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Research article

Nonlinear generalized semi-Jordan triple derivable mappings on completely distributive commutative subspace lattice algebras

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Abstract: In this note we proved that each nonlinear generalized semi-Jordan triple derivable mapping on completely distributive commutative subspace lattice algebras is an additive derivation.

Keywords: completely distributive commutative subspace lattice algebras; generalized semi-Jordan triple derivable mapping; Jordan triple derivation; strong convergence

1. Introduction

Let $\mathcal{A}$ be an associative algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a *derivation*, *Jordan derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$, $\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$ hold for all $A, B \in \mathcal{A}$, respectively, where $A \circ B = AB + BA$ is the usual Jordan product. Also, δ is called *Jordan triple derivation* if $\delta(A \circ B \circ C) = \delta(A) \circ B \circ C + A \circ \delta(B) \circ C + A \circ B \circ \delta(C)$ for all $A, B, C \in \mathcal{A}$. If there is no assumption of additivity for δ in the above definitions, then δ is said to be nonlinear. δ is called *Jordan derivable mapping* if δ satisfies $\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$, for every $A, B \in \mathcal{A}$ with $A \circ B \in \Omega$ where Ω is a set which satisfies some conditions. δ is called *nonlinear generalized semi-Jordan triple derivable mapping* if there is no assumption of additivity for δ but δ satisfies

$$\delta(ABC + BAC) = \delta(A)BC + A\delta(B)C + AB\delta(C) + \delta(B)AC + B\delta(A)C + BA\delta(C)$$

for all $A, B, C \in \mathcal{A}$ with $ABC \in \Omega$. Clearly, every derivation is a Jordan derivation as well as triple derivation, and every triple derivation is a Jordan triple derivation. The converse is not true in general(see [1–3]).

The standard problem is to find out whether (under some conditions) a Jordan (triple) derivation is necessarily a derivation. In 1957, Herstein [4] proved that every Jordan derivation on 2-torsion free prime rings is a derivation, and it is the first result in this direction. Then, many mathematicians studied this problem and obtained abundant results. Zhang [5] extended Herstein's result to the triangular algebra. Later, Ma [2] proved that each generalized Jordan derivation from the upper triangular matrix algebra into its bimodule can be uniquely decomposed into the sum of a generalized derivation and an anti-derivation. With the development of research, many achievements have been obtained that linear (or nonlinear) mappings on operator algebras are derivations, such as Jordan triple derivable mappings. Li [6] proved that every Jordan derivable mapping on nest algebras is a derivation. Ashraf and Jabeen [7] showed that each nonlinear Jordan triple higher derivable mapping of triangular algebras is an additive derivation. Zhao and Li in [8] proved that every nonlinear Jordan triple $*$-derivation on von Neumann algebras with no central summands of type I_1 is an additive $*$-derivation, and Darvish [9] extended the result to $*$-algebra. An and He in [10] study (m,n)-Jordan derivable mappings at zero on generalized matrix algebras. Recently, Fei and Zhang in [11] proved that every nonlinear nonglobal semi-Jordan triple derivable mapping on triangular algebras is an additive derivation. For more details see [12–18] and references therein.

Let $\mathcal{H}$ be a Hilbert space over real or complex field $\mathcal{F}$ and $\mathcal{L}$ be the subspace lattice of $\mathcal{H}$. A subspace lattice $\mathcal{L}$ is called a *commutative subspace lattice*(CSL) if each pair of projections in $\mathcal{L}$ commutes, and $Alg\mathcal{L} = \{T \in B(\mathcal{H}) : T(L) \subseteq L, \forall L \in \mathcal{L}\}$ is the associated subspace lattice algebra in $\mathcal{L}$, which is called *CSL algebra*. A totally ordered subspace lattice is called a *nest*. Recall that a subspace lattice is called *completely distributive* if $e = \bigvee\{L \in \mathcal{L} : N_- \not\supseteq L\}$ for every $0 \neq e \in \mathcal{L}$, where $N_- = \bigvee\{P \in \mathcal{L} : P \not\supseteq N\}$. Accordingly, its associated subspace lattice algebra is called *completely distributive CSL algebra*(CDC algebra). For standard definitions concerning completely distributive subspace lattice algebras see [19,20].

