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Research article

Common fixed and coincidence point theorems for nonlinear self-mappings in cone b-metric spaces using φ-mapping

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Abstract: In this paper, by means of a mapping $\varphi \in \Phi(P, P_1)$, some new common fixed and coincidence point theorems for four and six nonlinear self-mappings in cone b-metric spaces are established, respectively. Also, some examples are given to prove the effectiveness of our results. And with some remarks stating that our results complement and sharply improve some related results in the literature.

Keywords: coincidence point; common fixed point; φ-mapping; nonlinear contraction; cone b-metric space

1. Introduction

Coincidence points and their applications for two mappings in metric spaces were first studied in 1996 by Jungck [1, 2]. From then on, the coincidence point theorems for various different nonlinear mappings is established by some authors in metric spaces, b-metric spaces and cone metric spaces, respectively. For example, the authors of [3–6] establish some common fixed and coincidence point theorems for commuting and noncommuting mappings in metric spaces, the authors of [7–10] establish some coincidence point theorems for noncontinuity and Prešić-Reich type mappings in cone metric spaces and the authors of [11–14] establish some common fixed point theorems for weakly T-Chatterjea (T-Kannan) and four mappings in b-metric spaces. In addition, some fixed point theorems for KKM and contractive mappings are established by the authors of [15, 16] in cone b-metric spaces. Further, some common fixed theories are also discussed by the authors of [17–19] in some extended b-metric spaces.

Recently, Abbas et al. [20], Han et al. [21], Rangamma et al. [22], Malhotra et al. [23] and Dubey et al. [24] proved some the existence and uniqueness of coincidence points for three or four nonlinear mappings in cone metric spaces, respectively. Malhotra et al. [23] unify and generalize the results of [22] and [24] with a new type of contractive condition by introducing a mapping $\varphi \in \Phi(P, P_1)$

(See Definition 2.10). Liu et al. [25] proved some new the existence and uniqueness of common fixed point (CFP) for six self-mappings in b-metric spaces. In particular, it is not difficult to see that in the conclusions obtained by the authors of [20–25], the restriction of coefficients for the nonlinear mappings must satisfy the inequality $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. The necessity of the inequality condition is a question worthy of study.

Inspired by the work of [23, 25], in this manuscript we discuss the existence and uniqueness of coincidence point and CFP for four and six nonlinear self-mappings using a φ-mapping in cone b-metric spaces. In Sections 2 and 3, we first introduce the new class of mappings $\Phi(P, P_1)$. Then, by using a mapping $\varphi \in \Phi(P, P_1)$, some new coincidence point and CFP theorems for four and six nonlinear self-mappings in cone b-metric spaces are established, respectively. Also, several important corollaries are given. Finally, two examples are given to show the validity of our results which indicate that our results complement and sharply improve some related results in [7,8,10,13,16,20,21,23–27].

2. Preliminaries

Throughout this paper the following notation and lemmas will be used which were taken from [7,8,10,15,16,20,26,27].

Let E be a real Banach space (RBS). A cone $P \subset E$ is defined by

1) $P \neq \{\theta\}$, $P \neq \emptyset$ and P is closed;
2) $t, s \in \mathbb{R}^+ = [0, +\infty)$, $x, y \in P$ implies $tx + sy \in P$;
3) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, the partial ordering $\preccurlyeq$ induced by P is defined as $x \preccurlyeq y$ if and only if $y - x \in P$ where $x, y \in E$. For any $x, y \in E$, if $x \preccurlyeq y$ and $x \neq y$ we abbreviated as $x \prec y$. More, $x \ll y$ indicate that $y - x \in \mathrm{int}P$ where $\mathrm{int}P$ denotes the interior of P. If $\mathrm{int}P \neq \emptyset$ then P is called a solid cone. For all $\theta \preccurlyeq x \preccurlyeq y$, if there is $k > 0$ such that $\|x\| \preccurlyeq k\|y\|$ then P is called a normal cone (see [15, 16, 25]).

In the following, E is always assumed as the RBS. θ_E denotes the zero element. $P \subset E$ is a solid cone. The notation of $\preccurlyeq$ is the partial ordering with respect to P.

Definition 2.1. *[26] Let $X \neq \emptyset$. For all $x, y, z \in X$, suppose the mapping $d : X \times X \to E$ satisfies:*
(cm$_1$) $\theta_E \preccurlyeq d(x, y)$ and $d(x, y) = \theta_E \Leftrightarrow x = y$;
(cm$_2$) $d(x, y) = d(y, x)$;
(cm$_3$) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$.
Then, the pair (X, d) is called a cone metric space (CMS for short), and d is called a cone metric on X.

Definition 2.2. *[15] Let $X \neq \emptyset$ and $s \geqslant 1$. For all x, y, $z \in X$, suppose the mapping $d : X \times X \to E$ satisfies:*
(cbm$_1$) $\theta_E \preccurlyeq d(x, y)$ and $d(x, y) = \theta_E \Leftrightarrow x = y$;
(cbm$_2$) $d(x, y) = d(y, x)$;
(cbm$_3$) $d(x, y) \preccurlyeq s[d(x, z) + d(z, y)]$.
Then, the pair (X, d) is called a cone b-metric space (CbMS for short), d is called a cone b-metric on X, and s is called the coefficient of (X, d).

Remark 2.3. *The class of CbMSs is effectively larger than that of CMSs. Indeed, a cone b-metric is a cone metric as $s = 1$ but the converse is not true. For the counter-example see [15, 16].*

The remark on the cones is given below.

Remark 2.4. *[4] Let $P \subset E$ be a cone, $\mathbb{N} = \{1, 2, \cdots\}$ and x, y, z, x_n, $y_n \in E$.*

(a) If $x \preccurlyeq y$ and $y \ll z$ then $x \ll z$.
(b) For all $x \in intP$, if $\theta_E \preccurlyeq y \ll x$ then $y = \theta_E$.
(c) If $x \in intP$ and $y_n \to \theta_E$ then there is an $n_0 \in \mathbb{N}$ such that $y_n \ll x$ for all $n > n_0$.
(d) If $x_n \to x, y_n \to y$ and $\theta_E \preccurlyeq x_n \preccurlyeq y_n$ for all $n \in \mathbb{N}$, then $x \preccurlyeq y$.
(e) If there is $\lambda \in [0, 1)$ such that $x \preccurlyeq \lambda x$ then $x = \theta_E$.

Definition 2.5. *[15] Let (X, d) be a CbMS, $x \in X$ and $\{x_n\}$ be a sequence in X.*
(i) If for every $y \in P$ with $\theta_E \ll y$ there is $N \in \mathbb{N}$ such that $d(x_n, x) \ll y$ for all $n > N$ then $\{x_n\}$ is said to be converges to x. Abbreviated as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii) If for every $y \in P$ with $\theta_E \ll y$ there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll y$ for all $n, m > N$ then $\{x_n\}$ is called a Cauchy sequence in X.
(iii) If every Cauchy sequence in X is convergent in X then (X, d) is called a complete CbMS.

Lemma 2.6. *[15, 16] Let (X, d) be a CbMS, $P \subset E$ be a solid cone and let $\{x_n\} \subset X$. We have*
1) $\{x_n\}$ converges to $x \in X$ iff $d(x_n, x) \to \theta_E$ as $n \to \infty$;
2) $\{x_n\}$ is a Cauchy sequence iff $d(x_n, x_m) \to \theta_E$ as $n, m \to \infty$.

Lemma 2.7. *[15, 16] Let (X, d) be a CbMS, P be a solid cone and $\{x_n\} \subset X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$.*

Definition 2.8. *[7] Let $X \neq \emptyset$. Suppose that f and g are two self-mappings defined on X. An element $x \in X$ is called to a coincidence point of f and g if $fx = gx = w \in X$. At this time, w is a point of coincidence of f and g.*

Definition 2.9. *[1] Two self-mappings f and g of a CMS (X, d) are said to be weakly compatible if $fgx = gfx$ whenever $fx = gx$ for some $x \in X$.*

Definition 2.10. *Let E and E_1 be two RBS and $P \subset E$ and $P_1 \subset E_1$ be two solid cones. The notations of $\preccurlyeq$ and $\leq$ are two partial orderings with respect to P and P_1 respectively. Let a mapping $\varphi : P \to P_1$ satisfying the following properties:*
(Φ-1) there is a constant $K \geqslant 1$ such that $\varphi(x) \leq K\varphi(y)$ for all $x, y \in P$ with $x \preccurlyeq y$;
(Φ-2) there is a constant $\sigma > 0$ such that $\varphi(sx) \leq s^\sigma \varphi(x)$ for all $x \in P$, $s \geqslant 1$;
(Φ-3) there is a constant $\omega \geqslant 1$ such that $\varphi(x + y) \leq \omega[\varphi(x) + \varphi(y)]$ for all $x, y \in P$;
(Φ-4) φ is sequentially continuous, i.e., if $x_n, x \in P$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$;
(Φ-5) if $\varphi(x_n) \to \theta_{E_1}$, then $x_n \to \theta_E$ where θ_E and θ_{E_1} are the zero vectors of E and E_1 respectively.
The set of the maps that satisfy all the above properties are represented by $\Phi(P, P_1)$.

Remark 2.11. *It is clear that $\varphi(x_n) \to \theta_{E_1}$ if and only if $x_n \to \theta_E$. Let (X, d) be a CbMS with the coefficient $s \geqslant 1$ and $P \subset E$ be a solid cone and $\varphi \in \Phi(P, P_1)$. Since $d(x, y) \preccurlyeq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$, we have*

$$\varphi(d(x, y)) \leq K\omega s^\sigma [\varphi(d(x, z)) + \varphi(d(z, y))]. \tag{2.1}$$

In addition, if $\omega = 1 = s = \sigma$, P and P_1 are normal cones then $\Phi(P, P_1)$ in Definition 2.10 is reduced to the original definition of a CMS in [23].

The following are some examples of $\Phi(P, P_1)$ defined above.

