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Research article

A discrete second-order Hamiltonian system with asymptotically linear conditions

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Abstract: This paper deals with a non-autonomous discrete second-order Hamiltonian system under asymptotically linear conditions. The existence of a periodic solution is obtained via the saddle point theorem.

Keywords: asymptotically linear; second-order Hamiltonian system; saddle point theorem; periodic solution; existence

1. Introduction and main results

We are interested in the following discrete second-order Hamiltonian system (SOHS in short):

$$\Delta^2 u(n-1) = -\nabla F(n, u(n)), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $u(n) \in \mathbb{R}^N$, $\Delta^2 u(n) = \Delta(\Delta u(n))$, $\Delta u(n) = u(n+1) - u(n)$, and $F(n, x)$ is continuously differentiable about the first variable. $F(n+T, x) = F(n, x), (n, x) \in \mathbb{Z} \times \mathbb{R}^N$, with the positive integer $T \geq 2$.

The study of nonlinear Hamiltonian systems (HS) is one of the important research directions in mathematics, and it is related to many mathematical physics fields. Many scholars are committed to studying the continuous HS and have obtained a lot of results on nonlinear HS in studies of past decades via different critical point theories (see [1, 2] and references therein). The critical point theories also have important applications in nonlocal elliptic problems [3].

Guo and Yu [4–6] first studied the discrete SOHS by using variational method and aroused much research enthusiasm in this topic. Specially, in [4] the authors obtained the existence of periodic solutions with the help of the saddle point theorem with $F(n, x)$ satisfying superlinear conditions at the origin and infinity.

In [7] Tang and Zhang investigated a discrete SOHS under sublinear conditions. Wang, Zhang and Chen [8] introduced a control function $h(s)$ and discussed a class of non-autonomous SOHS via the least action principle.

Tang and Xiao [9] obtained the existence of a nontrivial homoclinic solution for continuous HS via the mountain pass theorem. Tang and Xue [10] investigated the multiplicity of periodic solutions for a discrete SOHS under superquadratic conditions via the operator theory. Gradually, more and more scholars have devoted themselves to studying the non-autonomous SOHS under quadratic conditions. Some solvable results have been for the non-autonomous SOHS by the minimax methods, such as in the papers [11–13]. Among them, [11, 13] considered the cases with subquadratic conditions, while in [12], it was treated with superquadratic conditions. However, Xie, Li and Luo in [14] studied a continuous SOHS with the help of the linking theorem.

Zhao, Yang and Chen considered an asymptotically linear case for the SOHS in [15], that is, $F(t,x) = W(t,x) - K(t,x)$, where W satisfies the asymptotically linear condition at infinity, and K satisfies the coupling condition. Chen, Guo and Liu [16] demonstrated continuous HS with asymptotically linear terms, which further extended the previous results under coupling conditions.

HS with asymptotically linear terms have been extensively studied in the continuous case, e.g., [17–19], whereas few results have been obtained in the discrete case. Inspired by the above literature, we will discuss a non-autonomous SOHS with asymptotically linear terms in discrete cases. The difference with [16] is that we construct a new workspace to estimate the minimax level associated with the energy functional.

We first write the nonlinear term $F(n,x)$ in (1.1) in the form $F(n,x) = -G(n,x) + H(n,x)$ with the following conditions:

(G_1) : for any $(n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N$, there exist $b > 0$ and $g_1(n) \in \mathbb{R}$ satisfying

$$G(n,x) \geq -b|x|^2 + g_1(n);$$

(G_2) : for any $n \in \mathbb{Z}[1,T]$, there exists $K_1 > 0$ such that

$$(\nabla G(n,x), x) \leq 2G(n,x), \quad |x| \geq K_1;$$

(H_1) : for any $(n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N$, there exist $g_2(n) \in \mathbb{R}$ and

$$d \in (0, \frac{-bT^2 + b + 3}{T^2 - 1})$$

such that

$$H(n,x) \leq d|x|^2 + g_2(n);$$

(H_2) : $(\nabla H(n,x), x) - 2H(n,x) \to +\infty$ uniformly for $n \in \mathbb{Z}[1,T]$, when $|x| \to +\infty$;

