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On regularity criteria for MHD system in anisotropic Lebesgue spaces

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Abstract: This paper concerns the regularity criteria of the three-dimensional magnetohydrodynamic (MHD) system in anisotropic Lebesgue spaces. Two regularity results were proved under additional assumptions on the horizontal components of the velocity field $\mathbf{u}$ and the magnetic field $\mathbf{B}$, or directions of Elsässer's variables $\mathbf{u} \pm \mathbf{B}$.

Keywords: MHD system; regularity criteria; anisotropic Lebesgue space; sum spaces; Elsässer's variables

1. Introduction

We consider the following MHD system

$$\begin{cases} \mathbf{u}_t - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{B} \cdot \nabla)\mathbf{B} + \nabla \left(p + \frac{1}{2}|\mathbf{B}|^2\right) = 0, \\ \mathbf{B}_t - \Delta \mathbf{B} + (\mathbf{u} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} = 0, \\ \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{B} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0, \quad \mathbf{B}(x,0) = \mathbf{B}_0, \end{cases} \tag{1.1}$$

in $\mathbb{R}^3$. Here $\mathbf{u}$ represents the velocity field, $\mathbf{B}$ represents the magnetic field and p represents the pressure. $\mathbf{u}_0$ and $\mathbf{B}_0$ are given initial datum satisfying $\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{B}_0 = 0$ in $\mathbb{R}^3$.

The MHD system has been extensively studied. For studies on the existence of weak and strong solutions, see [1,2]. Sermange and Teman [2] also studied the smoothness of strong solutions and the so-called squeezing property of the trajectories. The regularity criteria of weak solutions were studied by many authors. For the fundamental Serrin-type regularity criteria, we refer to [3–6] and references therein. For more results on MHD and related systems, see [7–20] and references therein.

This paper concerns the regularity criteria of three-dimensional MHD system in anisotropic Lebesgue space. For the Navier-Stokes equations, Zheng [21] first studied anisotropic regularity criterion in terms of one velocity component. Years later, Qian [22], Guo et al. [23] and Guo et al. [24] further studied the regularity condition in anisotropic Lebesgue spaces for the Leary-Hopf weak solutions of Navier-Stokes equations. Guo et al. [25] also studied the regularity condition in anisotropic Lebesgue spaces for MHD system. By considering different weights in spatial variables, they proved that if $\partial_3 u_3$ and $\mathbf{B}$ satisfy certain space-time integrable conditions in anisotropic Lebesgue spaces, then the weak solution is indeed regular.

We are concerned with the regularity criteria of MHD system in anisotropic Lebesgue space under conditions on $\mathbf{u}_h = (u_1, u_2, 0)$ and $\mathbf{B}_h = (b_1, b_2, 0)$, or the directions of $\mathbf{u} \pm \mathbf{B}$. For the Navier-stokes system, Montgomery-Smith [26] proved the logarithmically improved regularity criteria for the Navier-Stokes equations, which says that condition

$$\int_0^T \frac{\|\mathbf{u}(t)\|_{L^q}^p}{1 + \log^+ \|\mathbf{u}(t)\|_{L^q}} dt < \infty \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 1 \quad \text{and} \quad 3 < q < \infty$$

implies that $\mathbf{u}$ is regular on $(0, T] \times \mathbb{R}^3$. Vasseur [27] showed that the condition

$$\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \leq \frac{1}{2} \quad \text{and} \quad 6 \leq q \leq \infty$$

ensures the smooth of weak solution. In [28], Miller extended the Ladyzhenskaya-Prodi-Serrin regularity criterion to the Lebesgue sum space and proved that if the maximum existence time T of the local smooth solution is finite, then, it holds that

$$\int_0^T \|\mathbf{v}\|_{L^p}^q \, dt + \int_0^T \|\sigma\|_{L^\infty} \, dt = \infty \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1 \quad \text{and} \quad 3 < p < \infty,$$

where $\mathbf{u} = \mathbf{v} + \sigma$. In [29], Wu improves the above results [26–28] in anisotropic Lebesgue spaces. More precisely, let $\mathbf{u}_h = \xi + \sigma$ and $\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) = f + g$, Wu proved that if $q > 0$ then either

$$\int_o^T \frac{\|\xi(t)\|_{L^{\vec{p}}}^q + \|\sigma(t)\|_{L^\infty}^2}{1 + \ln(e + \|\mathbf{u}_h\|_{L^{\vec{s}}})} dt < \infty$$