In [21], they proved that the collection of finite sums of rank-one operators in a CDC algebra is strongly dense. This result will be frequently used in the study of CDC algebra. Let $Alg\mathcal{L}$ be a CDC algebra. Set $\mathcal{U}(\mathcal{L}) = \{e \in \mathcal{L} : e \neq 0, e_- \neq H\}$. We say $e, e' \in \mathcal{U}(\mathcal{L})$ are connected if there exist finitely many projections $e_1, e_2, ..., e_n \in \mathcal{U}(\mathcal{L})$, such that e_i and e_{i+1} are comparable for each $i = 0, 1, \dots, n$, where $e_0 = e, e_{n+1} = e'$. $C \subseteq \mathcal{U}(\mathcal{L})$ is called a connected component if each pair in C is connected and any element in $\mathcal{U}(\mathcal{L})\setminus C$ is not connected with any element in C. Recall that a CDC algebra $Alg\mathcal{L}$ is *irreducible* if and only if the commutant is trivial, i.e. $(Alg\mathcal{L})' = \mathcal{F}I$, which is also equivalent to the condition that $\mathcal{L}\bigcap\mathcal{L}^\perp = \{0, I\}$, where $\mathcal{L}^\perp = \{e^\perp : e \in \mathcal{L}\}$. Clearly, Nest algebra is irreducible. In [22,23], it turns out that any CDC algebra can be decomposed into the direct sum of irreducible CDC algebras.

Lemma 1.1[22, 23]**.** Let $Alg\mathcal{L}$ be a CDC algebra on a separable Hitbert space $\mathcal{H}$. Then, there are no more than countably many connected components $C_n : n \in \Lambda$ of $\mathcal{E}(\mathcal{L})$ such that $\mathcal{E}(\mathcal{L}) = \cup\{e : e \in C_n, n \in \Lambda\}$. Let $e_n = \vee\{e : e \in C_n, n \in \Lambda\}$. Then, $\{e_n, n \in \Lambda\} \subseteq \mathcal{L}\cap\mathcal{L}^\perp$ is a subset of pairwise orthogonal projections, and the algebra $Alg\mathcal{L}$ can be written as a direct sum:

$$Alg\mathcal{L} = \sum_{n\in\Lambda} \oplus (Alg\mathcal{L})e_n,$$

where each $(Alg\mathcal{L})e_n$ viewed as a subalgebra of operators acting on the range of e_n is an irreducible CDC algebra. Thus, all convergence means strong convergence.

From the definition of e_n, we know that its linear span is Hilbert space $\mathcal{H}$, and pairwise orthogonal projection. It follows that the identity and center of $Alg\mathcal{L}$ are $I = \sum_{n\in\Lambda} \oplus e_n$ and $\mathcal{Z}(Alg\mathcal{L}) = \sum_{n\in\Lambda} \oplus \lambda_n e_n$, respectively, where $\lambda_n \in \mathcal{F}$. In [24], they prove that each Jordan isomorphism between irreducible CDC algebras is the sum of an isomorphism and an anti-isomorphism.

Lemma 1.2[24]**.** Let $Alg\mathcal{L}$ be a non-trivially irreducible CDC algebra on a complex Hilbert space

$\mathcal{H}$. Then, there exists a non-trivial projection $e \in \mathcal{L}$, such that $e(Alg\mathcal{L})e^{\perp}$ is faithful $Alg\mathcal{L}$ bimodule, i.e., for all $A \in Alg\mathcal{L}$, if $Ae(Alg\mathcal{L})e^{\perp} = \{0\}$, then $Ae = 0$ and if $e(Alg\mathcal{L})e^{\perp}A = \{0\}$, then $e^{\perp}A = 0$.

Let I be the identity operator on $\mathcal{H}$. If $\mathcal{L}$ is non-trivial, by Lemma 1.2, there exists a non-trivial projection $e \in \mathcal{L}$, such that $e(Alg\mathcal{L})e^{\perp}$ is faithful $Alg\mathcal{L}$ bimodule. Set $e_1 = e, e_2 = I - e_1$. Then, e_1, e_2 are the projections of $Alg\mathcal{L}$. Thus, for every A in irreducible CDC algebra $Alg\mathcal{L}$, A can be decomposed as: $A = e_1Ae_1 + e_1Ae_2 + e_2Ae_2$. Set $\mathcal{A}_{ij} = e_i(Alg\mathcal{L})e_j$. Then $Alg\mathcal{L}$ can be decomposed as

$$Alg\mathcal{L} = e_1(Alg\mathcal{L})e_1 \oplus e_1(Alg\mathcal{L})e_2 \oplus e_2(Alg\mathcal{L})e_2 = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{22}. \tag{1.1}$$

In the present note, we pursue nonlinear generalized semi-Jordan triple derivable mappings on completely distributive commutative subspace lattice algebras. Without loss of generality, we assume that any algebra is 2-torsion free.

2. Nonlinear generalized semi-Jordan triple derivable mappings on irreducible completely distributive commutative subspace lattice algebras

In this section, we begin with the irreducible case.