Example 2.12. *Let E be a RBS with $P \subset E$ is a cone. Define $\varphi : P \to P$ by $\varphi(x) = x$, for all $x \in P$. Then, $\varphi \in \Phi(P, P)$ with $E = E_1, P = P_1, K = 1, \sigma = 1$ and $\omega = 1$.*

Example 2.13. *Let E be a RBS with $P \subset E$ is a normal cone and normal constant $k \geqslant 1$. Define $\varphi : P \to [0, +\infty)$ by $\varphi(x) = \|x\|^\alpha$, for all $x \in P$ where $1 \geqslant \alpha > 0$. Then, $\varphi \in \Phi(P, P_1)$ with $E_1 = \mathbb{R}, P_1 = [0, +\infty), K = k^\alpha, \sigma = \alpha$ and $\omega = 1$.*

In fact, the validity of (Φ-1), (Φ-2), (Φ-4) and (Φ-5) is evident. Note that $(\gamma + t)^\alpha \leqslant \gamma^\alpha + t^\alpha$ $(\gamma, t \geqslant 0, 1 \geqslant \alpha > 0)$. For all $x, y \in P$, we have

$$\|x + y\|^\alpha \leqslant (\|x\| + \|y\|)^\alpha \leqslant \|x\|^\alpha + \|x\|^\alpha$$

which implies $\omega = 1$ and (Φ-3) holds.

Example 2.14. *Let E be a RBS with $P \subset E$ is a normal cone and normal constant $k \geqslant 1$. Define $\varphi : P \to [0, +\infty)$ by $\varphi(x) = \|x\|^\alpha$, for all $x \in P$ where $1 < \alpha$. Then, $\varphi \in \Phi(P, P_1)$ with $E_1 = \mathbb{R}, P_1 = [0, +\infty), K = k^\alpha, \sigma = \alpha$ and $\omega = 2^{[\alpha]}$.*

In fact, the validity of (Φ-1), (Φ-2), (Φ-4) and (Φ-5) is evident. Note that $(\gamma + t)^\alpha \leqslant 2^{[\alpha]}(\gamma^\alpha + t^\alpha)$, where $\gamma, t \geqslant 0, 1 < \alpha$ and $[\alpha]$ is the integral function. For all $x, y \in P$, we have

$$\|x + y\|^\alpha \leqslant (\|x\| + \|y\|)^\alpha \leqslant 2^{[\alpha]}(\|x\|^\alpha + \|y\|^\alpha)$$

which implies $\omega = 2^{[\alpha]}$ and (Φ-3) holds.

Example 2.15. *Let $E = \mathbb{R}^2$ and $E_1 = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x, y \in \mathbb{R} \right\}$. It is clear that E and E_1 are two RBSs.*

Suppose that $P = \{(x, y) : x, y \geqslant 0\}$ and $P_1 = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x, y \geqslant 0 \right\}$ then $P \subset E$ and $P_1 \subset E_1$ are two normal cones. Define $\varphi : P \to P_1$ by

$$\varphi((x, y)) = \begin{pmatrix} x^2 & 2y^2 \\ 0 & x^2 \end{pmatrix}, \text{ for all } (x, y) \in P.$$

Then, $\varphi \in \Phi(P, P_1)$, where $K = 1$ and $\sigma = 2 = \omega$.

Example 2.16. *Let $E = C([0, 1])$ with the supremum norm and $P = \{x \in E : x(t) \geqslant 0, \forall t \in [0, 1]\}$. Then, P is a normal cone where normal constant $k = 1$. For any $1 \geqslant \alpha > 0$, define $\varphi : P \to P$ by*

$$\varphi(x(t)) = \frac{x^\alpha(t)}{1 + x^\alpha(t)}, \text{ for all } x(t) \in P.$$

We can prove that $\varphi \in \Phi(P, P)$ with $K = 1, \sigma = \alpha$ and $\omega = 1$. In fact, we only need to verify (Φ-3) in Definition 2.10. Note that $1 \geqslant \alpha > 0$. For all $x(t), y(t) \in P$, we have

$$\frac{(x(t) + y(t))^\alpha}{1 + (x(t) + y(t))^\alpha} \leqslant \frac{x^\alpha(t) + y^\alpha(t)}{1 + x^\alpha(t) + y^\alpha(t)} \leqslant \frac{x^\alpha(t)}{1 + x^\alpha(t)} + \frac{y^\alpha(t)}{1 + y^\alpha(t)},$$

which shows $\omega = 1$ and (Φ-3) holds.

If $E = C^2([0, 1])$ with the norm $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$ and $P = \{x \in E : x(t) \geqslant 0, \forall t \in [0, 1]\}$. Then, P is a non-normal cone. But $\varphi : P \to P$ in Example 2.16 also is an element of $\Phi(P, P)$ with $K = 1, \sigma = \alpha$ and $\omega = 1$.

Example 2.17. *Let $E = C([0,1])$ with the supremum norm and $P = \{x \in E : x(t) \geqslant 0, \forall t \in [0,1]\}$.*

Define $\varphi : P \to P$ by

$$\varphi(x(t)) = \ln(x^2(t) + 1), \text{ for all } x(t) \in P.$$

Then, $\varphi \in \Phi(P,P)$ where $K = 1, \sigma = 2$ and $\omega = 2$. In fact, the validity of (Φ-1), (Φ-4) and (Φ-5) is evident. We only need to verify (Φ-2) and (Φ-3) in Definition 2.10. Note that $s^2(\gamma^2+1)^{s^2-1} - s^2 \geqslant 0$ for all $\gamma \geqslant 0, s \geqslant 1$. We have $(\gamma^2+1)^{s^2} - s^2\gamma^2 - 1 \geqslant 0$ for all $\gamma \geqslant 0, s \geqslant 1$ which implies that $\ln((sx(t))^2+1) \leqslant s^2 \ln(x^2(t)+1)$, i.e., $\sigma = 2$.

Further, for any $\gamma, t \geqslant 0$, according to the fact that

$$\gamma^2 + t^2 + 2\gamma t + 1 \leqslant 2(\gamma^2+t^2) + 1 \leqslant ((\gamma^2+t^2)+1)^2 \leqslant (\gamma^2+t^2+\gamma^2t^2+1)^2,$$

we know that

$$\ln((x(t)+y(t))^2+1) \leqslant 2[\ln(x^2(t)+1) + \ln(y^2(t)+1)], \ \forall\, x(t), y(t) \in P$$

which shows $\omega = 2$ and (Φ-3) holds.

3. Main results

First, with the aid of a mapping $\varphi \in \Phi(P, P_1)$ we establish a new coincidence point and CFP theorem for four nonlinear self-mappings in a CbMS.

Theorem 3.1. *Let (X,d) be a CbMS with the coefficient $s \geqslant 1$ and P be a solid cone. Suppose four mappings $f, g, S, T : X \to X$ satisfy the following conditions:*

(a) *$f(X) \subset T(X), g(X) \subset S(X)$ with one of $f(X), g(X), S(X)$ or $T(X)$ is a complete subspace of X;*

(b) *there exists a mapping $\varphi \in \Phi(P, P_1)$ such that*

$$\begin{aligned}\varphi(d(fx,gy)) \leq{}& A_1(x,y)\varphi(d(Sx,Ty)) + A_2(x,y)\varphi(d(fx,Sx)) + A_3(x,y)\varphi(d(gy,Ty))\\ &+A_4(x,y)\varphi(d(fx,Ty)) + A_5(x,y)\varphi(d(Sx,gy)), \forall x,y \in X \end{aligned} \tag{3.1}$$

where A_1, A_2, A_3, A_4, A_5 are five functions from $X \times X$ to $[0,+\infty)$ such that

(i) *$A_1(x,y) + A_4(x,y) + A_5(x,y) < 1$ for all $x, y \in X$;*

(ii) *$\inf_{x,y\in X}\{1 - A_3(x,y) - K\omega s^\sigma A_5(x,y)\} = a > 0$, $\inf_{x,y\in X}\{1 - A_2(x,y) - K\omega s^\sigma A_4(x,y)\} = b > 0$,*

$\sup_{x,y\in X}\{A_1(x,y) + A_2(x,y) + K\omega s^\sigma A_5(x,y)\} = A$, $\sup_{x,y\in X}\{A_1(x,y) + A_3(x,y) + K\omega s^\sigma A_4(x,y)\} = B$, with $\frac{A}{a} \cdot \frac{B}{b} < \frac{1}{\omega^2 s^{2\sigma}}$ and $K \geqslant 1, \omega \geqslant 1, \sigma > 0$ are some constants as in Definition 2.10;

(iii) *$K\omega s^\sigma(1-a) < 1$ and $K\omega s^\sigma(1-b) < 1$.*

Then, there is a unique $u \in X$ that is the point of coincidence of $\{g,T\}$ and $\{f,S\}$. Moreover, if $\{g,T\}$ and $\{f,S\}$ are weakly compatible then f, g, S and T have a unique CFP.

Proof. In X, arbitrarily take an element x_0. For this x_0, since $f(X) \subset T(X), g(X) \subset S(X)$ there is $x_1, x_2 \in X$ such that $fx_0 = Tx_1$ and $gx_1 = Sx_2$. By induction we produce two sequences $\{x_m\}$ and $\{y_m\}$ of points of X such that

$$y_{2n+1} = fx_{2n} = Tx_{2n+1}, \ \ y_{2n+2} = gx_{2n+1} = Sx_{2n+2} \ \ (n = 0,1,2,\cdots).$$

Now, we will first verify that the sequence $\{y_m\}$ is a Cauchy sequence in X.

