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Research article

Projective class rings of the category of Yetter-Drinfeld modules over the 2-rank Taft algebra

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Abstract: In this paper, all simple Yetter-Drinfeld modules and indecomposable projective Yetter-Drinfeld modules over the 2-rank Taft algebra $\bar{\mathcal{A}}$ are construted and classified by Radford's method of constructing Yetter-Drinfeld modules over a Hopf algebra. Furthermore, the projective class ring of the category of Yetter-Drinfeld modules over $\bar{\mathcal{A}}$ is described explicitly by generators and relations.

Keywords: Yetter-Drinfeld module; tensor product; the projective class ring

1. Introduction

In 1998, Andruskiewitsch and Schneider (see [1]) introduced the *lifting method* which was extensively used in the classification of finite dimensional pointed and copointed Hopf algebras. In this process, Yetter-Drinfeld modules over a Hopf algebra (see [2]) play important roles. See for example [3–12]. Therefore, constructing Yetter-Drinfeld modules over a particular interesting Hopf algebra is very important.

In 2003, Radford [13] introduced a general technique to construct simple Yetter-Drinfeld modules. By this idea, one can get all simple Yetter-Drinfeld modules over a finite dimensional Hopf algebra. For example, Zhu and Chen [14] described all simple Yetter-Drinfeld modules over the Hopf Ore extension of the dihedral group $\mathcal{D}_n$. Xiong [15] (see also Zhang [16] in somewhat different idea) constructed all simple (resp. indecomposable projective) Yetter-Drinfeld modules and then described their corresponding projective class rings over a non-semisimple and non-pointed Hopf algebra H_{4n} which was firstly introduced in [17] and reconstructed via Hopf Ore extension of automorphism type in [18] by Yang and Zhang. It is remarked that all finite dimensional Hopf algebras over H_8 and H_{12} were given in [7] and [8] respectively. In [19], all Yetter-Drinfeld modules and their corresponding Grothendieck rings for a $2n^2$-dimensional semisimple Hopf algebra H_{2n^2} of Kac-Paljutkin type were constructed. In [20], all simple (resp. indecomposable projective) Yetter-Drinfeld modules and their corresponding projective class rings for a family of non-semisimple $8m$-dimensional Hopf algebra H_{8m}

of tame type with even number $m \geq 2$.

Motivated by the above works, we will firstly construct and classify all simple (resp. indecomposable projective) Yetter-Drinfeld modules over $\bar{\mathcal{A}}$ by the Radford technique where $\bar{\mathcal{A}}$ is the 2-rank Taft algebra which Green ring was constructed in [21]. It is remarked that $\bar{\mathcal{A}}$ is isomorphic to $H_2(0,-1)$ as Hopf algebras given by $f(x_1) = b, f(x_2) = c, f(y_1) = ab, f(y_2) = cd$ where a, b, c, d are generators of $H_n(p,q)$. Here, $H_n(p,q)$ is firstly constructed in [22] which is a family of n^4-dimensional Hopf algebras where $n \geq 2$ is an integer, p and q are scalars in the ground field and q is a primitive n-th root of unity. The second task of this paper is to describe the projective class rings of the category of the Yetter-Drinfeld modules over $\bar{\mathcal{A}}$. It is pointed that the projective class rings of $H_n(0,q)$ and $H_n(1,q)$ for $n > 2$ in [23] are desribed. Also, we note that the n-rank Taft algebra $\bar{\mathcal{A}}_q(n)$ based on the work in [24] was firstly introdcued in [25] in which its Drinfeld double $\mathcal{D}(\bar{\mathcal{A}}_q(n))$ as well as all simple modules of $\mathcal{D}(\bar{\mathcal{A}}_q(n))$ by the different method from the Radford technique, were constructed.

We organize the paper as follows. In Section 2, we recall some definitions and results. In Section 3, all simple (or indecomposable projective) Yetter-Drinfeld modules over $\bar{\mathcal{A}}$ are constructed. In Section 4, the indecomposable projective Yetter-Drinfeld modules corresponding to the simple Yetter-Drinfeld modules of $\bar{\mathcal{A}}$ are constructed and the tensor products of arbitrary simple and indecomposable projective Yetter-Drinfeld modules are established. Furthermore, the projective class ring of the category of the Yetter-Drinfeld modules over $\bar{\mathcal{A}}$ is described.

We firstly fix some notations. Throughout the paper, $\mathbb{K}$ is assumed to be an algebraically closed field of characteristic zero and $[\![a,b]\!] = \{a, a+1, \cdots, b\}$ for $a \leq b \in \mathbb{Z}$. The Sweedler notation

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$$

for a Hopf algebra is used and some other notations are referred to [26].

2. Preliminaries

The 2-rank Taft algebra $\bar{\mathcal{A}}$ was introduced in [21,22] which is generated by $x_i, y_j, i, j = 1, 2$, subject to the relations

$$x_i^2 = 1, \quad y_j^2 = 0, \quad x_1x_2 = x_2x_1, \quad y_1y_2 = -y_2y_1, \quad x_iy_j = -y_jx_i.$$

$\bar{\mathcal{A}}$ is a Hopf algebra with the coalgebra structure and the antipode given by

$$\begin{aligned} \Delta(x_i) &= x_i \otimes x_i, & \epsilon(x_i) &= 1, & S(x_i) &= x_i, \\ \Delta(y_i) &= y_i \otimes 1 + x_i \otimes y_i, & \epsilon(y_i) &= 0, & S(y_i) &= y_ix_i. \end{aligned}$$

Note that $\dim\bar{\mathcal{A}} = 16$ and $\left\{x_1^kx_2^ly_1^sy_2^t | k, l, s, t = 0, 1\right\}$ forms a $\mathbb{K}$-basis for $\bar{\mathcal{A}}$.

For the definition of n-rank Taft algebra $\bar{\mathcal{A}}_q(n)$, the authors are referred to [25]. It is well-known that among n-rank Taft algebras, the only 2-rank one is of tame type. From the viewpoint of representation theory, we are more interested in describing all simple (resp. indecomposable projective) modules in ${}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD}$. Note that $\bar{\mathcal{A}} = \bar{\mathcal{A}}_{-1}(2)$.

Lemma 2.1. [21] *There are* 4 *non-isomorphic* 1*-dimensional simple* $\bar{\mathcal{A}}$*-modules* $S(s_1, s_2)$ *with the basis* $\{v^{s_1,s_2}\}$. *The actions are given by*

$$x_i \cdot v^{s_1,s_2} = s_i v^{s_1,s_2}, \qquad y_i \cdot v^{s_1,s_2} = 0$$

where $i = 1, 2$ *and* $s_1, s_2 = \pm 1$.

Let ${}_H\mathcal{M}$ be the category of left H-modules. Recall that a left Yetter-Drinfeld H-module M for a finite dimensional Hopf algebra H is a left H-module $(M, \cdot)$ and a left H-comodule (M, ρ) satisfying

$$\rho(h \cdot m) = \sum h_{(1)} m_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}, \quad \forall m \in M, h \in H$$

where S is the antipode of H and $\rho(m) = \sum_{(m)} m_{(-1)} \otimes m_{(0)}$.

The category of left Yetter-Drinfeld H-modules is denoted by ${}_H^H\mathcal{YD}$, whose morphisms are both H-linear and H-colinear maps (see [2]). Let $V \in {}_H^H\mathcal{YD}$, the left dual V^* is defined by

$$\langle h \cdot f, v \rangle = \langle f, S(h)v \rangle, \quad f_{(-1)} \langle f_{(0)}, v \rangle = S^{-1}(v_{(-1)}) \langle f, v_{(0)} \rangle. \tag{2.1}$$

According to Radford's results in [13], we have the following results.

Proposition 2.2. [13] *If* $V, W \in {}_H^H\mathcal{YD}$ *then* $V \otimes W \in {}_H^H\mathcal{YD}$. *The actions and coactions are as follows:*

$$h \cdot (v \otimes \omega) = \sum_{(h)} h_{(1)} \cdot v \otimes h_{(2)} \cdot \omega, \quad \rho(v \otimes \omega) = \sum_{(v),(\omega)} v_{(-1)} \omega_{(-1)} \otimes v_{(0)} \otimes \omega_{(0)}$$

where $v \in V, \omega \in W, h \in H$.

Lemma 2.3. [13] *Let* $L \in {}_H\mathcal{M}$. *Then, we have*

(1) $H \otimes L \in {}_H^H\mathcal{YD}$; *the module and comodule actions are given by*

$$g \cdot (h \otimes l) = \sum_{(g)} g_{(1)} h S(g_{(3)}) \otimes g_{(2)} \cdot l, \tag{2.2}$$

$$\rho(h \otimes l) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes l, \quad \forall h, g \in H, l \in L. \tag{2.3}$$

(2) If M *is a simple Yetter-Drinfeld* H*-module, then* $M = H \cdot N$ *for some simple subcomodule* N *of* $H \otimes L$ *where* L *is a simple left* H*-module.*

Applying Proposition 2.2 and Lemma 2.3, we can construct all simple Yetter-Drinfeld modules over a specific Hopf algebra H, in particular, for $H = \bar{\mathcal{A}}$.