with

$$\frac{2}{q} + \sum_{i=1}^{3} \frac{1}{p_i} = 1, \quad \sum_{i=1}^{3} \frac{1}{s_i} = \frac{1}{2}, \quad 2 < p_i \leq \infty, \quad 2 < s_i < \infty, \quad \text{for } i = 1, 2, 3$$

or

$$\int_0^T \|f\|_{L^{\vec{p}}}^q \, dt + \int_0^T \|g\|_{L^\infty}^4 \, dt < \infty$$

with

$$\frac{2}{q} + \sum_{i=1}^{3} \frac{1}{p_i} = \frac{1}{2}, \quad 2 < p_i \leq \infty$$

is sufficient to ensure the smoothness of $\mathbf{u}$.

Motivated by the work of Wu [29], we study the regularity of MHD systems in the framework of anisotropic Lebesgue space.

2. Preliminaries and main results

Let us first recall the definition of anisotropic Lebesgue spaces introduced by Benedek and Panzone [30].

Definition 2.1. For a given $\vec{p} = (p_1, p_2, p_3) \in [1, \infty)^3$, the anisotropic Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^3\right)$ is defined to be the space consisting of all measurable functions $f : \mathbb{R}^3 \to \mathbb{R}$ such that the norm

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^3)} = \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x)|^{p_1}\,dx_1\right)^{\frac{p_2}{p_1}}\,dx_2\right)^{\frac{p_3}{p_2}}\,dx_3\right)^{\frac{1}{p_3}} < \infty.$$

We also write the norm as $\|f\|_{L_1^{p_1}L_2^{p_2}L_3^{p_3}}$.

Now we state our main results as follows.

Theorem 2.2. *Suppose that* $(\mathbf{u}_0, \mathbf{B}_0) \in H^1\left(\mathbb{R}^3\right)$ *and* $\operatorname{div}\mathbf{u}_0 = \operatorname{div}\mathbf{B}_0 = 0$. *Let* $(\mathbf{u}, \mathbf{B})$ *be a weak solution to* (1.1) *on* $[0, T]$. *Assume that* $\mathbf{u}_h = (u_1, u_2, 0) = \xi_1 + \sigma_1$ *and* $\mathbf{B}_h = (B_1, B_2, 0) = \xi_2 + \sigma_2$ *such that*

$$\int_0^T \frac{\|\xi_1\|_{L^{\vec{p}}}^q + \|\xi_2\|_{L^{\vec{k}}}^l + \|\sigma_1\|_{L^\infty}^2 + \|\sigma_2\|_{L^\infty}^2}{1 + \ln(e + \|\mathbf{u}_h\|_{L^{\vec{s}}} + \|\mathbf{B}_h\|_{L^{\vec{r}}})}\,dt < \infty,$$

with $2 < p_i, k_i \le \infty$, $\frac{2}{q} = 1 - \sum_{i=1}^3 \frac{1}{p_i} > 0$, $\frac{2}{l} = 1 - \sum_{i=1}^3 \frac{1}{k_i} > 0$ *and* $2 < s_i, r_i < \infty$, $\sum_{i=1}^3 \frac{1}{s_i} = \sum_{i=1}^3 \frac{1}{r_i} = \frac{1}{2}$, *then,* $(\mathbf{u}, \mathbf{B})$ *is regular on* $(0, T]$.

Theorem 2.3. *Suppose that* $(\mathbf{u}_0, \mathbf{B}_0) \in H^1(\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{u}_0 = \operatorname{div}\mathbf{B}_0 = 0$. *Let* $(\mathbf{u}, \mathbf{B})$ *be a weak solution of* (1.1) *on* $[0, T]$. *Assume that* $\operatorname{div}(\frac{\mathbf{u}+\mathbf{B}}{|\mathbf{u}+\mathbf{B}|}) = f_1 + g_1$ *and* $\operatorname{div}\left(\frac{\mathbf{u}-\mathbf{B}}{|\mathbf{u}-\mathbf{B}|}\right) = f_2 + g_2$ *such that*

$$\int_0^T (\|f_1\|_{L^{\vec{p}}}^q + \|f_2\|_{L^{\vec{r}}}^s)\,dt + \int_0^T (\|g_1\|_{L^\infty}^4 + \|g_2\|_{L^\infty}^4)\,dt < \infty,$$

with $2 < p_i, r_i \le \infty$, $\frac{2}{q} = \frac{1}{2} - \sum_{i=1}^3 \frac{1}{p_i} > 0$, $\frac{2}{s} = \frac{1}{2} - \sum_{i=1}^3 \frac{1}{r_i} > 0$, *then,* $(\mathbf{u}, \mathbf{B})$ *is regular on* $(0, T]$.