If $y_m = y_{m+1}$ for some m, e.g., if $y_{2n} = y_{2n+1}$ then applying Definition 2.2 and Remark 2.11 from (2.1) and (3.1) we can obtain

$$\begin{aligned}
&\varphi(d(y_{2n+1}, y_{2n+2})) = \varphi(d(fx_{2n}, gx_{2n+1}))\\
&\leq A_1(x_{2n}, x_{2n+1})\varphi(d(Sx_{2n}, Tx_{2n+1})) + A_2(x_{2n}, x_{2n+1})\varphi(d(fx_{2n}, Sx_{2n})) + A_3(x_{2n}, x_{2n+1})\\
&\quad \varphi(d(gx_{2n+1}, Tx_{2n+1})) + A_4(x_{2n}, x_{2n+1})\varphi(d(fx_{2n}, Tx_{2n+1})) + A_5(x_{2n}, x_{2n+1})\varphi(d(Sx_{2n}, gx_{2n+1}))\\
&= A_1(x_{2n}, x_{2n+1})\varphi(d(y_{2n}, y_{2n+1})) + A_2(x_{2n}, x_{2n+1})\varphi(d(y_{2n+1}, y_{2n})) + A_3(x_{2n}, x_{2n+1})\varphi(d(y_{2n+2}, y_{2n+1}))\\
&\quad +A_4(x_{2n}, x_{2n+1})\varphi(d(y_{2n+1}, y_{2n+1})) + A_5(x_{2n}, x_{2n+1})\varphi(d(y_{2n}, y_{2n+2}))\\
&\leq A_1(x_{2n}, x_{2n+1})\varphi(d(y_{2n}, y_{2n+1})) + A_2(x_{2n}, x_{2n+1})\varphi(d(y_{2n+1}, y_{2n})) + A_3(x_{2n}, x_{2n+1})\varphi(d(y_{2n+2}, y_{2n+1}))\\
&\quad +A_5(x_{2n}, x_{2n+1})K\omega s^{\sigma}[\varphi(d(y_{2n}, y_{2n+1})) + \varphi(d(y_{2n+1}, y_{2n+2}))]\\
&= [A_1(x_{2n}, x_{2n+1}) + A_2(x_{2n}, x_{2n+1}) + K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1})]\varphi(\theta_E)\\
&\quad +[A_3(x_{2n}, x_{2n+1}) + K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1})]\varphi(d(y_{2n+2}, y_{2n+1}))\\
&= [A_3(x_{2n}, x_{2n+1}) + K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1})]\varphi(d(y_{2n+2}, y_{2n+1})).
\end{aligned}$$

Note that as $\inf_{x,y\in X}\{1 - A_3(x,y) - K\omega s^{\sigma}A_5(x,y)\} = a > 0$, we have $0 \leqslant A_3(x_{2n}, x_{2n+1}) + K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1}) < 1$. From part (e) of Remark 2.4 we know that $\varphi(d(y_{2n+1}, y_{2n+2})) = \theta_{E_1}$. By $\varphi \in \Phi(P, P_1)$, we see that $d(y_{2n+1}, y_{2n+2}) = \theta_E$, i.e., $y_{2n+2} = y_{2n+1}$. Similarly, we obtain $y_{2n} = y_{2n+1} = y_{2n+2} = \cdots$. Therefore, $\{y_m\}$ is a Cauchy sequence.

Suppose that $y_m \neq y_{m+1}$ for all m. Then, for $n = 0, 1, 2, \cdots$ from (2.1), (3.1) and Definition 2.2 it follows that

$$\begin{aligned}
&\varphi(d(y_{2n+1}, y_{2n+2})) = \varphi(d(fx_{2n}, gx_{2n+1}))\\
&\leq A_1(x_{2n}, x_{2n+1})\varphi(d(Sx_{2n}, Tx_{2n+1})) + A_2(x_{2n}, x_{2n+1})\varphi(d(fx_{2n}, Sx_{2n})) + A_3(x_{2n}, x_{2n+1})\cdot\\
&\quad \varphi(d(gx_{2n+1}, Tx_{2n+1})) + A_4(x_{2n}, x_{2n+1})\varphi(d(fx_{2n}, Tx_{2n+1})) + A_5(x_{2n}, x_{2n+1})\varphi(d(Sx_{2n}, gx_{2n+1}))\\
&= A_1(x_{2n}, x_{2n+1})\varphi(d(y_{2n}, y_{2n+1})) + A_2(x_{2n}, x_{2n+1})\varphi(d(y_{2n+1}, y_{2n})) + A_3(x_{2n}, x_{2n+1})\varphi(d(y_{2n+2}, y_{2n+1}))\\
&\quad +A_4(x_{2n}, x_{2n+1})\varphi(d(y_{2n+1}, y_{2n+1})) + A_5(x_{2n}, x_{2n+1})\varphi(d(y_{2n}, y_{2n+2}))\\
&\leq A_1(x_{2n}, x_{2n+1})\varphi(d(y_{2n}, y_{2n+1})) + A_2(x_{2n}, x_{2n+1})\varphi(d(y_{2n+1}, y_{2n})) + A_3(x_{2n}, x_{2n+1})\varphi(d(y_{2n+2}, y_{2n+1}))\\
&\quad +A_5(x_{2n}, x_{2n+1})K\omega s^{\sigma}[\varphi(d(y_{2n}, y_{2n+1})) + \varphi(d(y_{2n+1}, y_{2n+2}))]\\
&= [A_1(x_{2n}, x_{2n+1}) + A_2(x_{2n}, x_{2n+1}) + K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1})]\varphi(d(y_{2n}, y_{2n+1}))\\
&\quad +[A_3(x_{2n}, x_{2n+1}) + K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1})]\varphi(d(y_{2n+2}, y_{2n+1}))
\end{aligned}$$

which implies that

$$\begin{aligned}
\varphi(d(y_{2n+1}, y_{2n+2})) &\leq \frac{A_1(x_{2n}, x_{2n+1}) + A_2(x_{2n}, x_{2n+1}) + K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1})}{1 - A_3(x_{2n}, x_{2n+1}) - K\omega s^{\sigma}A_5(x_{2n}, x_{2n+1})}\varphi(d(y_{2n}, y_{2n+1}))\\
&\leq \frac{\sup_{x,y\in X}\{A_1(x,y) + A_2(x,y) + K\omega s^{\sigma}A_5(x,y)\}}{\inf_{x,y\in X}\{1 - A_3(x,y) - K\omega s^{\sigma}A_5(x,y)\}}\varphi(d(y_{2n}, y_{2n+1})).
\end{aligned}$$

Form condition (ii), we have

$$\varphi(d(y_{2n+1}, y_{2n+2})) \leq \frac{A}{a}\varphi(d(y_{2n}, y_{2n+1})). \tag{3.2}$$

A similar method can get

$$\begin{aligned}
&\varphi(d(y_{2n+1}, y_{2n})) = \varphi(d(fx_{2n}, gx_{2n-1}))\\
&\leq A_1(x_{2n}, x_{2n-1})\varphi(d(Sx_{2n}, Tx_{2n-1})) + A_2(x_{2n}, x_{2n-1})\varphi(d(fx_{2n}, Sx_{2n})) + A_3(x_{2n}, x_{2n-1})\\
&\quad \varphi(d(gx_{2n-1}, Tx_{2n-1})) + A_4(x_{2n}, x_{2n-1})\varphi(d(fx_{2n}, Tx_{2n-1})) + A_5(x_{2n}, x_{2n-1})\varphi(d(Sx_{2n}, gx_{2n-1}))\\
&= A_1(x_{2n}, x_{2n-1})\varphi(d(y_{2n}, y_{2n-1})) + A_2(x_{2n}, x_{2n-1})\varphi(d(y_{2n+1}, y_{2n})) + A_3(x_{2n}, x_{2n-1})\varphi(d(y_{2n}, y_{2n-1}))\\
&\quad +A_4(x_{2n}, x_{2n-1})\varphi(d(y_{2n+1}, y_{2n-1})) + A_5(x_{2n}, x_{2n-1})\varphi(d(y_{2n}, y_{2n}))\\
&\leq [A_1(x_{2n}, x_{2n-1}) + A_3(x_{2n}, x_{2n-1}) + K\omega s^{\sigma}A_4(x_{2n}, x_{2n-1})]\varphi(d(y_{2n}, y_{2n-1}))\\
&\quad +[A_2(x_{2n}, x_{2n-1}) + K\omega s^{\sigma}A_4(x_{2n}, x_{2n-1})]\varphi(d(y_{2n}, y_{2n+1})),
\end{aligned}$$

which implies that

$$\begin{aligned}
\varphi(d(y_{2n}, y_{2n+1})) &\leq \frac{A_1(x_{2n}, x_{2n-1}) + A_3(x_{2n}, x_{2n-1}) + K\omega s^{\sigma}A_4(x_{2n}, x_{2n-1})}{1 - A_2(x_{2n}, x_{2n-1}) - K\omega s^{\sigma}A_4(x_{2n}, x_{2n-1})}\varphi(d(y_{2n}, y_{2n-1}))\\
&\leq \frac{\sup_{x,y\in X}\{A_1(x, y) + A_3(x, y) + K\omega s^{\sigma}A_4(x, y)\}}{\inf_{x,y\in X}\{1 - A_2(x, y) - K\omega s^{\sigma}A_4(x, y)\}}\varphi(d(y_{2n}, y_{2n-1}))\\
&= \frac{B}{b}\varphi(d(y_{2n}, y_{2n-1})).
\end{aligned} \tag{3.3}$$

For $n = 1, 2, \cdots$, using inequalities (3.2) and (3.3) we easily get

$$\begin{aligned}
\varphi(d(y_{2n+1}, y_{2n+2})) \leq \frac{A}{a}\varphi(d(y_{2n}, y_{2n+1})) &\leq \frac{A}{a}\cdot\frac{B}{b}\varphi(d(y_{2n-1}, y_{2n}))\\
&\leq \cdots \leq \left(\frac{A}{a}\cdot\frac{B}{b}\right)^n \varphi(d(y_2, y_1))
\end{aligned} \tag{3.4}$$

and

$$\varphi(d(y_{2n+3}, y_{2n+2})) \leq \frac{B}{b}\varphi(d(y_{2n+2}, y_{2n+1})) \leq \frac{B}{b}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^n \varphi(d(y_2, y_1)). \tag{3.5}$$