3. Simple Yetter-Drinfeld modules over $\bar{\mathcal{A}}$

Let $U(k, l, s_1, s_2) := \mathbb{K}\left\{x_1^k x_2^l \otimes v^{s_1,s_2}\right\}$ where $s_1, s_2 = \pm 1, k, l \in \mathbb{Z}_2$. We have

Lemma 3.1. *For* $s_1, s_2 = \pm 1$, *the set*

$$\{U(k, l, s_1, s_2) | k, l \in \mathbb{Z}_2\}$$

forms a complete set of all simple subcomodules of $\bar{\mathcal{A}} \otimes S(s_1, s_2)$.

Proof. Since $\dim(S(s_1,s_2)) = 1$, $\bar{\mathcal{A}} \otimes S(s_1,s_2) \cong \bar{\mathcal{A}}$ as (left) $\bar{\mathcal{A}}$-comodules. Since $\bar{\mathcal{A}}$ is pointed, any simple subcomodule of $\bar{\mathcal{A}}$ is equal to $\mathbb{K}g$ for some group-like element g in $\bar{\mathcal{A}}$ by [26, Corollary 5.1.8]. Thus, the lemma follows.

Let

$$I = \{(k,l,s_1,s_2) | k,l \in \mathbb{Z}_2, s_1, s_2 = \pm 1\}$$

and

$$\begin{aligned}
I_0 &= \left\{(k,l,s_1,s_2) | (k,l,s_1,s_2) \in I, s_1 = s_2 = (-1)^{k+l}\right\},\\
I_1 &= \left\{(k,l,s_1,s_2) | (k,l,s_1,s_2) \in I, s_1 = -s_2 = (-1)^{k+l}\right\},\\
I_2 &= \left\{(k,l,s_1,s_2) | (k,l,s_1,s_2) \in I, s_1 = -s_2 = (-1)^{k+l+1}\right\},\\
I_3 &= \left\{(k,l,s_1,s_2) | (k,l,s_1,s_2) \in I, s_1 = s_2 = (-1)^{k+l+1}\right\}.
\end{aligned}$$

It's easy to see that $I = \bigcup_{t=0}^{3} I_t$.

Now, we consider the Yetter-Drinfeld module $\bar{\mathcal{A}} \cdot U(k,l,s_1,s_2)$ by the rules given in Lemma 2.3 where $(k,l,s_1,s_2) \in I$.

(a) Assume that $(k,l,s_1,s_2) \in I_0$. In this case

$$\begin{aligned}
&x_i\left(x_1^k x_2^l \otimes v^{s_1,s_2}\right) = s_i x_1^k x_2^l \otimes v^{s_1,s_2}, \qquad y_i\left(x_1^k x_2^l \otimes v^{s_1,s_2}\right) = 0,\\
&\rho\left(x_1^k x_2^l \otimes v^{s_1,s_2}\right) = x_1^k x_2^l \otimes x_1^k x_2^l \otimes v^{s_1,s_2}.
\end{aligned}$$

We get 1-dimensional Yetter-Drinfeld modules $M_0(k,l) := \mathbb{K}\left\{\theta^{k,l}\right\}$ where $\theta^{k,l} = x_1^k x_2^l \otimes v^{(-1)^{k+l},(-1)^{k+l}}, k,l \in \mathbb{Z}_2$. The actions and the coactions are given by

$$x_i \cdot \theta^{k,l} = (-1)^{k+l}\theta^{k,l}, \qquad y_i \cdot \theta^{k,l} = 0, \qquad \rho(\theta^{k,l}) = x_1^k x_2^l \otimes \theta^{k,l}$$

where $i = 1,2$.

(b) Assume that $(k,l,s_1,s_2) \in I_1$. In this case

$$\begin{aligned}
x_i\left(x_1^k x_2^l \otimes v^{s_1,s_2}\right) &= (-1)^{k+l+i-1} x_1^k x_2^l \otimes v^{s_1,s_2},\\
y_i\left(x_1^k x_2^l \otimes v^{s_1,s_2}\right) &= \left(1 + (-1)^i\right)(-1)^{k+l} x_1^k x_2^l y_i \otimes v^{s_1,s_2},\\
x_i\left(x_1^k x_2^l y_2 \otimes v^{s_1,s_2}\right) &= (-1)^{k+l+i} x_1^k x_2^l y_2 \otimes v^{s_1,s_2},\\
y_i\left(x_1^k x_2^l y_2 \otimes v^{s_1,s_2}\right) &= 0,\\
\rho\left(x_1^k x_2^l \otimes v^{s_1,s_2}\right) &= x_1^k x_2^l \otimes x_1^k x_2^l \otimes v^{s_1,s_2},\\
\rho\left(x_1^k x_2^l y_2 \otimes v^{s_1,s_2}\right) &= x_1^k x_2^l y_2 \otimes x_1^k x_2^l \otimes v^{s_1,s_2} + x_1^k x_2^{l+1} \otimes x_1^k x_2^l y_2 \otimes v^{s_1,s_2}.
\end{aligned}$$

We get 2-dimensional Yetter-Drinfeld modules $M_1(k,l) := \mathbb{K}\left\{\mu_0^{k,l}, \mu_1^{k,l}\right\}$ where $\mu_0^{k,l} = x_1^k x_2^l \otimes v^{s_1,s_2}, \mu_1^{k,l} = x_1^k x_2^l y_2 \otimes v^{s_1,s_2}$, $s_1 = -s_2 = (-1)^{k+l}, k,l \in \mathbb{Z}_2$. The actions and the coactions are given by

$$x_i \cdot \mu_0^{k,l} = (-1)^{k+l+i-1}\mu_0^{k,l}, \qquad y_i \cdot \mu_0^{k,l} = 2\delta_{i,2}(-1)^{k+l}\mu_1^{k,l},$$

$$x_i \cdot \mu_1^{k,l} = (-1)^{k+l+i}\mu_1^{k,l}, \qquad y_i \cdot \mu_1^{k,l} = 0,$$
$$\rho(\mu_0^{k,l}) = x_1^k x_2^l \otimes \mu_0^{k,l}, \qquad \rho(\mu_1^{k,l}) = x_1^k x_2^l y_2 \otimes \mu_0^{k,l} + x_1^k x_2^{l+1} \otimes \mu_1^{k,l}$$

where $i = 1, 2$.

Similarly, assume that $(k, l, s_1, s_2) \in I_2$ and we get 2-dimensional Yetter-Drinfeld modules $M_2(k, l) := \mathbb{K}\left\{\nu_0^{k,l}, \nu_1^{k,l}\right\}$ where $\nu_0^{k,l} = x_1^k x_2^l \otimes \upsilon^{s_1,s_2}, \nu_1^{k,l} = x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_2}$, $s_1 = -s_2 = (-1)^{k+l+1}, k, l \in \mathbb{Z}_2$. The actions and the coactions are given by

$$x_i \cdot \nu_0^{k,l} = (-1)^{k+l+i}\nu_0^{k,l}, \qquad y_i \cdot \nu_0^{k,l} = 2\delta_{i,1}(-1)^{k+l}\nu_1^{k,l},$$
$$x_i \cdot \nu_1^{k,l} = (-1)^{k+l+i-1}\nu_1^{k,l}, \qquad y_i \cdot \nu_1^{k,l} = 0,$$
$$\rho(\nu_0^{k,l}) = x_1^k x_2^l \otimes \nu_0^{k,l}, \qquad \rho(\nu_1^{k,l}) = x_1^k x_2^l y_1 \otimes \nu_0^{k,l} + x_1^{k+1} x_2^l \otimes \nu_1^{k,l}$$

where $i = 1, 2$.

(c) Assume that $(k, l, s_1, s_2) \in I_3$. In this case

$$\begin{aligned}
x_i\left(x_1^k x_2^l \otimes \upsilon^{s_1,s_2}\right) &= (-1)^{k+l+1} x_1^k x_2^l \otimes \upsilon^{s_1,s_2},\\
y_i\left(x_1^k x_2^l \otimes \upsilon^{s_1,s_2}\right) &= 2(-1)^{k+l} x_1^k x_2^l y_i \otimes \upsilon^{s_1,s_2},\\
x_i\left(x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_2}\right) &= (-1)^{k+l} x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_2},\\
y_i\left(x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_2}\right) &= 2(-1)^{k+l+1} x_1^k x_2^l y_1 y_i \otimes \upsilon^{s_1,s_2},\\
x_i\left(x_1^k x_2^l y_2 \otimes \upsilon^{s_1,s_2}\right) &= (-1)^{k+l} x_1^k x_2^l y_2 \otimes \upsilon^{s_1,s_2},\\
y_i\left(x_1^k x_2^l y_2 \otimes \upsilon^{s_1,s_2}\right) &= 2(-1)^{k+l} x_1^k x_2^l y_i y_2 \otimes \upsilon^{s_1,s_2},\\
x_i\left(x_1^k x_2^l y_1 y_2 \otimes \upsilon^{s_1,s_2}\right) &= (-1)^{k+l+1} x_1^k x_2^l y_1 y_2 \otimes \upsilon^{s_1,s_2},\\
y_i\left(x_1^k x_2^l y_1 y_2 \otimes \upsilon^{s_1,s_2}\right) &= 0,\\
\rho\left(x_1^k x_2^l \otimes \upsilon^{s_1,s_2}\right) &= x_1^k x_2^l \otimes x_1^k x_2^l \otimes \upsilon^{s_1,s_2},\\
\rho\left(x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_2}\right) &= x_1^k x_2^l y_1 \otimes x_1^k x_2^l \otimes \upsilon^{s_1,s_2} + x_1^{k+1} x_2^l \otimes x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_2},\\
\rho\left(x_1^k x_2^l y_2 \otimes \upsilon^{s_1,s_2}\right) &= x_1^k x_2^l y_2 \otimes x_1^k x_2^l \otimes \upsilon^{s_1,s_2} + x_1^k x_2^{l+1} \otimes x_1^k x_2^l y_2 \otimes \upsilon^{s_1,s_2},\\
\rho\left(x_1^k x_2^l y_1 y_2 \otimes \upsilon^{s_1,s_2}\right) &= x_1^k x_2^l y_1 y_2 \otimes x_1^k x_2^l \otimes \upsilon^{s_1,s_2} - x_1^k x_2^{l+1} y_1 \otimes x_1^k x_2^l y_2 \otimes \upsilon^{s_1,s_2}\\
&\quad + x_1^{k+1} x_2^l y_2 \otimes x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_2} + x_1^{k+1} x_2^{l+1} \otimes x_1^k x_2^l y_1 y_2 \otimes \upsilon^{s_1,s_2}.
\end{aligned}$$

