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Research article

Fredholm inversion around a singularity: Application to autoregressive time series in Banach space

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Abstract: This paper considers inverting a holomorphic Fredholm operator pencil. Specifically, we provide necessary and sufficient conditions for the inverse of a holomorphic Fredholm operator pencil to have a simple pole and a second order pole. Based on these results, a closed-form expression of the Laurent expansion of the inverse around an isolated singularity is obtained in each case. As an application, we also obtain a suitable extension of the Granger-Johansen representation theorem for random sequences taking values in a separable Banach space. Due to our closed-form expression of the inverse, we may fully characterize solutions to a given autoregressive law of motion except a term that depends on initial values.

Keywords: operator pencils; analytic Fredholm theorem; linear time-invariant dynamical systems; Granger-Johansen representation theorem

1. Introduction

The Granger-Johansen representation theorem (see [1]) is a result on the existence and representation of solutions to a given autoregressive law of motion. Due to the crucial contributions by [2–5], we already have a well developed representation theory in finite dimensional Euclidean space. It is worth mentioning that, in the latter two works, the representation theorem is obtained in the framework of analytic function theory; [4] obtains a necessary and sufficient condition for a matrix-valued function of a single complex variable (matrix pencil), which characterizes an autoregressive law of motion, to have a simple pole at one and shows that this leads to nonstationary I(1) solutions, which become stationary through first-order differencing. The monograph of [5] provides a systematic reworking and extension of [4] and contains a representation theorem associated with nonstationary I(2) solutions, which become stationary through second-order differencing; see also [6, 7] for more general results on this topic.

More recently, the Granger-Johansen representation theorem was extended to infinite dimensional function spaces (see, e.g., [8]). As in [4], it turns out that the desired representation can be obtained by inverting the operator pencil which characterizes the autoregressive law of motion at an isolated singularity; the reader is referred to [9, 10] for general Hilbert-valued time series, [11] for density-valued time series, and [12] for general Banach-valued time series. Of course, this is certainly not the only example where inversion of operator pencils can be useful in applied fields.

In this paper, we consider inverting holomorphic Fredholm operator pencils around an isolated singularity. Specifically, we first obtain necessary and sufficient conditions for the inverse of a holomorphic Fredholm pencil to have a simple pole and a second order pole. We then obtain a closed-form expression of the inverse by deriving a recursive formula that determines all the coefficients in the Laurent expansion of the inverse around an isolated singularity. We apply our theoretical results to obtain a suitable version of the Granger-Johansen representation theorem; our version of the theorem distinguishes itself from existing ones by placing a stronger emphasis on presenting more detailed mathematical expressions of I(1) and I(2) solutions, rather than focusing on their cointegration properties, which have already been developed in the aforementioned literature. Of course, this is achieved through the use of our closed-form expression of the inverse of a holomorphic Fredholm pencil. We believe that this application demonstrates the usefulness of our theoretical results.

While the local behavior around an isolated singularity of the inverse of a Fredholm operator pencil has been explored in the context of the Granger-Johansen representation theorem in a Hilbert space (see, e.g., [9, 10]), this paper appears to be the first to provide a full characterization of the inverse specifically around a pole of order 1 and 2. Considering the recent extension of the Granger-Johansen representation theorem to incorporate function-valued highly integrated processes (see [13]), obtaining a closed-form expression of the inverse around a pole of an arbitrary order would be important. Furthermore, our closed-form expression is derived by leveraging some special spectral properties of Fredholm operator pencils. It would also be interesting to explore whether a similar characterization can be achieved for more general non-Fredholm operator pencils. However, these pursuits fall outside the scope of this paper and are left for future research.

The remainder of the paper is organized as follows. In Section 2, we review some essential mathematics. In Section 3, we study in detail inversion of a holomorphic Fredholm pencil based on the analytic Fredholm theorem; our main results are obtained in this section. Section 4 contains a suitable extension of the Granger-Johansen representation theorem as an application of our inversion results. The conclusion follows in Section 5.

2. Preliminaries

2.1. Review of Banach spaces

Let $\mathcal{B}$ be a separable Banach space over the complex plane $\mathbb{C}$ with norm $\|\cdot\|$, and let $\mathcal{L}_\mathcal{B}$ denote the Banach space of bounded linear operators on $\mathcal{B}$ with the usual operator norm $\|A\|_{\mathcal{L}_\mathcal{B}} = \sup_{\|x\|\leq 1}\|Ax\|$. We also let $\mathrm{id}_\mathcal{B} \in \mathcal{L}_\mathcal{B}$ denote the identity map on $\mathcal{B}$. Given a subspace $V \subset \mathcal{B}$, let $A|_V$ denote the restriction of an operator $A \in \mathcal{L}_\mathcal{B}$ to V. Given $A \in \mathcal{L}_\mathcal{B}$, we define two important subspaces of $\mathcal{B}$ as follows:

$$\ker A = \{x \in \mathcal{B} \mid Ax = 0\},$$

$$\operatorname{ran} A = \{Ax \mid x \in \mathcal{B}\}.$$

Let $V_1, V_2, \ldots, V_k$ be subspaces of $\mathcal{B}$. The algebraic sum of $V_1, V_2, \ldots, V_k$ is defined by

$$\sum_{j=1}^{k} V_j = \{v_1 + v_2 + \ldots, v_k : v_j \in V_j \text{ for each } j\}.$$

We say that $\mathcal{B}$ is the (internal) direct sum of $V_1, V_2, \ldots, V_k$ and write $\mathcal{B} = \oplus_{j=1}^{k} V_j$, if $V_1, V_2, \ldots, V_k$ are closed subspaces satisfying $V_j \cap \sum_{j' \neq j} V_{j'} = \{0\}$ and $\sum_{j=1}^{k} V_j = \mathcal{B}$. For any $V \subset \mathcal{B}$, we let $V^c \subset \mathcal{B}$ denote a subspace (if it exists) such that $\mathcal{B} = V \oplus V^c$. Such a subspace V^c is called a complementary subspace of V. It turns out that a subspace V allows a complementary subspace V^c if and only if there exists a unique bounded projection onto V^c along V (Theorem 3.2.11 of [14]). In general, a complementary subspace is not uniquely determined. In the case where $\mathcal{B}$ is a Hilbert space, any closed subspace V can be complemented by its orthogonal complement $V^{\perp}$.

For any subspace $V \subset \mathcal{B}$, the cosets of V are the collection of the following sets:

$$x + V = \{x + v : v \in V\}, \quad x \in \mathcal{B}.$$

The quotient space $\mathcal{B}/V$ is the vector space whose elements are equivalence classes of the cosets of V, with the equivalence relation $\simeq$ given by

$$x + V \simeq y + V \quad \Leftrightarrow \quad x - y \in V.$$

When $V = \operatorname{ran} A$ for some $A \in \mathcal{L}_{\mathcal{B}}$, the dimension of $\mathcal{B}/V$ is called the defect of A.

2.2. *Fredholm operators*

An operator $A \in \mathcal{L}_{\mathcal{B}}$ is said to be a Fredholm operator if $\ker A$ and $\mathcal{B}/\operatorname{ran} A$ are finite dimensional. The index of a Fredholm operator A is the integer given by $\dim(\ker A) - \dim(\mathcal{B}/\operatorname{ran} A)$. It turns out that a bounded linear operator with finite defect has a closed range (Lemma 4.38 of [15]); that is, for $A \in \mathcal{L}_{\mathcal{B}}$, $\operatorname{ran} A$ is closed if $\dim(\mathcal{B}/\operatorname{ran} A) < \infty$. Therefore, $\operatorname{ran} A$ is closed if A is a Fredholm operator. Fredholm operators are invariant under compact perturbation; if A is a Fredholm operator, and K is a compact operator, $A + K$ is a Fredholm operator of the same index. In this paper, we mainly consider Fredholm operators of index zero, so we let $\mathcal{F}_0(\subset \mathcal{L}_{\mathcal{B}})$ denote the collection of such operators.

2.3. *Generalized inverse operators*

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and $\mathcal{L}_{\mathcal{B}_1,\mathcal{B}_2}$ denote the space of bounded linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. In the subsequent discussion, we need a notion of a generalized inverse operator of $A \in \mathcal{L}_{\mathcal{B}_1,\mathcal{B}_2}$. Suppose that $\mathcal{B}_1 = \ker A \oplus (\ker A)^c$ and $\mathcal{B}_2 = \operatorname{ran} A \oplus (\operatorname{ran} A)^c$. Given the direct sum conditions, the generalized inverse of A, denoted by A^g, is defined as the unique linear extension of $(A|_{(\ker A)^c})^{-1}$(defined on $\operatorname{ran} A$) to $\mathcal{B}$. Specifically, A^g is given by

$$A^g = (A|_{(\ker A)^c})^{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{(\operatorname{ran} A)^c}), \tag{2.1}$$

where P_{V^c} denotes the bounded projection onto V^c along V. It can be shown (Section 1 of [16]) that the generalized inverse A^g has the following properties:

$$AA^gA = A, \quad A^gAA^g = A^g, \quad AA^g = (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{(\operatorname{ran} A)^c}), \quad A^gA = \mathrm{P}_{(\ker A)^c}.$$

Since complementary subspaces are not uniquely determined, A^g depends on our choice of them.

The considered notion of a generalized inverse in this section is studied in the frame of Banach algebras by [17]. In a Hilbert space setting, $(\ker A)^c$ (resp. $(\operatorname{ran} A)^c$) can be set to the orthogonal complement $(\ker A)^\perp$ (resp. $(\operatorname{ran} A)^\perp$) of $\ker A$ (resp. $\operatorname{ran} A$), and then A^g becomes identical to the Moore-Penrose inverse of A.

2.4. *Operator pencils*

Let U be an open connected subset of $\mathbb{C}$. A map $A : U \to \mathcal{L}_{\mathcal{B}}$ is called an operator pencil. An operator pencil A is holomorphic at $z_0 \in U$ if the limit

$$A^{(1)}(z_0) := \lim_{z \to z_0} \frac{A(z) - A(z_0)}{z - z_0}$$

exists in the uniform operator topology. If A is holomorphic for all $z \in D \subset U$ for an open connected set D, then we say that A is holomorphic on D. A holomorphic operator pencil A on D allows the Taylor series for every $z_0 \in D$ (see pages 7 and 8 of [18]).

An operator pencil A is said to be meromorphic on U if there exists a discrete set $U_0 \subset U$ such that $A : U \setminus U_0 \to \mathcal{L}_{\mathcal{B}}$ is holomorphic, and the following Laurent expansion is allowed in a punctured neighborhood of $z_0 \in U_0$:

$$A(z) = \sum_{j=-m}^{-1} A_j (z - z_0)^j + \sum_{j=0}^{\infty} A_j (z - z_0)^j,$$

where the first term is called the principal part, and the second term is called the holomorphic part of the Laurent series. A finite positive integer m is called the order of the pole at z_0. When $m = 1$ (resp. $m = 2$), we simply say that $A(z)$ has a simple pole (resp. second order pole) at z_0. If $A_{-m}, \ldots, A_{-1}$ are finite rank operators, we say that $A(z)$ is finitely meromorphic at z_0. In addition, $A(z)$ is said to be finitely meromorphic on U if it is finitely meromorphic at each of its poles.