Then, for any $n < k$ using Definition 2.2 and (Φ-1)–(Φ-3) in Definition 2.10 from (3.4) and (3.5) we have

$$d(y_{2n+1}, y_{2k+1}) \leq sd(y_{2n+1}, y_{2n+2}) + s^2d(y_{2n+2}, y_{2n+3}) + \cdots + s^{2(k-n)-1}d(y_{2k-1}, y_{2k}) + s^{2(k-n)-1}d(y_{2k}, y_{2k+1})$$

and

$$\begin{aligned}
&\varphi(d(y_{2n+1}, y_{2k+1}))\\
&\leq K\varphi(sd(y_{2n+1}, y_{2n+2}) + s^2d(y_{2n+2}, y_{2n+3}) + \cdots + s^{2(k-n)-1}d(y_{2k-1}, y_{2k}) + s^{2(k-n)-1}d(y_{2k}, y_{2k+1}))\\
&\leq K\omega s^{\sigma}\varphi(d(y_{2n+1}, y_{2n+2})) + K\omega\varphi(s^2d(y_{2n+2}, y_{2n+3}) + \cdots + s^{2(k-n)-1}d(y_{2k-1}, y_{2k}) + s^{2(k-n)-1}d(y_{2k}, y_{2k+1}))\\
&\leq K\omega s^{\sigma}\varphi(d(y_{2n+1}, y_{2n+2})) + K\omega^2s^{2\sigma}\varphi(d(y_{2n+2}, y_{2n+3})) + K\omega^2s^{2\sigma}\varphi(\cdots + s^{2(k-n)-1}d(y_{2k-1}, y_{2k})\\
&\quad +s^{2(k-n)-1}d(y_{2k}, y_{2k+1}))
\end{aligned}$$

$$\leq K\omega s^{\sigma}\varphi(d(y_{2n+1},y_{2n+2}))+K(\omega s^{\sigma})^{2}\varphi(d(y_{2n+2},y_{2n+3}))+\cdots+K(\omega s^{\sigma})^{2(k-n)-1}\varphi(d(y_{2k-1},y_{2k}))$$
$$+K(\omega s^{\sigma})^{2(k-n)-1}\varphi(d(y_{2k},y_{2k+1}))$$
$$\leq K\omega s^{\sigma}\varphi(d(y_{2n+1},y_{2n+2}))+K(\omega s^{\sigma})^{2}\varphi(d(y_{2n+2},y_{2n+3}))+\cdots+K(\omega s^{\sigma})^{2(k-n)-1}\varphi(d(y_{2k-1},y_{2k}))$$
$$+K(\omega s^{\sigma})^{2(k-n)}\varphi(d(y_{2k},y_{2k+1}))$$
$$\leq K\left[\sum_{i=n}^{k-1}(\omega s^{\sigma})^{2(i-n)+1}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^{i}+\frac{B}{b}\sum_{i=n}^{k-1}(\omega s^{\sigma})^{2(i-n)+2}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^{i}\right]\varphi(d(y_2,y_1))$$
$$\leq K\left[\left(1+\omega s^{\sigma}\frac{B}{b}\right)\cdot\sum_{i=n-1}^{\infty}(\omega s^{\sigma})^{2(i-n)+1}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^{i}\right]\varphi(d(y_2,y_1)).$$

Similarly, we can obtain

$$\varphi(d(y_{2n},y_{2k+1}))\leq K\left[\left(1+\omega s^{\sigma}\frac{B}{b}\right)\cdot\sum_{i=n-1}^{\infty}(\omega s^{\sigma})^{2(i-n)+1}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^{i}\right]\varphi(d(y_2,y_1)),$$
$$\varphi(d(y_{2n},y_{2k}))\leq K\left[\left(1+\omega s^{\sigma}\frac{B}{b}\right)\cdot\sum_{i=n-1}^{\infty}(\omega s^{\sigma})^{2(i-n)+1}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^{i}\right]\varphi(d(y_2,y_1))$$

and

$$\varphi(d(y_{2n+1},y_{2k}))\leq K\left[\left(1+\omega s^{\sigma}\frac{B}{b}\right)\cdot\sum_{i=n-1}^{\infty}(\omega s^{\sigma})^{2(i-n)+1}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^{i}\right]\varphi(d(y_2,y_1)).$$

Thus, for any $p>m>0$ there is a natural number n with $\frac{m-1}{2}\leqslant n\leqslant\frac{m}{2}$ such that

$$\varphi(d(y_m,y_p))\leq K\left[\left(1+\omega s^{\sigma}\frac{B}{b}\right)\cdot\sum_{i=n-1}^{\infty}(\omega s^{\sigma})^{2(i-n)+1}\left(\frac{A}{a}\cdot\frac{B}{b}\right)^{i}\right]\varphi(d(y_2,y_1)).\tag{3.6}$$

Let $\theta_{E_1}\ll c$ be given. Since $P_1\subset E_1$ is a solid cone, we can take $\delta>0$ such that $c+N_\delta(\theta)\subset P_1$ where $N_\delta(\theta)=\{y\in E_1:\|y\|<\delta\}$. Note that $0\leqslant\frac{A}{a}\cdot\frac{B}{b}<\frac{1}{\omega^2s^{2\sigma}}$. It follows from Cauchy's root test that $\sum(\omega s^{\sigma})^{2n-1}(\frac{A}{a}\cdot\frac{B}{b})^n$ is convergent. Then, there exists a natural number N such that

$$K[(1+\omega s^{\sigma}\frac{B}{b})\cdot\sum_{i=n-1}^{\infty}(\omega s^{\sigma})^{2(i-n)+1}(\frac{A}{a}\cdot\frac{B}{b})^{i}]\varphi(d(y_2,y_1))\in N_\delta(\theta),\ \forall n\geqslant N.$$

This shows that $K[(1+\omega s^{\sigma}\frac{B}{b})\cdot\sum_{i=n-1}^{\infty}(\omega s^{\sigma})^{2(i-n)+1}(\frac{A}{a}\cdot\frac{B}{b})^{i}]\varphi(d(y_2,y_1))\ll c$, for all $n\geqslant N$. From (3.6), we have $\varphi(d(y_m,y_p))\ll c$, for all $m\geqslant 2N$ which proves that $\{y_m\}$ is a Cauchy sequence in X via Remark 2.11 and Definition 2.10.

Next, to prove the existence of the point of coincidence we discuss it in two cases:

Case 1: Suppose that $T(X)$ is complete. So, we know that $y_{2n+1}=fx_{2n}=Tx_{2n+1}\to u\in T(X)$, and there is $v\in X$ such that $Tv=u$. (If $f(X)$ is complete, there is $u\in f(X)\subset T(X)$ which means to a similar conclusion.)

Now we shall show that $gv=Tv=u$. If $gv\neq Tv$, for $n=0,1,2,\cdots$, applying Definition 2.2 and (3.1) by Remark 2.11 we have

$$\varphi(d(y_{2n+1},gv))=\varphi(d(fx_{2n},gv))$$
$$\leq A_1(x_{2n},v)\varphi(d(Sx_{2n},Tv))+A_2(x_{2n},v)\varphi(d(fx_{2n},Sx_{2n}))+A_3(x_{2n},v)\varphi(d(gv,Tv))$$

$$
\begin{aligned}
&+A_4(x_{2n},v)\varphi(d(fx_{2n},Tv))+A_5(x_{2n},v)\varphi(d(Sx_{2n},gv))\\
&=A_1(x_{2n},v)\varphi(d(y_{2n},u))+A_2(x_{2n},v)\varphi(d(y_{2n+1},y_{2n}))+A_3(x_{2n},v)\varphi(d(gv,u))\\
&\quad+A_4(x_{2n},v)\varphi(d(y_{2n+1},u))+A_5(x_{2n},v)\varphi(d(y_{2n},gv))\\
&\leq A_1(x_{2n},v)\varphi(d(y_{2n},u))+A_2(x_{2n},v)\varphi(d(y_{2n+1},y_{2n}))+A_3(x_{2n},v)\varphi(d(gv,u))\\
&\quad+A_4(x_{2n},v)\varphi(d(y_{2n+1},u))+K\omega s^{\sigma}A_5(x_{2n},v)[\varphi(d(y_{2n},u))+\varphi(d(u,gv))]
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\varphi(d(y_{2n+1},gv))\leq{}&[A_1(x_{2n},v)+K\omega s^{\sigma}A_5(x_{2n},v)]\varphi(d(y_{2n},u))+A_2(x_{2n},v)\varphi(d(y_{2n+1},y_{2n}))\\
&+A_4(x_{2n},v)\varphi(d(y_{2n+1},u))+[A_3(x_{2n},v)+K\omega s^{\sigma}A_5(x_{2n},v)]\varphi(d(gv,u)). \qquad (3.7)
\end{aligned}
$$

Since $y_{2n+1}\to u, y_{2n}\to u, d(y_{2n+1},y_{2n})\to\theta_E$ as $n\to\infty$ and $\varphi\in\Phi(P,P_1)$, letting $n\to\infty$ in (3.7) we get

$$\limsup_{n\to\infty}\varphi(d(y_{2n+1},gv))\leq\sup_{x,y\in X}[A_3(x,y)+K\omega s^{\sigma}A_5(x,y)]\varphi(d(gv,u)).$$

Note that

$$\varphi(d(u,gv))\leq K\omega s^{\sigma}[\varphi(d(u,y_{2n+1}))+\varphi(d(y_{2n+1},gv))].$$

So, we have

$$\varphi(d(u,gv))\leq K\omega s^{\sigma}\sup_{x,y\in X}[A_3(x,y)+K\omega s^{\sigma}A_5(x,y)]\varphi(d(gv,u))=K\omega s^{\sigma}(1-a)\varphi(d(gv,u)).$$

Using part the condition (iii) and (e) of Remark 2.4 we obtain that $\varphi(d(u,gv)))=\theta_{E_1}$, i.e., $d(u,gv)=\theta_E$ which is not possible. Therefore, $Tv=gv=u$.

Since $u=gv\in g(X)\subset S(X)$, there is $\omega\in X$ such that $S\omega=u$. By (3.1), we have

$$
\begin{aligned}
&\varphi(d(f\omega,S\omega))=\varphi(d(f\omega,u))=\varphi(d(f\omega,gv))\\
&\leq A_1(\omega,v)\varphi(d(S\omega,Tv))+A_2(\omega,v)\varphi(d(f\omega,S\omega))+A_3(\omega,v)\varphi(d(gv,Tv))\\
&\quad+A_4(\omega,v)\varphi(d(f\omega,Tv))+A_5(\omega,v)\varphi(d(S\omega,gv))\\
&=[A_2(\omega,v)+A_4(\omega,v)]\varphi(d(f\omega,S\omega))\leq\sup_{x,y\in X}\{A_2(x,y)+A_4(x,y)\}\varphi(d(f\omega,S\omega)).
\end{aligned}
$$

Since $\inf_{x,y\in X}\{1-A_2(x,y)-K\omega s^{\sigma}A_4(x,y)\}=b>0$, $K\geqslant 1,\omega\geqslant 1$ and $s^{\sigma}\geqslant 1$, we have $\sup_{x,y\in X}\{A_2(x,y)+A_4(x,y)\}<1$. Hence, from Remarks 2.4 and 2.11 we can obtain $f\omega=S\omega$. Therefore, $f\omega=S\omega=gv=Tv=u$.

