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Research article

Synchronization of heterogeneous harmonic oscillators for generalized uniformly jointly connected networks

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Abstract: The synchronization problem for heterogeneous harmonic oscillators is investigated. In practice, the communication network among oscillators might suffer from equipment failures or malicious attacks. The connection may switch extremely frequently without dwell time, and can thus be described by generalized uniformly jointly connected networks. We show that the presented typical control law is strongly robust against various unreliable communications. Combined with the virtual output approach and generalized Krasovskii-LaSalle theorem, the stability is proved with the help of its cascaded structure. Numerical examples are presented to show the correctness of the control law.

Keywords: generalized uniform joint connectivity; harmonic oscillators; dwell time; generalized Krasovskii-LaSalle theorem

1. Introduction

Over the last two decades, synchronization has been a popular topic in the study of cooperative control theory; see [1–7] and the reference therein. In the problem of oscillator synchronization, all coupled oscillators are controlled to reach a common state. Each oscillator independently handles the information of their neighbors. In different contexts, the communication networks are assumed to operate under different connecting conditions. Among the communication networks, the most common and trivial one is the static connected network without switching [8, 9]. This condition requires the connection among the oscillators to be maintained online and reliable over the time. In practice, the communication network may suffer device failure or a malicious attack, and the controller design based on a static network may fail. The conditions of switching networks such as the uniformly jointly connected (UJC) switching network, which is often combined with a dwell-time constraint [9] have been proposed.

A dwell-time constraint [10] is usually assumed in switching networks to guarantee a common joint Lyapunov function in the stability analysis. It requires the connection to maintain unchanged over a time interval. The dwell-time condition is different from the frequently occurring instantaneous link failures. The generalized UJC (GUJC) network was newly proposed in [11] to avoid the dwell-time constraint.

Switched closed-loop systems usually have higher complexity [12, 13] that is created by the control scheme of more loosely switching networks, Different analysis tools [14, 15] have been developed to deal with the stability such as non-smooth analysis and the generalized Barbalat lemma [16]. Among them, two categories of techniques have been developed to deal with the UJC networks with dwell time. One is transition matrix analysis [17] which is dependent on the dwell-time constraint to make the transition matrix properly defined. The other is Lyapunov analysis based on the system's stable states with dwell time [14].

With numerous applications in the real world such as repetitive control, mapping, sampling movements, the synchronization of harmonic oscillators is among the most fundamental topics in cooperative control theory [9, 18, 19]. Considering controller design such that all harmonic oscillators achieve synchrony is meaningful in both theory and application. Cyber-attacks or connection equipment failures will cause very fast network switching. A question naturally arises whether the controller design is still valid for the switching network if the dwell-time assumption is not applied. In real world application, compared with homogeneous harmonic oscillators, the heterogeneous harmonic oscillators are more common due to the individual differences and parameter uncertainties. For example, to generate sine waves at the frequency of the leading frequency, one can use the synchronization control of heterogeneous harmonic oscillators. The heterogeneity makes the static distributed synchronization [9] algorithm invalid. New synchronization methods such as event-triggered control protocols [20] and asynchronous sampled-data protocols [21] are applied for the synchronization of heterogeneous harmonic oscillators.

Motivated by the above observations, in contrast with the existing work with static network topology [9, 20, 21] or UJC network topology with dwell time [22], in this paper, an unreliable networked scenario subject to the GUJC condition and heterogeneous harmonic oscillators synchronization is investigated. The main contributions are two-fold.

(i) We extend the study on the heterogeneous harmonic oscillator synchronization problem to more loosely switching networks. The dwell-time assumption is avoided. The network assumption allows for the instantaneous change of communication connection and fast switching which makes its stability analysis much more challenging.

(ii) We prove that the distributed controller is strongly robust against various unreliable communications. The common Lyapunov function techniques that rely on the system trajectory is invalid for the fast switching communication topology. Without a strict negative Lyapunov function, new methods should be introduced to study the control problem. To overcome the difficulty in analyzing the stability, the technical contribution of this paper is presented as follows: the virtual output method and the generalized Krasovskii-LaSalle theorem [23] are applied by using a limiting zeroing-output solution to describe the stable states.

Notations $\mathbb{R}^p$ denotes Euclidean space with p dimensions and $\mathbb{R}^{p\times q}$ denotes all real entry $p \times q$ matrices. I_n represents the $n \times n$ identity matrix. For matrix A and matrix B, $||A||$ is the Euclidean norm and $A \otimes B$ is the Kronecker product. A continuous increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{K}$ function if $f(0) = 0$.

2. Preliminaries and problem

2.1. Graph theory

Let $\lambda : \mathbb{R}^+ \to \Lambda$ be a switching signal where Λ is a finite set and Θ is the set of switching signals. Denote $\bar{\mathcal{V}} = \{0, 1, \dots, N\}$ and

$$\delta^\xi(t) = \begin{cases} 0, & \text{if } \lambda(t) \neq \xi, \\ 1 & \text{if } \lambda(t) = \xi. \end{cases} \tag{2.1}$$

Let $[s, t)$ be a time interval with $s < t$. Denote

$$\lambda_\tau[s, t) = \{\xi \in \Lambda : \int_s^t \delta^\xi(u)\mathrm{d}u \geq \tau\}$$

for any $\tau > 0$. Let

$$\bar{\mathcal{G}}^\tau_\lambda([s, t)) = (\bar{\mathcal{V}}, \bigcup_{\xi \in \lambda_\tau[s,t)} \bar{\mathcal{E}}_\xi)$$

be the τ-joint graphs over $[s, t)$. Denote the adjacency matrices of $\mathcal{G}_\xi$ as

$$\mathcal{A}_\xi = [a^\xi_{ij}]^N_{i,j=1}$$

and the adjacency matrices of $\bar{\mathcal{G}}_\xi$ as $\bar{\mathcal{A}}_\xi = [a^\xi_{ij}]^N_{i,j=0}$. Let

$$\mathcal{H}_\xi = \mathcal{L}_\xi + \mathrm{diag}[a^\xi_{10}, \dots, a^\xi_{N0}]$$

where $\mathcal{L}_\xi$ is the Laplacian of $\mathcal{G}_\xi$. For $\xi \in \Lambda$, $a^\xi_{ij} = a^\xi_{ji}$.