We get 4-dimensional Yetter-Drinfeld modules $M_3(k, l) := \mathbb{K}\left\{\eta_0^{k,l}, \eta_1^{k,l}, \eta_2^{k,l}, \eta_3^{k,l}\right\}$ where $\eta_0^{k,l} = x_1^k x_2^l \otimes \upsilon^{s_1,s_1}, \eta_1^{k,l} = x_1^k x_2^l y_1 \otimes \upsilon^{s_1,s_1}, \eta_2^{k,l} = x_1^k x_2^l y_2 \otimes \upsilon^{s_1,s_1}, \eta_3^{k,l} = x_1^k x_2^l y_1 y_2 \otimes \upsilon^{s_1,s_1}$, $s_1 = (-1)^{k+l+1}, k, l \in \mathbb{Z}_2$. The actions and the coactions are given by

$$x_i \cdot \eta_0^{k,l} = (-1)^{k+l+1}\eta_0^{k,l}, \qquad y_i \cdot \eta_0^{k,l} = 2(-1)^{k+l}\eta_i^{k,l},$$
$$x_i \cdot \eta_1^{k,l} = (-1)^{k+l}\eta_1^{k,l}, \qquad y_i \cdot \eta_1^{k,l} = 2\delta_{i,2}(-1)^{k+l+1}\eta_3^{k,l},$$
$$x_i \cdot \eta_2^{k,l} = (-1)^{k+l}\eta_2^{k,l}, \qquad y_i \cdot \eta_2^{k,l} = 2\delta_{i,1}(-1)^{k+l}\eta_3^{k,l},$$
$$x_i \cdot \eta_3^{k,l} = (-1)^{k+l+1}\eta_3^{k,l}, \qquad y_i \cdot \eta_3^{k,l} = 0,$$
$$\rho\left(\eta_0^{k,l}\right) = x_1^k x_2^l \otimes \eta_0^{k,l},$$

$$\rho\left(\eta_1^{k,l}\right) = x_1^k x_2^l y_1 \otimes \eta_0^{k,l} + x_1^{k+1} x_2^l \otimes \eta_1^{k,l},$$
$$\rho\left(\eta_2^{k,l}\right) = x_1^k x_2^l y_2 \otimes \eta_0^{k,l} + x_1^k x_2^{l+1} \otimes \eta_2^{k,l},$$
$$\rho\left(\eta_3^{k,l}\right) = x_1^k x_2^l y_1 y_2 \otimes \eta_0^{k,l} + x_1^{k+1} x_2^l y_2 \otimes \eta_1^{k,l} - x_1^k x_2^{l+1} y_1 \otimes \eta_2^{k,l} + x_1^{k+1} x_2^{l+1} \otimes \eta_3^{k,l}$$

where $i = 1, 2$.

In summary, we have

Theorem 3.2. *The set*

$$\{M_t(k,l) \,|\, k, l \in \mathbb{Z}_2, t \in [\![0,3]\!]\}$$

forms a complete list of non-isomorphic simple Yetter-Drinfeld modules over $\bar{\mathcal{A}}$.

Proof. (1) Let $f_0 : M_0(k,l) \to M_0(k',l'), \theta^{k,l} \mapsto a\theta^{k',l'}$ be a Yetter-Drinfeld module isomorphism where $k, l, k', l' \in \mathbb{Z}_2$ and $0 \neq a \in \mathbb{K}$. Then, for $i = 1, 2$ we have

$$x_i \cdot f_0(\theta^{k,l}) = (-1)^{k'+l'} a\theta^{k',l'} = f_0(x_i \cdot \theta^{k,l}) = (-1)^{k+l} a\theta^{k',l'},$$
$$(\mathrm{id} \otimes f_0)\rho(\theta^{k,l}) = a x_1^k x_2^l \otimes \theta^{k',l'} = \rho(f_0(\theta^{k,l})) = a x_1^{k'} x_2^{l'} \otimes \theta^{k',l'}.$$

Hence, $k = k', l = l'$. Therefore, $M_0(k,l) \cong M_0(k',l')$ if and only if $k = k', l = l'$.

(2) Let $0 \neq L$ be a simple Yetter-Drinfeld submodule of $M_1(k,l)$ and $0 \neq v = a_0\mu_0^{k,l} + a_1\mu_1^{k,l} \in L$ where $k, l \in \mathbb{Z}_2, a_0, a_1 \in \mathbb{K}$. Then, we have

$$\rho(v) = (a_0 x_1^k x_2^l + a_1 x_1^k x_2^l y_2) \otimes \mu_0^{k,l} + a_1 x_1^k x_2^{l+1} \otimes \mu_1^{k,l}.$$

Since $a_0 x_1^k x_2^l + a_1 x_1^k x_2^l y_2$ and $x_1^k x_2^{l+1}$ are linearly independent, $\mu_0^{k,l} \in L$. Moreover,

$$y_2 \cdot \mu_0^{k,l} = 2(-1)^{k+l}\mu_1^{k,l} \in L.$$

Thus, $L = M_1(k,l)$. Therefore, $M_1(k,l)$ is a simple Yetter-Drinfeld module. Similarly, $M_2(k,l)$ is also a simple Yetter-Drinfeld module.

Let

$$\begin{aligned} f_1 : & M_1(k,l) \to M_1(k',l') \\ & \mu_0^{k,l} \mapsto b_0\mu_0^{k',l'} + b_1\mu_1^{k',l'} \\ & \mu_1^{k,l} \mapsto b_2\mu_0^{k',l'} + b_3\mu_1^{k',l'} \end{aligned}$$

be a Yetter-Drinfeld module isomorphism where $k, l, k', l' = 0, 1$ and $b_0, b_1, b_2, b_3 \in \mathbb{K}$. Then, we have

$$y_2 \cdot f_1(\mu_1^{k,l}) = 2(-1)^{k'+l'} b_2\mu_1^{k',l'} = f_1(y_2 \cdot \mu_1^{k,l}) = 0,$$
$$\begin{aligned} (\mathrm{id} \otimes f_1)\rho(\mu_0^{k,l}) &= x_1^k x_2^l \otimes (b_0\mu_0^{k',l'} + b_1\mu_1^{k',l'}) = \rho(f_1(\mu_0^{k,l})) \\ &= \left(b_0 x_1^{k'} x_2^{l'} + b_1 x_1^{k'} x_2^{l'} y_2\right) \otimes \mu_0^{k',l'} + b_1 x_1^{k'} x_2^{l'+1} \otimes \mu_1^{k',l'}. \end{aligned}$$

Hence, $b_1 = b_2 = 0, b_0, b_3 \neq 0, k = k', l = l'$. Therefore, $M_1(k,l) \cong M_1(k',l')$ if and only if $k = k', l = l'$. Similarly, we have $M_2(k,l) \cong M_2(k',l')$ if and only if $k = k', l = l'$.

Let

$$\begin{aligned} f_2 : & M_1(k,l) \to M_2(k',l') \\ & \mu_0^{k,l} \mapsto c_0\nu_0^{k',l'} + c_1\nu_1^{k',l'} \\ & \mu_1^{k,l} \mapsto c_2\nu_0^{k',l'} + c_3\nu_1^{k',l'} \end{aligned}$$

be a Yetter-Drinfeld module isomorphism where $k, l, k', l' = 0, 1$ and $c_0, c_1, c_2, c_3 \in \mathbb{K}$. Then, for $i = 1, 2$ we have

$$\begin{aligned} y_1 \cdot f_2(\mu_1^{k,l}) &= 2(-1)^{k'+l'} c_2 \nu_1^{k',l'} = f_2(y_1 \cdot \mu_1^{k,l}) = 0, \\ (\mathrm{id} \otimes f_2)\rho(\mu_0^{k,l}) &= x_1^k x_2^l \otimes (c_0\nu_0^{k',l'} + c_1\nu_1^{k',l'}) = \rho(f_2(\mu_0^{k,l})) \\ &= \left(c_0 x_1^{k'} x_2^{l'} + c_1 x_1^{k'} x_2^{l'} y_1\right) \otimes \nu_0^{k',l'} + c_1 x_1^{k'+1} x_2^{l'} \otimes \nu_1^{k',l'}. \end{aligned}$$

Hence, $c_1 = c_2 = 0$ and

$$\begin{aligned} (\mathrm{id} \otimes f_2)\rho(\mu_1^{k,l}) &= c_0 x_1^k x_2^l y_2 \otimes \nu_0^{k',l'} + c_3 x_1^k x_2^{l+1} \otimes \nu_1^{k',l'} = \rho(f_2(\mu_1^{k,l})) \\ &= c_3 \left(x_1^{k'} x_2^{l'} y_1 \otimes \nu_0^{k',l'} + x_1^{k'+1} x_2^{l'} \otimes \nu_1^{k',l'}\right). \end{aligned}$$

Then, $c_0 = c_3 = 0$. Therefore, $M_1(k,l) \not\cong M_2(k',l')$ for $k, l, k', l' \in \mathbb{Z}_2$.