(W_1) : for all $n \in \mathbb{Z}[1,T]$, there exists $K_2 > 0$, such that

$$\max_{|x|=a} G(n,x) < \min_{|x|=a} H(n,x), \quad a \geq K_2;$$

(W_2) : for any $n \in \mathbb{Z}[1,T]$, there exists a constant $K_3 > 0$, such that

$$\begin{cases} \nabla F(n,x) \not\equiv 0 & \text{for all } |x| \leq K_3, \\ \sum_{n=1}^{T} F(n,x) > \sum_{n=1}^{T} [g_2(n) - g_1(n)] & \text{for all } |x| > K_3. \end{cases}$$

Theorem 1.1. *If* $(G_1), (G_2), (H_1), (H_2), (W_1)$ *and* (W_2) *hold, then system* (1.1) *has a nontrivial T-periodic solution.*

2. Preliminaries

Define the Hilbert space

$$H_T = \{\{u(n)\} : u(n) \in \mathbb{R}^N, u(n+T) = u(n), n \in \mathbb{Z}\}$$

with its inner product

$$\langle u, v\rangle = \sum_{n=1}^{T}[(\Delta u(n), \Delta v(n)) + (u(n), v(n))], \quad \forall u, v \in H_T,$$

and the corresponding norm

$$\|u\| = \Big(\sum_{n=1}^{T}[|\Delta u(n)|^2 + |u(n)|^2]\Big)^{\frac{1}{2}}, \quad \forall u \in H_T.$$

The corresponding functional of the equation is

$$\varphi(u) = \frac{1}{2}\sum_{n=1}^{T}|\Delta u(n)|^2 + \sum_{n=1}^{T}G(n, u(n)) - \sum_{n=1}^{T}H(n, u(n)), \quad \forall u \in H_T, \tag{2.1}$$

which is continuously differentiable. So,

$$\langle \varphi^{'}(u), v\rangle = \sum_{n=1}^{T}(\Delta u(n), \Delta v(n)) + \sum_{n=1}^{T}(\nabla G(n, u(n)), v(n)) - \sum_{n=1}^{T}(\nabla H(n, u(n)), v(n)), \quad \forall u, v \in H_T. \tag{2.2}$$

Clearly, the critical point of functional (2.1) is the T-periodic solution of problem (1.1).

Proposition 2.1. *[16] Suppose that $H(n, x)$ satisfies (H_2), $G(n, x)$ satisfies (G_2), for any $n \in \mathbb{Z}[1, T]$. Then, there is a sufficiently large $M > 0$, such that*

$$H(n, x) \geq \frac{|x|^2}{M^2}\min_{|x|=M} H(n, x), \quad |x| \geq M, \tag{2.3}$$

$$G(n, x) \leq \frac{|x|^2}{M^2}\max_{|x|=M} G(n, x), \quad |x| \geq M. \tag{2.4}$$

Recall that (PS) condition in [8], a sequence $\{u_p\} \subset H_T$ has a convergent sequence when $\varphi(u_p)$ is bounded, and $\|\varphi^{'}(u_p)\| \to 0,\ p \to +\infty$. Similarly, we can see (C) condition in [16], if $\varphi(u_p)$ is bounded, and $\|\varphi^{'}(u_p)\|(1 + \|u_p\|) \to 0,\ p \to +\infty$, then the sequence $\{u_p\} \subset H_T$ has a convergent sequence.

Lemma 2.2. *If G satisfies (G_1) and (G_2), H satisfies (H_1) and (H_2), then the functional φ fulfills the (C) condition.*

Proof. Assume that $\{u_p\} \subset H_T$ is a (C) sequence, i.e.,

$$\sup_{p\in N^*}\{|\varphi(u_p)|\} < +\infty, \quad (1 + \|u_p\|)\|\varphi^{'}(u_p)\| \to 0,\ p \to +\infty.$$

In addition, there exists $M_1 > 0$ sufficiently large, such that

$$|\varphi(u_p)| \leq M_1, \quad (1 + \|u_p\|)\|\varphi^{'}(u_p)\| \leq M_1, \ p \to +\infty.$$

We claim that $\{u_p\}$ is bounded. Conversely, we obtain $\|u_{p_k}\| \to +\infty$, as $k \to +\infty$. We replace $\{u_{p_k}\}$ with $\{u_p\}$.