Case 2: Suppose $S(X)$ is complete. We have $y_{2n+2}=gx_{2n+1}=Sx_{2n+2}\to u\in S(X)$. Then, there is $\omega\in X$ such that $S\omega=u$. (If $g(X)$ is complete, there is $u\in g(X)\subset S(X)$, which means to a similar conclusion.) Now we shall show that $f\omega=u$. For $n=0,1,2,\cdots$, applying Definition 2.2 and (3.1), by Remark 2.11, we get

$$
\begin{aligned}
&\varphi(d(f\omega,y_{2n+2}))=\varphi(d(f\omega,gx_{2n+1}))\\
&\leq A_1(\omega,x_{2n+1})\varphi(d(S\omega,Tx_{2n+1}))+A_2(\omega,x_{2n+1})\varphi(d(f\omega,S\omega))+A_3(\omega,x_{2n+1})\varphi(d(gx_{2n+1},Tx_{2n+1}))\\
&\quad+A_4(\omega,x_{2n+1})\varphi(d(f\omega,Tx_{2n+1}))+A_5(\omega,x_{2n+1})\varphi(d(S\omega,gx_{2n+1})
\end{aligned}
$$

$$= A_1(\omega, x_{2n+1})\varphi(d(u, y_{2n+1})) + A_2(\omega, x_{2n+1})\varphi(d(f\omega, u)) + A_3(\omega, x_{2n+1})\varphi(d(y_{2n+2}, y_{2n+1}))$$

$$+A_4(\omega, x_{2n+1})\varphi(d(f\omega, y_{2n+1})) + A_5(\omega, x_{2n+1})\varphi(d(u, y_{2n+2})). \tag{3.8}$$

Since $y_{2n+2} \to u, y_{2n+1} \to u, d(y_{2n+1}, y_{2n+2}) \to \theta_E$ as $n \to \infty$ and $\varphi \in \Phi(P, P_1)$. Therefore, letting $n \to \infty$ in (3.8) we get

$$\limsup_{n\to\infty} \varphi(d(y_{2n+2}, f\omega))) \leq \sup_{x,y\in X}[A_2(x, y) + A_4(x, y)]\varphi(d(u, f\omega)).$$

From

$$\varphi(d(u, f\omega)) \leq K\omega s^{\sigma}[\varphi(d(u, y_{2n+2})) + \varphi(d(y_{2n+2}, f\omega))],$$

$K \geqslant 1, \omega \geqslant 1$ and $s^{\sigma} \geqslant 1$, we obtain

$$\varphi(d(u, f\omega)) \leq K\omega s^{\sigma} \sup_{x,y\in X}[A_2(x, y) + A_4(x, y)]\varphi(d(u, f\omega)) \leq K\omega s^{\sigma}(1 - b)\varphi(d(u, f\omega)).$$

Using part the condition (iii) and (e) of Remark 2.4 we obtain that $\varphi(d(u, f\omega)) = \theta_{E_1}$, i.e., $d(u, f\omega) = \theta_E$. Therefore, $S\omega = f\omega = u$. Then, according to $u = f\omega \in f(X) \subset T(X)$ there is $v \in X$ such that $Tv = u$. From (3.1), we have

$$\begin{aligned}\varphi(d(u, gv)) = \varphi(d(Tv, gv)) = \varphi(d(f\omega, gv)) &\leq A_1(\omega, v)\varphi(d(S\omega, Tv)) + A_2(\omega, v)\varphi(d(f\omega, S\omega))\\ &+A_3(\omega, v)\varphi(d(gv, Tv)) + A_4(\omega, v)\varphi(d(f\omega, Tv)) + A_5(\omega, v)\varphi(d(S\omega, gv))\\ &\leq \sup_{x,y\in X}\{A_3(x, y) + A_5(x, y)\}d(u, gv).\end{aligned}$$

Noticing that $K \geqslant 1, \omega \geqslant 1, s^{\sigma} \geqslant 1$ and $\inf_{x,y\in X}\{1 - A_3(x, y) - K\omega s^{\sigma} A_5(x, y)\} = a > 0$. We know that $\sup_{x,y\in X}\{A_3(x, y) + A_5(x, y)\} < 1$. Hence, from Remarks 2.4 and 2.11 we have $u = gv$. Therefore, $f\omega = S\omega = gv = Tv = u$.

Now, we will prove the uniqueness of the point of coincidence of f and g. To that end, assume that there is another u^* in X such that $f\omega^* = S\omega^* = gv^* = Tv^* = u^*$. Thus, by (3.1) we can obtain

$$\begin{aligned}\varphi(d(u, u^*)) = \varphi(d(f\omega, gv^*) &\leq A_1(\omega, v^*)\varphi(d(S\omega, Tv^*)) + A_2(\omega, v^*)\varphi(d(f\omega, S\omega))\\ &+A_3(\omega, v^*)\varphi(d(gv^*, Tv^*)) + A_4(\omega, v^*)\varphi(d(f\omega, Tv^*)) + A_5(\omega, v^*)\varphi(d(S\omega, gv^*))\\ &= [A_1(\omega, v^*) + A_4(\omega, v^*) + A_5(\omega, v^*)]\varphi(d(u, u^*)).\end{aligned}$$

Note that $A_1(x, y) + A_4(x, y) + A_5(x, y) < 1$ $(\forall x, y \in X)$. Thus, $\varphi(d(u, u^*)) = \theta_{E_1}$, i.e., $u = u^*$.

Moreover, for $f\omega = S\omega = gv = Tv = u$ if $\{f, S\}$ and $\{g, T\}$ are weakly compatible then $Sf\omega = fS\omega = fu = Su$ and $Tgv = gTv = gu = Tu$. From (3.1), we have

$$\begin{aligned}\varphi(d(fu, u)) = \varphi(d(ff\omega, gv) &\leq A_1(f\omega, v)\varphi(d(Sf\omega, Tv)) + A_2(f\omega, v)\varphi(d(ff\omega, Sf\omega))\\ &+A_3(f\omega, v)\varphi(d(gv, Tv)) + A_4(f\omega, v)\varphi(d(ff\omega, Tv)) + A_5(f\omega, v)\varphi(d(Sf\omega, gv))\\ &= [A_1(f\omega, v) + A_4(f\omega, v) + A_5(f\omega, v)]\varphi(d(fu, u)).\end{aligned}$$

Note that $A_1(x, y)+A_4(x, y)+A_5(x, y) < 1$ for all $x, y \in X$. Then, $\varphi(d(fu, u)) = \theta_{E_1}$, i.e., $fu = u = Su$. Similarly, we can prove $Tu = gu = u$. These show that u is a CFP of f, g, S and T. In addition, if $\overline{u} = f\overline{u} = g\overline{u} = S\overline{u} = T\overline{u}$ then $\overline{u}$ is also a point of coincidence of $\{f, S\}$ and $\{g, T\}$ and therefore $\overline{u} = u$ by uniqueness. The uniqueness is proved and we complete the proof of the theorem.

From the given examples in Section 2, we know that the mapping φ in the class $\Phi(P, P_1)$ is quite general. Choosing the different mappings in $\Phi(P, P_1)$, we can obtain different varieties of Theorem 3.1 which are significative. For example, taking $\varphi(a) = a$ in Theorem 3.1 (See Example 2.12) we can obtain the following:

Theorem 3.2. *Let (X, d) be a CbMS with the coefficient $s \geqslant 1$ and P be a solid cone. Suppose four mappings $f, g, S, T : X \to X$ satisfy the condition* (a) *in Theorem 3.1 and*
(b) *for all $x, y \in X$,*

$$d(fx, gy) \preccurlyeq A_1(x,y)d(Sx,Ty) + A_2(x,y)d(fx,Sx) + A_3(x,y)d(gy,Ty)$$
$$+A_4(x,y)d(fx,Ty) + A_5(x,y)d(Sx,gy) \tag{3.9}$$

where A_1, A_2, A_3, A_4, A_5 are five functions from $X \times X$ to $[0, +\infty)$ such that

(i) $A_1(x,y) + A_4(x,y) + A_5(x,y) < 1$ *for all $x, y \in X$;*

(ii) $\inf_{x,y\in X}\{1 - A_3(x,y) - sA_5(x,y)\} = a > 0, \quad \inf_{x,y\in X}\{1 - A_2(x,y) - sA_4(x,y)\} = b > 0,$
$\sup_{x,y\in X}\{A_1(x,y) + A_2(x,y) + sA_5(x,y)\} = A, \quad \sup_{x,y\in X}\{A_1(x,y) + A_3(x,y) + sA_4(x,y)\} = B,$
with $\frac{A}{a} \cdot \frac{B}{b} < \frac{1}{s^2}$;

(iii) $s(1-a) < 1$ *and* $s(1-b) < 1$.
Then, the conclusions of Theorem 3.1 are equally valid.

Corollary 3.3. *Let (X, d) be a CbMS with the coefficient $s \geqslant 1$ and P be a solid cone. Suppose four mappings $f, g, S, T : X \to X$ satisfy the condition* (a) *in Theorem 3.1 and*

(b) *there is a mapping $\varphi \in \Phi(P, P_1)$ such that for all $x, y \in X$,*

$$\varphi(d(fx, gy)) \leq a_1\varphi(d(Sx,Ty)) + a_2\varphi(d(fx,Sx)) + a_3\varphi(d(gy,Ty))$$
$$+a_4\varphi(d(fx,Ty)) + a_5\varphi(d(Sx,gy)) \tag{3.10}$$

where $a_i \geqslant 0$ $(i = 1,2,3,4,5)$ are constants with $a_1 + a_4 + a_5 < 1, K\omega s^\sigma(a_2 + K\omega s^\sigma a_4) < 1, K\omega s^\sigma(a_3 + K\omega s^\sigma a_5) < 1$ and $\frac{(a_1+a_2+K\omega s^\sigma a_5)(a_1+a_3+K\omega s^\sigma a_4)}{(1-a_2-K\omega s^\sigma a_4)(1-a_3-K\omega s^\sigma a_5)} < \frac{1}{\omega^2 s^{2\sigma}}$ and $K \geqslant 1, \omega \geqslant 1, \sigma > 0$ are some constants as in Definition 2.10. Then, the conclusions of Theorem 3.1 still hold.

Proof. Let $A_1(x,y) = a_1, A_2(x,y) = a_2, A_3(x,y) = a_3, A_4(x,y) = a_4, A_5(x,y) = a_5$ for all $x, y \in X$. It is evident that $a_1 + a_4 + a_5 < 1$, $0 < 1 - a_2 - K\omega s^\sigma a_4$, and $0 < 1 - a_3 - K\omega s^\sigma a_5$. Then, we easily see the conditions (i), (ii) and (iii) of Theorem 3.1 are valid. Therefore, by Theorem 3.1 Corollary 3.3 is proved.