(3) Let $0 \neq U$ be a simple Yetter-Drinfeld submodule of $M_3(k,l)$ and $0 \neq u = \sum_{t=0}^{3} d_t \eta_t^{k,l} \in U$ where $d_0, d_1, d_2, d_3 \in \mathbb{K}, k, l = 0, 1$. Then,

$$\begin{aligned} \rho(u) = &(d_0 x_1^k x_2^l + d_1 x_1^k x_2^l y_1 + d_2 x_1^k x_2^l y_2 + d_3 x_1^k x_2^l y_1 y_2) \otimes \eta_0^{k,l} \\ &+ (d_1 x_1^{k+1} x_2^l + d_3 x_1^{k+1} x_2^l y_2) \otimes \eta_1^{k,l} \\ &+ (d_2 x_1^k x_2^{l+1} - d_3 x_1^k x_2^{l+1} y_1) \otimes \eta_2^{k,l} \\ &+ d_3 x_1^{k+1} x_2^{l+1} \otimes \eta_3^{k,l}. \end{aligned}$$

Since $u \neq 0$, the vector $(d_0, d_1, d_2, d_3) \neq 0$, we have $\eta_0^{k,l} \in U$. Hence for $i = 1, 2$, $y_i \cdot \eta_0^{k,l} = 2(-1)^{k+l}\eta_i^{k,l} \in U$ and $y_1 \cdot \eta_2^{k,l} = 2(-1)^{k+l}\eta_3^{k,l} \in U$. Therefore, $U = M_3(k,l)$ and $M_3(k,l)$ is a simple Yetter-Drinfeld module.

Let $f_3 : M_3(k,l) \to M_3(k',l'), \eta_s^{k,l} \mapsto \sum_{t=0}^{3} c_{s,t}\eta_t^{k',l'}$ be a Yetter-Drinfeld module isomorphism where $s \in [\![0, 3]\!]$, $k, l, k', l' \in \mathbb{Z}_2$ and $c_{s,t} \in \mathbb{K}$. Hence, for $i = 1, 2$ we have

$$y_i \cdot f_3(\eta_0^{k,l}) = f_3(y_i \cdot \eta_0^{k,l}), \tag{3.1}$$

$$y_i \cdot f_3(\eta_1^{k,l}) = f_3(y_i \cdot \eta_1^{k,l}), \tag{3.2}$$

$$(\mathrm{id} \otimes f_3)\rho(\eta_1^{k,l}) = \rho\left(f_3(\eta_1^{k,l})\right). \tag{3.3}$$

By (3.1) we have

$$\begin{aligned} y_i f_3\left(\eta_0^{k,l}\right) &= y_i \sum_{t=0}^{3} c_{0,t}\eta_t^{k',l'} \\ &= c_{0,0} 2(-1)^{k'+l'}\eta_i^{k',l'} + \left(c_{0,1} 2(-1)^{k'+l'+1}(i-1) + c_{0,2} 2(-1)^{k'+l'}(2-i)\right)\eta_3^{k',l'} \end{aligned}$$

$$= f_3\left(y_i \cdot \eta_0^{k,l}\right) = 2(-1)^{k+l} \sum_{t=0}^{3} c_{i,t}\eta_t^{k',l'}.$$

Then,

$$\begin{aligned}
&c_{1,0} = c_{1,2} = c_{2,0} = c_{2,1} = 0, \quad c_{0,0}(-1)^{k'+l'} = c_{1,1}(-1)^{k+l} = c_{2,2}(-1)^{k+l},\\
&c_{0,2}(-1)^{k'+l'} = c_{1,3}(-1)^{k+l}, \qquad c_{0,1}(-1)^{k'+l'+1} = c_{2,3}(-1)^{k+l}.
\end{aligned}$$

By (3.2) we have

$$\begin{aligned}
y_i \cdot f_3(\eta_1^{k,l}) &= c_{1,1}2(-1)^{k'+l'+1}(i-1)\eta_3^{k',l'}\\
&= f_3\left(y_i \cdot \eta_1^{k,l}\right) = 2(-1)^{k+l+1}(i-1)\sum_{t=0}^{3}\eta_t^{k',l'}.
\end{aligned}$$

Then,

$$c_{3,0} = c_{3,1} = c_{3,2} = 0, \quad c_{1,1}(-1)^{k'+l'} = c_{3,3}(-1)^{k+l}.$$

By (3.3) we have

$$\begin{aligned}
(\mathrm{id}\otimes f_3)\rho\left(\eta_1^{k,l}\right) &= c_{0,0}x_1^k x_2^l y_1 \otimes \eta_0^{k',l'} + \left(c_{0,1}x_1^k x_2^l y_1 + c_{1,1}x_1^{k+1}x_2^l\right)\otimes \eta_1^{k',l'}\\
&\quad + c_{0,2}x_1^k x_2^l y_1 \otimes \eta_2^{k',l'} + \left(c_{0,3}x_1^k x_2^l y_1 + c_{1,3}x_1^{k+1}x_2^l\right)\otimes \eta_3^{k',l'}\\
&= \rho\left(f_3\left(\eta_1^{k,l}\right)\right)\\
&= \left(c_{1,1}x_1^{k'}x_2^{l'}y_1 + c_{1,3}x_1^{k'}x_2^{l'}y_1y_2\right)\otimes \eta_0^{k',l'} + \left(c_{1,1}x_1^{k'+1}x_2^{l'} + c_{1,3}x_1^{k'+1}x_2^{l'}y_2\right)\otimes \eta_1^{k',l'}\\
&\quad - c_{1,3}x_1^{k'}x_2^{l'+1}y_1 \otimes \eta_2^{k',l'} + c_{1,3}x_1^{k'+1}x_2^{l'+1}\otimes \eta_3^{k',l'}.
\end{aligned}$$

Then, $c_{0,1} = c_{0,2} = c_{0,3} = c_{1,3} = c_{2,3} = 0, c_{0,0} = c_{1,1} = c_{2,2} = c_{3,3} \neq 0, k = k', l = l'$. Therefore, $M_3(k,l) \cong M_3(k',l')$ if and only if $k = k', l = l'$.

By (1)–(3), the set

$$\{M_t\,(k,l)\,|k,l \in \mathbb{Z}_2, t \in [\![0,3]\!]\}$$

forms a complete list of non-isomorphic simple Yetter-Drinfeld modules over $\bar{\mathcal{A}}$.

Corollary 3.3. *(1) $M_0(k,l)^* = M_0(k,l),\ M_i(k,l)^* = M_i(k+i-1, l+i)$ for $k,l \in \mathbb{Z}_2, i = 1,2$.*
(2) $M_3(k,l)^* = M_3(k+1,l+1)$ for $k,l \in \mathbb{Z}_2$.

Proof. We only give the proof of (2), the other cases are similar.

Let $\left\{\varphi_t^{k,l} | t \in [\![0,3]\!]\right\}$ be the dual basis of $\left\{\eta_t^{k,l} | t \in [\![0,3]\!]\right\}$ such that

$$\begin{array}{llll}
\varphi_0^{k,l}(\eta_0^{k,l}) = 0, & \varphi_0^{k,l}(\eta_1^{k,l}) = 0, & \varphi_0^{k,l}(\eta_2^{k,l}) = 0, & \varphi_0^{k,l}(\eta_3^{k,l}) = 1,\\
\varphi_1^{k,l}(\eta_0^{k,l}) = 0, & \varphi_1^{k,l}(\eta_1^{k,l}) = 0, & \varphi_1^{k,l}(\eta_2^{k,l}) = (-1)^{k+l}, & \varphi_1^{k,l}(\eta_3^{k,l}) = 0,\\
\varphi_2^{k,l}(\eta_0^{k,l}) = 0, & \varphi_2^{k,l}(\eta_1^{k,l}) = (-1)^{k+l-1}, & \varphi_2^{k,l}(\eta_2^{k,l}) = 0, & \varphi_2^{k,l}(\eta_3^{k,l}) = 0,\\
\varphi_3^{k,l}(\eta_0^{k,l}) = 1, & \varphi_3^{k,l}(\eta_1^{k,l}) = 0, & \varphi_3^{k,l}(\eta_2^{k,l}) = 0, & \varphi_3^{k,l}(\eta_3^{k,l}) = 0.
\end{array}$$