Theorem 2.1. Let $Alg\mathcal{L}$ be an irreducible CDC algebra on a complex Hilbert space $\mathcal{H}$ and δ : $Alg\mathcal{L} \to Alg\mathcal{L}$ be a mapping without the additivity assumption and satisfy

$$\delta(ABC + BAC) = \delta(A)BC + A\delta(B)C + AB\delta(C) + \delta(B)AC + B\delta(A)C + BA\delta(C) \tag{2.1}$$

for all $A, B, C \in \mathcal{A}$ with $ABC \in \Omega = \{A \in Alg\mathcal{L} : A^2 = 0\}$. Then, δ is an additive derivation.

Assume that $Alg\mathcal{L}$ is an irreducible CDC algebra, $e_1 \in Alg\mathcal{L}$ is an associated non-trivial projection, $e_2 = I - e_1$, $\mathcal{A}_{ij} = e_i(Alg\mathcal{L})e_j (i, j = 1, 2)$, and δ is a mapping which satisfies (2.1). It is easy to obtain that $\delta(0) = 0$(taking $A = B = C = 0$ in (2.1)). Moreover, we have the following result.

Lemma 2.1. For every $A_{ij} \in \mathcal{A}_{ij}$, we have $\delta(A_{11}) \in \mathcal{A}_{11} + \mathcal{A}_{12}$, $\delta(A_{12}) \in \mathcal{A}_{12}$ and $\delta(A_{22}) \in \mathcal{A}_{12} + \mathcal{A}_{22}$. Moreover, $e_1\delta(A_{11})e_2 = A_{11}\delta(e_1) = -A_{11}\delta(e_2)$ and $e_1\delta(A_{22})e_2 = \delta(e_2)A_{22} = -\delta(e_1)A_{22}$.

Proof. Put $A = B = e_1, C = e_2$ in (2.1), and note that $e_1e_1e_2 = 0 \in \Omega$. Thus,

$$\begin{aligned} 0 &= \delta(e_1e_1e_2 + e_1e_1e_2) = 2\delta(e_1)e_1e_2 + 2e_1\delta(e_1)e_2 + 2e_1e_1\delta(e_2) \\ &= 2e_1(\delta(e_1) + \delta(e_2))e_2 + 2e_1\delta(e_2)e_1. \end{aligned}$$

By the definition of e_1, e_2, we have $e_1\delta(e_2)e_1 = 0$ and $e_1\delta(e_1)e_2 + e_1\delta(e_2)e_2 = 0$. Similarly, taking $A = e_1, B = C = e_2$ in (2.1) we can obtain $e_2\delta(e_1)e_2 = 0$.

For every $A_{12} \in \mathcal{A}_{12}$, putting $A = e_2, B = A_{12}, C = e_2$ in (2.1) and combining $e_2A_{12}e_2 = 0 \in \Omega$ we have

$$\begin{aligned} \delta(A_{12}) &= \delta(e_2A_{12}e_2 + A_{12}e_2e_2) \\ &= \delta(e_2)A_{12} + e_2\delta(A_{12})e_2 + \delta(A_{12})e_2 + A_{12}\delta(e_2)e_2 + A_{12}\delta(e_2). \end{aligned} \tag{2.2}$$

Multiplying left by e_1 and right by e_2 in (2.2) and combining $e_1\delta(e_2)e_1 = 0$, we have $2A_{12}e_2\delta(e_2)e_2 = 0$. Similarly we can obtain $2e_1\delta(e_2)e_1A_{12} = 0$. By Lemma 1.2, we have $e_1\delta(e_1)e_1 = e_2\delta(e_2)e_2 = 0$ and $\delta(e_1) = -\delta(e_2) \in \mathcal{A}_{12}$.

For any $A_{11} \in \mathcal{A}_{11}$, putting $A = A_{11}, B = C = e_2$ in (2.1) by $A_{11}e_2e_2 = 0 \in \Omega$ and $\delta(e_2) \in \mathcal{A}_{12}$ we have

$$0 = \delta(A_{11}e_2e_2 + e_2A_{11}e_2) = \delta(A_{11})e_2 + A_{11}\delta(e_2)e_2 + e_2\delta(A_{11})e_2. \tag{2.3}$$