Corollary 3.4. *Let (X, d) be a CMS, and P be a solid cone. Suppose four mappings $f, g, S, T : X \to X$ satisfy the condition* (a) *in Theorem 3.2 and*
(b) *for all $x, y \in X$,*

$$d(fx, gy) \preccurlyeq a_1d(Sx,Ty) + a_2d(fx,Sx) + a_3d(gy,Ty) + a_4d(fx,Ty) + a_5d(Sx,gy) \tag{3.11}$$

where $a_i \geqslant 0$ $(i = 1,2,3,4,5), \delta \geqslant 0$ with $\sum_{i=1}^{5} a_5 = 1 + \delta, a_1 + a_4 + a_5 < 1, a_2 + a_4 < 1, a_3 + a_5 < 1$ and $(a_2 - a_3)(a_5 - a_4) > 2\delta$. Then, the conclusions of Theorem 3.2 still hold.

Proof. Note that $a_1 < 1, (a_2 - a_3)(a_5 - a_4) > 2\delta$. We have

$$a_1(1+\delta) + a_2a_4 + a_3a_5 \leqslant a_1 + \delta + a_2a_4 + a_3a_5 < a_1 - \delta + a_2a_5 + a_3a_4.$$

By $\sum_{i=1}^{5} a_5 = 1 + \delta$, we can obtain

$$\begin{aligned} &a_1(\textstyle\sum_{i=1}^{5} a_5) + a_2a_4 + a_3a_5 + a_2a_3 + a_4a_5 \\ &< a_1 - \delta + a_2a_5 + a_3a_4 + a_2a_3 + a_4a_5 = (1-a_2-a_4)(1-a_3-a_5). \end{aligned}$$

Notice that (X, d) is a CMS with the coefficient $s = 1$. We have $\frac{(a_1+a_2+a_5)(a_1+a_3+a_4)}{(1-a_2-a_4)(1-a_3-a_5)} < 1 = \frac{1}{s^2}$. Thus the conditions in Theorem 3.2 are valid. Therefore, by Theorem 3.2 Corollary 3.4 is proved.

Remark 3.5. *The numbers a_i $(i = 1, 2, 3, 4, 5)$ and δ in Corollary 3.4 exist. For example, if we take $\delta = \frac{1}{40}, a_1 = \frac{1}{20}, a_2 = \frac{2}{5}, a_3 = \frac{1}{20}, a_4 = \frac{1}{40}, a_5 = \frac{1}{2}$ then $\sum_{i=1}^{5} a_5 = 1 + \frac{1}{40}, a_1 + a_4 + a_5 = \frac{23}{40} < 1, a_2 + a_4 = \frac{17}{40} < 1, a_3 + a_5 = \frac{11}{20} < 1$ and $(a_2 - a_3)(a_5 - a_4) = \frac{133}{800} > \frac{1}{20}$, i.e., the conditions in Corollary 3.4 are satisfied. If we take $\delta = 0, a_1 = \frac{1}{10}, a_2 = \frac{7}{15}, a_3 = \frac{1}{15}, a_4 = \frac{23}{90}, a_5 = \frac{1}{9}$, then $\sum_{i=1}^{5} a_5 = 1, a_1 + a_4 + a_5 < 1, a_2 + a_4 < 1, a_3 + a_5 < 1$, and $(a_2 - a_3)(a_5 - a_4) > 0$, i.e., the conditions in Corollary 3.4 are also satisfied. These two examples show that $a_1 + a_2 + a_3 + a_4 + a_5 \geqslant 1$ is admissible (see below Example 3.6) in Theorem 3.2 as (X, d) is a CMS.*

In addition, Theorem 2.8 in [20] is a special case of Theorem 3.2 as (X, d) is a CMS with $a_4 = a_5$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. In fact, by $a_1 + a_2 + a_3 + 2a_4 < 1$, we have

$$\frac{(a_1+a_2+a_5)(a_1+a_3+a_4)}{(1-a_2-a_4)(1-a_3-a_5)} < \frac{(a_1+a_2+a_4)(a_1+a_3+a_5)}{(a_1+a_3+a_5)(a_1+a_2+a_4)} = 1$$

which shows that the conditions in Theorem 3.2 are valid satisfied. Further, Theorem 2.1 in [21] is a special case of Theorem 3.2 as (X, d) is a CMS with $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < 1$ or $a_1 + \max\{a_2, a_3\} + a_4 + a_5 < 1$. These show that Theorem 3.2 sharply improves the main results of Abbas et al. [20] and Han et al. [21].

Example 3.6. *Let $X = \{0, 1, 2\}, E = C^1_{\mathbb{R}}[0, 1]$, and let $P = \{\psi \in E : \psi(t) \geqslant 0, t \in [0, 1]\}$. The mapping $d : X \times X \to P$ is defined by $d(x, y)(t) = \psi(t), x \neq y$ and $d(x, y)(t) = 0, x = y$ where $\psi \in P$ is a fixed function, for example $\psi(t) = e^t$. Then, (X, d) is a complete CMS with normal cone P. Define four mappings f, g, S, T: $X \to X$ as follows:*

$$f(0) = f(1) = f(2) = 0, \quad \text{and} \quad g(0) = g(1) = 0, g(2) = 1,$$

$S = T = I$ (the identity mapping on X). Then $g(X)$ is a complete subspace. Now taking $A_1 = \frac{1}{40}, A_2 = \frac{1}{40}, A_3 = \frac{19|x-y|+9}{40|x-y|+20}, A_4 = \frac{1}{2}$ and $A_5 = \frac{1}{40}$ in Theorem 3.2, we have $A_1 + A_4 + A_5 = \frac{39}{40} < 1$, $1 - A_2 - A_4 = \frac{19}{40} > 0, \inf_{x,y\in X}\{1 - A_3 - A_5\} = \frac{101}{200} > 0$, $A_1 + A_2 + A_5 = \frac{11}{20}$, $\sup_{x,y\in X}\{A_1 + A_3 + A_4\} = \frac{199}{200}$ and $(\frac{199}{200} \cdot \frac{11}{20})/(\frac{39}{40} \cdot \frac{19}{20}) = \frac{2189}{3705} < 1$ which imply that conditions (i)–(iii) *in Theorem 3.2 as $s = 1$ are satisfied .*

Moreover, if $x = 0$ and $y = 2$ then

$$\begin{aligned} d(f(0), g(2))(t) = e^t &\leqslant \tfrac{1}{40}d(0,2))(t) + \tfrac{1}{40}d(f(0),0)(t) + \tfrac{19|0-2|+9}{40|0-2|+20}d(g(2),2)(t) \\ &\quad + \tfrac{1}{2}d(f(0),2)(t) + \tfrac{1}{40}d(0,g(2))(t) \\ &= (\tfrac{1}{20} + 0 + \tfrac{47}{100} + \tfrac{1}{2} + \tfrac{1}{40})e^t = \tfrac{209}{200}e^t. \end{aligned}$$

If $x = 1$ and $y = 2$, then $d(f(1), g(2))(t) = e^t \leqslant \frac{61}{60}e^t$. If $x = 2$ and $y = 2$, then $d(f(2), g(2))(t) = e^t \leqslant \frac{41}{40}e^t$. If $x = 0, 1, 2$ and $y = 0, 1$, then $d(f(x), g(y))(t) = 0$. Hence, f, g, S, T satisfy the conditions in Theorem

3.2 and they have a unique CFP in X. But for any non-negative real numbers a_1, a_2, a_3, a_4, a_5 *with* $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, *we have*

$$\begin{aligned}&a_1 d(1,2)(t) + a_2(f(1),1)(t) + a_3 d(g(2),2)(t) + a_4 d(f(1),2)(t) + a_5 d(1,g(2))(t)\\ &= (a_1 + a_2 + a_3 + a_4)e^t \ll e^t = d(f(1), g(2))(t).\end{aligned}$$

Thus, f, g, s, t *cannot satisfy the relation about coefficients* $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

In the same way, taking $\varphi(a) = \|a\|$ in Theorem 3.1 (see the above Example 2.13) we can obtain Theorem 3.7.

Theorem 3.7. *Let* (X, d) *be a CbMS with the coefficient* $s \geqslant 1$ *and P be a normal cone with normal constant* $k \geqslant 1$. *Suppose four mappings* $f, g, S, T : X \to X$ *satisfy* (a) *in Theorem 3.1 and*
(b) *for all* $x, y \in X$,

$$\begin{aligned}\|d(fx, gy)\| \leqslant{}& A_1(x,y)\|d(Sx,Ty)\| + A_2(x,y)\|d(fx,Sx)\| + A_3(x,y)\|d(gy,Ty)\|\\ &+A_4(x,y)\|d(fx,Ty)\| + A_5(x,y)\|d(Sx,gy)\| \qquad (3.12)\end{aligned}$$

where A_1–A_5 *are five functions from* $X \times X$ *to* $[0, +\infty)$ *such that*

(i) $A_1(x,y) + A_4(x,y) + A_5(x,y) < 1$ *for all* $x, y \in X$;

(ii) $\inf_{x,y\in X}\{1 - A_3(x,y) - ksA_5(x,y)\} = a > 0$, $\inf_{x,y\in X}\{1 - A_2(x,y) - ksA_4(x,y)\} = b > 0$,

$\sup_{x,y\in X}\{A_1(x,y) + A_2(x,y) + ksA_5(x,y)\} = A$, $\sup_{x,y\in X}\{A_1(x,y) + A_3(x,y) + ksA_4(x,y)\} = B$,

with $\frac{A}{a} \cdot \frac{B}{b} < \frac{1}{s^2}$;

(iii) $ks(1-a) < 1$ *and* $ks(1-b) < 1$.

Then, the conclusions of Theorem 3.1 still hold.

As direct consequences of the above nonlinear results, we can obtain the corresponding linear results.