By (2.1) we have

$$\begin{aligned}
&x_i\cdot\varphi_0^{k,l}=(-1)^{k+l+1}\varphi_0^{k,l}, && y_i\cdot\varphi_0^{k,l}=2(-1)^{k+l}\varphi_i^{k,l},\\
&x_i\cdot\varphi_1^{k,l}=(-1)^{k+l}\varphi_1^{k,l}, && y_i\cdot\varphi_1^{k,l}=2\delta_{i,2}(-1)^{k+l+1}\varphi_3^{k,l},\\
&x_i\cdot\varphi_2^{k,l}=(-1)^{k+l}\varphi_2^{k,l}, && y_i\cdot\varphi_2^{k,l}=2\delta_{i,1}(-1)^{k+l}\varphi_3^{k,l},\\
&x_i\cdot\varphi_3^{k,l}=(-1)^{k+l+1}\varphi_3^{k,l}, && y_i\cdot\varphi_3^{k,l}=0,\\
&\rho\left(\varphi_0^{k,l}\right)=x_1^{k+1}x_2^{l+1}\otimes\varphi_0^{k,l},\\
&\rho\left(\varphi_1^{k,l}\right)=x_1^{k+1}x_2^{l+1}y_1\otimes\varphi_0^{k,l}+x_1^kx_2^{l+1}\otimes\varphi_1^{k,l},\\
&\rho\left(\varphi_2^{k,l}\right)=x_1^{k+1}x_2^{l+1}y_2\otimes\varphi_0^{k,l}+x_1^{k+1}x_2^{l}\otimes\varphi_2^{k,l},\\
&\rho\left(\varphi_3^{k,l}\right)=x_1^{k+1}x_2^{l+1}y_1y_2\otimes\varphi_0^{k,l}+x_1^kx_2^{l+1}y_2\otimes\varphi_1^{k,l}-x_1^{k+1}x_2^ly_1\otimes\varphi_2^{k,l}+x_1^kx_2^l\otimes\varphi_3^{k,l}
\end{aligned}$$

where $i=1,2$. Hence $M_3(k,l)^*=M_3(k+1,l+1)$.

4. Projective class rings of ${}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD}$

Assume that H is a finite dimensional Hopf algebra. Let $F(H)$ be the free abelian group generated by the isomorphic classes $[M]$ of H-modules M. Then, the abelian group $F(H)$ becomes a ring equipped with a multiplication given by the tensor product $[M][N]=[M\otimes N]$. The Green ring $r(H)$ is defined to be the quotient ring of $F(H)$ module the relations $[M\oplus N]=[M]+[N]$. The projective class ring of H is the subring of $r(H)$ generated by its simple and projective modules. As is known, ${}^H_H\mathcal{YD}\cong{}_{\mathcal{D}(H^{cop})}\mathcal{M}$. where ${}_{\mathcal{D}(H^{cop})}\mathcal{M}$ is the category of the left modules of Drinfeld double $\mathcal{D}(H^{\rm cop})$. For this reason, we have the projective class ring of ${}^H_H\mathcal{YD}$ which is denoted by $r_P({}^H_H\mathcal{YD})$.

In this section, we briefly set $\mathcal{D}:=\mathcal{D}(\bar{\mathcal{A}}^{\rm cop})$. Let $\mathcal{P}(V)$ be the projective cover of a simple $\mathcal{D}$-module V, or equivalently, a simple Yetter-Drinfeld module $V\in{}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD}$. Let $\mathrm{Irr}(\mathcal{D})$ be the set of isomorphism classes of simple $\mathcal{D}$-modules. One sees that

$$\mathcal{D}\cong\bigoplus_{V\in\mathrm{Irr}(\mathcal{D})}\mathcal{P}(V)^{\oplus\dim V}$$

and $\mathcal{D}$ is unimodular and quasi-triangular (see [26,27]).

For convenience, we let

$$A=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad C_i=\begin{pmatrix}0&0&0&0\\\delta_{i,1}&0&0&0\\\delta_{i,2}&0&0&0\\0&-\delta_{i,2}&\delta_{i,1}&0\end{pmatrix},$$

$$D=\begin{pmatrix}1&0&0&0\\y_1&x_1&0&0\\y_2&0&x_2&0\\y_1y_2&x_1y_2&-x_2y_1&x_1x_2\end{pmatrix},\quad E=\begin{pmatrix}x_1x_2y_1&0&0&0\\0&-x_2y_1&0&0\\x_1x_2y_1y_2&0&-x_1y_1&0\\0&-x_2y_1y_2&0&y_1\end{pmatrix},$$

$$F=\begin{pmatrix}x_1x_2y_2&0&0&0\\-x_1x_2y_1y_2&-x_2y_2&0&0\\0&0&-x_1y_2&0\\0&0&-x_1y_1y_2&y_2\end{pmatrix},\quad G=\begin{pmatrix}x_1x_2y_1y_2&0&0&0\\0&x_2y_1y_2&0&0\\0&0&x_1y_1y_2&0\\0&0&0&y_1y_2\end{pmatrix},$$

$$A' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B'_i = \begin{pmatrix} x_i & 0 & 0 & 0 \\ -x_iy_{3-i} & x_1x_2 & 0 & 0 \\ x_iy_i & 0 & 1 & 0 \\ x_iy_iy_{3-i} & x_1x_2y_i & y_{3-i} & x_1^{i-1}x_2^i \end{pmatrix},$$

$$C' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D'_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ y_{3-i} & x_1^{i-1}x_2^i & 0 & 0 \\ y_i & 0 & x_i & 0 \\ -y_iy_{3-i} & x_1^{i-1}x_2^iy_i & -x_iy_{3-i} & x_1x_2 \end{pmatrix}$$

where $i \in [\![1, 2]\!]$.

By a complex computation, we can obtain the following three classes of Yetter-Drinfeld modules for $\bar{\mathcal{A}}$.

(1) $\mathcal{P}_0(k, l), k, l \in \mathbb{Z}_2$: it is of a basis $\{u_0, u_1, \cdots, u_{15}\}$, the matrices of the action and coaction of $\bar{\mathcal{A}}$ on $\mathcal{P}_0(k, l)$ are given as follows.

$$[x_i] = (-1)^{k+l} \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -A & 0 & 0 \\ 0 & 0 & -A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}, \quad [y_i] = 2(-1)^{k+l} \begin{pmatrix} -C_i & 0 & 0 & 0 \\ \delta_{i,1}B & C_i & 0 & 0 \\ \delta_{i,2}B & 0 & C_i & 0 \\ 0 & -\delta_{i,2}B & \delta_{i,1}B & -C_i \end{pmatrix},$$

$$\rho(W) = x_1^kx_2^l \begin{pmatrix} x_1x_2D & 0 & 0 & 0 \\ E & x_2D & 0 & 0 \\ F & 0 & x_1D & 0 \\ G & x_1F & -x_2E & D \end{pmatrix} \otimes W$$

where $W = (u_0, u_1, \cdots, u_{15})^T$.

(2) $\mathcal{P}(k, l, j), k, l \in \mathbb{Z}_2, j \in [\![1, 2]\!]$: it is of a basis $\{p_0, p_1, \cdots, p_7\}$, the matrices of the action and coaction of $\bar{\mathcal{A}}$ on $\mathcal{P}(k, l, j)$ are given as follows.

$$[x_i] = (-1)^{k+l+i+j} \begin{pmatrix} -A & 0 \\ 0 & A \end{pmatrix}, \qquad [y_i] = (-1)^{k+l} \begin{pmatrix} 2\delta_{i,3-j}A' & \delta_{i,j}C' \\ 0 & 2\delta_{i,3-j}A' \end{pmatrix},$$

$$\rho(P) = x_1^kx_2^l \begin{pmatrix} B'_j & 0 \\ 0 & D'_j \end{pmatrix} \otimes P$$

where $P = (p_0, p_1, \cdots, p_7)^T$.

(3) $T_j(k, l), k, l \in \mathbb{Z}_2, j \in [\![1, 2]\!]$: it is of a basis $\{v_0^j, v_1^j, v_2^j, v_3^j\}$, the matrices of the action and coaction of $\bar{\mathcal{A}}$ on $T_j(k, l)$ are given as follows.

$$[x_i] = (-1)^{k+l-1}A, \quad [y_i] = \delta_{i,3-j}(-1)^{k+l} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(U) = x_1^kx_2^l \begin{pmatrix} x_{3-j} & 0 & 0 & 0 \\ x_{3-j}y_{3-j} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y_{3-j} & x_{3-j} \end{pmatrix} \otimes U$$

where $U = (v_0^j, v_1^j, v_2^j, v_3^j)^T$.