To discuss the improvement of network conditions, we list the UJC condition and dwell-time condition, respectively.

Assumption 1. *[16][UJC] There is a sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_0 = 0$ and $t_{i+1} - t_i < \chi$ for some $\chi > 0$; the joint graph $\bar{\mathcal{G}}_\lambda([s, t)) = (\bar{\mathcal{V}}, \bigcup_{\xi \in \lambda[t_i, t_{i+1})} \bar{\mathcal{E}}_\xi)$ contains a spanning tree rooted at the node 0.*

Assumption 2. *[24][Dwell time] There exist a constant $\tau_0 > 0$ and a sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_0 = 0$ such that $t_{i+1} - t_i \geq \tau_0$ satisfied $\lambda(t_{i+1}) \neq \lambda(t_i)$ and $\lambda(t_i) = \lambda(t)$ for $t_i \leq t < t_{i+1}$.*

This paper uses the following GUJC network without any dwell-time.

Assumption 3. *[23][GUJC] There exists $0 < \tau \leq T$ for any $t \geq 0$ and any $\lambda \in \Theta$; $\bar{\mathcal{G}}^\tau_\lambda([t, t + T))$ contains a spanning tree rooted at the node 0.*

Remark 1. *Dwell-time Assumption 2 describes the networks in which switching does not occur too often. In the literatures, the dwell-time assumption is usually adopted when the switching topology satisfies the UJC condition. Assumption 3 is less strict than the standard UJC Assumption 1 combined with dwell-time Assumption 2. It rules out the dwell-time requirement and accommodates an instantaneous attack or link failures. Since it uses the characteristic function in $\lambda_\tau[s, t)$, the network can be switched at any frequency at any time within a zero measure time subset of $[s, t)$. For example, if a network switches at all rational numbers within a time interval $[s, t)$, the switching network can not be modeled by applying the UJC condition with the dwell-time assumption. For more details, we refer the reader to [14, Remark 15].*

2.2. Some concepts and a theorem

We recall some concepts and a theorem of switched systems mainly from [23]. Consider the following switching system:

$$\dot{x} = \mathbf{A}_\lambda x + \mathbf{B}_\lambda u, \tag{2.2a}$$

$$y = \mathbf{C}_\lambda x, \tag{2.2b}$$

where $\mathbf{A}_\lambda, \mathbf{B}_\lambda \in \mathbb{R}^{m\times n}$, $\mathbf{C}_\lambda \in \mathbb{R}^{n\times m}$ and $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ represents the state and the output, respectively. The initial time of x and λ is set to be zero.

Definition 1. *[23] System* (2.2) *is said to be in the output-injection form if there is a function $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ which is continuous and satisfies the following:*
1) $\zeta(0) = 0$;
2) for any $\xi \in \Lambda$, $\|\mathbf{B}_\xi x\| \leq \zeta\left(\|\mathbf{C}_\xi x\|\right)$.

Denote all possible forward complete solution pair sets as $\Phi(\Theta) = \{(x, \lambda) | x \in \mathbb{R}^p, \lambda \in \Theta\}$. It satisfies the following, for all $t \geq 0$:

$$x(t) = x(0) + \int_0^t (\mathbf{A}_{\lambda(\tau)} + \mathbf{B}_{\lambda(\tau)}) x(\tau) \mathrm{d}\tau.$$

Definition 2. *[23] If it holds that $\{(\eta_n, \lambda_n)\} \subseteq \Phi(\Theta)$, $t_n \geq 2n$ and the following is true: 1) $\{\eta_n(\cdot + t_n) : [-n, n] \mapsto \mathbb{R}^p\}$ satisfies*

$$\lim_{n\to\infty} \eta_n(t + t_n) = \bar{\eta}(t)$$

uniformly on $\mathbb{R}$; 2) for almost all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} C_{\lambda_n(t+t_n)} \bar{\eta}(t) = 0;$$

then $\bar{\eta}(t)$ is a limiting zeroing-output solution of (2.2) *w.r.t.* $\Phi(\Theta)$.

The limiting zeroing-output solution satisfies

$$\bar{\eta}(t) = \bar{\eta}(0) + \lim_{n\to\infty} \int_0^t \mathbf{A}_{\lambda_n(\tau+t_n)} \bar{\eta}(\tau) \mathrm{d}\tau,$$

for all $t_n \geq 2n$ and all $t \in \mathbb{R}$. We list the stability concepts of the system (2.2) as follows.

Definition 3. *[23] If there is a function ν of class $\mathcal{K}$ that satisfies*

$$\|x(t)\| \leq \nu(x(s))$$

for any $(x, \lambda) \in \Phi(\Theta)$ and any $0 < s < t$, then system (2.2) *is uniformly globally stable at the origin w.r.t.* $\Phi(\Theta)$.

Definition 4. *[23] If there exist $\gamma_1 > 0$, $\gamma_2 > 0$,*

$$\|x(t)\| \leq \gamma_1 e^{-\gamma_2(t-s)} \|x(s)\|$$

for all $(x, \lambda) \in \Phi(\Theta)$ and $0 < s < t$, then system (2.2) *is uniformly globally exponentially stable at the origin w.r.t.* $\Phi(\Theta)$.