Before proving our main results, we recall some fundamental mixed norm inequalities. By successive applications of Hölder's inequality, we immediately have

Lemma 2.4. *For* $\vec{p} = (p_1, p_2, p_3)$ *and* $\vec{q} = (q_1, q_2, q_3)$ *with*

$$\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}, \quad 2 \le p_i, q_i \le \infty, i = 1, 2, 3,$$

it holds that

$$\|fg\|_{L^2(\mathbb{R}^3)} \le \|f\|_{L^{\vec{p}}(\mathbb{R}^3)}\|g\|_{L^{\vec{q}}(\mathbb{R}^3)}.$$

The following proposition is a direct consequence of Lemma 3 in [23], see also [21, 22].

Proposition 2.5. *For* $p_1, p_2, p_3 \in [2, \infty)$ *and* $0 \le \sum_{i=1}^3 \frac{1}{p_i} - \frac{1}{2} \le 1$, *there exists a positive constant* C *such that*

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^3)} \le C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2} - \sum_i^3 \frac{1}{p_i}}\|f\|_{L^2(\mathbb{R}^3)}^{\sum_i^3 \frac{1}{p_i} - \frac{1}{2}}.$$

We also need the following special case of the Sobolev embedding theorem in anisotropic spaces proved in [31, p.181], see also [32, Theorem 2.1].

Lemma 2.6. *Let* $\vec{s} = (s_1, s_2, \dots, s_n) \in [2, \infty]^n$ *satisfy*

$$\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n} = \frac{n}{2} - 1, \quad s_n \in (2, \infty).$$

Then, there exists a constant $C = C(\vec{s})$ *such that*

$$\|u\|_{L^{\vec{s}}(\mathbb{R}^n)} \le C\|u\|_{W^{1,2}(\mathbb{R}^n)}, \quad \forall u \in W^{1,2}(\mathbb{R}^n).$$

3. Proof of Theorem 2.2

We first deduce an *a priori* estimate for $(\mathbf{u}, \mathbf{B})$.

Multiplying the first equation of (1.1) with $\mathbf{u}$ and multiplying the second equation of (1.1) with $\mathbf{B}$, then integrating by parts over $\mathbb{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 - \int(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\mathbf{u}\,dx = 0, \tag{3.1}$$

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{B}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2 + \int(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\mathbf{u}\,dx = 0. \tag{3.2}$$

Multiplying both sides of the first equation of system (1.1) with $-\Delta\mathbf{u}$ and the second equation of system (1.1) with $-\Delta\mathbf{B}$, then integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2\,dx + \int_{\mathbb{R}^3}|\nabla^2\mathbf{u}|^2\,dx = \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,dx - \int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\triangle\mathbf{u}\,dx, \tag{3.3}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\mathbf{B}|^2\,dx + \int_{\mathbb{R}^3}|\nabla^2\mathbf{B}|^2\,dx = \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla)\mathbf{B}\cdot\Delta\mathbf{B}\,dx - \int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{u}\cdot\triangle\mathbf{B}\,dx. \tag{3.4}$$

Integrating by parts, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\triangle\mathbf{u}\,dx = -\int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)(\triangle\mathbf{u})\cdot\mathbf{B}\,dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}B_i\cdot\partial_i\partial_k\partial_k u_j\cdot B_j\,dx = \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i\partial_k u_j\cdot(B_j\partial_k B_i + B_i\partial_k B_j))\,dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j\cdot\partial_k(B_j\partial_k B_i + B_i\partial_k B_j))\,dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j\cdot(2\partial_k B_i\cdot\partial_k B_j + B_j\partial_{kk}B_i + B_i\partial_{kk}B_j)\,dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_j\cdot B_j\cdot\partial_{kk}B_i + 2\partial_i u_j\partial_k B_i\partial_k B_j)\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j\cdot B_i\cdot\partial_{kk}B_j\,dx
\end{aligned}$$

$$= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_j \cdot B_j \cdot \partial_{kk}B_i + 2\partial_i u_j \partial_k B_i \partial_k B_j)\,dx - \int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{u}\cdot\triangle\mathbf{B}\,dx.$$