Lemma 4.1. *For $k, l \in \mathbb{Z}_2$, $j \in [\![1, 2]\!]$, $\mathcal{P}_0(k, l)$, $\mathcal{P}(k, l, j)$ and $T_j(k, l)$ are indecomposable Yetter-Drinfeld modules over $\bar{\mathcal{A}}$.*

Proof. Suppose that $\mathcal{P}(k, l, j)$ is not indecomposable. Then, there exist two non-trivial submodules M and N such that $\mathcal{P}(k, l, j) = M \oplus N$. We claim that $p_7 \notin M$ and $p_7 \notin N$. If $p_7 \in M$ then $p_4, p_5, p_6 \in M$ since

$$\rho(p_7) = -x_1^k x_2^l y_j y_{3-j} \otimes p_4 + x_1^{k+j-1} x_2^{l+j} y_j \otimes p_5 - x_1^k x_2^l x_j y_{3-j} \otimes p_6 + x_1^{k+j-1} x_2^{l+j} x_j \otimes p_7$$

which implies that

$$\begin{aligned} p_0 &= \frac{(-1)^{k+l}}{2} y_j \cdot p_4 \in M, \\ p_1 &= \frac{(-1)^{k+l-1}}{2} y_j \cdot p_5 \in M, \\ p_2 &= (-1)^{k+l} y_j \cdot p_6 \in M, \\ p_3 &= (-1)^{k+l-1} y_j \cdot p_7 \in M. \end{aligned}$$

Then, $M = \mathcal{P}(k, l, j)$. It's a contradiction. Similarly, if $p_7 \in N$ then $N = \mathcal{P}(k, l, j)$, a contradiction too. Therefore, we may assume $p = \sum_{i=0}^{6} \alpha_i p_i + p_7 \in M$ for some $\alpha_i \in \mathbb{K}$, $i \in [\![0, 6]\!]$. Then,

$$\begin{aligned} \rho(p) = {} & x_1^k x_2^l x_j \otimes (\alpha_0 p_0 + \alpha_6 p_6) - x_1^k x_2^l x_j y_{3-j} \otimes (\alpha_1 p_0 + p_6) \\ & + x_1^k x_2^l x_j y_j \otimes \alpha_2 p_0 + x_1^k x_2^l x_j y_j y_{3-j} \otimes \alpha_3 p_0 \\ & + x_1^{k+j-1} x_2^{l+j} x_j \otimes (\alpha_1 p_1 + p_7) + x_1^{k+j-1} x_2^{l+j} x_j y_j \otimes \alpha_3 p_1 \\ & + x_1^k x_2^l \otimes (\alpha_2 p_2 + \alpha_4 p_4) + x_1^k x_2^l y_{3-j} \otimes (\alpha_3 p_2 + \alpha_5 p_4) \\ & + x_1^{k+j-1} x_2^{l+j} \otimes (\alpha_3 p_3 + \alpha_5 p_5) + x_1^k x_2^l y_j \otimes \alpha_6 p_4 \\ & - x_1^k x_2^l y_j y_{3-j} \otimes p_4 + x_1^{k+j-1} x_2^{l+j} y_j \otimes p_5. \end{aligned}$$

Hence, $p_5, \alpha_1 p_1 + p_7 \in M$ which implies that

$$p_7 = (\alpha_1 p_1 + p_7) - \alpha_1 p_1 = (\alpha_1 p_1 + p_7) - \frac{(-1)^{k+l-1}}{2} \alpha_1 y_j \cdot p_5 \in M.$$

It is a contradiction. Therefore, $\mathcal{P}(k, l, j)$ is an indecomposable Yetter-Drinfeld module over $\bar{\mathcal{A}}$. Similarly, one can check that $\mathcal{P}_0(k, l)$ and $T_j(k, l)$ are also indecomposable Yetter-Drinfeld modules over $\bar{\mathcal{A}}$.

The results are followed.

It is easy to see $\mathcal{P}(k, l, j)$, $\mathcal{P}_0(k, l)$ and $T_j(k, l)$, $k, l \in \mathbb{Z}_2$, $j = 1, 2$ are non-isomorphic to each other. Denote $\mathcal{P}_i(k, l) = \mathcal{P}(k + i, l + i - 1, i)$ for $i \in [\![1, 2]\!]$, $k, l \in \mathbb{Z}_2$.

Lemma 4.2. *For $k, l \in \mathbb{Z}_2$, $i \in [\![0, 2]\!]$, $\mathcal{P}(M_i(k, l)) \cong \mathcal{P}_i(k, l)$ and $\mathcal{P}(M_3(k, l)) \cong M_3(k, l)$.*

Proof. It is well known that $\mathcal{D}$ is a symmetric algebra [28] and every projective module is injective. Then, $\mathcal{P}(M_0(k, l)) = E(M_t(k, l))$ for some $t \in \mathbb{Z}_4$ and the socle and top of $\mathcal{P}(M_0(k, l))$ coincides. Therefore,

$\mathcal{P}(M_0(k,l)) \cong E(M_0(k,l))$. By Lemma 4.1, $\mathcal{P}_0(k,l)$ is an indecomposable module with $\mathrm{Soc}\mathcal{P}_0(k,l) \cong M_0(k,l)$. Thus, $\mathcal{P}_0(k,l)$ embeds in $E(M_0(k,l))$ which implies that $\dim \mathcal{P}(M_0(k,l)) \geq \dim \mathcal{P}_0(k,l) = 16$.

Similarly, $\dim \mathcal{P}(M_i(k,l)) \geq \dim \mathcal{P}_i(k,l) = 8$ since $\mathrm{Soc}\mathcal{P}_i(k,l) \cong M_i(k,l)$ for $i = 1,2$.

It follows that

$$\begin{aligned}\dim\mathcal{D} &= \sum_{k,l=0}^{1} \dim\mathcal{P}(M_0(k,l)) + 2\sum_{k,l=0}^{1}\sum_{i=1}^{2} \dim\mathcal{P}(M_i(k,l)) + 4\sum_{k,l=0}^{1} \dim\mathcal{P}(M_3(k,l)) \\ &\geq 4\times 16 + 2\times 8\times 8 + 4\times 4\times 4 = 256.\end{aligned}$$

Hence, we only have $\dim \mathcal{P}(M_0(k,l)) = 16$, $\dim \mathcal{P}(M_1(k,l)) = \dim \mathcal{P}(M_2(k,l)) = 8$, $\dim \mathcal{P}(M_3(k,l)) = 4$.

Therefore, $\mathcal{P}(M_i(k,l)) \cong \mathcal{P}_i(k,l)$ and $\mathcal{P}(M_3(k,l)) \cong M_3(k,l)$ for $i \in [\![0,2]\!]$.

Corollary 4.3. *We have*

$$_{\mathcal{D}}\mathcal{D} \cong \bigoplus_{k,l=0}^{1} \left(\mathcal{P}_0(k,l) \oplus \mathcal{P}_1(k,l)^{\oplus 2} \oplus \mathcal{P}_2(k,l)^{\oplus 2} \oplus M_3(k,l)^{\oplus 4}\right).$$

By Corollary 4.3, it is obvious that $T_i(k,l)$ is not projective for $i = 1,2$.

Note that the tensor products of Yetter-Drinfeld modules are commutative since $\mathcal{D}$ is quasi-triangular.

Lemma 4.4. *For $k,k',l,l' \in \mathbb{Z}_2$, we have*

(1) $M_0(k,l)\otimes M_t(k',l') \cong M_t(k+k',l+l')$ where $t \in [\![0,3]\!]$.
(2) $M_0(k,l)\otimes \mathcal{P}_i(k',l') \cong \mathcal{P}_i(k+k',l+l')$ where $i \in [\![0,2]\!]$.

Proof. (1) The results of (1) can be obtained by a direct computation.

(2) By (1) we have

$$\mathcal{P}_0(k+k',l+l') = \mathcal{P}(M_0(k,l)\otimes M_0(k',l')) \subseteq M_0(k,l)\otimes \mathcal{P}_0(k',l').$$

By Lemma 4.2 we have $\dim \mathcal{P}_0(k+k',l+l') = \dim(M_0(k,l)\otimes \mathcal{P}_0(k',l'))$ and $M_0(k,l)\otimes \mathcal{P}_0(k',l')$ is projective. Hence, $M_0(k,l)\otimes \mathcal{P}_0(k',l') \cong \mathcal{P}(k+k',l+l')$. Similarly, we have $M_0(k,l)\otimes \mathcal{P}_1(k',l') \cong \mathcal{P}_1(k+k',l+l')$ and $M_0(k,l)\otimes \mathcal{P}_2(k',l') \cong \mathcal{P}_2(k+k',l+l')$.

Lemma 4.5. *For $k,k',k'',l,l',l'' \in \mathbb{Z}_2, i = 1,2$, we have*

(1) $M_1(k,l)\otimes M_2(k',l') \cong M_3(k+k',l+l')$.
(2) $M_i(k,l)\otimes M_i(k',l') \cong T_i(k+k',l+l')$.
(3) $M_i(k,l)\otimes M_i(k',l')\otimes M_i(k'',l'')$
$\cong M_i(k+k'+k''+i-1,l+l'+l''+i)^{\oplus 2} \oplus M_i(k+k'+k'',l+l'+l'')^{\oplus 2}$.
(4) $M_i(k,l)\otimes M_3(k',l') \cong \mathcal{P}_{3-i}(k+k'+i-1,l+l'+i)$.
(5) $M_3(k,l)\otimes M_3(k',l') \cong \mathcal{P}_0(k+k'+1,l+l'+1)$.

Proof. The statements (1), (2) can be obtained by a direct computation.

On the other hand, by Lemma 4.4 (1) it suffices to prove the cases for $M_i(0,0)^{\otimes 3}$, $M_i(0,0)\otimes M_3(0,0)$ and $M_3(0,0)\otimes M_3(0,0)$.