The set of complex numbers $z \in U$ at which the operator $A(z)$ is noninvertible is called the spectrum of A and denoted by $\sigma(A)$. It turns out that the spectrum is always a closed set (see, e.g., page 56 of [19]).

If $A(z)$ is a Fredholm operator of index zero for $z \in U$, we hereafter simply call it an $\mathcal{F}_0$-pencil.

2.5. *Fredholm theorem*

We provide a crucial input, called the analytic Fredholm theorem, for the subsequent discussion.

Analytic Fredholm Theorem. *(Corollary 8.4 in [18]) Let $A : U \to \mathcal{L}_{\mathcal{B}}$ be a holomorphic Fredholm operator pencil, i.e., $A(z)$ is a Fredholm operator for $z \in U$, and $A^{(1)}(z_0) := \lim_{z \to z_0} \frac{A(z)-A(z_0)}{z-z_0}$ exists for all $z_0 \in U$, and assume that $A(z)$ is invertible for some element $z \in U$. Then, the following hold.*

(i) *$\sigma(A)$ is a discrete set.*

(ii) *In a punctured neighborhood of $z_0 \in \sigma(A)$,*

$$A(z)^{-1} = \sum_{j=-m}^{\infty} A_j (z - z_0)^j,$$

where A_0 is a Fredholm operator of index zero, and $A_{-m}, \ldots, A_{-1}$ are finite rank operators.

That is, the analytic Fredholm theorem implies that if the inverse of a holomorphic Fredholm pencil exists, it is finitely meromorphic.

2.6. Random elements of Banach space

We briefly introduce Banach-valued random variables, called $\mathcal{B}$-random variables. More detailed discussion on this subject can be found in, e.g., Chapter 1 of [20]. We let $\mathcal{B}'$ denote the topological dual of $\mathcal{B}$.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be an underlying probability triple. A $\mathcal{B}$-random variable is defined as a measurable map $X : \Omega \to \mathcal{B}$, where $\mathcal{B}$ is understood to be equipped with its Borel $\sigma-$field. X is said to be integrable if $E\|X\| < \infty$. If X is integrable, there exists a unique element $EX \in \mathcal{B}$ such that for all $f \in \mathcal{B}'$,

$$E[f(X)] = f(EX).$$

Let $L^2_{\mathcal{B}}$ denote the space of $\mathcal{B}$-random variables X such that $EX = 0$ and $E\|X\|^2 < \infty$.

2.7. I(1) and I(2) sequences in Banach space

Let $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ be an independent and identically distributed sequence in $L^2_{\mathcal{B}}$ such that $E\varepsilon_t = 0$ and $0 < E\|\varepsilon_t\|^2 < \infty$. In this paper, ε is simply called a strong white noise.

For some $t_0 \in \mathbb{Z} \cup \{-\infty\}$, let $X = (X_t, t \geq t_0)$ be a stochastic process taking values in $\mathcal{B}$ satisfying

$$X_t = \sum_{j=0}^{\infty} A_j \varepsilon_{t-j},$$

where $(A_j, j \geq 0)$ is a sequence in $\mathcal{L}_{\mathcal{B}}$ satisfying $\sum_{j=0}^{\infty} \|A_j\|_{\mathcal{L}_{\mathcal{B}}} < \infty$. We call the sequence $(X_t, t \geq t_0)$ a standard linear process. In this case $\sum_{j=0}^{\infty} A_j$ is convergent in $\mathcal{L}_{\mathcal{B}}$.

A sequence in $L^2_{\mathcal{B}}$ is said to be I(0) if it is a standard linear process with $\sum_{j=0}^{\infty} A_j \neq 0$. For $d \in \{1, 2\}$, let $X = (X_t, t \geq -d + 1)$ be a sequence in $L^2_{\mathcal{B}}$. We say $(X_t, t \geq 0)$ is I(d) if its d-th difference $\Delta^d X = (\Delta^d X_t, t \geq 1)$ is I(0) (see, e.g., [12]).

3. Inversion of a holomorphic $\mathcal{F}_0$-pencil around an isolated singularity

Throughout this section, we employ the following assumption.

Assumption 3.1. $A : U \to \mathcal{L}_{\mathcal{B}}$ is a holomorphic Fredholm pencil, and $z_0 \in \sigma(A)$ is an isolated element.

Under the above assumption, $A(z)^{-1}$ exists in a punctured neighborhood of $z = z_0$, and its properties around z_0 have been studied in various contexts (see, e.g., [9, 10, 18]). In the case where $z_0 = 1$ and $A(z)$ is understood as a linear filter, Assumption 3.1 may be understood as a generalization of the standard unit root assumption in time series analysis (see [12]). Since $A(z)$ is holomorphic, it allows the Taylor series around z_0 as follows:

$$A(z) = \sum_{j=0}^{\infty} A_j (z - z_0)^j, \tag{3.1}$$

where $A_0 = A(z_0)$, $A_j = A^{(j)}(z_0)/j!$ for $j \geq 1$, and $A^{(j)}(z)$ denotes the j-th complex derivative of $A(z)$. Furthermore, we know from the analytic Fredholm theorem that $N(z) := A(z)^{-1}$ allows the Laurent series expansion in a punctured neighborhood of z_0 as follows:

$$N(z) = \sum_{j=-m}^{-1} N_j(z - z_0)^j + \sum_{j=0}^{\infty} N_j(z - z_0)^j, \quad 1 \leq m < \infty. \tag{3.2}$$

Our first goal is to find necessary and sufficient conditions for $m = 1$ and 2. We then provide a recursive formula to obtain N_j for $j \geq -m$. Before stating our main assumptions and results of this section, we provide some preliminary results.

First, it can be shown that $A : U \to \mathcal{L}_{\mathcal{B}}$ in Assumption 3.1 is in fact an $\mathcal{F}_0$-pencil.

Lemma 3.1. *Under Assumption 3.1, $A : U \to \mathcal{L}_{\mathcal{B}}$ is an $\mathcal{F}_0$-pencil.*

Proof. Since z_0 is an isolated element, it implies that there exists some point in U where the operator pencil is invertible. It turns out that the index of $A(z)$ does not depend on $z \in U$ given that U is connected, and Fredholm operators of nonzero index are not invertible (Section 2 of [21]). We thus find that $A(z)$ has index zero for $z \in U$.

In view of Lemma 3.1, it may also be deduced that the analytic Fredholm theorem provided in Section 2.5 is only for $\mathcal{F}_0$-pencils. The following is another important observation implied by Assumption 3.1.

Lemma 3.2. *Under Assumption 3.1,*

(i) $\operatorname{ran} A(z)$ *allows a complementary subspace for* $z \in U$.
(ii) $\ker A(z)$ *allows a complementary subspace for* $z \in U$.
(iii) *For any finite dimensional subspace V,* $\operatorname{ran} A(z) + V$ *allows a complementary subspace for* $z \in U$

Proof. Since $A(z)$ is a Fredholm operator, we know that $\operatorname{ran} A(z)$ is closed, and $\mathcal{B}/\operatorname{ran} A(z)$ is finite dimensional. Given any closed subspace V, it turns out that V allows a complementary subspace if $\mathcal{B}/V$ is finite dimensional (Theorem 3.2.18 of [14]). Thus, (i) is proved. Our proof of (ii) follows from the fact that every finite dimensional subspace allows a complementary subspace (Theorem 3.2.18 of [14]). (iii) follows from the fact that the algebraic sum $\operatorname{ran} A(z) + V$ is a closed subspace and $\mathcal{B}/(\operatorname{ran} A(z) + V)$ is finite dimensional since $\operatorname{ran} A(z)$ is closed and V is finite dimensional.

In a Hilbert space, a closed subspace allows a complementary subspace, which can always be chosen as the orthogonal complement. We therefore know that $\operatorname{ran} A(z)$ and $\operatorname{ran} A(z) + V$ allow complementary subspaces in a Hilbert space if $\operatorname{ran} A(z)$ is closed. However, in a Banach space, the closedness of a certain subspace does not guarantee the existence of a complementary subspace. The reader is referred to [14] for a detailed discussion on this subject.

3.1. Simple poles of holomorphic $\mathcal{F}_0$ inverses

Due to Lemma 3.2, we know that $\operatorname{ran} A_0$ and $\ker A_0$ are complemented, meaning that we may find their complementary subspaces, as well as the associated bounded projections. Depending on our choice of complementary subspaces, we may also define the corresponding generalized inverse of A_0 as in (2.1). To simplify expressions, we let

$$\mathbb{1}_{j=0} = \begin{cases} \mathrm{id}_{\mathcal{B}} & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$G_j(\ell, m) = \sum_{k=-m}^{j-1} N_k A_{j+\ell-k}, \quad \ell = 0, 1, 2, \ldots,$$
$$\mathrm{R}_0 = \operatorname{ran} A_0,$$
$$\mathrm{K}_0 = \ker A_0,$$
$$\mathrm{K}_1 = \{x \in \mathrm{K}_0 : A_1 x \in \mathrm{R}_0\},$$
$$\mathrm{R}_0^c = \text{a complementary subspace of } \operatorname{ran} A_0,$$
$$\mathrm{K}_0^c = \text{a complementary subspace of } \ker A_0,$$
$$\mathrm{P}_{\mathrm{R}_0^c} = \text{the bounded projection onto } \mathrm{R}_0^c \text{ along } \mathrm{R}_0,$$
$$\mathrm{P}_{\mathrm{K}_0^c} = \text{the bounded projection onto } \mathrm{K}_0^c \text{ along } \mathrm{K}_0,$$
$$S_{\mathrm{R}_0^c} = \mathrm{P}_{\mathrm{R}_0^c} A_1|_{\mathrm{K}_0} : \mathrm{K}_0 \to \mathrm{R}_0^c,$$
$$(A_0)^g_{\{\mathrm{R}_0^c, \mathrm{K}_0^c\}} = \text{the generalized inverse of } A_0,$$

where R_0^c and K_0^c are not uniquely determined, and thus we need to be careful with $\mathrm{P}_{\mathrm{R}_0^c}, \mathrm{P}_{\mathrm{K}_0^c}$ and $S_{\mathrm{R}_0^c}$ depending on our choice of those complementary subspaces; however, if R_0^c and K_0^c are specified, $\mathrm{P}_{\mathrm{R}_0^c}, \mathrm{P}_{\mathrm{K}_0^c}$ and $S_{\mathrm{R}_0^c}$ are uniquely defined. Similarly, the subscript $\{\mathrm{R}_0^c, \mathrm{K}_0^c\}$ of the generalized inverse underscores its dependence on R_0^c and K_0^c.

We provide another useful lemma.

Lemma 3.3. *Suppose that Assumption 3.1 is satisfied. Then, invertibility (or noninvertibility) of $S_{\mathrm{R}_0^c}$ does not depend on the choice of R_0^c.*

Proof. Let V_0 and W_0 be two different choices of R_0^c. Then, it is trivial to show that

$$\ker S_{V_0} = \ker S_{W_0} = \mathrm{K}_1. \tag{3.3}$$

Moreover, we know due to Lemma 3.1 that $A(z)$ satisfying Assumption 3.1 is in fact an $\mathcal{F}_0$-pencil, which implies that $\dim(\mathcal{B}/\operatorname{ran} A_0) = \dim(\ker A_0) < \infty$. Since a complementary subspace of $\operatorname{ran} A_0$ is isomorphic to $\mathcal{B}/\operatorname{ran} A_0$ (Corollary 3.2.16 of [14]), we have

$$\dim(V_0) = \dim(W_0) = \dim(\mathrm{K}_0) < \infty. \tag{3.4}$$

Any injective linear map between finite dimensional vector spaces of the same dimension is also bijective. Therefore, in view of (3.4), $\mathrm{K}_1 = \{0\}$ is necessary and sufficient for S_{V_0} (and S_{W_0}) to be invertible. Therefore, if either one is invertible (resp. noninvertible), then the other is also invertible (resp. noninvertible).