Therefore, together with the fact that $\operatorname{div}\mathbf{B} = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\triangle\mathbf{u}\,dx + \int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{u}\cdot\triangle\mathbf{B}\,dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_j \cdot B_j \cdot \partial_{kk}B_i + 2\partial_i u_j \partial_k B_i \partial_k B_j)\,dx\\
&= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k(\partial_i u_j \cdot B_j)\cdot\partial_k B_i\,dx - 2\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j \partial_k B_i \partial_k B_j\,dx\\
&= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k\partial_i u_j \cdot B_j \cdot\partial_k B_i\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j \partial_k B_i \partial_k B_j\,dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_j \cdot \partial_i(B_j\partial_k B_i)\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j \partial_k B_i \partial_k B_j\,dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_j \cdot \partial_i B_j \cdot \partial_k B_i\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j \partial_k B_i \partial_k B_j\,dx.\\
&=I_1 + I_2.
\end{aligned}\tag{3.5}$$

To estimate I_1, we rewrite it as follows

$$\begin{aligned}
&-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_j \cdot \partial_i B_j \partial_k B_i\,dx\\
&= -\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_k u_j \cdot \partial_i B_j \cdot \partial_k B_i\,dx - \sum_{k=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_k u_j \cdot \partial_3 B_j \cdot \partial_k B_3\,dx\\
&\quad + \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3 \cdot (\partial_1 B_1 + \partial_2 B_2)\cdot\partial_k B_3\,dx\\
&\le C\int_{\mathbb{R}^3}|\mathbf{B}_h|(|\nabla\mathbf{u}||\nabla^2\mathbf{B}| + |\nabla\mathbf{B}||\nabla^2\mathbf{u}|)\,dx.
\end{aligned}\tag{3.6}$$

Similarly, we have the following estimate for I_2:

$$-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j \partial_k B_i \partial_k B_j\,dx \le C\int_{\mathbb{R}^3}(|\mathbf{B}_h| + |\mathbf{u}_h|)(|\nabla\mathbf{u}||\nabla^2\mathbf{B}| + |\nabla\mathbf{B}||\nabla^2\mathbf{u}|)\,dx.\tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\triangle\mathbf{u}\,dx + \int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{u}\cdot\triangle\mathbf{B}\,dx\\
&\le C\int_{\mathbb{R}^3}(|\mathbf{B}_h| + |\mathbf{u}_h|)(|\nabla\mathbf{u}||\nabla^2\mathbf{B}| + |\nabla\mathbf{B}||\nabla^2\mathbf{u}|)\,dx.
\end{aligned}\tag{3.8}$$

Similarly, we can get the following estimates

$$\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\triangle\mathbf{u}\,dx\le C\int_{\mathbb{R}^3}|\mathbf{u}_h||\nabla\mathbf{u}||\nabla^2\mathbf{u}|\,dx, \tag{3.9}$$

and

$$\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla)\mathbf{B}\cdot\triangle\mathbf{B}\,dx\le C\int_{\mathbb{R}^3}|\mathbf{u}_h||\nabla\mathbf{B}||\nabla^2\mathbf{B}|\,dx+C\int_{\mathbb{R}^3}|\mathbf{B}_h|(|\nabla\mathbf{u}||\nabla^2\mathbf{B}|+|\nabla\mathbf{B}||\nabla^2\mathbf{u}|)\,dx. \tag{3.10}$$

Combined (3.3)–(3.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{B}\|_{L^2}^2)+\|\Delta\mathbf{u}\|_{L^2}^2+\|\Delta\mathbf{B}\|_{L^2}^2\\
=&\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,dx+\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla)\mathbf{B}\cdot\triangle\mathbf{B}\,dx\\
&-\int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\Delta\mathbf{u}\,dx-\int_{\mathbb{R}^3}(\mathbf{B}\cdot\nabla)\mathbf{u}\cdot\triangle\mathbf{B}\,dx\\
\le&C\int_{\mathbb{R}^3}|\mathbf{u}_h||\nabla\mathbf{u}||\nabla^2\mathbf{u}|\,dx+C\int_{\mathbb{R}^3}|\mathbf{u}_h||\nabla\mathbf{B}||\nabla^2\mathbf{B}|\,dx\\
&+C\int_{\mathbb{R}^3}|\mathbf{u}_h||\nabla\mathbf{u}||\nabla^2\mathbf{B}|\,dx+C\int_{\mathbb{R}^3}|\mathbf{u}_h||\nabla\mathbf{B}||\nabla^2\mathbf{u}|\,dx\\
&+C\int_{\mathbb{R}^3}|\mathbf{B}_h||\nabla\mathbf{u}||\nabla^2\mathbf{B}|\,dx+C\int_{\mathbb{R}^3}|\mathbf{B}_h||\nabla\mathbf{B}||\nabla^2\mathbf{u}|\,dx\\
=&J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned} \tag{3.11}$$