Let $s_p = u_p/\|u_p\|$, and obviously, we know that $\|s_p\| = 1$. There exists $s \in H_T$, such that $s_p \to s$ in H_T.

As

$$\|s_p\| - \|s\| \leq \|s_p - s\| \to 0, \quad \|s\| - \|s_p\| \leq \|s_p - s\| \to 0,$$

we have $\|s\| = \|s_p\| = 1$. Therefore, $s \not\equiv 0$.

As $s \not\equiv 0$, we set

$$L = \{n \in \mathbb{Z}[1, T] : |s(n)| > 0\}.$$

From the previously described results, we obtain

$$|u_p(n)| = \|u_p\||s_p(n)| \to +\infty, \quad p \to +\infty. \tag{2.5}$$

When $p \in \mathbb{N}^*, \lambda > \max\{K_1, M\}$, set

$$M_2 = T \max_{n\in\mathbb{Z}[1,T]} \max_{|x|\leq\lambda}\{2|G(n, x)| + |\nabla G(n, x)||x|, 2|H(n, x)| + |\nabla H(n, x)||x|\},$$

by (G_2), we have

$$\begin{aligned}
&\sum_{n=1}^{T}[2G(n, u_p(n)) - (\nabla G(n, u_p(n)), u_p(n))] \\
&\quad \geq \sum_{\{n\in\mathbb{Z}[1,T]:|u_p(n)|\leq\lambda\}} [2G(n, u_p(n)) - (\nabla G(n, u_p(n)), u_p(n))] \\
&\quad \geq -M_2.
\end{aligned} \tag{2.6}$$

We set

$$L^c = \mathbb{Z}[1, T]\backslash L.$$

By (H_2), we obtain

$$\begin{aligned}
&\sum_{L^c}[(\nabla H(n, u_p(n)), u_p(n)) - 2H(n, u_p(n))] \\
&\quad \geq \sum_{L^c\cap\{n\in\mathbb{Z}[1,T]:|u_p(n)|\leq\lambda\}} [(\nabla H(n, u_p(n)), u_p(n)) - 2H(n, u_p(n))] \\
&\qquad + \sum_{L^c\cap\{n\in\mathbb{Z}[1,T]:|u_p(n)|>\lambda\}} [(\nabla H(n, u_p(n)), u_p(n)) - 2H(n, u_p(n))] \\
&\quad \geq \sum_{L^c\cap\{n\in\mathbb{Z}[1,T]:|u_p(n)|\leq\lambda\}} [(\nabla H(n, u_p(n)), u_p(n)) - 2H(n, u_p(n))] \\
&\quad \geq -M_2.
\end{aligned} \tag{2.7}$$

Combining (H_2) with (2.5), we obtain

$$\sum_{L}[(\nabla H(n,u_p(n)),u_p(n))-2H(n,u_p(n))]\to+\infty,\quad p\to+\infty. \tag{2.8}$$

As $|\varphi(u_p)|\le M_1$, one has

$$\|\langle\varphi'(u_p),u_p\rangle\|\le\|\varphi'(u_p)\|\|u_p\|\le\|\varphi'(u_p)\|(1+\|u_p\|)\le M_1. \tag{2.9}$$

Therefore, we have

$$2\varphi(u_p)-\langle\varphi'(u_p),u_p\rangle\le 3M_1. \tag{2.10}$$

Combining (2.1) and (2.2), we have

$$\begin{aligned}2\varphi(u_p)-\langle\varphi'(u_p),u_p\rangle=&\sum_{n=1}^{T}[(\nabla H(n,u_p(n)),u_p(n))-2H(n,u_p(n))]\\&+\sum_{n=1}^{T}[2G(n,u_p(n))-(\nabla G(n,u_p(n)),u_p(n))].\end{aligned} \tag{2.11}$$