We next provide necessary and sufficient conditions for $A(z)^{-1}$ to have a simple pole at z_0 and its closed form expression in a punctured neighborhood of z_0.

Proposition 3.1. *Suppose that Assumption 3.1 is satisfied. Then, the following conditions are equivalent to each other.*

(i) $m = 1$ *in the Laurent series expansion* (3.2).
(ii) $\mathcal{B} = \mathrm{R}_0 \oplus A_1\mathrm{K}_0$.
(iii) *For all possible choices of* R_0^c, $S_{\mathrm{R}_0^c} : \mathrm{K}_0 \to \mathrm{R}_0^c$ *is invertible.*
(iv) *For some choice of* R_0^c, $S_{\mathrm{R}_0^c} : \mathrm{K}_0 \to \mathrm{R}_0^c$ *is invertible.*

Under any of these conditions and any choice of R_0^c *and* K_0^c*, the coefficients* ($N_j \geq -1$) *in* (3.2) *are given by the following recursive formula.*

$$N_{-1} = S_{\mathrm{R}_0^c}^{-1}\mathrm{P}_{\mathrm{R}_0^c}, \tag{3.5}$$

$$N_j = (\mathbb{1}_{j=0} - G_j(0,1))(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}(\mathrm{id}_{\mathcal{B}} - A_1 S_{\mathrm{R}_0^c}^{-1}\mathrm{P}_{\mathrm{R}_0^c}) - G_j(1,1)S_{\mathrm{R}_0^c}^{-1}\mathrm{P}_{\mathrm{R}_0^c}, \tag{3.6}$$

where each N_j *is understood as a map from* $\mathcal{B}$ *to* $\mathcal{B}$ *without restriction of the codomain.*

Proof. We first show the claimed equivalence between conditions (i)–(iv) and then verify the recursive formula.

Equivalence between (i)–(iv) : Due to the analytic Fredholm theorem, we know that $A(z)^{-1}$ admits the Laurent series expansion (3.2) in a punctured neighborhood z_0. Moreover, $A(z)$ is holomorphic and thus admits the Taylor series as in (3.1). Combining (3.1) and (3.2), we obtain the identity expansion $\mathrm{id}_{\mathcal{B}} = A(z)^{-1}A(z)$ as follows:

$$\mathrm{id}_{\mathcal{B}} = \sum_{k=-m}^{\infty}\left(\sum_{j=0}^{m+k} N_{k-j}A_j\right)(z-z_0)^k. \tag{3.7}$$

Since (iii) $\Leftrightarrow$ (iv) is deduced from Lemma 3.3, we demonstrate equivalence between (i)–(iv) by showing (ii)$\Rightarrow$(i)$\Rightarrow$(iv)$\Rightarrow$(ii).

Now, we show that (ii)$\Rightarrow$(i). Suppose that $m > 1$. Collecting the coefficients of $(z-z_0)^{-m}$ and $(z-z_0)^{-m+1}$ in (3.7), we obtain

$$N_{-m}A_0 = 0, \tag{3.8}$$

$$N_{-m+1}A_0 + N_{-m}A_1 = 0. \tag{3.9}$$

Eq (3.8) implies that $N_{-m}\mathrm{R}_0 = \{0\}$, and further, (3.9) implies that $N_{-m}A_1\mathrm{K}_0 = \{0\}$. Therefore, if the direct sum decomposition (ii) is true, we necessarily have $N_{-m} = 0$. Note that $N_{-m} = 0$ holds for any $2 \leq m < \infty$. We therefore conclude that $m = 1$, which proves (ii)$\Rightarrow$(i).

We next show that (i)$\Rightarrow$(iv). Collecting the coefficients of $(z-z_0)^{-1}$ and $(z-z_0)^0$ in (3.7) when $m = 1$, we have

$$N_{-1}A_0 = 0, \tag{3.10}$$

$$N_{-1}A_1 + N_0A_0 = \mathrm{id}_{\mathcal{B}}. \tag{3.11}$$

Since A_0 is a Fredholm operator, we know from Lemma 3.2 that R_0 allows a complementary subspace V_0, and there exists the associated projection operator P_{V_0}. Then, Eq (3.10) implies that

$$N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = 0 \quad \text{and} \quad N_{-1} = N_{-1}\mathrm{P}_{V_0}. \tag{3.12}$$

Moreover, (3.11) implies $\mathrm{id}_{\mathcal{B}}|_{\mathrm{K}_0} = N_{-1}A_1|_{\mathrm{K}_0}$. In view of (3.12), it is apparent that

$$\mathrm{id}_{\mathcal{B}}|_{\mathrm{K}_0} = N_{-1}S_{V_0}. \tag{3.13}$$

Eq (3.13) implies that S_{V_0} is an injection. Moreover, due to Lemma 3.1, we know $A_0 \in \mathcal{F}_0$. Using the same arguments we used to establish (3.4), we obtain

$$\dim(V_0) = \dim(\mathcal{B}/\mathrm{R}_0) = \dim(\mathrm{K}_0) < \infty. \tag{3.14}$$

Eqs (3.13) and (3.14) together imply that $S_{V_0} : \mathrm{K}_0 \to V_0$ is an injective linear map between finite dimensional vector spaces of the same dimension. Therefore, we conclude that $S_{V_0} : \mathrm{K}_0 \to V_0$ is a bijection.

To show (iv)⇒(ii), suppose that our direct sum condition (ii) is false. We first consider the case where $\mathrm{R}_0 \cap A_1\mathrm{K}_0 \neq \{0\}$. If there exists a nonzero element x in $\mathrm{R}_0 \cap A_1\mathrm{K}_0$, we have, for any arbitrary choice of R_0^c, $S_{\mathrm{R}_0^c}x = 0$. This implies that $S_{\mathrm{R}_0^c}$ cannot be injective. We next consider the case where $\mathcal{B} \neq \mathrm{R}_0 + A_1\mathrm{K}_0$ even if $\mathrm{R}_0 \cap A_1\mathrm{K}_0 = \{0\}$ holds. In this case, clearly, $\mathrm{R}_0 \oplus A_1\mathrm{K}_0$ is a strict subspace of $\mathcal{B}$. On the other hand, since R_0^c is a complementary subspace of R_0, it is deduced that

$$\dim(A_1\mathrm{K}_0) < \dim(\mathrm{R}_0^c). \tag{3.15}$$

Note that $S_{\mathrm{R}_0^c}$ can be viewed as the composition of $\mathrm{P}_{\mathrm{R}_0^c}$ and $A_1|_{\mathrm{K}_0}$. From the rank-nullity theorem, $\dim(S_{\mathrm{R}_0^c}\mathrm{K}_0)$ must be at most equal to $\dim(A_1\mathrm{K}_0)$. In view of (3.15), this implies that $S_{\mathrm{R}_0^c}$ cannot be surjective for any arbitrary choice of R_0^c . Therefore, we conclude that (iv)⇒(ii).

Recursive formula for $(N_j, j \geq -1)$: Assume that V_0 as a choice of R_0^c and W_0 as a choice of K_0^c are fixed. We first verify the claimed formulas (3.5) and (3.6) for this specific choice of complementary subspaces.

At first, we consider the claimed formula for N_{-1}. In our demonstration of (i)⇒(iii) above, we obtained (3.13). Since the codomain of S_{V_0} is restricted to V_0, (3.13) can be written as

$$\mathrm{id}_{\mathcal{B}}|_{\mathrm{K}_0} = N_{-1}|_{V_0}S_{V_0}. \tag{3.16}$$

Moreover, we know that $S_{V_0} : \mathrm{K}_0 \to V_0$ is invertible. We therefore have $N_{-1}|_{V_0} = S_{V_0}^{-1}$, and note that we still need to restrict the domain of N_{-1} to V_0. By composing both sides of (3.16) with P_{V_0}, we obtain $N_{-1}\mathrm{P}_{V_0} = S_{V_0}^{-1}\mathrm{P}_{V_0}$. Recalling (3.12), which implies that $N_{-1} = N_{-1}\mathrm{P}_{V_0}$, we find that

$$N_{-1} = S_{V_0}^{-1}\mathrm{P}_{V_0}. \tag{3.17}$$

Since the codomain of $S_{V_0}^{-1}$ is K_0, the map (3.17) is the formula for N_{-1} with the restricted codomain. However, it can be understood as a map from $\mathcal{B}$ to $\mathcal{B}$ by composing both sides of (3.17) with a proper embedding.

Now, we verify the recursive formula for $(N_j, j \geq 0)$. Collecting the coefficients of $(z-1)^j$ and $(z-1)^{j+1}$ in the identity expansion (3.7), the following can be shown:

$$G_j(0,1) + N_jA_0 = \mathbb{1}_{j=0}, \tag{3.18}$$

$$G_j(1,1) + N_jA_1 + N_{j+1}A_0 = 0. \tag{3.19}$$

Since $\mathrm{id}_{\mathcal{B}} = (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) + \mathrm{P}_{V_0}$, N_j can be written as the sum of $N_j(\mathrm{id}_{\mathcal{B}} - P_{V_0})$ and $N_j\mathrm{P}_{V_0}$. We will obtain an explicit formula for each summand.

Given complementary subspaces V_0 and W_0, we may define $(A_0)^g_{\{V_0,W_0\}} : \mathcal{B} \to W_0$. Since we have $A_0(A_0)^g_{\{V_0,W_0\}} = \mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}$, (3.18) implies that

$$N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = \mathbb{1}_{j=0}(A_0)^g_{\{V_0,W_0\}} - G_j(0,1)(A_0)^g_{\{V_0,W_0\}}. \tag{3.20}$$

Moreover, by restricting the domain of both sides of (3.19) to K_0, we have

$$G_j(1,1)|_{\mathrm{K}_0} + N_jA_1|_{\mathrm{K}_0} = 0. \tag{3.21}$$

Since $N_j = N_j\mathrm{P}_{V_0} + N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})$, it is easily deduced from (3.21) that

$$N_jS_{V_0} = -G_j(1,1)|_{\mathrm{K}_0} - N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})A_1|_{\mathrm{K}_0}. \tag{3.22}$$

Substituting (3.20) into (3.22), we obtain

$$N_jS_{V_0} = -G_j(1,1)|_{\mathrm{K}_0} - \mathbb{1}_{j=0}(A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_0} + G_j(0,1)(A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_0}. \tag{3.23}$$

Since $S_{V_0} : \mathrm{K}_0 \to V_0$ is invertible, it is deduced that $\operatorname{ran} S^{-1}_{V_0} = \mathrm{K}_0$, and $S_{V_0}S^{-1}_{V_0} = \mathrm{id}_{\mathcal{B}}|_{V_0}$. We therefore obtain the following equation from (3.23):

$$N_j|_{V_0} = -G_j(1,1) - \mathbb{1}_{j=0}(A_0)^g_{\{V_0,W_0\}}A_1S^{-1}_{V_0} + G_j(0,1)(A_0)^g_{\{V_0,W_0\}}A_1S^{-1}_{V_0}. \tag{3.24}$$

Composing both sides of (3.24) with P_{V_0}, we obtain an explicit formula for $N_j\mathrm{P}_{V_0}$. Combining this result with (3.20), we obtain the formula for N_j, which is similar to (3.6), in terms of P_{V_0}, $(A_0)^g_{\{V_0,W_0\}}$, S_{V_0}, $G_j(0,1)$ and $G_j(1,1)$ after a little algebra. Of course, the resulting operator N_j should be understood as a map from $\mathcal{B}$ to $\mathcal{B}$.