For the statement (3), note that $T_j(0,0) = M_j(0,0) \otimes M_j(0,0)$, $j = 1,2$. Let $\{u_0^j, u_1^j\}$ be a basis of $M_j(0,0)$. The actions on the basis $\{u_0^j, u_1^j\}$ and the coactions are given by

$$\begin{aligned} x_i \cdot u_0^j &= (-1)^{i+j} u_0^j, & y_i \cdot u_0^j &= 2\delta_{i,3-j} u_1^j, \\ x_i \cdot u_1^j &= (-1)^{i+j-1} u_1^j, & y_i \cdot u_1^j &= 0, \\ \rho(u_0^j) &= 1 \otimes u_0^j, & \rho(u_1^j) &= y_{3-j} \otimes u_0^j + x_{3-j} \otimes u_1^j. \end{aligned}$$

where $i = 1,2$. Let

$$\begin{aligned} \lambda_0^j &= u_0^j \otimes v_0^j, & \lambda_1^j &= u_1^j \otimes v_0^j, \\ \lambda_2^j &= u_0^j \otimes v_1^j + u_1^j \otimes v_0^j, & \lambda_3^j &= u_1^j \otimes v_1^j, \\ \lambda_4^j &= u_0^j \otimes v_2^j, & \lambda_5^j &= u_1^j \otimes v_2^j - u_0^j \otimes v_0^j, \\ \lambda_6^j &= u_1^j \otimes v_2^j - u_0^j \otimes v_3^j, & \lambda_7^j &= u_1^j \otimes v_3^j - u_0^j \otimes v_1^j - u_1^j \otimes v_0^j \end{aligned}$$

and $L_t = \mathbb{K}\lambda_{2t}^j + \mathbb{K}\lambda_{2t+1}^j$ where $t \in [\![0,3]\!]$. By a direct computation, we have $L_0 \cong L_3 \cong M_j(j-1, j)$, $L_1 \cong L_2 \cong M_j(0,0)$. Therefore,

$$M_j(0,0)^{\otimes 3} \cong M_j(j-1,j)^{\oplus 2} \oplus M_j(0,0)^{\oplus 2}.$$

For the statement (4), we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(M_i(0,0) \otimes M_3(0,0),\ M_{3-i}(k,l)) &\cong \mathrm{Hom}_{\mathcal{D}}(M_3(0,0), M_{3-i}(k,l) \otimes M_i(0,0)^*) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(M_3(0,0), M_{3-i}(k,l) \otimes M_i(i-1,i)) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(M_3(0,0), M_3(k+i-1, l+i)) \end{aligned}$$

where $i = 1,2$. By Schur's lemma, $\mathrm{Hom}_{\mathcal{D}}(M_i(0,0) \otimes M_3(0,0), M_{3-i}(k,l)) \neq 0$ if and only if $k = i-1, l = i$. Since $M_i(0,0) \otimes M_3(0,0)$ is projective and $\dim(M_i(0,0) \otimes M_3(0,0)) = \dim \mathcal{P}_{3-i}(0,1) = 8$, we get that $M_i(0,0) \otimes M_3(0,0) \cong \mathcal{P}_{3-i}(i-1,i)$.

Similarly for the statement (5) we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(M_3(0,0) \otimes M_3(0,0),\ M_0(k,l)) &\cong \mathrm{Hom}_{\mathcal{D}}(M_3(0,0), M_0(k,l) \otimes M_3(0,0)^*) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(M_3(0,0), M_0(k,l) \otimes M_3(1,1)) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(M_3(0,0), M_3(k+1, l+1)). \end{aligned}$$

Hence, $M_3(0,0) \otimes M_3(0,0) \cong \mathcal{P}_0(1,1)$.

Lemma 4.6. *For $k, k', k'', l, l', l'' \in \mathbb{Z}_2$, $i = 1,2$ we have*

(1) $M_i(k,l) \otimes \mathcal{P}_i(k',l') \cong \mathcal{P}_0(k+k'+i-1, l+l'+i)$.
(2) $M_i(k,l) \otimes \mathcal{P}_{3-i}(k',l') \cong M_3(k+k', l+l')^{\oplus 2} \oplus M_3(k+k'+i-1, l+l'+i)^{\oplus 2}$.

Proof. It suffices to prove the lemma for $M_i(0,0) \otimes \mathcal{P}_i(0,0)$ and $M_i(0,0) \otimes \mathcal{P}_{3-i}(0,0)$ by Lemma 4.4 (1) (2), where $i = 1,2$.

(1) We have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(M_i(0,0)\otimes\mathcal{P}_i(0,0),\ M_0(k,l)) \cong&\mathrm{Hom}_{\mathcal{D}}(\mathcal{P}_i(0,0), M_0(k,l)\otimes M_i(0,0)^*)\\ \cong&\mathrm{Hom}_{\mathcal{D}}(\mathcal{P}_i(0,0), M_0(k,l)\otimes M_i(i-1,i))\\ \cong&\mathrm{Hom}_{\mathcal{D}}(\mathcal{P}_i(0,0), M_i(k+i-1,l+i)).\end{aligned}$$

Hence, $M_i(0,0)\otimes\mathcal{P}_i(0,0)\cong\mathcal{P}_0(i-1,i)$.

(2) Let $N_{-1}=0, N_t=\sum\limits_{s=0}^{t}\mathbb{K}v_s^i$ for $t\in[\![0,3]\!]$, where $i\in[\![1,2]\!]$. One can check that

$$0=N_{-1}\subset N_0\subset N_1\subset N_2\subset N_3=T_i(k,l)$$

is a Yetter-Drinfeld submodules chain of $T_i(k,l)$ such that N_t/N_{t-1} is a one dimensional Yetter-Drinfeld module and

$$N_0\cong N_3/N_2\cong M_0(k+i-1,l+i),\quad N_1/N_0\cong N_2/N_1\cong M_0(k,l).$$

Hence, by Lemma 4.5 (4) we have

$$\begin{aligned}M_i(0,0)\otimes\mathcal{P}_{3-i}(0,0)\cong& M_i(0,0)\otimes M_i(0,0)\otimes M_3(i-1,i)\cong T_i(0,0)\otimes M_3(i-1,i)\\ \cong&\left(M_0(i-1,i)^{\oplus 2}\oplus M_0(0,0)^{\oplus 2}\right)\otimes M_3(i-1,i)\\ \cong& M_3(0,0)^{\oplus 2}\oplus M_3(i-1,i)^{\oplus 2}.\end{aligned}$$

Lemma 4.7. *For $k,k',k'',l,l',l''\in\mathbb{Z}_2, i=1,2$ we have*

(1) $M_3(k,l)\otimes\mathcal{P}_i(k',l')\cong M_{3-i}(k,l)\otimes\mathcal{P}_0(k'+1,l'+1)$
$\cong\mathcal{P}_{3-i}(k+k'+1,l+l'+1)^{\oplus 2}\oplus\mathcal{P}_{3-i}(k+k'+i-1,l+l'+i)^{\oplus 2}.$

(2) $\mathcal{P}_i(k,l)\otimes\mathcal{P}_i(k',l')\cong\mathcal{P}_0(k+k'+1,l+l'+1)^{\oplus 2}\oplus\mathcal{P}_0(k+k'+i-1,l+l'+i)^{\oplus 2}.$

(3) $\mathcal{P}_1(k,l)\otimes\mathcal{P}_2(k',l')\cong M_3(k,l)\otimes\mathcal{P}_0(k',l')\cong\bigoplus\limits_{s,t=0}^{1}M_3(s,t)^{\oplus 4}.$

(4) $\mathcal{P}_i(k,l)\otimes\mathcal{P}_0(k',l')\cong\bigoplus\limits_{s,t=0}^{1}\mathcal{P}_i(s,t)^{\oplus 4}.$

(5) $\mathcal{P}_0(k,l)\otimes\mathcal{P}_0(k',l')\cong\bigoplus\limits_{s,t=0}^{1}\mathcal{P}_0(s,t)^{\oplus 4}.$

Proof. It suffices to prove the lemma for $M_3(0,0)\otimes\mathcal{P}_i(0,0)$, $M_3(0,0)\otimes\mathcal{P}_0(0,0)$, $\mathcal{P}_i(0,0)\otimes\mathcal{P}_i(0,0)$, $\mathcal{P}_1(0,0)\otimes\mathcal{P}_2(0,0)$, $\mathcal{P}_i(0,0)\otimes\mathcal{P}_0(0,0)$ and $\mathcal{P}_0(0,0)\otimes\mathcal{P}_0(0,0)$ by Lemma 4.4 (1) (2).