This implies that $e_2\delta(A_{11})e_2 = 0$, and then $\delta(A_{11}) \in \mathcal{A}_{11} + \mathcal{A}_{12}$. Furthermore, multiplying left by e_1 and right by e_2 in (2.3) and following from $\delta(e_1) = -\delta(e_2) \in \mathcal{A}_{12}$, we can obtain

$$e_1\delta(A_{11})e_2 = -A_{11}\delta(e_2) = A_{11}\delta(e_1).$$

For any $A_{22} \in \mathcal{A}_{22}$, putting $A = B = e_1, C = A_{22}$ in (2.1) it follows from $e_1e_1A_{22} = 0 \in \Omega$ that

$$0 = \delta(e_1e_1A_{22} + e_1e_1A_{22}) = 2e_1(\delta(e_1)A_{22} + \delta(A_{22}))e_2 + 2e_1\delta(A_{22})e_1.$$

This implies that $e_1\delta(A_{22})e_2 = -\delta(e_1)A_{22} = \delta(e_2)A_{22}$, $e_1\delta(A_{22})e_1 = 0$, and $\delta(A_{22}) \in \mathcal{A}_{12} + \mathcal{A}_{22}$.

For any $A_{12} \in \mathcal{A}_{12}$, noting that $A_{12}e_1e_2 = 0 \in \Omega$, putting $A = A_{12}, B = e_1, C = e_2$ in (2.1) and combining $\delta(e_1) \in \mathcal{A}_{12}$ we have

$$\delta(A_{12}) = \delta(A_{12}e_1e_2 + e_1A_{12}e_2) = e_1\delta(A_{12})e_2 \in \mathcal{A}_{12}.$$

The proof is completed. □

Lemma 2.2. For any $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$, we have $\delta(A_{ij}B_{kl}) = \delta(A_{ij})B_{kl} + A_{ij}\delta(B_{kl})$, for all $i, j, k, l = 1, 2$ and $i \leq j \leq k \leq l$.

Proof. Consider the case when $i = j = 1,\ k = l = 2$. Since $A_{11}A_{22}e_2 = 0 \in \Omega$, putting $A = A_{11}, B = A_{22}, C = e_2$ in (2.1) it then follows from Lemma 2.1 that

$$0 = \delta(A_{11}A_{22}) = \delta(A_{11}A_{22}e_2 + A_{22}A_{11}e_2) = \delta(A_{11})A_{22} + A_{11}\delta(A_{22}).$$

Consider the case when $i = j = k = 1,\ l = 2$. Since $A_{12}A_{11}e_2 = 0 \in \Omega$, putting $A = A_{12}, B = A_{11}, C = e_2$ in (2.1) by $\delta(e_2) \in \mathcal{A}_{12}$ and Lemma 2.1 we can obtain that

$$\delta(A_{11}A_{12}) = \delta(A_{12}A_{11}e_2 + A_{11}A_{12}e_2) = \delta(A_{11})A_{12} + A_{11}\delta(A_{12}). \tag{2.4}$$

Consider the case when $i = 1, j = k = l = 2$. Since $A_{12}A_{11}e_2 = 0 \in \Omega$, taking $A = A_{22}, B = A_{12}, C = e_2$ in (2.1) by $\delta(e_2) \in \mathcal{A}_{12}$ and Lemma 2.1 we can obtain that

$$\delta(A_{12}A_{22}) = \delta(A_{22}A_{12}e_2 + A_{12}A_{22}e_2) = \delta(A_{12})A_{22} + A_{12}\delta(A_{22}). \tag{2.5}$$

Consider the case when $i = j = k = l = 1$. By (2.4) we know that

$$\delta(A_{11}B_{11}A_{12}) = \delta((A_{11}B_{11})A_{12}) = \delta(A_{11}B_{11})A_{12} + A_{11}B_{11}\delta(A_{12}), \tag{2.6}$$

and

$$\delta(A_{11}B_{11}A_{12}) = \delta(A_{11}(B_{11}A_{12})) = \delta(A_{11})B_{11}A_{12} + A_{11}\delta(B_{11})A_{12} + A_{11}B_{11}\delta(B_{12}). \tag{2.7}$$

Comparing (2.6) and (2.7), we get

$$(\delta(A_{11}B_{11}) - \delta(A_{11})B_{11} - A_{11}\delta(B_{11}))A_{12} = 0.$$

It follows from Lemma 2.1 that

$$e_1\delta(A_{11}B_{11})e_1 = e_1\delta(A_{11})B_{11} + A_{11}\delta(B_{11})e_1 = \delta(A_{11})B_{11} + A_{11}\delta(B_{11})e_1.$$

Furthermore, by Lemma 2.1 we have

$$e_1\delta(A_{11}B_{11})e_2 = A_{11}B_{11}\delta(e_1) = A_{11}\delta(B_{11})e_2.$$