The formula for each N_j that we have obtained seems to depend on our choice of complementary subspaces, especially due to P_{V_0}, S_{V_0} and $(A_0)^g_{\{V_0,W_0\}}$. However, if a Laurent series exists, it is unique. Even if the aforementioned operators are differently defined due to any changes in our choice, we can still obtain a recursive formula for $(N_j, j \geq -1)$ in terms of those operators. Such a newly obtained formula cannot be different from what we have obtained from a fixed choice of complementary subspaces because of the uniqueness of the Laurent series. Therefore, it is easily deduced that our recursive formula for N_j derived in Proposition 3.1 does not depend on a specific choice of complementary subspaces.

3.2. Second order poles of holomorphic $\mathcal{F}_0$ inverses

To simplify expressions, we let

$$\begin{aligned}
\mathrm{R}_1 &= \operatorname{ran} A_0 + A_1 \ker A_0,\\
\mathrm{R}_1^c &= \text{a complementary subspace of } \operatorname{ran} A_0 + A_1 \ker A_0,\\
\mathrm{K}_1^c &= \text{a complementary subspace of } \mathrm{K}_1 \text{ in } \mathrm{K}_0,\\
\mathrm{P}_{\mathrm{R}_1^c} &= \text{the bounded projection onto } \mathrm{R}_1^c \text{ along } \mathrm{R}_1,\\
\mathrm{P}_{\mathrm{K}_1^c} &= \text{the bounded projection onto } \mathrm{K}_1^c \text{ along } \mathrm{K}_1.
\end{aligned}$$

We know from Lemma 3.2 that R_0, K_0 and R_1 are complemented, so we may find complementary subspaces R_0^c, K_0^c, and R_1^c, as well as the bounded projections $\mathrm{P}_{\mathrm{R}_0^c}$, $\mathrm{P}_{\mathrm{K}_0^c}$ and $\mathrm{P}_{\mathrm{R}_1^c}$. Given R_0^c, R_1^c is not uniquely determined in general. We require our choice to satisfy

$$\mathrm{R}_1^c \subset \mathrm{R}_0^c, \tag{3.25}$$

so that

$$\mathrm{R}_0^c = S_{\mathrm{R}_0^c}\mathrm{K}_0 \oplus \mathrm{R}_1^c, \tag{3.26}$$

$$\mathrm{P}_{\mathrm{R}_0^c}\mathrm{P}_{\mathrm{R}_1^c} = \mathrm{P}_{\mathrm{R}_1^c}\mathrm{P}_{\mathrm{R}_0^c} = \mathrm{P}_{\mathrm{R}_1^c}. \tag{3.27}$$

For any choice of R_0^c, R_1^c satisfying (3.25) always exists, and such a subspace can easily be obtained as follows:

Lemma 3.4. *Suppose that Assumption 3.1 is satisfied. Given* R_0^c*, let* V_1 *be a specific choice of* R_1^c*. Then,* $\mathrm{P}_{\mathrm{R}_0^c}V_1 \subset \mathrm{R}_0^c$ *is also a complementary subspace of* R_1.

Proof. Let V_0 be a given choice of R_0^c. If $\mathcal{B} = \mathrm{R}_0 + A_1\mathrm{K}_0$, then $V_1 = \{0\}$, and then our statement trivially holds. Now, consider the case when V_1 is a nontrivial subspace. Since $\mathrm{R}_0 + A_1\mathrm{K}_0 = \mathrm{R}_0 \oplus \mathrm{P}_{V_0}A_1\mathrm{K}_0$ holds, it is deduced that $\mathcal{B} = \mathrm{R}_0 \oplus \mathrm{P}_{V_0}A_1\mathrm{K}_0 \oplus V_1$. This implies that $M := \mathrm{P}_{V_0}A_1\mathrm{K}_0 \oplus V_1$ is a complementary subspace of R_0. Since $\mathrm{P}_{V_0}\mathcal{B} = \mathrm{P}_{V_0}M$, clearly $\mathrm{P}_{V_0}|_M : M \to V_0$ must be a surjection, so we have

$$\mathrm{P}_{V_0}M = V_0. \tag{3.28}$$

Moreover, both M and V_0 are complementary subspaces of R_0, and we know due to Lemma 3.1 that $A_0 \in \mathcal{F}_0$. Then, it is deduced from similar arguments to those we used to derive (3.4) that

$$\dim(\mathcal{B}/\mathrm{R}_0) = \dim(V_0) = \dim(M).$$

Thus, $\mathrm{P}_{V_0}|_M : M \to \mathrm{R}_0^c$ is a surjection between vector spaces of the same finite dimension, meaning that it is also an injection. We therefore obtain $\mathrm{P}_{V_0}A_1\mathrm{K}_0 \cap \mathrm{P}_{V_0}V_1 = \{0\}$, which implies that $\mathrm{P}_{V_0}M = \mathrm{P}_{V_0}A_1\mathrm{K}_0 \oplus \mathrm{P}_{V_0}V_1$. Combining this with (3.28), it is deduced that

$$\mathcal{B} = \mathrm{R}_0 \oplus \mathrm{P}_{V_0}A_1\mathrm{K}_0 \oplus \mathrm{P}_{V_0}V_1.$$

Clearly, $\mathrm{P}_{V_0}V_1$ is a complementary subspace of R_1.

Due to Lemma 3.4, we know how to make an arbitrary choice of R_1^c satisfy the requirement (3.25) and thus may assume that our choice of R_1^c satisfies (3.25) in the subsequent discussion.

Under any choice of our complementary subspaces satisfying (3.25), we define

$$A^{\dagger}_{2\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}} = A_2 - A_1(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}A_1,$$

$$S^{\dagger}_{\{\mathrm{R}_0^c,\mathrm{K}_0^c,\mathrm{R}_1^c\}} = \mathrm{P}_{\mathrm{R}_1^c}A^{\dagger}_{2\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}|_{\mathrm{K}_1} : \mathrm{K}_1 \to \mathrm{R}_1^c,$$

where the subscripts indicate the sets of complementary subspaces upon which the corresponding operators depend.

In this section, we consider the case $\mathrm{K}_1 \neq \{0\}$. Then, $S_{\mathrm{R}_0^c}$ is not invertible since $\ker S_{\mathrm{R}_0^c} = \mathrm{K}_1$. However, note that $S_{\mathrm{R}_0^c}$ is a linear map between finite dimensional subspaces, so we can always define its generalized inverse as follows:

$$(S_{\mathrm{R}_0^c})^g_{\{\mathrm{R}_1^c,\mathrm{K}_1^c\}} = \left(S_{\mathrm{R}_0^c}|_{\mathrm{K}_1^c}\right)^{-1} (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{\mathrm{R}_1^c})|_{\mathrm{R}_0^c}. \tag{3.29}$$

Before stating our main result of this section, we first establish the following preliminary result.

Lemma 3.5. *Suppose that Assumption 3.1 is satisfied. Let V_0 and $\widetilde{V}_0$ be arbitrary choices of R_0^c and $V_1 \subset V_0$ and $\widetilde{V}_1 \subset \widetilde{V}_0$ be arbitrary choices of R_1^c. Then,*

$$\dim(V_1) = \dim(\widetilde{V}_1) = \dim(\mathrm{K}_1).$$

Proof. For V_0 and $\widetilde{V}_0$, we have the two operators $S_{V_0} : \mathrm{K}_0 \to V_0$ and $S_{\widetilde{V}_0} : \mathrm{K}_0 \to \widetilde{V}_0$. We established that $\ker S_{V_0} = \ker S_{\widetilde{V}_0} = \mathrm{K}_1$ in (3.3). From Lemma 3.1, we know $A_0 \in \mathcal{F}_0$, so it is easily deduced that

$$\dim(V_0) = \dim(\mathrm{K}_0) = \dim(S_{V_0}\mathrm{K}_0) + \dim(\mathrm{K}_1), \tag{3.30}$$

$$\dim(\widetilde{V}_0) = \dim(\mathrm{K}_0) = \dim(S_{\widetilde{V}_0}\mathrm{K}_0) + \dim(\mathrm{K}_1). \tag{3.31}$$

In each of (3.30) and (3.31), the first equality is deduced from the same arguments as those we used to derive (3.4), and the second equality is justified by the rank-nullity theorem. Moreover, the following direct sum decompositions are allowed:

$$V_0 = S_{V_0}\mathrm{K}_0 \oplus V_1, \tag{3.32}$$

$$\widetilde{V}_0 = S_{\widetilde{V}_0}\mathrm{K}_0 \oplus \widetilde{V}_1. \tag{3.33}$$

To see why (3.32) and (3.33) are true, first note that we have $\mathrm{R}_0 + A_1\mathrm{K}_0 = \mathrm{R}_0 \oplus S_{V_0}\mathrm{K}_0 = \mathrm{R}_0 \oplus S_{\widetilde{V}_0}\mathrm{K}_0$. We thus have $\mathcal{B} = \mathrm{R}_0 \oplus S_{V_0}\mathrm{K}_0 \oplus V_1 = \mathrm{R}_0 \oplus S_{\widetilde{V}_0}\mathrm{K}_0 \oplus \widetilde{V}_1$. These direct sum conditions imply that $S_{V_0}\mathrm{K}_0 \oplus V_1$ and $S_{\widetilde{V}_0}\mathrm{K}_0 \oplus \widetilde{V}_1$ are complementary subspaces of R_0. Since $V_1 \subset V_0$ and $\widetilde{V}_1 \subset \widetilde{V}_0$, (3.32) and (3.33) are established. Now, it is deduced from (3.32) and (3.33) that

$$\dim(V_0) = \dim(S_{V_0}\mathrm{K}_0) + \dim(V_1), \tag{3.34}$$

$$\dim(\widetilde{V}_0) = \dim(S_{\widetilde{V}_0}\mathrm{K}_0) + \dim(\widetilde{V}_1). \tag{3.35}$$

Comparing (3.30) and (3.34), we obtain $\dim(\mathrm{K}_1) = \dim(V_1)$. Additionally, from (3.31) and (3.35), we obtain $\dim(\mathrm{K}_1) = \dim(\widetilde{V}_1)$.

Now, we provide necessary and sufficient conditions for $A(z)^{-1}$ to have a second order pole at z_0 and its closed-form expression in a punctured neighborhood of z_0.