In this paper, with respect to $\Phi(\Theta)$ and with respect to Θ are omitted for convenience. The generalized Krasovskii-LaSalle theorem is introduced as follows.

Lemma 1. *[23] Let $\lambda \in \Theta$. Assume that the following is true:*
1) system (2.2) *is in the output injection form and uniformly globally stable at the origin;*
2) there is a continuous function $\mu : \mathbb{R}^+ \mapsto \mathbb{R}^+$ that satisfies

$$\int_s^{+\infty} \|C_{\lambda(\tau)} x(\tau)\|^2 \mathrm{d}\tau \leq \mu(\|x(s)\|)$$

for any $s \geq 0$;
3) each limiting zeroing-output solution $\bar{\eta}$ of system (2.2) *which is bounded satisfies*

$$\inf_{t \in \mathbb{R}} \|\bar{\eta}(t)\| = 0;$$

then, system (2.2) *is uniformly globally exponentially stable at the origin.*

2.3. Problem formulation

We consider N heterogeneous harmonic oscillators and a leader oscillator under Assumption 3. Let the leader oscillator associate with node 0 and the i-th oscillator associates with node i. The dynamics of the i-th oscillator can be described by

$$\begin{aligned} \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= -\beta_i x_{1i} + u_i, \end{aligned} \tag{2.3}$$

where $\beta_i > 0$ is the square of the frequency of the i-th harmonic oscillator, $x_i = [x_{1i}, x_{2i}]^\top \in \mathbb{R}^2$ is the state of the i-th oscillator and the input is $u_i(t) \in \mathbb{R}$. The dynamics of the leader oscillator can be described by

$$\begin{aligned} \dot{x}_{10} &= x_{20}, \\ \dot{x}_{20} &= -\beta x_{10}, \end{aligned} \tag{2.4}$$

where $\beta > 0$ is the square of the frequency of the leader harmonic oscillator and $x_0 = [x_{10}, x_{20}]^\top$ is the state of the leader oscillator.

Definition 5. *The synchronization control problem for the GUJC network is to find the control $u_i(t)$ for each oscillator $i \in \mathcal{V}$,*

$$\lim_{t \to \infty} (x_{1i}(t) - x_{10}(t)) = 0$$

and

$$\lim_{t \to \infty} (x_{2i}(t) - x_{20}(t)) = 0.$$

3. Main results

Assume that every oscillator can only use the state information of the neighbors in the switching graph. We adopt the following control law for the i-th oscillator:

$$u_i = -k_1 x_{1i} - k_2 x_{2i} + (\beta_i - \beta + k_1)\eta_{1i} + k_2 \eta_{2i}, \tag{3.1a}$$

$$\dot{\eta}_i = \Upsilon\eta_i - \mu\sum_{j\in\bar{\mathcal{V}}} a^{\lambda}_{ij}\left(\eta_i - \eta_j\right), \tag{3.1b}$$

where $\eta_0 = x_0$ and $\eta_i = [\eta_{1i}, \eta_{2i}]^\top \in \mathbb{R}^2$ is a dynamic compensator with the leader's state for the i-th oscillator; the parameters μ, k_1, k_2 are arbitrary positive constants. Moreover,

$$\Upsilon = \begin{bmatrix} 0 & 1 \\ -\beta & 0 \end{bmatrix}.$$

Our main result is that the synchronization control problem can be achieved by applying the distributed controller (3.1) under Assumption 3. For $j = 1, 2$, denote the synchronization error as

$$e_{ji} = x_{ji} - x_{j0}$$

and the state error as

$$\tilde{e}_{ji} = \eta_{ji} - x_{j0}.$$

Additionally,

$$e_i = [e_{1i}, e_{2i}]^\top,$$
$$\tilde{e}_i = [\tilde{e}_{1i}, \tilde{e}_{2i}]^\top.$$

Then, let

$$e = [e_1^\top, e_2^\top, \cdots, e_N^\top]^\top$$

and

$$\tilde{e} = [\tilde{e}_1^\top, \tilde{e}_2^\top, \cdots, \tilde{e}_N^\top]^\top.$$

Denote

$$\Psi_{ai} = \begin{bmatrix} 0 & 1 \\ -k_1 - \beta_i & -k_2 \end{bmatrix}$$

and

$$\Psi_{bi} = \begin{bmatrix} 0 & 0 \\ \beta - k_1 - \beta_i & -k_2 \end{bmatrix}$$

and

$$\Psi_a = diag[\Psi_{a1}, \Psi_{a2}, \cdots, \Psi_{aN}],$$
$$\Psi_b = diag[\Psi_{b1}, \Psi_{b2}, \cdots, \Psi_{bN}].$$

It is worth to noting that Ψ_a is then a Hurwitz matrix. In fact, for the i-th oscillator, the eigenvalues λ_{1i} and λ_{2i} of Ψ_{ai} satisfy that $\lambda_{1i} + \lambda_{2i} = -k_2 < 0$ and $\lambda_{1i}\lambda_{2i} = k_1 + \beta_i > 0$. Thus, for arbitrary $k_1 > 0$ and $k_2 > 0$, we have that $\lambda_{1i} < 0$ and $\lambda_{2i} < 0$.