We first estimate J_3. By Lemma 2.4, Proposition 2.5 and Young's inequality, we have

$$\begin{aligned}
J_3=&C\int_{\mathbb{R}^3}|\mathbf{u}_h||\nabla\mathbf{u}||\nabla^2\mathbf{B}|\,dx\le C\int_{\mathbb{R}^3}|\xi_1||\nabla\mathbf{u}||\nabla^2\mathbf{B}|\,dx+C\int_{\mathbb{R}^3}|\sigma_1||\nabla\mathbf{u}||\nabla^2\mathbf{B}|\,dx\\
\le&C\|\xi_1\|_{L^{\vec{p}}}\|\nabla\mathbf{u}\|_{L_1^{\frac{2p_1}{p_1-2}}L_2^{\frac{2p_2}{p_2-2}}L_3^{\frac{2p_3}{p_3-2}}}\|\nabla^2\mathbf{B}\|+C\|\sigma_1\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{B}\|_{L^2}\\
\le&C\|\xi_1\|_{L^{\vec{p}}}\|\nabla\mathbf{u}\|_{L^2}^{1-\sum_{i=1}^3\frac{1}{p_i}}\|\nabla^2\mathbf{u}\|_{L^2}^{\sum_{i=1}^3\frac{1}{p_i}}\|\nabla^2\mathbf{B}\|_{L^2}+C\|\sigma_1\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{B}\|_{L^2}\\
\le&C\|\xi_1\|_{L^{\vec{p}}}^2\|\nabla\mathbf{u}\|_{L^2}^{2-\sum_{i=1}^3\frac{2}{p_i}}\|\nabla^2\mathbf{u}\|_{L^2}^{\sum_{i=1}^3\frac{2}{p_i}}+\frac{\varepsilon}{2}\|\nabla^2\mathbf{B}\|_{L^2}^2+C\|\sigma_1\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2+\frac{\varepsilon}{2}\|\nabla^2\mathbf{B}\|_{L^2}^2\\
\le&C\|\xi_1\|_{L^{\vec{p}}}^{\frac{2}{1-\sum_{i=1}^3\frac{1}{p_i}}}\|\nabla\mathbf{u}\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2+C\|\sigma_1\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{B}\|_{L^2}^2.
\end{aligned}$$

Recall the assumption that $\frac{2}{q}+\sum_{i=1}^3\frac{1}{p_i}=1$, we have

$$q=\frac{2}{1-\sum_{i=1}^3\frac{1}{p_i}},$$

hence,

$$J_3\le C(\|\xi_1\|_{L^{\vec{p}}}^q+\|\sigma_1\|_{L^\infty}^2)\|\nabla\mathbf{u}\|^2+\varepsilon(\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{B}\|_{L^2}^2).$$

The estimates of J_1, J_2, J_4, J_5 and J_6 can be obtained in a similar way. Indeed, if we replace $\nabla^2\mathbf{B}$ by $\nabla^2\mathbf{u}$ in the estimate process for J_3, we immediately get the estimate for J_1:

$$J_1 \le C(\|\xi_1\|_{L^{\vec{p}}}^q + \|\sigma_1\|_{L^\infty}^2)\|\nabla\mathbf{u}\|^2 + \varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2.$$

Replace $\nabla\mathbf{u}$ by $\nabla\mathbf{B}$ (and then $\nabla^2\mathbf{u}$ by $\nabla^2\mathbf{B}$) in the estimate process for J_3 and we can get the estimate for J_2:

$$J_2 \le C(\|\xi_1\|_{L^{\vec{p}}}^q + \|\sigma_1\|_{L^\infty}^2)\|\nabla\mathbf{B}\|_{L^2}^2 + \varepsilon\|\nabla^2\mathbf{B}\|_{L^2}^2.$$