Proposition 3.2. *Suppose that Assumption 3.1 is satisfied, and $\mathrm{K}_1 \neq \{0\}$. Then, the following conditions are equivalent to each other.*

(i) *$m = 2$ in the Laurent series expansion* (3.2).
(ii) *For some choice of R_0^c and K_0^c, we have*

$$\mathcal{B} = \mathrm{R}_1 \oplus A^{\dagger}_{2\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}\mathrm{K}_1.$$

(iii) *For all possible choices of* R_0^c, K_0^c *and* R_1^c *satisfying* (3.25)*,* $S^{\dagger}_{\{\mathrm{R}_0^c,\mathrm{K}_0^c,\mathrm{R}_1^c\}} : \mathrm{K}_1 \to \mathrm{R}_1^c$ *is invertible.*

(iv) *For some choice of* R_0^c, K_0^c *and* R_1^c *satisfying* (3.25)*,* $S^{\dagger}_{\{\mathrm{R}_0^c,\mathrm{K}_0^c,\mathrm{R}_1^c\}} : \mathrm{K}_1 \to \mathrm{R}_1^c$ *is invertible.*

Under any of these conditions and any choice of complementary subspaces satisfying (3.25)*, the coefficients* $(N_j \geq -2)$ *in* (3.2) *are given by the following recursive formula.*

$$N_{-2} = (S^{\dagger}_{\{\mathrm{R}_0^c,\mathrm{K}_0^c,\mathrm{R}_1^c\}})^{-1}\mathrm{P}_{\mathrm{R}_1^c}, \tag{3.36}$$

$$\begin{aligned} N_{-1} = {} & \Big(Q^R_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}(S_{\mathrm{R}_0^c})^g_{\{\mathrm{R}_1^c,\mathrm{K}_1^c\}}\mathrm{P}_{\mathrm{R}_0^c} - N_{-2}A_1(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}\Big)Q^L_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}} \\ & - Q^R_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}A_1N_{-2} - N_{-2}A^{\dagger}_{3\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}N_{-2}, \end{aligned} \tag{3.37}$$

$$\begin{aligned} N_j = {} & \Big(G_j(1,2)(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}A_1 - G_j(2,2)\Big)N_{-2} \\ & + \Big(\mathbb{1}_{j=0} - G_j(0,2)\Big)(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}\Big(\mathrm{id}_{\mathcal{B}} - A_1(S_{\mathrm{R}_0^c})^g_{\{\mathrm{R}_1^c,\mathrm{K}_1^c\}}\mathrm{P}_{\mathrm{R}_0^c}\Big)Q^L_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}} \\ & - G_j(1,2)(S_{\mathrm{R}_0^c})^g_{\{\mathrm{R}_1^c,\mathrm{K}_1^c\}}\mathrm{P}_{\mathrm{R}_0^c}Q^L_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} A^{\dagger}_{3\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}} &= A_3 - A_1(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}A_1(A_0)^g_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}A_1, \\ Q^L_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}} &= \mathrm{id}_{\mathcal{B}} - A^{\dagger}_{2\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}N_{-2}, \\ Q^R_{\{\mathrm{R}_0^c,\mathrm{K}_0^c\}} &= \mathrm{id}_{\mathcal{B}} - N_{-2}A^{\dagger}_{2\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}. \end{aligned}$$

Each N_j *is understood as a map from* $\mathcal{B}$ *to* $\mathcal{B}$ *without restriction of the codomain.*

Proof. We first establish some results that are repeatedly mentioned in the subsequent proof. Given any choice of complementary subspaces satisfying (3.25), the following identity decomposition is easily deduced from (3.27):

$$\mathrm{id}_{\mathcal{B}} = (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{\mathrm{R}_0^c}) + (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{\mathrm{R}_1^c})\mathrm{P}_{\mathrm{R}_0^c} + \mathrm{P}_{\mathrm{R}_1^c}. \tag{3.39}$$

Since we have $\mathrm{R}_1 = \mathrm{R}_0 + A_1\mathrm{K}_0 = \mathrm{R}_0 \oplus S_{\mathrm{R}_0^c}\mathrm{K}_0$, our direct sum condition (ii) is equivalent to

$$\mathcal{B} = \mathrm{R}_0 \oplus S_{\mathrm{R}_0^c}\mathrm{K}_0 \oplus A^{\dagger}_{2\,\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}\mathrm{K}_1. \tag{3.40}$$

Moreover, we may obtain the following expansion of the identity from (3.1) and (3.2):

$$\mathrm{id}_{\mathcal{B}} = \sum_{k=-m}^{\infty}\left(\sum_{j=0}^{m+k} N_{k-j}A_j\right)(z-z_0)^k \tag{3.41}$$

$$= \sum_{k=-m}^{\infty}\left(\sum_{j=0}^{m+k} A_jN_{k-j}\right)(z-z_0)^k. \tag{3.42}$$

Equivalence between (i)–(iv) : Since (iii) $\Rightarrow$ (iv) is trivial, we will show that (ii)$\Rightarrow$(i)$\Rightarrow$(iii) and (iv) $\Rightarrow$ (ii).

To show (ii)⇒(i), let V_0 (resp. W_0) be a choice of R_0^c (resp. K_0^c), and the direct sum condition (ii) holds for V_0 and W_0. Since $\ker S_{V_0} = \mathrm{K}_1 \neq \{0\}$, S_{V_0} cannot be invertible. Therefore, $m \neq 1$ by Proposition 3.1. Therefore, suppose that $2 \leq m < \infty$ in (3.2). Collecting the coefficients of $(z - z_0)^{-m}$, $(z - z_0)^{-m+1}$ and $(z - z_0)^{-m+2}$ in (3.41) and (3.42), we obtain

$$N_{-m}A_0 = A_0N_{-m} = 0, \tag{3.43}$$

$$N_{-m}A_1 + N_{-m+1}A_0 = A_1N_{-m} + A_0N_{-m+1} = 0, \tag{3.44}$$

$$N_{-m}A_2 + N_{-m+1}A_1 + N_{-m+2}A_0 = 0. \tag{3.45}$$

We may define the generalized inverse $(A_0)^g_{\{V_0,W_0\}}$ for V_0 and W_0. Composing both sides of (3.43) with $(A_0)^g_{\{V_0,W_0\}}$, we obtain

$$N_{-m}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = 0 \quad \text{and} \quad N_{-m} = N_{-m}\mathrm{P}_{V_0}. \tag{3.46}$$

From (3.44) and (3.46), it is deduced that

$$N_{-m}A_1|_{\mathrm{K}_0} = N_{-m}\mathrm{P}_{V_0}A_1|_{\mathrm{K}_0} = N_{-m}S_{V_0} = 0. \tag{3.47}$$

Restricting the domain of both sides of (3.45) to K_1, we find that

$$N_{-m}A_2|_{\mathrm{K}_1} + N_{-m+1}A_1|_{\mathrm{K}_1} = 0. \tag{3.48}$$

Moreover, (3.44) trivially implies that

$$N_{-m+1}A_0 = -N_{-m}A_1 \quad \text{and} \quad A_0N_{-m+1} = -A_1N_{-m}. \tag{3.49}$$

By composing each of (3.49) with $(A_0)^g_{\{V_0,W_0\}}$, it can be deduced that

$$N_{-m+1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = -N_{-m}A_1(A_0)^g_{\{V_0,W_0\}}, \tag{3.50}$$

$$\mathrm{P}_{W_0}N_{-m+1} = -(A_0)^g_{\{V_0,W_0\}}A_1N_{-m}. \tag{3.51}$$

Composing both sides of (3.51) with P_{V_0} and using (3.46), we find that

$$\mathrm{P}_{W_0}N_{-m+1}\mathrm{P}_{V_0} = -(A_0)^g_{\{V_0,W_0\}}A_1N_{-m}. \tag{3.52}$$

From (3.52) and the identity decomposition $\mathrm{id}_{\mathcal{B}} = (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{W_0}) + \mathrm{P}_{W_0}$, we obtain

$$N_{-m+1}\mathrm{P}_{V_0} = -(A_0)^g_{\{V_0,W_0\}}A_1N_{-m} + (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{W_0})N_{-m+1}\mathrm{P}_{V_0}. \tag{3.53}$$

Summing both sides of (3.50) and (3.53) gives

$$N_{-m+1} = -N_{-m}A_1(A_0)^g_{\{V_0,W_0\}} - (A_0)^g_{\{V_0,W_0\}}A_1N_{-m} + (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{W_0})N_{-m+1}\mathrm{P}_{V_0}. \tag{3.54}$$

Therefore, (3.48) and (3.54) together imply that

$$\begin{aligned} 0 = {} & N_{-m}A_2|_{\mathrm{K}_1} - N_{-m}A_1(A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_1} - (A_0)^g_{\{V_0,W_0\}}A_1N_{-m}A_1|_{\mathrm{K}_1} \\ & + (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{W_0})N_{-m+1}\mathrm{P}_{V_0}A_1|_{\mathrm{K}_1}. \end{aligned} \tag{3.55}$$

From the definition of K_1, $P_{V_0}A_1|_{K_1} = 0$. Therefore, the last term in (3.55) is zero. Moreover, in view of (3.46), we have $N_{-m}(\mathrm{id}_{\mathcal{B}} - P_{V_0}) = 0$. This implies that the third term in (3.55) is zero, and (3.55) reduces to

$$N_{-m}A^{\dagger}_{2\{V_0,W_0\}}|_{K_1} = 0. \tag{3.56}$$

Given our direct sum condition (ii) (or equivalently (3.40)) with Eqs (3.46), (3.47) and (3.56), we conclude that $N_{-m} = 0$. The above arguments hold for any arbitrary choice of m such that $2 < m < \infty$, and we already showed that $m = 1$ is impossible. Therefore, m must be 2. This proves (ii)⇒(i).

Now, we show that (i)⇒(iii). We let V_0, W_0 and $V_1(\subset V_0)$ be our choice of R_0^c, K_0^c and R_1^c, respectively. Suppose that $S^{\dagger}_{\{V_0,W_0,V_1\}}$ is not invertible. Due to Lemma 3.5, we know $\dim(V_1) = \dim(K_1)$, meaning that $S^{\dagger}_{\{V_0,W_0,V_1\}}$ is not injective. Therefore, we know there exists an element $x \in K_1$ such that $S^{\dagger}_{\{V_0,W_0,V_1\}}x = 0$. Collecting the coefficients of $(z - z_0)^{-2}$, $(z - z_0)^{-1}$ and $(z - z_0)^0$ in (3.41) and (3.42), we have

$$\sum_{k=-m}^{-3} N_k A_{-2-k} + N_{-2}A_0 = 0, \tag{3.57}$$

$$\sum_{k=-m}^{-3} N_k A_{-1-k} + N_{-2}A_1 + N_{-1}A_0 = 0, \tag{3.58}$$

$$\sum_{k=-m}^{-3} N_k A_{-k} + N_{-2}A_2 + N_{-1}A_1 + N_0A_0 = \mathrm{id}_{\mathcal{B}}. \tag{3.59}$$

From (3.39), N_{-2} can be written as the sum of $N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_0})$, $N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_1})P_{V_0}$ and $N_{-2}P_{V_1}$. We will obtain an explicit formula for each summand. It is deduced from (3.57) that

$$N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_0}) = -\sum_{k=-m}^{-3} N_k A_{-2-k}(A_0)^g_{\{V_0,W_0\}}. \tag{3.60}$$

Restricting both sides of (3.58) to K_0, we obtain

$$N_{-2}A_1|_{K_0} = -\sum_{k=-m}^{-3} N_k A_{-1-k}|_{K_0}. \tag{3.61}$$

Since $N_{-2} = N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_0}) + P_{V_0}$, we obtain from (3.60) and (3.61) that

$$N_{-2}S_{V_0} = -\sum_{k=-m}^{-3} N_k A_{-1-k}|_{K_0} + \sum_{k=-m}^{-3} N_k A_{-2-k}(A_0)^g_{\{V_0,W_0\}}A_1|_{K_0}. \tag{3.62}$$