Corollary 3.8. *Let* (X, d) *be a CbMS with the coefficient* $s \geqslant 1$ *and P be a normal cone with* $k \geqslant 1$. *Suppose four mappings* $f, g, S, T : X \to X$ *satisfy* (a) *in Theorem 3.1 and* (b) *for all* $x, y \in X$,

$$\|d(fx, gy)\| \leqslant a_1\|d(Sx,Ty)\| + a_2\|d(fx,Sx)\| + a_3\|d(gy,Ty)\| + a_4\|d(fx,Ty)\| + a_5\|d(Sx,gy)\|, \quad (3.13)$$

where $a_i \geqslant 0$ $(i = 1,2,3,4,5)$ *with* $a_1 + a_4 + a_5 < 1, ks(a_2 + ksa_4) < 1, ks(a_3 + ksa_5) < 1$ *and* $\frac{(a_1+a_2+ksa_5)(a_1+a_3+ksa_4)}{(1-a_2-ksa_4)(1-a_3-ksa_5)} < \frac{1}{s^2}$. *Then, the conclusions of Theorem 3.7 still hold.*

Remark 3.9. *Theorem 2.4 in [8] is a special case of Corollary 3.8 as* (X, d) *is a CMS with* $a_1 < 1, a_2 = 0 = a_3 = a_4 = a_5$ *and* S, T *are continuous. Therefore, Corollary 3.8 complements and improves the result of Radenović [8].*

Next, by using Theorem 3.1, we prove that the following coincidence point and CFP theorems for six self-mappings in X.

Theorem 3.10. *Let (X, d) be a CbMS with the coefficient $s \geqslant 1$ and P be a solid cone. Suppose that six mappings $f, g, S, T, F, G : X \to X$ satisfy the following conditions:*

(a) *$f(X) \subset TG(X), g(X) \subset SF(X)$ with one of $f(X), g(X), SF(X)$ or $TG(X)$ is a complete subspace of X;*

(b) *$fF = Ff, SF = FS, gG = Gg, TG = GT$;*

(c) *there is a mapping $\varphi \in \Phi(P, P_1)$ such that*

$$\varphi(d(fx, gy)) \leq A_1(x, y)\varphi(d(SFx, TGy)) + A_2(x, y)\varphi(d(fx, SFx)) + A_3(x, y)\varphi(d(gy, TGy))$$

$$+A_4(x, y)\varphi(d(fx, TGy)) + A_5(x, y)\varphi(d(SFx, gy)),\ \forall x, y \in X \tag{3.14}$$

where A_1–A_5 are five functions from $X \times X$ to $[0, +\infty)$ such that (i), (ii) *and* (iii) *of Theorem 3.1 hold. Then, there is a unique $z \in X$ that is the point of coincidence of $\{f, SF\}$ and $\{g, TG\}$. Moreover, if $\{f, SF\}$ and $\{g, TG\}$ are weakly compatible then f, g, S, T, F and G have a unique CFP.*

Proof. Putting $P = SF, Q = TG$, we can see that conditions (a) and (c) of the theorem imply conditions (a) and (b) of Theorem 3.1. Therefore, $\{f, P = SF\}$ and $\{g, Q = TG\}$ have a unique point of coincidence in X via Theorem 3.1. Moreover, if $\{f, P = SF\}$ and $\{g, Q = TG\}$ are weakly compatible then f, g, P and Q have a unique CFP z in X, i.e.,

$$fz = gz = Pz = Qz = z. \tag{3.15}$$

Next, we will illustrate that z is also a CFP of S and F. By (3.15) and condition (b), we have $fFz = Ffz = Fz$ and $PFz = (SF)Fz = (FS)Fz = F(SF)z = FPz = Fz$. Thus, by condition (b) of Theorem 3.1 (putting $x = Fz$ and $y = z$ in (3.1)) we get

$$\begin{aligned}
&\varphi(d(Fz, z)) = \varphi(d(fFz, gz))\\
&\leq A_1(Fz, z)\varphi(d(SFFz, TGz)) + A_2(Fz, z)\varphi(d(fFz, SFFz)) + A_3(Fz, z)\\
&\quad \cdot\varphi(d(gz, TGz)) + A_4(Fz, z)\varphi(d(fFz, TGz)) + A_5(Fz, z)\varphi(d(SFFz, gz))\\
&= A_1(Fz, z)\varphi(d(Fz, z)) + A_2(Fz, z)\varphi(d(Fz, Fz)) + A_3(Fz, z)\varphi(d(z, z))\\
&\quad +A_4(Fz, z)\varphi(d(Fz, z)) + A_5(Fz, z)\varphi(d(Fz, z))\\
&= [A_1(Fz, z) + A_4(Fz, z) + A_5(Fz, z)]\varphi(d(Fz, z)).
\end{aligned}$$

From (i) of Theorem 3.1, Remarks 2.4 and 2.11 we can obtain $Fz = z$, and so $z = Pz = SFz = Sz$. Therefore, z is a CFP of S and F.

Similarly, we can prove that z is also a CFP of T and G. In fact, by (3.15) and condition (b) we have $gGz = Ggz = Gz$ and $QGz = (TG)Gz = (GT)Gz = G(TG)z = GQz = Gz$. Thus, by condition (b) of Theorem 3.1 (putting $x = z$ and $y = Gz$ in (3.1)), we get

$$\begin{aligned}
&\varphi(d(z, Gz)) = \varphi(d(fz, gGz))\\
&\leq A_1(z, Gz)\varphi(d(SFz, TGGz)) + A_2(z, Gz)\varphi(d(fz, SFz)) + A_3(z, Gz)\\
&\quad \cdot\varphi(d(gGz, TGGz)) + A_4(z, Gz)\varphi(d(fz, TGGz)) + A_5(z, Gz)\varphi(d(SFz, gGz))\\
&= A_1(z, Gz)\varphi(d(z, Gz)) + A_2(z, Gz)\varphi(d(z, z)) + A_3(z, Gz)\varphi(d(Gz, Gz))
\end{aligned}$$

$$+A_4(z,Gz)\varphi(d(z,Gz)) + A_5(z,Gz)\varphi(d(z,Gz))$$
$$= [A_1(z,Gz) + A_4(z,Gz) + A_5(z,Gz)]\varphi(d(z,Gz)).$$

From (i) of Theorem 3.1, Remarks 2.4 and 2.11, we know that $Gz = z$, and so $z = Qz = TGz = Tz$. This shows that z is also a CFP of T and G. Therefore, z is a CFP of f, g, S, T, F and G. Since z is a unique CFP of f, g, P and Q, it is easy to see that z is also a unique CFP of f, g, S, T, F and G.

Remark 3.11. *We also can obtain Theorem 3.1 by putting $F = G = I$ in Theorem 3.10. Therefore, Theorem 3.1 and Theorem 3.10 are equivalent.*

In the same way, we obtain the corresponding result.

Corollary 3.12. *Let (X,d) be a CbMS with the coefficient $s \geqslant 1$ and P be a solid cone. Suppose six mappings $f, g, S, T, F, G : X \to X$ satisfy the conditions* (a) *and* (b) *in Theorem 3.10 and*

(c) *there is a mapping $\varphi \in \Phi(P, P_1)$ such that*

$$\varphi(d(fx,gy)) \leq a_1\varphi(d(SFx,TGy)) + a_2\varphi(d(fx,SFx)) + a_3\varphi(d(gy,TGy))$$
$$+a_4\varphi(d(fx,TGy)) + a_5\varphi(d(SFx,gy)),\ \forall x,y \in X \tag{3.16}$$

where $a_i \geqslant 0$ $(i = 1,2,3,4,5)$ are constants with $a_1 + a_4 + a_5 < 1, K\omega s^\sigma(a_2 + K\omega s^\sigma a_4) < 1, K\omega s^\sigma(a_3 + K\omega s^\sigma a_5) < 1$ and $\frac{(a_1+a_2+K\omega s^\sigma a_5)(a_1+a_3+K\omega s^\sigma a_4)}{(1-a_2-K\omega s^\sigma a_4)(1-a_3-K\omega s^\sigma a_5)} < \frac{1}{\omega^2 s^{2\sigma}}$, and $K \geqslant 1, \omega \geqslant 1, \sigma > 0$ are some constants as in Definition 2.10. Then, the conclusions of Theorem 3.10 are equally valid.

Remark 3.13. *Obviously, taking (X,d) as a b-metric space and $\varphi(a) = a$ in Corollary 3.12 then we have $K = \omega = \sigma = 1$ which imply that $a_i \geqslant 0$ $(i = 1,2,3,4,5)$ are constants with $a_1 + a_4 + a_5 < 1, s(a_2 + sa_4) < 1, s(a_3 + sa_5) < 1$ and $\frac{(a_1+a_2+sa_5)(a_1+a_3+sa_4)}{(1-a_2-sa_4)(1-a_3-sa_5)} < \frac{1}{s^2}$.*

Let $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < \frac{1}{s^4}$. Note that $s \geqslant 1$. We have $a_1 + a_4 + a_5 < 1, s(a_2 + sa_4) < 1$ and $s(a_3 + sa_5) < 1$. Again by $s \geqslant 1$, we have $a_1 + a_2 + a_3 + 2a_4 < \frac{1}{s^4} < \frac{1}{s^2} \Rightarrow s(a_1 + a_3 + sa_4) < 1 - a_2 - sa_4$ and $a_1 + a_2 + a_3 + 2a_5 < \frac{1}{s^2} \Rightarrow s(a_1 + a_2 + sa_5) < 1 - a_3 - sa_5$, which show that $\frac{(a_1+a_2+sa_5)(a_1+a_3+sa_4)}{(1-a_2-sa_4)(1-a_3-sa_5)} < \frac{1}{s^2}$. Therefore, taking (X,d) as a b-metric space, $\varphi(a) = a$ and $a_i \geqslant 0$ $(i = 1,2,3,4,5)$ are constants with $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < \frac{1}{s^4}$ in Corollary 2.11 of [25] then the conclusion of Corollary 3.12 is still holds. Hence, Corollary 3.12 sharply improves Corollary 2.11 of [25] in its four aspects:

1) the b-metric space is replaced by the CbMS;

2) the contractive condition is replaced by the new contractive condition defined by (3.16)*;*

3) the continuity of function is not required;

4) one pair of maps is compatible and another is weak compatible decrease to the two pairs are both weak compatible.

Moreover, from Remark 2.13 in [25], we easily see that Corollary 3.12 also improves Theorem 2.7 of Roshan et al. [13].

In the following, we will give a specific example of Theorem 3.2.