(1) Let $V_{-1}=0, V_t=\sum_{s=0}^{t}(\mathbb{K}p_{2s}+\mathbb{K}p_{2s+1})$ for $t\in[\![0,3]\!]$. One can check that

$$0=V_{-1}\subset V_0\subset V_1\subset V_2\subset V_3=\mathcal{P}(k,l,i)=\mathcal{P}_i(k+i,l+i-1)$$

is a Yetter-Drinfeld submodules chain of $\mathcal{P}_i(k+i,l+i-1)$ such that V_t/V_{t-1} is a two dimensional Yetter-Drinfeld module and

$$V_0\cong V_3/V_2\cong M_i(k+i,l+i-1),\qquad V_1/V_0\cong V_2/V_1\cong M_i(k,l).$$

Hence,

$$M_3(0,0) \otimes \mathcal{P}_i(0,0) \cong M_3(0,0) \otimes \left(M_i(0,0)^{\oplus 2} \oplus M_i(i,i-1)^{\oplus 2}\right) \cong \mathcal{P}_{3-i}(i-1,i)^{\oplus 2} \oplus \mathcal{P}_{3-i}(1,1)^{\oplus 2}$$
$$\cong M_3(0,0) \otimes M_3(i,i-1) \otimes M_{3-i}(0,0) \cong \mathcal{P}_0(i-1,i) \otimes M_{3-i}(0,0).$$

Therefore,

$$M_3(0,0) \otimes \mathcal{P}_i(0,0) \cong M_0(i,i-1) \otimes \mathcal{P}_0(i-1,i) \otimes M_{3-i}(0,0) \cong \mathcal{P}_0(1,1) \otimes M_{3-i}(0,0)$$

since

$$\mathcal{P}_{3-i}(i-1,i)^{\oplus 2} \oplus \mathcal{P}_{3-i}(1,1)^{\oplus 2} \cong M_0(i,i-1) \otimes (\mathcal{P}_{3-i}(i-1,i)^{\oplus 2} \oplus \mathcal{P}_{3-i}(1,1)^{\oplus 2}).$$

(2) By Lemma 4.6 (1), we have

$$\mathcal{P}_i(0,0) \otimes \mathcal{P}_i(0,0) \cong \mathcal{P}_i(0,0) \otimes \left(M_i(0,0)^{\oplus 2} \oplus M_i(i,i-1)^{\oplus 2}\right) \cong \mathcal{P}_0(i-1,i)^{\oplus 2} \oplus \mathcal{P}_0(1,1)^{\oplus 2}.$$

(3) By Lemma 4.5 (1) (4) (5) and Lemma 4.6 (2), we have

$$\mathcal{P}_1(0,0) \otimes \mathcal{P}_2(0,0) \cong \mathcal{P}_1(0,0) \otimes \left(M_2(0,0)^{\oplus 2} \oplus M_2(0,1)^{\oplus 2}\right) \cong \bigoplus_{s,t=0}^{1} M_3(s,t)^{\oplus 2}$$
$$\cong M_2(1,0) \otimes M_3(0,0) \otimes M_1(0,1) \otimes M_3(0,0) \cong M_3(0,0) \otimes \mathcal{P}_0(0,0).$$

(4) By (1), we have

$$\mathcal{P}_i(0,0) \otimes \mathcal{P}_0(0,0) \cong \left(M_i(0,0)^{\oplus 2} \oplus M_i(i,i-1)^{\oplus 2}\right) \otimes \mathcal{P}_0(0,0) \cong \bigoplus_{s,t=0}^{1} \mathcal{P}_i(s,t)^{\oplus 4}.$$

(5) Let

$$\begin{array}{llll}
\kappa_0 = u_3 - u_6 + u_9 + u_{12}, & \kappa_1 = u_7 + u_{13}, & \kappa_2 = u_{11} + u_{14}, & \kappa_3 = u_{15}, \\
\kappa_4 = u_1 - u_4, & \kappa_5 = u_5, & \kappa_6 = u_8 - u_2, & \kappa_7 = u_{10}, \\
\kappa_8 = u_0, & \kappa_9 = u_6 - u_3, & \kappa_{10} = u_3 + u_9, & \kappa_{11} = u_{11} - u_{14}, \\
\kappa_{12} = u_1, & \kappa_{13} = u_2, & \kappa_{14} = u_7, & \kappa_{15} = u_6,
\end{array}$$

and $K_{-1} = 0$, $K_t = \sum_{s=0}^{t} \mathbb{K}\kappa_s$ for $t \in [\![0,15]\!]$. One sees that

$$0 = K_{-1} \subset K_0 \subset K_1 \subset \cdots \subset K_{15} = \mathcal{P}_0(k,l)$$

is a Yetter-Drinfeld submodules chain of $\mathcal{P}_0(k,l)$ such that K_t/K_{t-1} is a one dimensional Yetter-Drinfeld module and

$$\begin{array}{ll}
K_s/K_{s-1} \cong M_0(k,l), \qquad \text{if } s = 0,9,10,15; & K_s/K_{s-1} \cong M_0(k,l+1), \qquad \text{if } s = 2,4,11,12; \\
K_s/K_{s-1} \cong M_0(k+1,l), \text{ if } s = 1,6,13,14; & K_s/K_{s-1} \cong M_0(k+1,l+1), \text{ if } s = 3,5,7,8.
\end{array}$$

Hence,

$$\mathcal{P}_0(0,0) \otimes \mathcal{P}_0(0,0) \cong \mathcal{P}_0(0,0) \otimes \left(\bigoplus_{s,t=0}^{1} M_0(s,t)^{\oplus 4}\right) \cong \bigoplus_{s,t=0}^{1} \mathcal{P}_0(s,t)^{\oplus 4}.$$

The proof is finished.

Now, we can describe the projective class rings of ${}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD}$.

Let $a_1 = [M_0(1,0)], a_2 = [M_0(0,1)], b_1 = [M_1(0,0)], b_2 = [M_2(0,0)]$.

Lemma 4.8. *The following statements hold in $r_P({}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD})$.*

(1) For $k, l \in \mathbb{Z}_2, i = 1, 2$,

$$\begin{aligned}
&[M_0(k,l)] = a_1^k a_2^l, && [M_i(k,l)] = a_1^k a_2^l b_i, && [M_3(k,l)] = a_1^k a_2^l b_1 b_2,\\
&[\mathcal{P}_i(k,l)] = a_1^{k+i} a_2^{l+i+1} b_1^i b_2^{3-i}, && [\mathcal{P}_0(k,l)] = a_1^{k+1} a_2^{l+1} b_1^2 b_2^2. &&
\end{aligned}$$

(2) For $i = 1, 2$,

$$a_i^2 = 1, \quad b_i^3 = 2(1 + a_1^{i-1} a_2^i) b_i.$$

Proof. The results are easy to get from Lemma 4.4 and Lemma 4.5.

Corollary 4.9. *The following set is a $\mathbb{Z}$-basis of $r_P({}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD})$:*

$$\left\{a_1^k a_2^l b_1^s b_2^t | k, l \in [\![0, 1]\!], s, t \in [\![0, 2]\!]\right\}.$$

Proof. By Lemma 4.8(2), $a_i^2 = 1$ and $b_i^3 = 2(1 + a_1^{i-1} a_2^i) b_i$ for all $i \in [\![1, 2]\!]$. By Lemma 4.5(2) we have $b_i^3 = 2(1 + a_1^{i-1} a_2^i) b_i$ for $i = 1, 2$. Hence, the highest degree of b_i is 2. It is easy to check that the set

$$\left\{a_1^k a_2^l b_1^s b_2^t | k, l \in [\![0, 1]\!], s, t \in [\![0, 2]\!]\right\}$$

is an independent set since $\#\{a_1^k a_2^l b_1^s b_2^t | k, l, s, t \in \mathbb{Z}_2\} = 36$, the number of $\mathbb{Z}$-basis of $r_P({}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD})$. Therefore,

$$\left\{a_1^k a_2^l b_1^s b_2^t | k, l \in [\![0, 1]\!], s, t \in [\![0, 2]\!]\right\}$$

is a $\mathbb{Z}$-basis of $r_P({}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD})$.

The results of this section is as follows.

Theorem 4.10. *The projective class ring $r_P({}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD})$ is isomorphic to the quotient ring of the ring $\mathbb{Z}[x_1, x_2, y_1, y_2]$ module the ideal I generated by the following elements*

$$x_i^2 - 1, \quad y_i^3 - 2(1 + x_1^{i-1} x_2^i) y_i, \quad i = 1, 2. \tag{4.1}$$

Proof. By Corollary 4.9, there is a unique ring epimorphism

$$\Phi : \mathbb{Z}[x_1, x_2, y_1, y_2] \to r_P({}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD})$$

such that $\Phi(x_i) = a_i, \Phi(y_i) = b_i$ for $i = 1, 2$. By Lemma 4.8(2) we have

$$\begin{aligned}
&\Phi(x_i^2 - 1) = a_i^2 - 1 = 0,\\
&\Phi(y_i^3 - 2(1 + x_1^{i-1} x_2^i) y_i) = b_i^3 - 2(1 + a_1^{i-1} a_2^i) b_i = 0.
\end{aligned}$$

It follows that $\Phi(I) = 0$ and hence Φ induces a natural ring epimorphism

$$\overline{\Phi} : \mathbb{Z}[x_1, x_2, y_1, y_2]/I \to r_P({}^{\bar{\mathcal{A}}}_{\bar{\mathcal{A}}}\mathcal{YD})$$

such that $\overline{\Phi}(\overline{v}) = \Phi(v)$ for all $v \in \mathbb{Z}[x_1, x_2, y_1, y_2]$, where $\overline{v} = v + I$. It is straightforward to check that the ring $\mathbb{Z}[x_1, x_2, y_1, y_2]/I$ is $\mathbb{Z}$-spanned by

$$\left\{\overline{x_1}^k \overline{x_2}^l \overline{y_1}^s \overline{y_2}^t | k, l \in [\![0, 1]\!], s, t \in [\![0, 2]\!]\right\}.$$

This means the $\mathbb{Z}$-rank of $\mathbb{Z}[x_1, x_2, y_1, y_2]/I$ is 36. Hence, we get the ring isomorphism $\overline{\Phi}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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