The closed-loop is in compact form:

$$\dot{e} = \Psi_a e + \Psi_b \tilde{e}, \tag{3.2a}$$
$$\dot{\tilde{e}} = (I_N \otimes \Upsilon - \mu\mathcal{H}_\lambda \otimes I_2)\tilde{e}. \tag{3.2b}$$

We use the coordinate transform

$$\theta(t) = (I_N \otimes e^{-\Upsilon t})\tilde{e}(t).$$

Then, the origin does not need to be changed since

$$0 = (I_N \otimes e^{-\Upsilon t})0;$$

the initial state is also maintained since $\theta(0) = (I_N \otimes e^0)\tilde{e}(0) = \tilde{e}(0)$. Then, we have

$$\begin{aligned}\dot{\theta} &= -(I_N \otimes \Upsilon e^{-\Upsilon t})\tilde{e} + (I_N \otimes e^{-\Upsilon t})\dot{\tilde{e}} \\ &= -(I_N \otimes \Upsilon e^{-\Upsilon t})\tilde{e} + (I_N \otimes e^{-\Upsilon t})(I_N \otimes \Upsilon - \mu\mathcal{H}_\lambda \otimes I_2)\tilde{e} \\ &= -(I_N \otimes e^{-\Upsilon t})(\mu\mathcal{H}_\lambda \otimes I_2)\tilde{e} \\ &= -(\mu\mathcal{H}_\lambda \otimes I_2)\theta.\end{aligned} \tag{3.3}$$

We first prove the following theorem.

Theorem 1. *System* (3.3) *is uniformly globally exponentially stable at the origin under Assumption 3,.*

Proof: Now, for system (3.3), we define the virtual output as

$$Y = \sqrt{\mathcal{H}_\lambda \otimes I_2}\theta.$$

System (3.3) has the output-injection form given by (2.2); let $x = \theta$, $y = Y$ and

$$\mathbf{A}_\lambda = 0,$$

$$\mathbf{B}_\lambda = -(\mu\mathcal{H}_\lambda \otimes I_2),$$
$$\mathbf{C}_\lambda = \sqrt{\mu\mathcal{H}_\lambda \otimes I_2}$$

and let $\zeta(x)$ be given as

$$\zeta(x) = x^2.$$

Let $V = \theta^\top\theta$. Then, V is positive definite. Moreover,

$$\dot{V} = \dot{\theta}^\top\theta + \theta^\top\dot{\theta} = -2\theta^\top(\mu\mathcal{H}_\lambda \otimes I_2)\theta \le 0. \tag{3.4}$$

We obtain

$$V(t) \le V(s)$$

for $t > s$. System (3.3) is then uniformly globally stable at the origin.

By [14, Lemma 1], let $\alpha_1(t) = V(t)$, $\alpha_2(t) = \sqrt{2\mu\mathcal{H}_{\lambda(t)} \otimes I_2}\theta(t)$ and $\alpha_3(t) = 0$. We have

$$\int_s^{+\infty} \|Y(\tau)\|^2 \, \mathrm{d}\tau \le \alpha_1(s) + (1 + \alpha_1(s))\,\varpi(s) = \alpha_1(s) = V(s)$$

for any $s \ge 0$, where

$$\varpi(s) = \int_s^{+\infty} \alpha_3(\tau)\mathrm{d}\tau e^{\int_s^{+\infty} \alpha_3(\tau)\mathrm{d}\tau}.$$

We have checked that the condition 2 of Lemma 1 is satisfied. Assume that $\bar{\theta} : \mathbb{R} \to \mathbb{R}^{2N}$ is any bounded limiting zeroing-output solution of system (3.3); then there exist $t_n \ge 2n$ and $\{\lambda_n\} \subseteq \Theta$ such that

$$\lim_{n\to\infty} \sqrt{\mathcal{H}_{\lambda_n(t+t_n)} \otimes I_2}\bar{\theta}(t) = 0 \tag{3.5}$$

for almost all t in $\mathbb{R}$ and

$$\dot{\bar{\theta}} = 0. \tag{3.6}$$

Thus, by (3.6), we obtain that $\bar{\theta} = \bar{\theta}(0)$. Moreover, by (3.5), we obtain

$$\lim_{n\to\infty}(\mathcal{H}_{\lambda(t+t_n)} \otimes I_2)\bar{\theta}(t) = 0 \tag{3.7}$$

for almost all t in $\mathbb{R}$. Thus, we have

$$\lim_{n\to\infty}(\mathcal{H}_{\lambda_n(t+t_n)} \otimes I_2)\bar{\theta}(0) = 0.$$

Thus, there exists $c_\theta \in \mathbb{R}^{2N}$ that satisfies

$$\bar{\theta}(0) = c_\theta$$

and

$$\lim_{n\to\infty}(\mathcal{H}_{\lambda_n(t+t_n)} \otimes I_2)c_\theta = 0.$$

By Assumption 3, there exists a spanning tree of $\bar{\mathcal{G}}^\tau_\lambda([t, t+T))$ that is rooted at the node 0. By [23, Lemma 3], there is $\epsilon_1 > 0$ that satisfies

$$u^\top \left[\int_t^{t+T} \mathcal{H}_{\lambda(\tau)} \otimes I_2 \mathrm{d}\tau\right] u \geq \epsilon_1$$

for all $u \in \mathbb{R}^{2N}$ with $||u|| = 1$, all $\lambda \in \Theta$ and all $t \geq 0$. If $c_\theta \neq 0$, by (3.7), we have

$$\begin{aligned}\epsilon_1 &\leq \lim_{n\to\infty} \frac{c_\theta^\top}{||c_\theta||}\left(\int_{t_n}^{t_n+T}(\mathcal{H}_{\lambda_n(\tau)} \otimes I_2)\mathrm{d}\tau\right)\frac{c_\theta}{||c_\theta||} \\ &= \frac{c_\theta^\top}{||c_\theta||^2}(\int_0^T \lim_{n\to+\infty}(\mathcal{H}_{\lambda_n(\tau+t_n)} \otimes I_2)c_\theta \mathrm{d}\tau) \\ &= 0.\end{aligned}$$

