{\beta}{2}} \alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} S^6_{\alpha,\beta} t^6 \\
&= \frac{1}{5184} b_1^{\frac{\alpha}{2}} b_2^{\frac{\beta}{2}} \alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} S^6_{\alpha,\beta}(\frac{3}{2}t^4 - t^6).
\end{aligned}
$$

Based on $p'(t) = 0$, we can determine that $t = 1$. Therefore, there exists

$$
\Lambda = \max_{t>0} p(t) = \frac{1}{10368} b_1^{\frac{\alpha}{2}} b_2^{\frac{\beta}{2}} \alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} S^6_{\alpha,\beta}. \tag{4.19}
$$

On one hand, considering

$$
f(\alpha) = \alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} = \alpha^{\frac{\alpha}{2}} (6-\alpha)^{\frac{6-\alpha}{2}}
$$

we have

$$f(\alpha)_{min} = f(3).$$

Hence,

$$\Lambda_{min} = \frac{1}{384} b_1^{\frac{\alpha}{2}} b_2^{\frac{\beta}{2}} S_{\alpha,\beta}^{6}.$$

On the other hand, we can derive

$$\frac{\nu_i}{6} \geq \frac{1}{6} \times (\frac{\min(b_1, b_2) S_{\alpha,\beta}^{2}}{4})^3 = \frac{1}{384}[\min(b_1, b_2)]^3 S_{\alpha,\beta}^{6}.$$

Therefore, it is only when $\alpha = \beta = 3$ and $b_1 = b_2$ that

$$m \geq \frac{1}{6}\nu_i - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}} \geq \Lambda_{min} - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}} \tag{4.20}$$

which contradicts Lemma 4.4 (i).

Remark 4.2. *The reason for only considering the case where $\alpha = \beta = 3$ is that after extensive estimation, it is only when $\alpha = \beta = 3$ that Λ reaches its minimum value Λ_{min}. Therefore, the estimate satisfied by m contradicts Lemma 4.4 (i). For the cases in Lemma 4.4 (ii) and (iii), obtaining the result is challenging due to the mutual interaction of (u, v) adding complexity to our computations. This will be our main task in the following work.*

Moving forward, we will only consider the case where $a_1 = a_2 = 0$, $\alpha = \beta = 3$ and $b_1 = b_2$. We need to solve the lack of compactness problem from the region $\mathbb{R}^3$ which causes the invariance of translation.

For $R > 0$, define

$$\begin{aligned} \mu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^2 + |\nabla v_n|^2 dx, \\ \nu_\infty &= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} u_n^\alpha v_n^\beta dx. \end{aligned} \tag{4.21}$$

By concentration compactness principle, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 dx &= \int_{\mathbb{R}^3} d\mu + \mu_\infty, \\ \limsup_{n\to\infty} \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx &= \int_{\mathbb{R}^3} d\nu + \nu_\infty, \end{aligned} \tag{4.22}$$

and $S_{\alpha,\beta}\mu_\infty^{\frac{1}{3}} \leq \nu_\infty$. Next, we estimate ν_∞ and μ_∞. Assume that $\chi_R \in C_0^\infty(\mathbb{R}^3)$ satisfy $\chi_R \in [0, 1]$, we have

$$\chi_R(x) = 0, \ \ for \ \ |x| < \frac{R}{2}, \ \ \chi_R(x) = 1, \ \ for \ \ |x| > R$$

where $|\nabla\chi_R| < \frac{3}{R}$. It follows from $\langle(\Phi_\varepsilon'(u_n, v_n), (\chi_R u_n, 0)\rangle \to 0$ that

$$\begin{aligned} &(a_1 + b_1\|u_n\|^2)(\int_{\mathbb{R}^3} u_n \nabla u_n \nabla\chi_R dx + \int_{\mathbb{R}^3} |\nabla u_n|^2 \chi_R dx) \\ &= \tfrac{\alpha}{3} \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta \chi_R dx + \varepsilon \int_{\mathbb{R}^3} f\chi_R u_n dx. \end{aligned} \tag{4.23}$$