Example 3.14. *Let $E = \mathbb{R}^2, X = [0,1]$, and let $P = \{(x_1,x_2) \in E : x_1, x_2 \geqslant 0\}$. Define $d : X \times X \to P$ by*

$$d(x,y) = ((x-y)^2, \alpha(x-y)^2)$$

for all $x, y \in X$ where $\alpha > 0$. Then (X, d) is a complete CbMS with the coefficient $s = 2$. Suppose that four mappings f, g, S and T be defined by

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}], \\ \frac{1}{4}, & x \in (\frac{1}{2}, 1]; \end{cases} \qquad g(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}], \\ \frac{1}{2}, & x \in (\frac{1}{2}, 1]; \end{cases}$$

$$S(x) = \begin{cases} 0, & x = 0, \\ \frac{1}{2}, & x \in (0, \frac{1}{2}], \\ 1, & x \in (\frac{1}{2}, 1]; \end{cases} \qquad T(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ 1, & x \in (\frac{1}{2}, 1]. \end{cases}$$

It is easy to see that $f(X) \subset T(X)$, $g(X) \subset S(X)$, and $f(X), g(X), S(X)$ and $T(X)$ are some complete subspaces of X, and $\{f, S\}$ and $\{g, T\}$ are weakly compatible. Now taking $A_1(x, y) \equiv \frac{1}{40} \equiv A_2(x, y), A_3(x, y) \equiv \frac{9}{40} = A_4(x, y)$ and $A_5(x, y) \equiv \frac{1}{40}$ for all $x, y \in X$ in Theorem 3.2, we have $A_1 + A_4 + A_5 = \frac{11}{40} < 1$, $1 - A_2 - 2A_4 = \frac{21}{40} > 0, \inf_{x,y \in X}\{1 - A_3 - 2A_5\} = \frac{29}{40} > 0$, $A_1 + A_2 + 2A_5 = \frac{1}{10}$, $2(1 - \frac{21}{40}) = \frac{19}{20} < 1, 2(1 - \frac{29}{40}) = \frac{11}{20} < 1$, $\sup_{x,y \in X}\{A_1 + A_3 + 2A_4\} = \frac{7}{10}$ and $(\frac{1}{10} \cdot \frac{7}{10})/(\frac{29}{40} \cdot \frac{21}{40}) = \frac{16}{87} < \frac{1}{2^2}$, which imply that conditions (i)–(iii) *in Theorem 3.2 are satisfied.*

Moreover, by a simple calculation we can check that f, g, S and T are satisfying the condition (3.9) *of Theorem 3.2. For this purpose, we consider the following six cases:*

case 1. $x \in [0, \frac{1}{2}]$ *and* $y \in [0, \frac{1}{2}]$. *Obviously, we have*

$$d(fx, gy) = (0, 0) \preccurlyeq A_1 d(Sx, Ty) + A_2 d(fx, Sx) + A_3 d(gy, Ty) + A_4 d(fx, Ty) + A_5 d(Sx, gy).$$

case 2. $x = 0$ *and* $y \in (\frac{1}{2}, 1]$. *In this case, we have*

$$\begin{aligned} &A_1 d(Sx, Ty) + A_2 d(fx, Sx) + A_3 d(gy, Ty) + A_4 d(fx, Ty) + A_5 d(Sx, gy) \\ &= \tfrac{1}{40}(1, \alpha) + 0 + \tfrac{9}{40}(\tfrac{1}{4}, \alpha\tfrac{1}{4}) + \tfrac{9}{40}(1, \alpha) + \tfrac{1}{40}(\tfrac{1}{4}, \alpha\tfrac{1}{4}) = (\tfrac{5}{16}, \alpha\tfrac{5}{16}) \succcurlyeq (\tfrac{1}{4}, \alpha\tfrac{1}{4}) = d(fx, gy). \end{aligned}$$

case 3. $x \in (0, \frac{1}{2}]$ *and* $y \in (\frac{1}{2}, 1]$. *In this case, we have*

$$\begin{aligned} &A_1 d(Sx, Ty) + A_2 d(fx, Sx) + A_3 d(gy, Ty) + A_4 d(fx, Ty) + A_5 d(Sx, gy) \\ &= \tfrac{1}{40}(\tfrac{1}{4}, \alpha\tfrac{1}{4}) + \tfrac{1}{40}(\tfrac{1}{4}, \alpha\tfrac{1}{4}) + \tfrac{9}{40}(\tfrac{1}{4}, \alpha\tfrac{1}{4}) + \tfrac{9}{40}(1, \alpha) + 0 = (\tfrac{47}{160}, \alpha\tfrac{47}{160}) \succcurlyeq (\tfrac{1}{4}, \alpha\tfrac{1}{4}) = d(fx, gy). \end{aligned}$$

case 4. $x \in (\frac{1}{2}, 1]$ *and* $y = 0$. *In this case, we can obtain that*

$$\begin{aligned} &A_1 d(Sx, Ty) + A_2 d(fx, Sx) + A_3 d(gy, Ty) + A_4 d(fx, Ty) + A_5 d(Sx, gy) \\ &= \tfrac{1}{40}(1, \alpha) + \tfrac{1}{40}((\tfrac{3}{4})^2, \alpha(\tfrac{3}{4})^2) + 0 + \tfrac{9}{40}((\tfrac{1}{4})^2, \alpha(\tfrac{1}{4})^2) + \tfrac{1}{40}(1, \alpha) \\ &= (\tfrac{5}{64}, \alpha\tfrac{5}{64}) \succcurlyeq (\tfrac{1}{16}, \alpha\tfrac{1}{16}) = d(fx, gy). \end{aligned}$$

case 5. $x \in (\frac{1}{2}, 1]$ *and* $y \in (0, \frac{1}{2}]$. *In this case, by* $\frac{19}{40}y^2 - \frac{13}{80}y + \frac{5}{64} \geqslant \frac{1}{16}$ *for all* $y \in (0, \frac{1}{2}]$, *we can obtain that*

$$\begin{aligned} &A_1 d(Sx, Ty) + A_2 d(fx, Sx) + A_3 d(gy, Ty) + A_4 d(fx, Ty) + A_5 d(Sx, gy) \\ &= \tfrac{1}{40}((1-y)^2, \alpha(1-y)^2) + \tfrac{1}{40}((\tfrac{3}{4})^2, \alpha(\tfrac{3}{4})^2) + \tfrac{9}{40}(y^2, \alpha y^2) + \tfrac{9}{40}((\tfrac{1}{4} - y)^2, \alpha(\tfrac{1}{4} - y)^2) + \tfrac{1}{40}(1, \alpha) \\ &= (\tfrac{19}{40}y^2 - \tfrac{13}{80}y + \tfrac{5}{64}, \alpha[\tfrac{19}{40}y^2 - \tfrac{13}{80}y + \tfrac{5}{64}]) \succcurlyeq (\tfrac{1}{16}, \alpha\tfrac{1}{16}) = d(fx, gy). \end{aligned}$$

case 6. $x \in (\frac{1}{2}, 1]$ *and* $y \in (\frac{1}{2}, 1]$. *In this case, we can obtain that*

$$\begin{aligned} &A_1 d(Sx, Ty) + A_2 d(fx, Sx) + A_3 d(gy, Ty) + A_4 d(fx, Ty) + A_5 d(Sx, gy) \\ &= 0 + \tfrac{1}{40}((\tfrac{3}{4})^2, \alpha(\tfrac{3}{4})^2) + \tfrac{9}{40}(\tfrac{1}{4}, \alpha\tfrac{1}{4}) + \tfrac{9}{40}((\tfrac{3}{4})^2, \alpha(\tfrac{3}{4})^2) + \tfrac{1}{40}(\tfrac{1}{4}, \alpha\tfrac{1}{4}) = (\tfrac{13}{64}, \alpha\tfrac{13}{64}) \end{aligned}$$

$\geqslant (\frac{1}{16}, \alpha\frac{1}{16}) = d(fx, gy).$

Then, in all the above cases, f, g, S *and* T *are satisfying the condition* (3.9) *of Theorem 3.2. So all the conditions of Theorem 3.2 are valid. Obviously,* 0 *is the unique CFP for all of the mappings* f, g, S *and* T*. But for any non-negative real numbers* a_1, a_2, a_3, a_4, a_5 *with* $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < \frac{1}{2^4}$*, if* $x = 0, y \in (\frac{1}{2}, 1]$*, we have*

$$\begin{aligned}
&a_1 d(Sx, Ty) + a_2 d(fx, Sx) + a_3 d(gy, Ty) + a_4 d(fx, Ty) + a_5 d(Sx, gy) \\
=\ & a_1(1, \alpha) + 0 + a_3(\tfrac{1}{4}, \alpha\tfrac{1}{4}) + a_4(1, \alpha) + a_5(\tfrac{1}{4}, \alpha\tfrac{1}{4}) = (a_1 + \tfrac{a_3}{4} + a_4 + \tfrac{a_5}{4}, \alpha(a_1 + \tfrac{a_3}{4} + a_4 + \tfrac{a_5}{4})) \\
\preccurlyeq\ & (a_1 + a_3 + a_4 + a_5, \alpha(a_1 + a_3 + a_4 + a_5)) \preccurlyeq (\tfrac{1}{2^4}, \tfrac{\alpha}{2^4}) \preccurlyeq (\tfrac{1}{4}, \alpha\tfrac{1}{4}) = d(fx, gy).
\end{aligned}$$

Thus, f, g, S, T *cannot satisfy the relation about coefficients* $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < \frac{1}{2^4}$*. This also shows that our results improve and extend the results in [13, 25].*

Remark 3.15. *In Example 3.14, if we discuss it under the usual metric* $d(x, y) = |x - y|$*, it is not difficult to verify that the inequality does not hold in the cases 2–4 of Example 3.14. In particular, we are able to check that the inequality does not true under* $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < 1$*. These illustrate the validity of Example 3.14.*

4. Conclusions

It is well known that the study of fixed points of mappings satisfying a more weak nonlinear and linear contractive conditions has been at the center of vigorous research activity. Thus, a question arises of whether the restriction of coefficients $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ can be omitted or improved in [13, 16, 20–25].

In this paper, we give an affirmative answer to the above question. To this end, we first introduce the new class $\Phi(P, P_1)$ which is defined from a RBS into another RBS. By using the function $\varphi \in \Phi(P, P_1)$, we prove a coincidence point and CFP theorem for four self-mappings which satisfy new nonlinear conditions in a CbMS. Secondly, as applications of the main theorem, we consider some coincidence point and CFP theorems for six self-mappings satisfying new nonlinear conditions in a CbMS. Also, some examples are given to illustrate the validity of our results and the restriction of coefficients $a_1 + a_2 + a_3 + a_4 + a_5 \geq 1$ are allowed, which indicate that the conditions under which our theorem holds are clearly weaker than those found in [13, 16, 20–25]. For details, see Examples 3.6 and 3.14, and Remarks 3.5, 3.11 and 3.13 of this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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