A contradiction exists. Thus,

$$c_\theta = 0$$

and it implies that

$$\inf_{t\in\mathbb{R}} ||\bar{\theta}(t)|| = 0,$$

which verifies that condition 3 of Lemma 1 is satisfied; we have completed the proof. □

By Theorem 1, we obtain that subsystem (3.2b) is also uniformly globally exponentially stable at the origin. In fact, there exist $a > 0$, $b > 0$ such that

$$||\theta(t)|| < ae^{-b(t-s)}||\theta(s)||$$

for any $t > s > 0$. We can see that

$$||\tilde{e}(t)|| = ||(I_N \otimes e^{\Upsilon t})||||\theta(t)|| < a||(I_N \otimes e^{\Upsilon t})||e^{-b(t-s)}||\theta(s)|| < ace^{-b(t-s)}||\tilde{e}(s)||$$

for some $c > 0$ since Υ is marginally stable. The main theorem is given as follows.

Theorem 2. *Under Assumption 3, system* (3.2) *is uniformly globally exponentially stable at the origin. The synchronization control problem for a GUJC network can be achieved by the distributed controller* (3.1).

Proof: Since Ψ_a is Hurwitz by $k_1 > 0$ and $k_2 > 0$, there is a positive definite matrix P such that

$$\Psi_a^\top P + P\Psi_a \triangleq \Psi_c < 0.$$

We also denote

$$\Psi_d = \Psi_b^\top P + P\Psi_b.$$

Define

$$Y_1 = [(\sqrt{-\Psi_c}e)^\top, \tilde{e}^\top]^\top \tag{3.8}$$

as the virtual output of system (3.2). We claim that system (3.2) combined with (3.8) is in the output-injection form given by (2.2). Indeed, let $x = [e^\top, \tilde{e}^\top]^\top$, $y = Y_1$, and

$$\mathbf{A}_\lambda = 0,$$

$$\mathbf{B}_\lambda = \begin{bmatrix} \Psi_a & \Psi_b \\ 0 & I_N \otimes \Upsilon - \mu\mathcal{H}_\lambda \otimes I_2 \end{bmatrix},$$

and

$$\mathbf{C}_\lambda = \begin{bmatrix} \sqrt{-\Psi_c} \\ I_{2N} \end{bmatrix}^\top.$$

one has

$$\begin{aligned} &\|\mathbf{B}_\lambda x\| \\ &= \left\|\left[\Psi_a(\sqrt{-\Psi_c})^{-1}\sqrt{-\Psi_c}e + \Psi_b\tilde{e} + (I_N \otimes \Upsilon - \mu\mathcal{H}_\lambda \otimes I_2)\tilde{e}\right]\right\| \\ &\le \left\|\Psi_a(\sqrt{-\Psi_c})^{-1}\right\| \|\sqrt{-\Psi_c}e\| + \|\Psi_b\|\|\tilde{e}\| + \|I_N \otimes \Upsilon\|\|\tilde{e}\| + \max_{\xi\in\Lambda} \|\mu\mathcal{H}_\xi \otimes I_2\|\|\tilde{e}\| \\ &\le A(\|\sqrt{-\Psi_c}e\| + \|\tilde{e}\|) \\ &\le 2A(\|[\sqrt{-\Psi_c}e^\top, \tilde{e}^\top]^\top\|) \end{aligned}$$

with $A = \max\{\left\|\Psi_a(\sqrt{-\Psi_c})^{-1}\right\|\|, \|\Psi_b\|, \|I_N \otimes \Upsilon\|, \max_{\xi\in\Lambda}\|\mu\mathcal{H}_\xi \otimes I_2\|\}$. Then, system (3.2) combined with (3.8) is in the output injection form if we choose that $\zeta(x) = 2Ax$. Let $V_1 = e^\top Pe$; we obtain that

$$\dot{V}_1 = \dot{e}^\top Pe + e^\top P\dot{e} = e^\top\Psi_c e + e^\top\Psi_d\tilde{e}.$$

Thus, we have

$$\dot{V}_1 \le e^\top\Psi_c e + \frac{1}{2}(1 + V_1)\|\Psi_d\|ace^{-b(t-s)}\|\tilde{e}(s)\|.$$

Let

$$\gamma(t) = \frac{1}{2}\|\Psi_d\|ace^{-b(t-s)}\|\tilde{e}(s)\|.$$

We have

$$\int_s^{+\infty} \gamma(\tau)d\tau < \infty.$$

Therefore,

$$\dot{V}_1 \leq e^\top \Psi_c e + \gamma(1 + V_1).$$

Thus, let $\alpha_1(t) = V_1(t)$, $\alpha_2(t) = -e^\top(t)\Psi_c e(t)$ and $\alpha_3(t) = \gamma(t)$; we can apply [14, Lemma 1]. There exist $a_1 > 0$ and $b_1 > 0$ such that

$$V_1 \leq e^{a_1\gamma(s))(1+b_1\|e(s)\|^2} - 1$$

and then

$$\|e(t)\| \leq \sqrt{c_1(e^{a_1\gamma(s))(1+b_1\|e(s)\|^2} - 1)}$$

for some $c_1 > 0$. Thus,

$$\|[e^\top(t), \tilde{e}^\top(t)]^\top\| \leq \sqrt{c_1(e^{1/a_1\gamma(s))(1+b_1\|e(s)\|^2} - 1) + a^2c^2e^{2bs}\|\tilde{e}(s)\|^2}.$$