To be convenient, we denote

$$\sum_{n=1}^{T}[(\nabla H(n,u_p(n)),u_p(n))-2H(n,u_p(n))]$$

by I_1 and

$$\sum_{n=1}^{T}[2G(n,u_p(n))-(\nabla G(n,u_p(n)),u_p(n))]$$

by I_2. By (2.6), (2.7) and (2.8), we see that

$$\begin{aligned}I_1+I_2=&\sum_{L}[(\nabla H(n,u_p(n)),u_p(n))-2H(n,u_p(n))]\\&+\sum_{L^c}[(\nabla H(n,u_p(n)),u_p(n))-2H(n,u_p(n))]\\&+\sum_{n=1}^{T}[2G(n,u_p(n))-(\nabla G(n,u_p(n)),u_p(n))]\\\ge&\sum_{L}[(\nabla H(n,u_p(n)),u_p(n))-2H(n,u_p(n))]-2M_2\\\to&+\infty,\quad p\to+\infty.\end{aligned} \tag{2.12}$$

By (2.10)-(2.12), we have

$$3M_1\ge+\infty,$$

which contradicts the boundedness of M_1.

Therefore $\{u_p\}$ is bounded in H_T, and then we have $u_p\to u$ in H_T, which demonstrates that φ fulfills (C) condition. □

3. Proof of Theorem 1.1

Set

$$\bar{u} = \frac{1}{T}\sum_{n=1}^{T} u(n),\ \widetilde{u}(n) = u - \bar{u} \quad \text{for all } \ u \in H_T.$$

Let

$$\widetilde{H}_T = \{u \in H_T \mid \bar{u} = 0\},$$

and therefore

$$H_T = \widetilde{H}_T \bigoplus \mathbb{R}^N.$$

Just as in [7], we have

$$\|\widetilde{u}\|_\infty^2 \leq \frac{T^2-1}{6T}\sum_{n=1}^{T} |\Delta u(n)|^2. \tag{3.1}$$

Next, we will prove Theorem 1.1 in two steps.

Step 1 Set $\mathcal{W} = X \bigoplus Y$ as $\mathcal{H} = V \bigoplus X$ in [16, Lemma 4], where $X = \mathbb{R}^N$ and $Y = \widetilde{H}_T$. As is discussed in [16], (PS) condition can be replaced by (C) condition, and [16, Lemma 4] holds under (C) condition. Now, we will prove that (i) holds in [16, Lemma 4]. By (W_1) and Proposition 2.1, let $m_0 \in \mathbb{R}^N$, and $|m_0| = 1$. If $z \geq \max\{M, K_2\}$, we obtain

$$\begin{aligned}
\varphi(zm_0) &= \sum_{n=1}^{T}[G(n, zm_0) - H(n, zm_0)] \\
&\leq \frac{z^2}{M^2}\sum_{n=1}^{T}[\max_{|m|=M} G(n, m) - \min_{|m|=M} H(n, m)] \\
&\leq \frac{z^2 T}{M^2}\max_{n\in\mathbb{Z}[1,T]}[\max_{|m|=M} G(n, m) - \min_{|m|=M} H(n, m)] \\
&\to -\infty, \quad z \to +\infty.
\end{aligned} \tag{3.2}$$

There exist $r > 0$ large enough and a constant

$$\alpha := \sum_{n=1}^{T}[g_1(n) - g_2(n)] - 1,$$

such that

$$\varphi|_{\partial B_r(0)\cap X} \leq \alpha.$$

Step 2 We will show that (ii) holds in [16, Lemma 4]. By $(G_1), (H_1)$, and (3.1), for all $u \in \widetilde{H}_T$, we get

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\sum_{n=1}^{T} |\Delta(n)|^2 + \sum_{n=1}^{T}[G(n, u(n)) - H(n, u(n))] \\
&\geq \frac{1}{2}\sum_{n=1}^{T} |\Delta u(n)|^2 - (b + d)\sum_{n=1}^{T} |u(n)|^2 + \sum_{n=1}^{T}[g_1(n) - g_2(n)]
\end{aligned}$$

$$\geq \left(\frac{1}{2} - \frac{(b+d)(T^2-1)}{6}\right)\sum_{n=1}^{T}|\Delta u(n)|^2 + \sum_{n=1}^{T}[g_1(n) - g_2(n)]$$
$$\geq \sum_{n=1}^{T}[g_1(n) - g_2(n)]. \tag{3.3}$$