In this way, we can also have from $\langle(\Phi_\varepsilon'(u_n,v_n),(0,\chi_R v_n)\rangle \to 0$ that

$$\begin{aligned}&(a_2+b_2\|v_n\|^2)(\textstyle\int_{\mathbb{R}^3} v_n\nabla v_n\nabla\chi_R dx+\int_{\mathbb{R}^3}|\nabla v_n|^2\chi_R dx)\\ =&\tfrac{\beta}{3}\textstyle\int_{\mathbb{R}^3} u_n^\alpha v_n^\beta\chi_R dx+\varepsilon\int_{\mathbb{R}^3} g\chi_R v_n dx.\end{aligned}\tag{4.24}$$

By Hölder inequality, we have

$$\begin{aligned}B_1&=\lim_{R\to\infty}\limsup_{n\to\infty}(a_1+b_1\|u_n\|^2)|\textstyle\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\chi_R dx|\\ &\le \lim_{R\to\infty}\limsup_{n\to\infty} C(\textstyle\int_{\frac{R}{2}\le|x|\le R}|\nabla u_n|^2dx)^{\frac12}(\int_{\frac{R}{2}\le|x|\le R}|\nabla\chi_R|^2|u_n|^2dx)^{\frac12}\\ &\le \lim_{R\to\infty} C(\textstyle\int_{\frac{R}{2}\le|x|\le R}|\nabla\chi_R|^3dx)^{\frac13}(\int_{\frac{R}{2}\le|x|\le R}|u|^6dx)^{\frac16}\\ &\le \lim_{R\to\infty} C(\textstyle\int_{\frac{R}{2}\le|x|\le R}|u|^6dx)^{\frac16}=0\end{aligned}\tag{4.25}$$

and

$$B_2=\lim_{R\to\infty}\limsup_{n\to\infty}(a_2+b_2\|v_n\|^2)|\textstyle\int_{\mathbb{R}^3} v_n\nabla v_n\nabla\chi_R dx|=0.\tag{4.26}$$

Combining (4.21), we have

$$\begin{aligned}&\lim_{R\to\infty}\limsup_{n\to\infty}(a_1+b_1\|u_n\|^2)\textstyle\int_{\mathbb{R}^3}|\nabla u_n|^2\chi_R dx+(a_2+b_2\|v_n\|^2)\int_{\mathbb{R}^3}|\nabla v_n|^2\chi_R dx\\ \ge&\lim_{R\to\infty}\limsup_{n\to\infty}(a_1\textstyle\int_{\mathbb{R}^3}|\nabla u_n|^2\chi_R dx+a_2\int_{\mathbb{R}^3}|\nabla v_n|^2\chi_R dx)+[b_1(\int_{\mathbb{R}^3}|\nabla u_n|^2\chi_R dx)^2\\ &+b_2(\textstyle\int_{\mathbb{R}^3}|\nabla v_n|^2\chi_R dx)^2]\\ \ge&\lim_{R\to\infty}\limsup_{n\to\infty}(a_1\textstyle\int_{|x|>R}|\nabla u_n|^2\chi_R dx+a_2\int_{|x|>R}|\nabla v_n|^2\chi_R dx)\\ &+\tfrac12(\sqrt{b_1}\textstyle\int_{|x|>R}|\nabla u_n|^2\chi_R dx+\sqrt{b_2}\int_{|x|>R}|\nabla v_n|^2\chi_R dx)^2\\ \ge&\tfrac12 b_1 S_{\alpha,\beta}^2\nu_\infty^{\frac23}\end{aligned}\tag{4.27}$$

and

$$\begin{aligned}(\tfrac{\alpha}{3}+\tfrac{\beta}{3})\lim_{R\to\infty}\limsup_{n\to\infty}\textstyle\int_{\mathbb{R}^3} u_n^\alpha v_n^\beta\chi_R dx&=\lim_{R\to\infty}\limsup_{n\to\infty}2\textstyle\int_{|x|\ge\frac{R}{2}} u_n^\alpha v_n^\beta\chi_R dx\\ &\le\lim_{R\to\infty}2\textstyle\int_{|x|\ge\frac{R}{2}} u_n^\alpha v_n^\beta dx=2\nu_\infty.\end{aligned}\tag{4.28}$$