System (3.2) is then uniformly globally stable at the origin. By [14, Lemma 1] again,

$$\int_s^{+\infty} \|Y_1(\tau)\|^2 \,\mathrm{d}\tau = \int_s^{+\infty} e^\top(\tau)(-\Psi_c)e(\tau)\,\mathrm{d}\tau \leq \alpha_1(s) + (1 + \alpha_1(s))\,\varpi_1(s) \tag{3.9}$$

where

$$\varpi_1(s) = \int_s^{+\infty} \alpha_3(\tau)\mathrm{d}\tau e^{\int_s^{+\infty}\alpha_3(\tau)\mathrm{d}\tau}.$$

Therefore, by (3.9), the condition 2 in Lemma 1 is now satisfied. Next, we consider $[\bar{e}^\top, \bar{\tilde{e}}^\top]^\top : \mathbb{R} \to \mathbb{R}^{4N}$ as any bounded limiting zeroing-output solution of system (3.2). Then, there exist the sequences $\{\lambda_n\} \subseteq \Theta$ and $t_n \geq 2n$ such that

$$\begin{bmatrix} \sqrt{-\Psi_c}\bar{e}(t) \\ \bar{\tilde{e}}(t) \end{bmatrix} = 0. \tag{3.10}$$

for almost all t in $\mathbb{R}$. We have

$$\begin{bmatrix} \dot{\bar{e}} \\ \dot{\bar{\tilde{e}}} \end{bmatrix} = 0. \tag{3.11}$$

By (3.11), we obtain

$$\begin{bmatrix} \bar{e}(t) \\ \bar{\tilde{e}}(t) \end{bmatrix} = \begin{bmatrix} \bar{e}(0) \\ \bar{\tilde{e}}(0) \end{bmatrix}. \tag{3.12}$$

By (3.10), we have

$$\begin{bmatrix} \bar{e}(t) \\ \bar{\tilde{e}}(t) \end{bmatrix} = \begin{bmatrix} \sqrt{-\Psi_c}^{-1}\sqrt{-\Psi_c}\bar{e}(t) \\ \bar{\tilde{e}}(t) \end{bmatrix} = 0 \tag{3.13}$$

for almost all t in $\mathbb{R}$. Thus, by (3.12) and (3.13), one can obtain that

$$\begin{bmatrix} \bar{e}(t) \\ \bar{\tilde{e}}(t) \end{bmatrix} = 0. \tag{3.14}$$

Thus, the condition 3 of Lemma 1 is satisfied. Thus, system (3.2) is uniformly globally exponentially stable at the origin. Then, there exist $\gamma_1 > 0$, $\gamma_2 > 0$ such that

$$\|[e^\top(t), \tilde{e}^\top(t)]^\top\| \leq \gamma_1 e^{-\gamma_2(t-s)}\|[e^\top(s), \tilde{e}^\top(s)]^\top\|$$

for all $([e^\top, \tilde{e}^\top]^\top, \lambda) \in \Phi(\Theta)$ and $0 < s < t$. It follows that $\lim_{t\to\infty}(x_{1i}(t) - x_{10}(t)) = 0$ and $\lim_{t\to\infty}(x_{2i}(t) - x_{20}(t)) = 0$. In other words, the synchronization control is achieved by the distributed controller (3.1).

The proof is then completed. □

Remark 2. *Since the harmonic oscillators are heterogeneous in this paper, the classical controller in [9] is invalid. The controllers in [20, 21] were designed for a static network. In contrast, we consider the switching network in this paper. Controller* (3.1) *comes from controller (4) in [25] which is a typical distributed dynamic state feedback controller [25]*

$$\begin{cases} u_i = K_{1i}x_i + K_{2i}\eta_i \quad i = 1, \dots, N \\ \dot{\eta}_i = \Upsilon\eta_i - \mu \sum_{j\in\bar{\mathcal{V}}} a^{\lambda}_{ij}\left(\eta_i - \eta_j\right) \end{cases}, \tag{3.15}$$

where K_{1i} and K_{2i} are the gain matrices which can be determined as follows:

1, Select K_{1i} such that $A_i + B_iK_{1i}$ is Hurwitz.

2, Let K_{2i} be as follows:

$$K_{2i} = U_i - K_{1i}X_i, \quad i = 1, \dots, N$$

where X_i, U_i are the solutions of the linear matrix equations of (8) in [25]. In our paper, the corresponding gain matrices are

$$K_{1i} = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix}$$

and

$$K_{2i} = \begin{bmatrix} \beta_i - \beta + k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

We omit the computation for solving the linear matrix regulator equations. As compared to the stability result of [25] with dwell time in a switching network, our result has the advantage of uniform and exponential stability when subjected to a fast switching network.

Remark 3. *To show the stability with dwell time, it is possible to use the information of the system trajectory and find a common joint Lyapunov function [14] when the network is UJC with a dwell time. If there is no dwell time, only UJC network topology is assumed; then, the graph switching that occurs very fast makes the trajectory incomputable and the techniques relying on the trajectory cannot be used. The multi-systems lose controllability in the fast switching moment. Under the conditions of GUJC networks, V_1 cannot be a strict Lyapunov function and one cannot use the system trajectory since the network is not maintained where there is no dwell time; thus the stability analysis is not easy. With the help of the virtual output and newly developed stability analysis tool, we show the stability of system* (3.2). *In particular, the stable states are described by the limiting zeroing-output solution.*

4. Numerical examples

To show the correctness of the theoretical results, three examples are presented. Consider one leader oscillator and five heterogeneous harmonic oscillators. We set $\beta = 1$ and $\beta_i = i + 1$, $[x_1(0), x_2(0)] = [3, -1]$; also, $[x_{1i}(0), x_{2i}(0)] = [i, i]$, $[\eta_{1i}(0), \eta_{2i}(0)] = -[i, i]$ for all $1 \leq i \leq 5$.