We may define $(S_{V_0})^g_{\{V_1,W_1\}}$ as in (3.29). Composing both sides of (3.62) with $(S_{V_0})^g_{\{V_1,W_1\}}P_V$, we obtain

$$N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_1})P_{V_0} = -\sum_{k=-m}^{-3} N_k A_{-1-k}(S_{V_0})^g_{\{V_1,W_1\}}P_{V_0} + \sum_{k=-m}^{-3} N_k A_{-2-k}(A_0)^g_{\{V_0,W_0\}}A_1(S_{V_0})^g_{\{V_1,W_1\}}P_{V_0}. \tag{3.63}$$

Restricting both sides of (3.59) to K_1, we have

$$\sum_{k=-m}^{-3} N_k A_{-k}|_{K_1} + N_{-2}A_2|_{K_1} + N_{-1}A_1|_{K_1} = \mathrm{id}_{\mathcal{B}}|_{K_1}. \tag{3.64}$$

From (3.58), we can also obtain

$$N_{-1}(\mathrm{id}_{\mathcal{B}} - P_{V_0}) = -\sum_{k=-m}^{-3} N_k A_{-1-k}(A_0)^g_{\{V_0,W_0\}} - N_{-2}A_1(A_0)^g_{\{V_0,W_0\}}. \tag{3.65}$$

Since $A_1K_1 \subset R_0$, we have $N_{-1}A_1|_{K_1} = N_{-1}(\mathrm{id}_{\mathcal{B}} - P_{V_0})A_1|_{K_1}$. Substituting (3.65) into (3.64), the following can be obtained:

$$\sum_{k=-m}^{-3} N_k A_{-k}|_{K_1} - \sum_{k=-m}^{-3} N_k A_{-1-k}(A_0)^g_{\{V_0,W_0\}}A_1|_{K_1} + N_{-2}A^{\dagger}_{2\{V_0,W_0\}}|_{K_1} = \mathrm{id}_{\mathcal{B}}|_{K_1}.$$

Since $N_{-2} = N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_0}) + N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_1})P_{V_0} + N_{-2}P_{V_1}$, we have

$$\begin{aligned}\mathrm{id}_{\mathcal{B}}|_{K_1} = &\sum_{k=-m}^{-3} N_k A_{-k}|_{K_1} - \sum_{k=-m}^{-3} N_k A_{-1-k}(A_0)^g_{\{V_0,W_0\}}A_1|_{K_1} \\ &+ N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_0})A^{\dagger}_{2\{V_0,W_0\}}|_{K_1} + N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_1})P_{V_0}A^{\dagger}_{2\{V_0,W_0\}}|_{K_1} + N_{-2}S^{\dagger}_{\{V_0,W_0,V_1\}}.\end{aligned} \tag{3.66}$$

Note that if N_j is zero for every $j \leq -3$, then the first four terms of the right hand side of (3.66) are equal to zero, which can be easily deduced from the obtained formulas for $N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_0})$ and $N_{-2}(\mathrm{id}_{\mathcal{B}} - P_{V_1})P_{V_0}$ in (3.60) and (3.63). However, we showed that there exists some $x \in K_1$ such that $S^{\dagger}_{\{V_0,W_0,V_1\}}x = 0$, which implies that N_j for some $j \leq -3$ must not be zero. This shows (i)⇒(iii).

It remains to show (iv) ⇒ (ii). Suppose that (ii) does not hold. Then, for any arbitrary choice of R_0^c and K_0^c, we must have either

$$R_1 \cap A^{\dagger}_{2\{R_0^c,K_0^c\}}K_1 \neq \{0\} \tag{3.67}$$

or

$$R_1 + A^{\dagger}_{2\{R_0^c,K_0^c\}}K_1 \neq \mathcal{B}. \tag{3.68}$$

If (3.67) is true, then clearly $S^{\dagger}_{\{R_0^c,K_0^c,R_1^c\}}$ cannot be injective for any arbitrary choice of R_1^c satisfying (3.25). Moreover, if (3.68) is true, then we must have $\dim(A^{\dagger}_{2\{R_0^c,K_0^c\}}K_1) < \dim(R_1^c)$. This implies that $S^{\dagger}_{\{R_0^c,K_0^c,R_1^c\}}$ cannot be surjective for any arbitrary choice of R_1^c satisfying (3.25). Therefore, (iv) ⇒ (ii) is easily deduced.

Formulas for N_{-2} and N_{-1} : We let V_0, W_0, $V_1(\subset V_0)$ and W_1 be our choice of R_0^c, K_0^c, R_1^c and K_1^c, respectively. Collecting the coefficients of $(z - z_0)^{-2}, (z - z_0)^{-1}$ and $(z - z_0)^0$ from (3.41) and (3.42), we have

$$N_{-2}A_0 = A_0N_{-2} = 0,$$

$$N_{-2}A_1 + N_{-1}A_0 = A_1N_{-2} + A_0N_{-1} = 0,$$
$$N_{-2}A_2 + N_{-1}A_1 + N_0A_0 = 0.$$

From similar arguments and algebra to those in our demonstration of (ii)⇒(i), it can easily be deduced that

$$N_{-2}\mathrm{R}_1 = \{0\}, \tag{3.69}$$
$$N_{-2}A^{\dagger}_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1} = \mathrm{id}_{\mathcal{B}}|_{\mathrm{K}_1}. \tag{3.70}$$

Eq (3.69) implies that

$$N_{-2}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1}) = 0 \quad \text{and} \quad N_{-2} = N_{-2}\mathrm{P}_{V_1}. \tag{3.71}$$

Eqs (3.69) and (3.70) together imply that

$$N_{-2}|_{V_1}S^{\dagger}_{\{V_0,W_0,V_1\}} = \mathrm{id}_{\mathcal{B}}|_{\mathrm{K}_1}. \tag{3.72}$$

Composing both sides of (3.72) with $(S^{\dagger}_{\{V_0,W_0,V_1\}})^{-1}\mathrm{P}_{V_1}$, we obtain

$$N_{-2}\mathrm{P}_{V_1} = (S^{\dagger}_{\{V_0,W_0,V_1\}})^{-1}\mathrm{P}_{V_1}. \tag{3.73}$$

In view of (3.71), (3.73) is in fact equal to N_{-2} with the codomain restricted to K_1. Viewing this as a map from $\mathcal{B}$ to $\mathcal{B}$, we obtain (3.36) for our choice of complementary subspaces.

We next verify the claimed formula for N_{-1}. In view of the identity decomposition (3.39), N_{-1} may be written as the sum of $N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})$, $N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0}$ and $N_{-1}\mathrm{P}_{V_1}$. We will find an explicit formula for each summand. From (3.41) when $m = 2$, we obtain the coefficients of $(z - z_0)^{-1}$, $(z - z_0)^0$ and $(z - z_0)^1$ as follows.

$$N_{-2}A_1 + N_{-1}A_0 = 0, \tag{3.74}$$
$$N_{-2}A_2 + N_{-1}A_1 + N_0A_0 = \mathrm{id}_{\mathcal{B}}, \tag{3.75}$$
$$N_{-2}A_3 + N_{-1}A_2 + N_0A_1 + N_1A_0 = 0. \tag{3.76}$$

From (3.74) and the properties of the generalized inverse, it is easily deduced that

$$N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = -N_{-2}A_1(A_0)^{g}_{\{V_0,W_0\}}. \tag{3.77}$$

Restricting the domain of both sides of (3.75) to K_0, we obtain

$$N_{-1}A_1|_{\mathrm{K}_0} = \mathrm{id}_{\mathcal{B}}|_{\mathrm{K}_0} - N_{-2}A_2|_{\mathrm{K}_0}. \tag{3.78}$$

Using the identity decomposition $\mathrm{id}_{\mathcal{B}} = \mathrm{P}_{V_0} + (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})$, (3.78) can be written as

$$N_{-1}S_{V_0} = \mathrm{id}_{\mathcal{B}}|_{\mathrm{K}_0} - N_{-2}A_2|_{\mathrm{K}_0} - N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})A_1|_{\mathrm{K}_0}. \tag{3.79}$$

Substituting (3.77) into (3.79), we obtain

$$N_{-1}S_{V_0} = \left(\mathrm{id}_{\mathcal{B}} - N_{-2}A^{\dagger}_{2\,\{V_0,W_0\}}\right)|_{\mathrm{K}_0}. \tag{3.80}$$

Under our direct sum condition (ii), $S_{V_0} : \mathrm{K}_0 \to V_0$ is not invertible but allows a generalized inverse as in (3.29). From the construction of $(S_{V_0})^g_{\{V_1,W_1\}}$, we have $S_{V_0}(S_{V_0})^g_{\{V_1,W_1\}} = (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})|_{V_0}$. Composing both sides of (3.80) with $(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0}$, we obtain

$$N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0} = \left(\mathrm{id}_{\mathcal{B}} - N_{-2}A^{\dagger}_{2\{V_0,W_0\}}\right)(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0}. \tag{3.81}$$

Restricting the domain of both sides of (3.76) to K_1, we have

$$N_{-2}A_3|_{\mathrm{K}_1} + N_{-2}A_2 + N_0A_1|_{\mathrm{K}_1} = 0. \tag{3.82}$$

Composing both sides of (3.75) with $(A_0)^g_{\{V_0,W_0\}}$, it is deduced that

$$N_0(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = (\mathrm{id}_{\mathcal{B}} - N_{-2}A_2 - N_{-1}A_1)(A_0)^g_{\{V_0,W_0\}}. \tag{3.83}$$

From the definition of K_1, we have $A_1\mathrm{K}_1 \subset \mathrm{R}_0$. Therefore, it is easily deduced that

$$N_0A_1|_{\mathrm{K}_1} = N_0(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})A_1|_{\mathrm{K}_1}. \tag{3.84}$$

Combining (3.82), (3.83) and (3.84), we have

$$\left(N_{-2}A_3 + N_{-1}A_2 + (\mathrm{id}_{\mathcal{B}} - N_{-2}A_2 - N_{-1}A_1)(A_0)^g_{\{V_0,W_0\}}A_1\right)|_{\mathrm{K}_1} = 0. \tag{3.85}$$

Rearranging terms, (3.85) reduces to

$$N_{-1}A^{\dagger}_{2\{V_0,W_0\}}|_{\mathrm{K}_1} = -N_{-2}\left(A_3 - A_2(A_0)^g_{\{V_0,W_0\}}A_1\right)|_{\mathrm{K}_1} - (A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_1}. \tag{3.86}$$

Moreover, with trivial algebra, it can be shown that (3.86) is equal to

$$N_{-1}A^{\dagger}_{2\{V_0,W_0\}}|_{\mathrm{K}_1} = -N_{-2}\left(A^{\dagger}_{3\{V_0,W_0\}} - A^{\dagger}_{2\{V_0,W_0\}}(A_0)^g_{\{V_0,W_0\}}A_1\right)|_{\mathrm{K}_1} - (A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_1}. \tag{3.87}$$

From the identity decomposition (3.39), we have $N_{-1} = N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) + N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0} + N_{-1}\mathrm{P}_{V_1}$, so (3.87) can be written as follows:

$$\begin{aligned} N_{-1}S^{\dagger}_{\{V_0,W_0,V_1\}} = &- N_{-2}\left(A^{\dagger}_{3\{V_0,W_0\}} - A^{\dagger}_{2\{V_0,W_0\}}(A_0)^g_{\{V_0,W_0\}}A_1\right)|_{\mathrm{K}_1} - (A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_1} \\ &- N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})A^{\dagger}_{2\{V_0,W_0\}}|_{\mathrm{K}_1} - N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0}A^{\dagger}_{2\{V_0,W_0\}}|_{\mathrm{K}_1}. \end{aligned} \tag{3.88}$$

We obtained explicit formulas for $N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})$ and $N_{-1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0}$ in (3.77) and (3.81). Moreover, we proved that $S^{\dagger}_{\{V_0,W_0,V_1\}} : \mathrm{K}_1 \to \mathrm{R}_1$ is invertible. After some tedious algebra from (3.88), one can obtain the claimed formula for N_{-1}, (3.37), for our choice of complementary subspaces. Of course, the resulting N_{-1} needs to be understood as a map from $\mathcal{B}$ to $\mathcal{B}$.