Otherwise, we get

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3}(f\chi_R u_n+g\chi_R v_n)dx=\lim_{R\to\infty}\int_{\mathbb{R}^3}(f\chi_R u+g\chi_R v)dx=0.\tag{4.29}$$

Combining (4.23)–(4.29), we have

$$\nu_\infty\ge\frac14 b_1 S_{\alpha,\beta}^2\nu_\infty^{\frac23}.$$

We obtain one of the following two cases holds:

(1) $$\nu_\infty=0;\ \ \mu_\infty=0.$$

(2)

$$\begin{aligned}&\nu_\infty\ge(\tfrac{b_1S_{\alpha,\beta}^2+\sqrt{b_1^2S_{\alpha,\beta}^4}}{8})^3,\\ &\mu_\infty\ge 0.\end{aligned}$$

Suppose that case (2) holds. We deduce that

$$\begin{aligned} m =& \lim_{n\to\infty} \Phi_\varepsilon(u_n, v_n) - \frac{1}{4}\langle \Phi_\varepsilon'(u_n, v_n), (u_n, v_n)\rangle \\ \geq& \frac{a_1}{4}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{a_2}{4}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{4}\mu_\infty + \frac{1}{6}\nu_\infty - \frac{3\varepsilon}{4}\int_{\mathbb{R}^3}(fu + gv)dx. \end{aligned}$$

Considering as the same as (4.15)–(4.20), we get

$$m \geq \frac{1}{4}\mu_\infty + \frac{1}{6}\nu_\infty - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}} \geq \Lambda - \varepsilon^2(|f|_{\frac{6}{5}} + |g|_{\frac{6}{5}})S^{\frac{1}{4}}$$

which is a contradiction. Thus, case (1) holds.

Combining (4.5), (4.22) with $\Gamma = \emptyset$, we have:

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx = \int_{\mathbb{R}^3} u^\alpha v^\beta dx.$$

Applying Fatou's lemma, we obtain:

$$\int_{\mathbb{R}^3} u^\alpha v^\beta dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx \leq \limsup_{n\to\infty} \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx = \int_{\mathbb{R}^3} u^\alpha v^\beta dx.$$

Thus, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx = \int_{\mathbb{R}^3} u^\alpha v^\beta dx.$$

Set $\|u_n\| \to d$. Then, by $\lim_{n\to\infty} \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx = \int_{\mathbb{R}^3} u^\alpha v^\beta dx$, we have

$$\begin{aligned} 0 =& (\Phi_\varepsilon'(u_n, v_n), (u_n, 0)) + o(1) \\ =& (a_1 + b_1\|u_n\|^2)\|u_n\|^2 - \int_{\mathbb{R}^3} u_n^\alpha v_n^\beta dx - \varepsilon\int_{\mathbb{R}^3} f u_n dx + o(1) \\ =& (a_1 + b_1 d^2)d^2 - \int_{\mathbb{R}^3} u^\alpha v^\beta dx - \varepsilon\int_{\mathbb{R}^3} f u dx \end{aligned}$$

and

$$\begin{aligned} 0 =& (\Phi_\varepsilon'(u_n, v_n), (u, 0)) + o(1) \\ =& (a_1 + b_1\|u_n\|^2)\int_{\mathbb{R}^3} \nabla u_n \nabla u dx - \int_{\mathbb{R}^3} u_n^{\alpha-1} v_n^\beta u dx - \varepsilon\int_{\mathbb{R}^3} f u dx + o(1) \\ =& (a_1 + b_1 d^2)\|u\|^2 - \int_{\mathbb{R}^3} u^\alpha v^\beta dx - \varepsilon\int_{\mathbb{R}^3} f u dx \end{aligned}$$

which deduces $d = \|u\|$. Combining $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$, we obtain $u_n \to u$ in $D^{1,2}(\mathbb{R}^3)$. Following the same approach and steps, we can also establish that $v_n \to v$ in $D^{1,2}(\mathbb{R}^3)$. According to Remark 4.1 and Lemma 4.4, there exists a non-negative bounded sequence $(u_n, v_n) \subset E$ that satisfies

$$\Phi_\varepsilon(u_n, v_n) \to m < \Lambda, \qquad \Phi_\varepsilon'(u_n, v_n) \to 0.$$