The controller parameters are set as $\mu = 1$, $k_1 = 1$ and $k_2 = 2$. We consider the switching graphs $\bar{\mathcal{G}}_\xi$, with $\xi \in \{1, 2, 3, 4\}$ which are shown in Figure 1. It is worth noting that $\bar{\mathcal{G}}_4$ has an empty edge set and there is no connection among the oscillators. The arrow means that the two oscillators can communicate between each other. And we set

$$a^{\lambda}_{ij} = \begin{cases} 1, & \text{if oscillator } i \text{ connects with oscillator } j, \\ 0, & \text{if oscillator } i \text{ disconnects from oscillator } j. \end{cases}$$

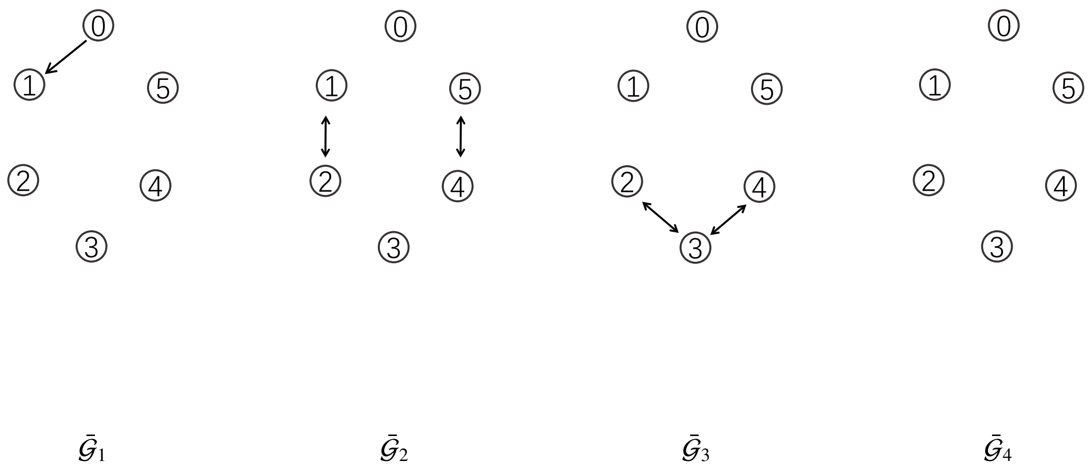

Figure 1. The communication topology for switching cases.

We investigate three cases of switching graphs as follows.

First, we consider the static graph case. The network graph is the joint graph of $\bar{\mathcal{G}}_\xi$ with $\xi \in \{1, 2, 3, 4\}$. The corresponding $\mathcal{H}$ is

$$\mathcal{H} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Figures 2 and 3 show the synchronization states and the errors, respectively.

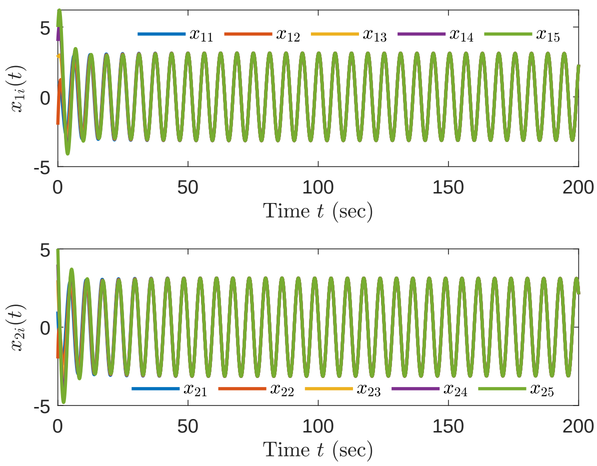

Figure 2. The synchronization states in the static graph case.

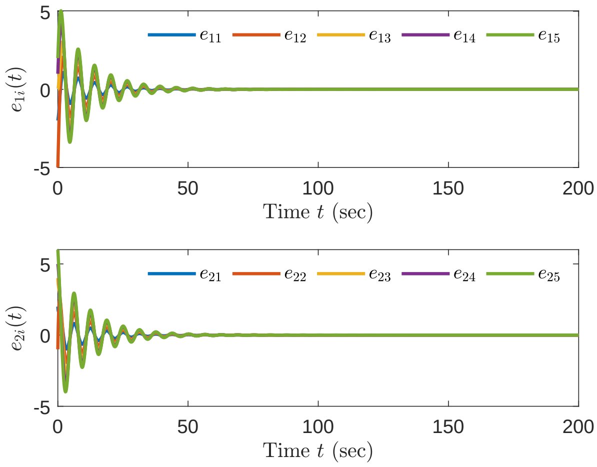

Figure 3. The synchronization errors in the static graph case.

Second, we adopt the following UJC network with a switching signal

$$\lambda_1(t) = \begin{cases} 1, & \text{if } nH \leq t < (n + 1/4)H \\ 2, & \text{if } (n + 1/4)H \leq t < (n + 2/4)H \\ 3, & \text{if } (n + 2/4)H \leq t < (n + 3/4)H \\ 4, & \text{if } (n + 3/4)H \leq t < (n + 1)H \end{cases}$$

with $H = 10$ and $n = 0, 1, 2, \ldots$. Then, the switching signal $\lambda_1(t)$ is UJC with a dwell time $\tau_0 = H/4$. The synchronization states and errors are shown in Figures 4 and 5, respectively.