Formulas for $(N_j, j \geq 0)$: Collecting the coefficients of $(z-1)^j$, $(z-1)^{j+1}$ and $(z-1)^{j+2}$ in the expansion of the identity (3.41) when $m = 2$, we have

$$G_j(0,2) + N_jA_0 = \mathbb{1}_{j=0}, \tag{3.89}$$

$$G_j(1,2) + N_jA_1 + N_{j+1}A_0 = 0, \tag{3.90}$$

$$G_j(2,2) + N_jA_2 + N_{j+1}A_1 + N_{j+2}A_0 = 0. \tag{3.91}$$

From the identity decomposition (3.39), the operator N_j can be written as the sum of $N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})$, $N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0}$ and $N_j\mathrm{P}_{V_1}$. We will find an explicit formula for each summand. First, from (3.89), it can be easily verified that

$$N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = \mathbb{1}_{j=0}(A_0)^g - G_j(0,2)(A_0)^g_{\{V_0,W_0\}}. \tag{3.92}$$

By restricting the domain of (3.90) to K_0, we obtain

$$N_jA_1|_{\mathrm{K}_0} = -G_j(1,2)|_{\mathrm{K}_0}. \tag{3.93}$$

Using the identity decomposition $\mathrm{id}_{\mathcal{B}} = \mathrm{P}_{V_0} + (\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})$ and (3.92), we may rewrite (3.93) as follows:

$$N_jS_{V_0} = -G_j(1,2)|_{\mathrm{K}_0} - \mathbb{1}_{j=0}(A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_0} + G_j(0,2)(A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_0}. \tag{3.94}$$

Composing both sides of (3.94) with $(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0}$, an explicit formula for $N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0}$ can be obtained as follows:

$$\begin{aligned} N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0} = &- G_j(1,2)(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0} - \mathbb{1}_{j=0}(A_0)^g_{\{V_0,W_0\}}A_1(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0} \\ &+ G_j(0,2)(A_0)^g_{\{V_0,W_0\}}A_1(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0}. \end{aligned} \tag{3.95}$$

Restricting the domain of (3.91) to K_1, we obtain

$$G_j(2,2)|_{\mathrm{K}_1} + N_jA_2|_{\mathrm{K}_1} + N_{j+1}A_1|_{\mathrm{K}_1} = 0. \tag{3.96}$$

Composing both sides of (3.90) with $(A_0)^g_{\{V_0,W_0\}}$, it is easily deduced that

$$N_{j+1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) = -G_j(1,2)(A_0)^g_{\{V_0,W_0\}} - N_jA_1(A_0)^g_{\{V_0,W_0\}}. \tag{3.97}$$

Note that we have $N_{j+1}A_1|_{\mathrm{K}_1} = N_{j+1}(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0})A_1|_{\mathrm{K}_1}$ from the definition of K_1. Combining this with (3.96) and (3.97), we obtain the following equation:

$$N_jA^\dagger_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1} = -G_j(2,2)|_{\mathrm{K}_1} + G_j(1,2)(A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_1}. \tag{3.98}$$

We know $N_j = N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_0}) + N_j(\mathrm{id}_{\mathcal{B}} - \mathrm{P}_{V_1})\mathrm{P}_{V_0} + N_j\mathrm{P}_{V_1}$ and already obtained explicit formulas for the last two terms. Substituting the obtained formulas into (3.98), we obtain

$$\begin{aligned} N_j\mathrm{P}_{V_1}A^\dagger_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1} = &-G_j(2,2)|_{\mathrm{K}_1} + G_j(1,2)(A_0)^g_{\{V_0,W_0\}}A_1|_{\mathrm{K}_1} - \mathbb{1}_{j=0}(A_0)^g_{\{V_0,W_0\}}A^\dagger_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1} \\ &+ G_j(0,2,)(A_0)^g_{\{V_0,W_0\}}A^\dagger_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1} + G_j(1,2)(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0}A^\dagger_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1} \\ &+ \mathbb{1}_{j=0}(A_0)^gA_1(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0}A^\dagger_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1} \\ &- G_j(0,2)(A_0)^g_{\{V_0,W_0\}}A_1(S_{V_0})^g_{\{V_1,W_1\}}\mathrm{P}_{V_0}A^\dagger_{2\,\{V_0,W_0\}}|_{\mathrm{K}_1}. \end{aligned} \tag{3.99}$$

Composing both sides of (3.99) with $(S^\dagger_{\{V_0,W_0,V_1\}})^{-1}\mathrm{P}_{V_1}$, we obtain the formula for $N_j\mathrm{P}_{V_1}$. Combining this formula with (3.92) and (3.95), one can verify the claimed formula (3.38) for our choice of complementary subspaces after some algebra.

Even though our recursive formula is obtained under a given choice of complementary subspaces V_0, W_0, V_1 and W_1, we know, due to the uniqueness of the Laurent series, that it does not depend on our choice of complementary subspaces.

3.3. Discussion

Let us narrow down our discussion to $\mathcal{H}$, a complex separable Hilbert space. In $\mathcal{H}$, there is a canonical notion of a complementary subspace, called the orthogonal complement, while we do not have such a notion in $\mathcal{B}$. We therefore may let the orthogonal complement $(\operatorname{ran} A_0)^\perp$ (resp. $(\ker A_0)^\perp$) be our choice of R_0^c (resp. K_0^c). In this case, $\mathrm{P}_{(\operatorname{ran} A_0)^\perp}$ and $\mathrm{P}_{(\ker A_0)^\perp}$ are orthogonal projections. Then, the generalized inverse $(A_0)^g_{\{(\operatorname{ran} A_0)^\perp,(\ker A_0)^\perp\}}$ has the following properties:

$$(A_0)^g_{\{(\operatorname{ran} A_0)^\perp,(\ker A_0)^\perp\}}A_0 = (\mathrm{id}_{\mathcal{H}} - \mathrm{P}_{(\operatorname{ran} A_0)^\perp}),$$
$$A_0(A_0)^g_{\{(\operatorname{ran} A_0)^\perp,(\ker A_0)^\perp\}} = \mathrm{P}_{(\ker A_0)^\perp}.$$

That is, both $(A_0)^g_{\{(\operatorname{ran} A_0)^\perp,(\ker A_0)^\perp\}}A_0$ and $A_0(A_0)^g_{\{(\operatorname{ran} A_0)^\perp,(\ker A_0)^\perp\}}$ are self-adjoint operators, meaning that $(A_0)^g_{\{(\operatorname{ran} A_0)^\perp,(\ker A_0)^\perp\}}$ is the Moore-Penrose inverse operator of A_0 (Section 1 of [16]). Moreover, we may let $(\operatorname{ran} A_0)^\perp \cap (S_{(\operatorname{ran} A_0)^\perp}\mathrm{K}_0)^\perp$ be our choice of R_1^c. This choice trivially satisfies (3.25), and it allows the orthogonal decomposition of $\mathcal{H}$ as follows:

$$\mathcal{H} = \mathrm{R}_0 \oplus S_{(\operatorname{ran} A_0)^\perp}\mathrm{K}_0 \oplus \mathrm{R}_1^c.$$

Letting $\mathrm{K}_1^\perp \cap \mathrm{K}_0$ be our choice of K_1^c, we can also make the generalized inverse of $S_{(\operatorname{ran} A_0)^\perp}$ become the Moore-Penrose inverse of $S_{(\operatorname{ran} A_0)^\perp}$. This specific choice of complementary subspaces appears to be standard in $\mathcal{H}$. Under this choice, [9] stated and proved theorems similar to our Propositions 3.1 and 3.2, without providing a recursive formula for N_j. The reader is referred to Theorems 3.1 and 3.2 of their paper for more details. On the other hand, we explicitly take all other possible choices of complementary subspaces into account and provide a recursive formula to obtain a closed-form expression of the Laurent series. Therefore, even if we restrict our concern to a Hilbert space setting, our propositions can be viewed as extended versions of those in [9].

4. Representation theory

4.1. Representations

In this section, we derive a suitable extension of the Granger-Johansen representation theorem, which will be given as an application of the results established in Section 3. Even if there are a few versions of this theorem developed in a possibly infinite dimensional Hilbert/Banach space (see, e.g., [9, 10, 12, 13]), ours seems to be the first that can provide a full characterization of I(1) and I(2) solutions (except a term depending on initial values) of a possibly infinite order autoregressive law of motion in a Banach space.

Let $A : \mathbb{C} \to \mathcal{L}_{\mathcal{B}}$ be a holomorphic operator pencil, and then it allows the following Taylor series:

$$A(z) = \sum_{j=0}^{\infty} A_{j,(0)} z^j,$$

where $A_{j,(0)}$ denotes the coefficient of z^j in the Taylor series of $A(z)$ around 0. Note that we use the additional subscript (0) to distinguish it from A_j denoting the coefficient of $(z-1)^j$ in the Taylor series of $A(z)$ around 1. As in the previous sections, we let $N(z)$ denote $A(z)^{-1}$ if it exists.

Let $D_r \subset \mathbb{C}$ denote the open disk centered at the origin with radius $r > 0$ and $\overline{D}_r$ be its closure. Throughout this section, we employ the following assumption:

Assumption 4.1.

(i) $A : \mathbb{C} \to \mathcal{L}_{\mathcal{B}}$ is a holomorphic Fredholm pencil.
(ii) $A(z)$ is invertible on $\overline{D}_1 \setminus \{1\}$.

A similar assumption is employed to derive the Granger-Johansen representation in a Hilbert space setting (see, e.g., [9] and [10]). Under the above assumption, $A(1)$ is not invertible, and in this case the local behavior of $A(z)^{-1}$ near $z = 1$ turns out to be crucial to characterize the behavior of X_t.

Now, we provide one of the main results of this section. To simplify expressions in the following propositions, we keep using the notations introduced in Section 3. Moreover, we introduce $\pi_j(k)$ for $j \geq 0$, which is given by

$$\pi_0(k) = 1, \quad \pi_1(k) = k, \quad \pi_j(k) = k(k-1)\cdots(k-j+1)/j!, \quad j \geq 2.$$

Proposition 4.1. *Suppose that $A(z)$ satisfies Assumption 4.1, and we have a sequence $(X_t, t \geq -p+1)$ satisfying*

$$\sum_{j=0}^{\infty} A_{j,(0)} X_{t-j} = \varepsilon_t, \tag{4.1}$$

where $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is a strong white noise. Then, the following conditions are equivalent to each other.