Consequently, by $u_n \to u$ in $D^{1,2}(\mathbb{R}^3)$ and $v_n \to v$ in $D^{1,2}(\mathbb{R}^3)$ we have

$$\Phi_\varepsilon(u_n, v_n) \to m = \Phi_\varepsilon(u, v), \quad \Phi'_\varepsilon(u_n, v_n) \to 0 = \Phi'_\varepsilon(u, v).$$

This completes the proof of the existence of the second solution.

5. Conclusions

In this paper, we first consider the existence of a local minimum solution for problem (1.1) by applying the Ekeland variational principle. Next, we investigate the existence of a second solution for problem (1.1) by applying the mountain pass lemma and the concentration compactness principle. To obtain the energy estimation of the associated functional Φ_ε for problem (1.1), we will need the explicit form of (λ_1, λ_2). Therefore, when $a_1 = a_2 = 0$ we have: Assume $\alpha = \beta = 3$ and $b_1 = b_2$, there exists $\varepsilon^{**} \in (0, \varepsilon^*]$ such that for any $\varepsilon \in (0, \varepsilon^{**})$, problem (1.1) has another solution. The value of ε^* is defined in Theorem 1.2. The reason for only considering the case where $\alpha = \beta = 3$ is that after extensive estimation, it is only when $\alpha = \beta = 3$ that Λ reaches its minimum value Λ_{min}. Therefore, the estimate satisfied by m contradicts Lemma 4.4 (i). For the cases in Lemma 4.4 (ii) and (iii), obtaining the result is challenging due to the mutual interaction of (u, v) adding complexity to our computations. This will be our main task in the following work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere gratitude to Prof. HaiYang He for her invaluable guidance throughout this research. Her expertise and insightful comments have greatly contributed to the quality of this work.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, *J. Differ. Geom.*, **11** (1976), 573–598.
2. G. Rosen, Minimum value for c in the Sobolev inequality $\left\|\phi^3\right\| \leq c\|\nabla\phi\|^3$, *SIAM J. Appl. Math.*, **21** (1971), 30–32. https://doi.org/10.1137/0121004
3. G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, **110** (1976), 353–372. https://doi.org/10.1007/BF02418013
4. G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, *Ann. Inst. H. Poincarè Anal., Non Lineairè*, **9** (1992), 281–304. https://doi.org/10.1016/S0294-1449(16)30238-4