Noting that $e_2\delta(A_{11}B_{11})e_2 = 0$ and $\delta(A_{11}) \in \mathcal{A}_{11} + \mathcal{A}_{12}$, we get

$$\begin{aligned}\delta(A_{11}B_{11}) &= (e_1 + e_2)\delta(A_{11}B_{11})(e_1 + e_2)\\ &= e_1\delta(A_{11}B_{11})e_1 + e_1\delta(A_{11}B_{11})e_2\\ &= \delta(A_{11})B_{11} + A_{11}\delta(B_{11})e_1 + A_{11}\delta(B_{11})e_2\\ &= \delta(A_{11})B_{11} + A_{11}\delta(B_{11}).\end{aligned}$$

Similarly, by (2.5) one can check that when $i = j = k = l = 2$,

$$\delta(A_{22}B_{22}) = \delta(A_{22})B_{22} + A_{22}\delta(B_{22}).$$

The proof is completed. □

Lemma 2.3. δ is an additive mapping on irreducible *CDC* algebra $Alg\mathcal{L}$.

Proof. We divide the proof into three claims.

Claim 1. For all $A_{ij} \in \mathcal{A}_{ij}$, $\delta(A_{11} + A_{12}) = \delta(A_{11}) + \delta(A_{12})$ and $\delta(A_{12} + A_{22}) = \delta(A_{12}) + \delta(A_{22})$.

For every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$, noting that $\delta(e_i) \in \mathcal{A}_{12} (i = 1, 2)$, $e_1\delta(A_{11})e_2 = -A_{11}\delta(e_2) \in \mathcal{A}_{12}$ and $e_2(A_{11} + A_{12})e_2 = 0 \in \Omega$. Then, putting $A = e_2, B = A_{11} + A_{12}, C = e_2$ in (2.1) we can obtain

$$\begin{aligned}\delta(A_{12}) &= \delta(e_2(A_{11} + A_{12})e_2 + (A_{11} + A_{12})e_2e_2)\\ &= e_2\delta(A_{11} + A_{12})e_2 + \delta(A_{11} + A_{12})e_2 + A_{11}\delta(e_2)e_2\\ &= e_2\delta(A_{11} + A_{12})e_2 + \delta(A_{11} + A_{12})e_2 - e_1\delta(A_{11})e_2.\end{aligned}$$

It follows from $\delta(A_{12}) = e_1\delta(A_{12})e_2$ that $e_2\delta(A_{11} + A_{12})e_2 = 0$ and

$$e_1\delta(A_{11} + A_{12})e_2 = e_1(\delta(A_{11}) + \delta(A_{12}))e_2. \tag{2.8}$$

Furthermore, since $B_{12}(A_{11} + A_{12})e_2 = 0 \in \Omega$, putting $A = B_{12}, B = A_{11} + A_{12}, C = e_2$ in (2.1) then from Lemma 2.1 we get

$$\delta(A_{11}B_{12}) = \delta(B_{12}(A_{11} + A_{12})e_2 + (A_{11} + A_{12})B_{12}e_2) = \delta(A_{11} + A_{12})B_{12} + A_{11}\delta(B_{12}).$$

By Lemma 2.2, based on $(\delta(A_{11} + A_{12}) - \delta(A_{11}))B_{12} = 0$ and Lemma 1.2, we have

$$e_1\delta(A_{11} + A_{12})e_1 = e_1\delta(A_{11})e_1. \tag{2.9}$$

Hence, by (2.8),(2.9) and Lemma 2.1 we get

$$\delta(A_{11} + A_{12}) = \delta(A_{11}) + \delta(A_{12}). \tag{2.10}$$

Putting $A = A_{12} + A_{22}, B = e_1, C = e_2$ in (2.1), one can check that

$$\delta(A_{12} + A_{22}) = \delta(A_{12}) + \delta(A_{22}). \tag{2.11}$$

Claim 2. For all $i, j = 1, 2$ and $i \leq j$, $\delta(A_{ij} + B_{ij}) = \delta(A_{ij}) + \delta(B_{ij})$.

Since $(B_{12} + e_2)(e_1 + A_{12})e_2 = 0 \in \Omega$, taking $A = (e_1 + A_{12}), B = (e_2 + B_{12}), C = e_2$ in (2.1), by (2.10), (2.11), Lemma 2.1 and $\delta(e_1) = -\delta(e_2) \in \mathcal{A}_{12}$ we have

$$\begin{aligned}\delta(A_{12} + B_{12}) &= \delta((B_{12} + e_2)(e_1 + A_{12})e_2 + (e_1 + A_{12})(B_{12} + e_2)e_2) \\ &= \delta(e_1 + A_{12})(B_{12} + e_2)e_2 + (e_1 + A_{12})\delta(B_{12} + e_2)e_2 + (e_1 + A_{12})(B_{12} + e_2)\delta(e_2) \\ &= \delta(e_1) + \delta(A_{12}) + \delta(B_{12}) + \delta(e_2) = \delta(A_{12}) + \delta(B_{12}).\end{aligned} \tag{2.12}$$