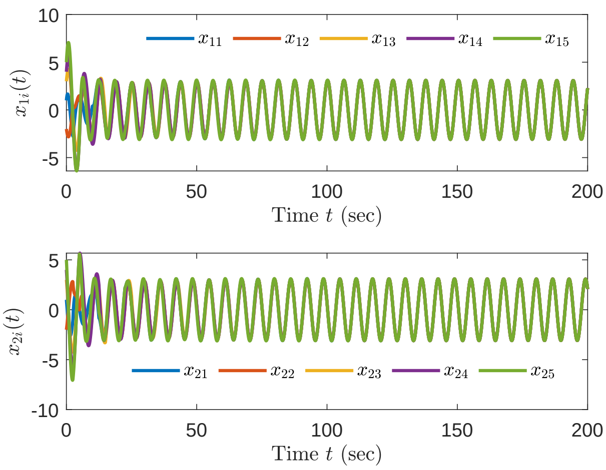

Figure 4. The synchronization states under the UJC condition with dwell time.

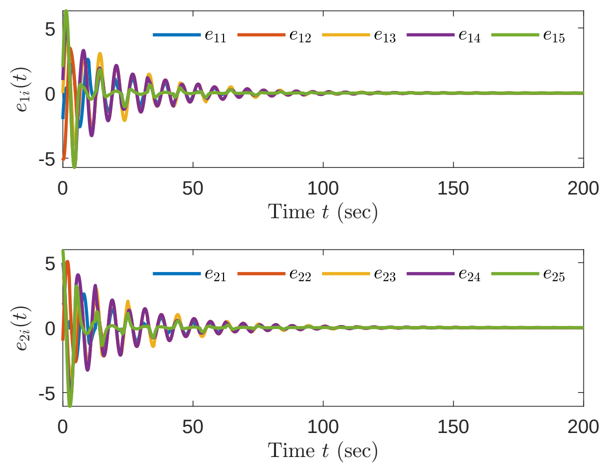

Figure 5. The synchronization errors under the UJC condition with dwell time.

Finally, we adopt the following GUJC network with a switching signal

$$\lambda_2(t) = \begin{cases} 1, & \text{if } (n + \frac{l}{m+1})H \le t < (n + \frac{l+1/4}{m+1})H \\ 2, & \text{if } (n + \frac{l+1/4}{m+1})H \le t < (n + \frac{l+2/4}{m+1})H \\ 3, & \text{if } (n + \frac{l+2/4}{m+1})H \le t < (n + \frac{l+3/4}{m+1})H \\ 4, & \text{if } (n + \frac{l+3/4}{m+1})H \le t < (n + \frac{l+1}{m+1})H \end{cases}$$

with $n = 0, 1, 2, \ldots$, $m = k$, $l = 0, 1, \ldots, m$ and $H = 10$. Then $\lambda_2(t)$ does not have any dwell time. The switching time slot converges to 0. The switching frequency tends to infinity. In other words, the network demonstrates Zeno switching, i.e., the switching happens infinitely in finite time. Assumption 3 holds with $\tau = H/8$ and $T = H$. The synchronization states and errors are shown in Figures 6 and 7, respectively.

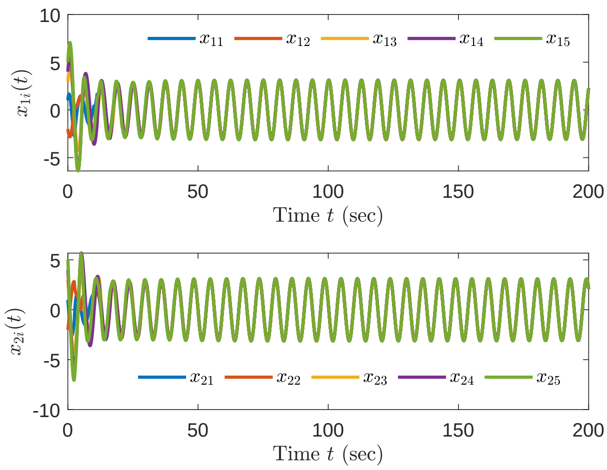

Figure 6. The synchronization states under the GUJC condition without dwell time.

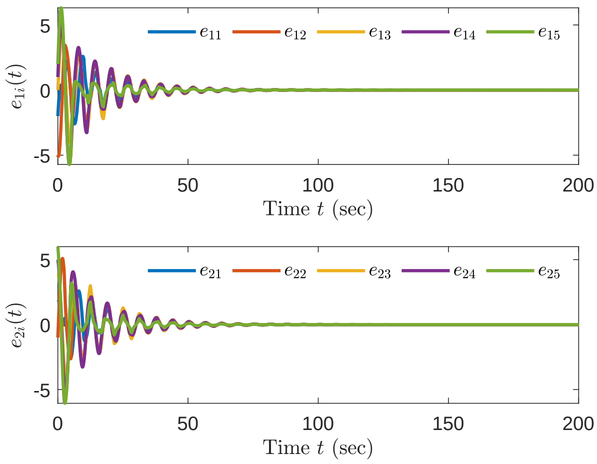

Figure 7. The synchronization errors under the GUJC condition without dwell time.

As shown in the simulation results, the control law (3.1) can solve the synchronization control problems in the three cases.

5. Conclusions

We have investigated the synchronization control problem for jointly connected switching networks for heterogeneous harmonic oscillators. The dwell-time assumption for the network is avoided, which makes the stability analysis challenging. By applying the generalized Krasovskii-LaSalle theorem, the stability is proved. In the future, we may further consider some more practical issues, such as control based on sampling in fast-switching networks.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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