5. J. Liu, J. F. Liao, C. L. Tang, Positive solutions for Kirchhoff-type equations with critical exponent in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **429** (2015), 1153–1172. https://doi.org/10.1016/j.jmaa.2015.04.066
6. M. Willem, *Minimax Theorems*, Birkhäuser Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1
7. P. G. Han, Mutiple positive solutions of nonhomogeneous elliptic systems involving critical Sobolev exponents, *Nonlinear Anal.*, **64** (2006), 869–886. https://doi.org/10.1016/j.na.2005.04.053
8. G. Kirchhoff, *Mechanik, Teubner, Leipzig*, 1883.
9. M. Chipot , B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, *Nonlinear Anal.*, **30** (1997), 4619–4627. https://doi.org/10.1016/S0362-546X(97)00169-7
10. S. Bernstein, Sur une class dẑquations fonctionnelles aux dẑrivẑes partielles, *Izv. Akad. Nauk SSSR Ser. Mat.*, **4** (1940), 17–26.
11. S. Pohoẑaev, A certain class of quasilinear hyperbolic equations, *Mat. Sb.*, **96** (1975), 152–166.
12. J. Lions, On some questions in boundary value problems of mathematical physics, in *North-Holland Mathematics Studies North-Holland* (eds. G. M. De La Penha, L. A. J. Medeiros), North-Holland, **30** (1978), 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3
13. C. O. Alves, F. J. S. A. Corrêa, G. M. Figueiredo, On a class of nonlocal elliptic problems with critical growth, *Differ. Equation Appl.*, **3** (2010), 409-417. https://doi.org/10.7153/dea-02-25
14. Q. Y. Dai, E. H. Lan, F. L. Shi, A priori bounds for positive solutions of Kirchhoff type equations, *Comput. Math. Appl.*, **76** (2018), 1525–1534. https://doi.org/10.1016/j.camwa.2018.07.004
15. G. M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, *J. Math. Anal. Appl.*, **401** (2013), 706–713. https://doi.org/10.1016/j.jmaa.2012.12.053
16. A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156–170. https://doi.org/10.1016/j.na.2013.08.011
17. S. Gupta, G. Dwivedi, Ground state solution to N-Kirchhoff equation with critical exponential growth and without Ambrosetti-Rabinowitz condition, *Rend. Circ. Mat. Palermo II. Ser.*, **2023** (2023), 1–12. https://doi.org/10.1007/s12215-023-00902-7
18. W. He, D. Qin, Q. Wu, Existence, multiplicity and nonexistence results for Kirchhoff type equations, *Adv. Nonlinear Anal.*, **10** (2021), 616–635. https://doi.org/10.1515/anona-2020-0154
19. S. H. Liang, S. Y. Shi, Soliton solutions to Kirchhoff type problems involving the critical growth in $\mathbb{R}^N$, *Nonlinear Anal.*, **81** (2013), 31-41. https://doi.org/10.1016/j.na.2012.12.003
20. S. H. Liang, J. H. Zhang, Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^3$, *Nonlinear Anal. Real World Appl.*, **17** (2014), 126–136. https://doi.org/10.1016/j.nonrwa.2013.10.011
21. P. Lv, G. Lin, X. Lv, The asymptotic behaviors of solutions for higher-order (m1, m2)-coupled Kirchhoff models with nonlinear strong damping, *Dem. Math.*, **56** (2023), 20220197. https://doi.org/10.1515/dema-2022-0197

22. T. Mukherjee, P. Pucci, M. Xiang, Combined effects of singular and exponential nonlinearities in fractional Kirchhoff problems, *Discrete Cont. Dyn. Syst.*, **2022** (2022), 163–187. https://doi.org/10.3934/dcds.2021111

23. D. Naimen, The critical problem of Kirchhoff type elliptic equations in dimension four, *J. Differ. Equations*, **257** (2014), 168–1193. https://doi.org/10.1016/j.jde.2014.05.002

24. E. Toscano, C. Vetro, D. Wardowski, Systems of Kirchhoff type equations with gradient dependence in the reaction term via subsolution-supersolution method, *Discrete Cont. Dyn. Syst. S*, **2023** (2023). https://doi.org/10.3934/dcdss.2023070

25. X. H. Tang, S. T. Chen, Ground state solutions of Nehari-Pohozaev type for Kirchhoff-type problems with general potentials, *Calc. Var.*, **110** (2017). https://doi.org/10.1007/s00526-017-1214-9

26. C. O. Alves, D. C. de Morais Filho, M. A. S. Souto, On systems of elliptic equations involving subcritical or critical Sobolev exponents, *Nonlinear Anal.*, **42** (2000), 771–787. https://doi.org/10.1016/S0362-546X(99)00121-2

27. J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, (1989).

28. B. Cheng, X. Wu, J. Liu, Multiple solutions for a class of Kirchhoff type problem with concave nonlinearity, *Nonlinear Differ. Equation Appl.*, **19** (2012), 521–537. https://doi.org/10.1007/s00030-011-0141-2

29. J. Chabrowski, J. Yang, On the Neumana problem for an elliptic system of equations involving the critical Sobolev exponent, *Colloq. Math.*, **90** (2001), 19–35. https://doi.org/10.4064/cm90-1-2