From (2.4), we know that

$$\delta((A_{11} + B_{11})A_{12}) = \delta(A_{11} + B_{11})A_{12} + (A_{11} + B_{11})\delta(A_{12}).$$

From (2.12) and (2.4), we have

$$\begin{aligned}\delta((A_{11} + B_{11})A_{12}) &= \delta(A_{11}A_{12} + B_{11}A_{12}) = \delta(A_{11}A_{12}) + \delta(B_{11}A_{12}) \\ &= \delta(A_{11})A_{12} + A_{11}\delta(A_{12}) + \delta(B_{11})A_{12} + B_{11}\delta(A_{12}).\end{aligned}$$

Combining above two equations, we can get $(\delta(A_{11}+B_{11})-\delta(A_{11})-\delta(B_{11}))A_{12} = 0$, for all $A_{12} \in \mathcal{A}_{12}$. From Lemma 1.2, we have

$$e_1\delta(A_{11} + B_{11})e_1 = e_1\delta(A_{11})e_1 + e_1\delta(B_{11})e_1.$$

It follows from Lemma 2.1 that

$$e_1\delta(A_{11} + B_{11})e_2 = (A_{11} + B_{11})\delta(e_1) = A_{11}\delta(e_1) + B_{11}\delta(e_1) = e_1\delta(A_{11})e_2 + e_1\delta(B_{11})e_2.$$

Therefore, it follows from above two equations and $e_2\delta(A_{11} + B_{11})e_2 = 0$ that

$$\delta(A_{11} + B_{11}) = \delta(A_{11}) + \delta(B_{11}). \tag{2.13}$$

Similarly, one can check that

$$\delta(A_{22} + B_{22}) = \delta(A_{22}) + \delta(B_{22}). \tag{2.14}$$

Claim 3. $\delta(A_{11} + A_{12} + A_{22}) = \delta(A_{11}) + \delta(A_{12}) + \delta(A_{22})$.

For any $A_{ij} \in \mathcal{A}_{ij}$, since $(A_{11} + A_{12} + A_{22})e_1e_2 = 0 \in \Omega$, putting $A = A_{11} + A_{12} + A_{22}, B = e_1, C = e_2$ in (2.1) it follows from $\delta(e_1) = -\delta(e_2) \in \mathcal{A}_{12}$ and Lemma 2.1 that

$$\begin{aligned}\delta(A_{12}) &= \delta((A_{11} + A_{12} + A_{22})e_1e_2 + e_1(A_{11} + A_{12} + A_{22})e_2) \\ &= \delta(e_1)A_{22} + e_1\delta(A_{11} + A_{12} + A_{22})e_2 + A_{11}\delta(e_2) + (A_{11}\delta(e_1) + A_{11}\delta(e_2)) \\ &= e_1\delta(A_{11} + A_{12} + A_{22})e_2 - e_1\delta(A_{11})e_2 - e_1\delta(A_{22})e_2.\end{aligned}$$

It follows from $\delta(A_{12}) \in \mathcal{A}_{12}$ that

$$e_1\delta(A_{11} + A_{12} + A_{22})e_2 = e_1(\delta(A_{11}) + \delta(A_{12}) + \delta(A_{22}))e_2. \tag{2.15}$$

For any $B_{12} \in \mathcal{A}_{12}$, since $(e_1(A_{11} + A_{12} + A_{22})B_{12})^2 = (A_{11}B_{12})^2 = 0$ which implies $e_1(A_{11} + A_{12} + A_{22})B_{12} \in \Omega$, putting $A = e_1, B = A_{11} + A_{12} + A_{22}, C = B_{12}$ in (2.1) it then follows from Lemma 2.1 and $\delta(e_1) \in \mathcal{A}_{12}$ that

$$\begin{aligned}\delta(2A_{11}B_{12}) &= \delta(e_1(A_{11} + A_{12} + A_{22})B_{12} + (A_{11} + A_{12} + A_{22})e_1B_{12}) \\ &= 2e_1\delta(A_{11} + A_{12} + A_{22})B_{12} + 2A_{11}\delta(B_{12}).\end{aligned}$$

It follows from Lemma 2.2 and Claim 1,2 that

$$\delta(2A_{11}B_{12}) = 2\delta(A_{11}B_{12}) = 2\delta(A_{11})B_{12} + 2A_{11}\delta(B_{12}).$$