(i) *$A(z)^{-1}$ has a simple pole at $z = 1$.*
(ii) *$\mathcal{B} = \mathrm{R}_0 \oplus A_1 \mathrm{K}_0$.*
(iii) *For any choice of R_0^c, $S_{\mathrm{R}_0^c} : \mathrm{K}_0 \to \mathrm{R}_0^c$ is invertible.*
(iv) *For some choice of R_0^c, $S_{\mathrm{R}_0^c} : \mathrm{K}_0 \to \mathrm{R}_0^c$ is invertible.*

Under any of these equivalent conditions, X_t allows the representation, for some τ_0 depending on initial values,

$$X_t = \tau_0 - N_{-1} \sum_{s=1}^{t} \varepsilon_s + \nu_t, \quad t \geq 1. \tag{4.2}$$

Moreover, $\nu_t \in L^2_{\mathcal{B}}$ and satisfies

$$\nu_t = \sum_{j=0}^{\infty} \Phi_j \varepsilon_{t-j}, \quad \Phi_j = \sum_{k=j}^{\infty} (-1)^{k-j} \pi_j(k) N_k, \tag{4.3}$$

where $(N_j, j \geq -1)$ can be explicitly obtained from Proposition 3.1.

Proof. Under Assumption 4.1, there exists $\eta > 0$ such that $A(z)^{-1}$ depends holomorphically on $z \in D_{1+\eta} \setminus \{1\}$. To see this, note that the analytic Fredholm theorem implies that $\sigma(A)$ is a discrete set. Since $\sigma(A)$ is closed (page 56 of [19]), it is deduced that $\sigma(A) \cap \overline{D}_{1+r}$ is a closed discrete subset of $\overline{D}_{1+r}$ for some $0 < r < \infty$. The fact that $\overline{D}_{1+r}$ is a compact subset of $\mathbb{C}$ implies that there are only finitely many elements in $\sigma(A) \cap \overline{D}_{1+r}$. Furthermore, since 1 is an isolated element of $\sigma(A)$, it can be easily deduced that there exists $\eta \in (0, r)$ such that $A(z)^{-1}$ depends holomorphically on $z \in D_{1+\eta} \setminus \{1\}$. Since $1 \in \sigma(A)$ is an isolated element, the equivalence of conditions (i)–(iv) is implied by Proposition 3.1.

Under any of the equivalent conditions, it is deduced from Proposition 3.1 that $N(z) = N_{-1}(z-1)^{-1} + N^H(z)$, where $N^H(z)$ denotes the holomorphic part of the Laurent series. Moreover, we can explicitly obtain the coefficients $(N_j, j \geq -1)$ using the recursive formula provided in Proposition 3.1. It is clear that $(1-z)N(z)$ can be holomorphically extended over 1, and we can rewrite it as

$$(1-z)N(z)^{-1} = -N_{-1} + (1-z)N^H(z). \tag{4.4}$$

Applying the linear filter induced by (4.4) to both sides of (4.1), we obtain

$$\Delta X_t := X_t - X_{t-1} = -N_{-1}\varepsilon_t + (\nu_t - \nu_{t-1}),$$

where $\nu_s = \sum_{j=0}^{\infty} N^H_{j,(0)}\varepsilon_{s-j}$, and $N^H_{j,(0)}$ denotes the coefficient of z^j in the Taylor series of $N^H(z)$ around 0. Clearly, the process

$$X_t^* = -N_{-1}\sum_{s=1}^{t}\varepsilon_s + \nu_t$$

is a solution, and the complete solution is obtained by adding the solution to $\Delta X_t = 0$, which is given by τ_0. We then show ν_s is convergent in L^2_H. Note that

$$\left\| \sum_{j=0}^{\infty} N^H_{j,(0)}\varepsilon_{s-j} \right\| \leq \sum_{j=0}^{\infty} \|N^H_{j,(0)}\|_{\mathcal{L}_\mathcal{B}} \|\varepsilon_{s-j}\| \leq C\sum_{j=0}^{\infty} \|N^H_{j,(0)}\|_{\mathcal{L}_\mathcal{B}}, \tag{4.5}$$

where C is some positive constant. The fact that $N^H(z)$ is holomorphic on $D_{1+\eta}$ implies that $\|N^H_{j,(0)}\|$ exponentially decreases as j goes to infinity. This shows that the right-hand side of (4.5) converges to a finite quantity, so ν_s converges in L^2_H.

It is easy to verify (4.3) from elementary calculus.

Remark 4.1. Given that ε_t is a strong white noise, the sequence $(\nu_t, t \in \mathbb{Z})$ in our representation (4.2) is a stationary sequence. Therefore, (4.2) shows that X_t can be decomposed into three different components: a random walk, a stationary process and a term that depends on initial values.

Proposition 4.2. *Suppose that $A(z)$ satisfies Assumption 4.1, and we have a sequence $(X_t, t \geq -p+1)$ satisfying* (4.1). *Then, the following conditions are equivalent to each other.*

(i) *$A(z)^{-1}$ has a second order pole at $z = 1$.*
(ii) *For some choice of R_0^c and K_0^c, we have*

$$\mathcal{B} = \mathrm{R}_1 \oplus A^{\dagger}_{2\{\mathrm{R}_0^c,\mathrm{K}_0^c\}}\mathrm{K}_1.$$

(iii) *For any choice of R_0^c, K_0^c, and R_1^c satisfying* (3.25), *$S^{\dagger}_{\{\mathrm{R}_0^c,\mathrm{K}_0^c,\mathrm{R}_1^c\}} : \mathrm{K}_1 \to \mathrm{R}_1^c$ is invertible.*
(iv) *For some choice of R_0^c, K_0^c, and R_1^c satisfying* (3.25), *$S^{\dagger}_{\{\mathrm{R}_0^c,\mathrm{K}_0^c,\mathrm{R}_1^c\}} : \mathrm{K}_1 \to \mathrm{R}_1^c$ is invertible.*

Under any of these equivalent conditions, X_t allows the representation, for some τ_0 and τ_1 depending on initial values,

$$X_t = \tau_0 + \tau_1 t + N_{-2}\sum_{\tau=1}^{t}\sum_{s=1}^{\tau}\varepsilon_s - N_{-1}\sum_{s=1}^{t}\varepsilon_t + \nu_t, \quad t \geq 1. \tag{4.6}$$

Moreover, $\nu_t \in L^2_{\mathcal{B}}$ and satisfies

$$\nu_t = \sum_{j=0}^{\infty} \Phi_j \varepsilon_{t-j}, \quad \Phi_j = \sum_{k=j}^{\infty} (-1)^{k-j} \pi_j(k) N_k, \tag{4.7}$$

where $(N_j, j \geq -2)$ can be explicitly obtained from Proposition 3.2.

Proof. As shown in Proposition 4.1, there exists $\eta > 0$ such that $A(z)^{-1}$ depends holomorphically on $z \in D_{1+\eta} \setminus \{1\}$. Due to Proposition 3.2, we know $N(z) = N_{-2}(z-1)^{-2} + N_{-1}(z-1)^{-1} + N^H(z)$, where $N^H(z)$ is the holomorphic part of the Laurent series.

$(1-z)^2 A(z)^{-1}$ can be holomorphically extended over 1 so that it is holomorphic on $D_{1+\eta}$. Then, we have

$$(1-z)^2 N(z)^{-1} = N_{-2} - N_{-1}(1-z) + (1-z)^2 N^H(z).$$

Applying the linear filter induced by $(1-z)^2 A(z)^{-1}$ to both sides of (4.1), we obtain

$$\Delta^2 X_t = N_{-2}\varepsilon_t - N_{-1}\Delta\varepsilon_t + (\Delta\nu_t - \Delta\nu_{t-1}),$$

where $\nu_t := \sum_j N^H_{j,(0)} \varepsilon_{t-j}$. From (4.5), we know that ν_t converges in $L^2_{\mathcal{B}}$. Clearly, the process

$$X_t^* = N_{-2} \sum_{\tau=1}^{t} \sum_{s=1}^{\tau} \varepsilon_s - N_{-1} \sum_{s=1}^{t} \varepsilon_t + \nu_t$$

is a solution. Since the solution to $\Delta^2 X_t = 0$ is given by $\tau_0 + \tau_1 t$, we obtain (4.6). It is also easy to verify (4.7) from elementary calculus.

Remark 4.2. The sequence $(\nu_t, t \in \mathbb{Z})$ in our representation (4.6) is stationary given that ε is a strong white noise. Then, the representation (4.6) shows that X_t can be decomposed into a cumulative random walk, a random walk, a stationary process and a term that depends on initial values.

4.2. Discussion

From the analytic Fredholm theorem, we know that the random walk component in our I(1) or I(2) representation takes values in a finite dimensional space, which is similar to the existing results by [9, 10]. For statistical inference on function-valued time series containing a random walk component, the component is often assumed to be finite dimensional, and the representation results presented by [9] and [10] are used to justify this assumption (see, e.g., [22, 23]).

Propositions 4.1 and 4.2 require the autoregressive law of motion to be characterized by a holomorphic operator pencil satisfying Assumption 4.1. We expect that a wide class of autoregressive processes considered in practice satisfies the requirement. For example, for $p \in \mathbb{N}$, let $\Phi_1, \ldots, \Phi_p$ be compact operators. Then, the autoregressive law of motion given by

$$X_t = \sum_{j=1}^{p} \Phi_j X_{t-j} + \varepsilon_t$$

satisfies the requirement (see Theorems 3.1 and 4.1 of [9]).

Even though we have assumed that ε is a strong white noise for simplicity, we may allow more general innovations in Propositions 4.1 and 4.2. For example, we could allow $\|\varepsilon_t\|$ to depend on t. Even in this case, if $\|\varepsilon_t\|$ is bounded by $a + |t|^b$ for some $a, b \in \mathbb{R}$, the right hand side of (4.5) is still bounded by a finite quantity, meaning that ν_t converges in L^2_H. Moreover, we have only considered a purely stochastic process in Propositions 4.1 and 4.2. However, the inclusion of a deterministic component does not cause significant difficulties. Suppose that we have the following autoregressive law of motion:

$$\sum_{j=0}^{\infty} A_{j,(0)} X_{t-j} = \gamma_t + \varepsilon_t, \quad t \geq 1,$$

where $(\gamma_t, t \in \mathbb{Z})$ is a deterministic sequence. In this case, we need an additional condition on γ_t for $\sum_j N^H_{j,(0)}(\gamma_{t-j} + \varepsilon_{t-j})$ to be convergent. We can assume that $\|\gamma_t\| \leq a + |t|^b$ for some $a, b \in \mathbb{R}$.

5. Conclusions

This paper considered inversion of a holomorphic Fredholm pencil based on the analytic Fredholm theorem. We obtained necessary and sufficient conditions for the inverse of a Fredholm operator pencil to have a simple pole and a second order pole and further derived a closed-form expression of the Laurent expansion of the inverse around an isolated singularity. Using the results, we obtained a suitable version of the Granger-Johansen representation theorem in a general Banach space setting, which fully characterizes I(1) (and I(2)) solutions except a term depending on initial values.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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