Let constant

$$\beta = \sum_{n=1}^{T}[g_1(n) - g_2(n)],$$

and then $\varphi|_Y \geq \beta$. From [16, Lemma 4], we know that φ has a critical value

$$c \geq \beta.$$

Therefore, there is a $u \in H_T$ such that

$$\sum_{n=1}^{T}(\Delta u(n), \Delta l(n)) - (\nabla F(n, u(n)), l(n)) = 0, \quad \forall l(n) \in H_T.$$

Finally, we can get our conclusion according to (W_2).

4. Example

We provide an example to demonstrate the rationality of Theorem 1.1.

Example 1 Suppose $T = 2$ and set

$$G, H : \mathbb{Z}[1, T] \times \mathbb{R}^N \to \mathbb{R}$$

with

$$G(n, x) = \frac{\cos^2 \pi n}{6}(D(n)x, x),$$

where

$$D(n) = diag((-1)^2, (-1)^3, ..., (-1)^{1+n}).$$

Let

$$H(n, x) = \frac{1 + \cos^2 \pi n}{5}|x|^2(1 - \frac{1}{\ln(e^{10} + |x|^2)}),$$

where $n \in \mathbb{Z}[1, T]$ and $x \in \mathbb{R}^N$.

First of all, set

$$g_1(n) \equiv 0, \quad b = \frac{1}{6}.$$

Then, we have

$$\frac{\cos^2 \pi n}{6}(D(n)x, x) \geq -\frac{1}{6}|x|^2,$$

and

$$(\nabla G(n, x), x) = \frac{2\cos^2 \pi n}{6}\sum_{i=1}^{n}(-1)^{1+i}X_i^2 \leq \frac{2\cos^2 \pi n}{6}\sum_{i=1}^{n}(-1)^{1+i}X_i^2 = 2G(n, x).$$

Then, G satisfies (G_1) and (G_2) conditions.

In the same way, let

$$g_2(n) \equiv 0, \quad d = \frac{1}{2} \in (0, \frac{5}{6}).$$

Then, we obtain

$$\frac{1+\cos^2 \pi n}{5}|x|^2(1 - \frac{1}{\ln(e^{10}+|x|^2)}) \le \frac{1}{2}|x|^2,$$

and

$$\begin{aligned}(\nabla H(n,x), x) - 2H(n,x) &= \frac{2(1+\cos^2 \pi n)|x|^2}{5}(1 - \frac{1}{\ln(e^{10}+|x|^2)}) \\ &\quad + \frac{2(1+\cos^2 \pi n)|x|^4}{5}(\frac{1}{(e^{10}+|x|^2)\ln^2(e^{10}+|x|^2)}) \\ &\quad - \frac{2(1+\cos^2 \pi n)}{5}|x|^2(1 - \frac{1}{\ln(e^{10}+|x|^2)}) \\ &= \frac{2(1+\cos^2 \pi n)|x|^4}{5(e^{10}+|x|^2)\ln^2(e^{10}+|x|^2)} \\ &\to +\infty.\end{aligned}$$

Obviously, $H(n,x)$ satisfies (H_1) and(H_2) conditions. Also,

$$\max_{|x|=a} G(n,x) \le \frac{1}{6}a^2 < \min_{|x|=a} H(n,x)$$

for any $a \in \mathbb{R}$ and $n \in \mathbb{Z}[1,T]$, so that (W_1) holds. In addition,

$$F(n,x) = -G(n,x) + H(n,x) \ge -\frac{1}{6}a^2 + H(n,x) > 0.$$

Therefore,

$$\sum_{n=1}^{T} F(n,x) > \sum_{n=1}^{T}[g_2(n) - g_1(n)],$$

which means that (W_2) holds.

According to Theorem 1.1, we can get the existence of a nontrivial 2-periodic solution for system (1.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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