Comparing above two equations, we obtain that $2e_1(\delta(A_{11} + A_{12} + A_{22}) - \delta(A_{11}))e_1B_{12} = 0$, and then by Lemma 1.2, we have

$$e_1\delta(A_{11} + A_{12} + A_{22})e_1 = e_1\delta(A_{11})e_1. \tag{2.16}$$

Similarly, one can check that

$$e_2\delta(A_{11} + A_{12} + A_{22})e_2 = e_2\delta(A_{22})e_2. \tag{2.17}$$

It follows from (2.15)–(2.17) that $\delta(A_{11} + A_{12} + A_{22}) = \delta(A_{11}) + \delta(A_{12}) + \delta(A_{22})$ and then δ is an additive mapping. The proof is completed. □

In the following, we give the completed proof of Theorem 2.1.

Proof of Theorem 2.1. Let $A = A_{11} + A_{12} + A_{22}$ and $B = B_{11} + B_{12} + B_{22}$ be arbitrary elements of irreducible *CDC* algebra $Alg\mathcal{L}$ where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. It follows from Lemmas 2.1–2.3 that

$$\begin{aligned}\delta(AB) &= \delta(A_{11}B_{11} + A_{11}B_{12} + A_{12}B_{22} + A_{22}B_{22}) \\ &= \delta(A_{11}B_{11}) + \delta(A_{11}B_{12}) + \delta(A_{12}B_{22}) + \delta(A_{22}B_{22}) \\ &= \delta(A_{11})B_{11} + A_{11}\delta(B_{11}) + \delta(A_{11})B_{12} + A_{11}\delta(B_{12}) \\ &+ \delta(A_{12})B_{22} + A_{12}\delta(B_{22}) + \delta(A_{22})B_{22} + A_{22}\delta(B_{22}) \\ &= \delta(A_{11} + A_{12} + A_{22})(B_{11} + B_{12} + B_{22}) + (A_{11} + A_{12} + A_{22})\delta(B_{11} + B_{12} + B_{22}) \\ &= \delta(A)B + A\delta(B).\end{aligned}$$

Therefore δ is an additive derivation on irreducible *CDC* algebra $Alg\mathcal{L}$. The proof is completed. □

3. Main results

In this section, we study nonlinear generalized semi-Jordan triple derivable mappings on completely distributive commutative subspace lattice algebras. The main result reads as follows.

Theorem 3.1. Let $Alg\mathcal{L}$ be an associated completely distributive commutative subspace lattice algebras on a complex Hilbert space $\mathcal{H}$ and $\delta : Alg\mathcal{L} \to Alg\mathcal{L}$ be a mapping without the additivity assumption and satisfy

$$\delta(ABC + BAC) = \delta(A)BC + A\delta(B)C + AB\delta(C) + \delta(B)AC + B\delta(A)C + BA\delta(C)$$

for all $A, B, C \in \mathcal{A}$ with $ABC \in \Omega = \{A \in Alg\mathcal{L} : A^2 = 0\}$. Then, δ is an additive derivation.

Proof. Let $e_n = \vee\{e : e \in C_n, n \in \Lambda\}$ be the projections of $\mathcal{L}$ as in Lemma 1.1. By Lemma 1.1, we know that $Alg\mathcal{L} = \sum_{n\in\Lambda} \oplus(Alg\mathcal{L})e_n$ is the irreducible decomposition of $Alg\mathcal{L}$. Fix an index n, it follows that e_n is also Hilbert space and

$$(Alg\mathcal{L})e_n = e_n(Alg\mathcal{L})e_n = Alg(e_n\mathcal{L}).$$

Then, for all $A \in Alg\mathcal{L}$ and e_n, $Alg(e_n\mathcal{L})$ is an irreducible CDC algebra on Hilbert space e_n. Let δ be a nonlinear generalized semi-Jordan triple derivable mapping from $Alg\mathcal{L}$ into itself. Then, it follows from Theorem 2.1 that there exists an additive derivation δ_n from $(Alg\mathcal{L})e_n$ into itself such that for all $A, B \in Alg(e_n\mathcal{L})$

$$\delta(AB) = \delta_n(AB) = \delta_n(A)B + A\delta_n(B).$$

In [23], they prove that $Alg\mathcal{L}$ is CDC algebra if and only if the linear span of the rank-one operators in $Alg\mathcal{L}$ is ultraweakly dense. Choose a set $E \in \mathcal{U}(L)$, then, for every $x \in E$, fix an element $y \in E_-^\perp$, and then $x \otimes y \in Alg\mathcal{L}$ is a rank-one operator. For every $u \otimes v \in Alg(e_n\mathcal{L})$ and $A \in Alg(e_n\mathcal{L})$, it follows from Theorem 2.1 that

$$\delta_n((u\otimes v)A(x\otimes y)) = \delta_n(u\otimes v)A(x\otimes y) + (u\otimes v)\delta_n(A)(x\otimes y) + (u\otimes v)A\delta_n(x\otimes y). \tag{3.1}$$

Assuming that $\{A_k\}, A \in Alg(e_n\mathcal{L})$ and $\{A_k\}$ strongly converge to A, it follows from (3.1) that

$$\begin{aligned} (u\otimes v)\delta_n(A_k)(x\otimes y) &= \delta_n((u\otimes v)A_k(x\otimes y)) - \delta_n(u\otimes v)A_k(x\otimes y) - (u\otimes v)A_k\delta_n(x\otimes y) \\ &\rightarrow \delta_n((u\otimes v)A(x\otimes y)) - \delta_n(u\otimes v)A(x\otimes y) - (u\otimes v)A\delta_n(x\otimes y) \\ &= (u\otimes v)\delta_n(A)(x\otimes y). \end{aligned}$$

This means that δ_n is strongly convergent.

Assume $\{A_k\}, \{B_k\}, \{C_k\} \in Alg\mathcal{L}$ and $\{A_k\}, \{B_k\}, \{C_k\}$ converge strongly to A, B, C, respectively. Since $Alg\mathcal{L} = \sum_{n\in\Lambda} \oplus(Alg\mathcal{L})e_n$, and e_n are pairwise orthogonal projection, for every e_i, $\{A_ke_i\}, \{B_ke_i\}, \{C_ke_i\}$ converge strongly to Ae_i, Be_i, Ce_i, respectively and

$$A_kB_kC_k = (\sum_{i\in\Lambda} \oplus A_ke_i)(\sum_{i\in\Lambda} \oplus B_ke_i)(\sum_{i\in\Lambda} \oplus C_ke_i) = \sum_{i\in\Lambda} \oplus A_kB_kC_ke_i.$$

Then, for every x in Hilbert space $\mathcal{H}$ and $A_kB_kC_k \in \Omega$, $\{A_k\}, \{B_k\}, \{C_k\}$ converging strongly to A, B, C implies that $ABC \in \Omega$. It follows from the proof of Theorem 2.1 that

$$\begin{aligned} \delta(A_kB_kC_k + B_kA_kC_k)x &= \delta(\sum_{n\in\Lambda} \oplus(A_kB_kC_k + B_kA_kC_k)e_n)x \\ &= \sum_{n\in\Lambda} \oplus\delta_n(A_ke_nB_ke_nC_ke_n + B_ke_nA_ke_nC_ke_n)x \\ &= \sum_{n\in\Lambda} \oplus(\delta_n(A_ke_n)B_ke_nC_ke_n + A_ke_n\delta_n(B_ke_n)C_ke_n + A_ke_nB_ke_n\delta_n(C_ke_n) \\ &+ \delta_n(B_ke_n)A_ke_nC_ke_n + B_ke_n\delta_n(A_ke_n)C_ke_n + B_ke_nA_ke_n\delta_n(C_ke_n))x \\ &\rightarrow \sum_{n\in\Lambda} \oplus(\delta_n(Ae_n)Be_nCe_n + Ae_n\delta_n(Be_n)Ce_n + Ae_nBe_n\delta_n(Ce_n) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \delta_n(Be_n)Ae_nCe_n + Be_n\delta_n(Ae_n)Ce_n + Be_nAe_n\delta_n(Ce_n))x \\
&= \quad \sum_{n\in\Lambda} \oplus\delta_n((ABC + BAC)e_n)x = \delta(ABC + BAC)x
\end{aligned}
$$

It means that δ is strongly convergent on CDC algebra $Alg\mathcal{L}$. Thus, for every $A, B \in Alg\mathcal{L}$ we obtain that

$$
\delta(AB) = \sum_{n\in\Lambda} \oplus\delta_n(Ae_nBe_n) = \sum_{i\in\Lambda} \oplus(\delta_n(Ae_n)Be_n + Ae_n\delta_n(Be_n)) = \delta(A)B + A\delta(B).
$$

The proof is completed. □

4. Conclusions

In this paper, we use the structure properties of completely distributive commutative subspace lattice algebras and decomposition of algebraic to study the derivable mapping on certain CSL algebra. We proved that every nonlinear generalized semi-Jordan triple derivable mapping on completely distributive commutative subspace lattice algebras is an additive derivation. Moreover, the purpose of this modification is to answer the classic problem of preserving derivable mappings of certain CSL algebra.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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