Replace $\mathbf{u}_h$ by $\mathbf{B}_h$ (and then ξ_1 by ξ_2, σ_1 by σ_2, p_i by k_i, q by l) in the estimate process for J_3 and we can get the estimate for J_5:

$$J_5 \le C(\|\xi_2\|_{L^{\vec{k}}}^l + \|\sigma_2\|_{L^\infty}^2)\|\nabla\mathbf{u}\|_{L^2}^2 + \varepsilon(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2).$$

Now we estimate J_4. By Lemma 2.4, Proposition 2.5 and Young's inequality, we have

$$\begin{aligned}
J_4 &= C\int_{\mathbb{R}^3} |\mathbf{u}_h||\nabla\mathbf{B}||\nabla^2\mathbf{u}|\,dx \le C\int_{\mathbb{R}^3} |\xi_1||\nabla\mathbf{B}||\nabla^2\mathbf{u}|\,dx + C\int_{\mathbb{R}^3} |\sigma_1||\nabla\mathbf{B}||\nabla^2\mathbf{u}|\,dx\\
&\le C\|\xi_1\|_{L^{\vec{p}}}\|\nabla\mathbf{B}\|_{L_1^{\frac{2p_1}{p_1-2}}L_2^{\frac{2p_2}{p_2-2}}L_3^{\frac{2p_3}{p_3-2}}}\|\nabla^2\mathbf{u}\|_{L^2} + C\|\sigma_1\|_{L^\infty}\|\nabla\mathbf{B}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}\\
&\le C\|\xi_1\|_{L^{\vec{p}}}\|\nabla\mathbf{B}\|_{L^2}^{1-\sum_{i=1}^3\frac{1}{p_i}}\|\nabla^2\mathbf{B}\|_{L^2}^{\sum_{i=1}^3\frac{1}{p_i}}\|\nabla^2\mathbf{u}\|_{L^2} + C\|\sigma_1\|_{L^\infty}\|\nabla\mathbf{B}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}\\
&\le C\|\xi_1\|_{L^{\vec{p}}}^2\|\nabla\mathbf{B}\|_{L^2}^{2-\sum_{i=1}^3\frac{2}{p_i}}\|\nabla^2\mathbf{B}\|_{L^2}^{\sum_{i=1}^3\frac{2}{p_i}} + \frac{\varepsilon}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\|\sigma_1\|_{L^\infty}^2\|\nabla\mathbf{B}\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2\\
&\le C\|\xi_1\|_{L^{\vec{p}}}^{\frac{2}{1-\sum_{i=1}^3\frac{1}{p_i}}}\|\nabla\mathbf{B}\|_{L^2}^2 + \varepsilon\|\nabla^2\mathbf{B}\|_{L^2}^2 + C\|\sigma_1\|_{L^\infty}^2\|\nabla\mathbf{B}\|_{L^2}^2 + \varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2,
\end{aligned}$$

since

$$q = \frac{2}{1-\sum_{i=1}^3\frac{1}{p_i}},$$

we have

$$J_4 \le C(\|\xi_1\|_{L^{\vec{p}}}^q + \|\sigma_1\|_{L^\infty}^2)\|\nabla\mathbf{B}\|^2 + \varepsilon(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2).$$

Replace $\mathbf{u}_h$ by $\mathbf{B}_h$ (and then ξ_1 by ξ_2, σ_1 by σ_2, p_i by k_i, q by l) in the estimate process for J_4 and we can get the estimate for J_6:

$$J_6 \le C(\|\xi_2\|_{L^{\vec{k}}}^l + \|\sigma_2\|_{L^\infty}^2)\|\nabla\mathbf{B}\|_{L^2}^2 + \varepsilon(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2).$$

Substituting the above estimates of J_i, $i = 1, 2, \cdots, 6$ into (3.11), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2) + \|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2\\
&\le C\left(\|\xi_1\|_{L^{\vec{p}}}^q + \|\sigma_1\|_{L^\infty}^2\right)(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2)\\
&+ C\left(\|\xi_2\|_{L^{\vec{k}}}^l + \|\sigma_2\|_{L^\infty}^2\right)(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2) + 6\epsilon(\|\Delta\mathbf{u}\|_{L^2}^2 + \|\Delta\mathbf{B}\|_{L^2}^2).
\end{aligned} \tag{3.12}$$

Using Lemma 2.6, we find

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla \mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2)+C(\|\nabla^2 \mathbf{u}\|_{L^2}^2+\|\nabla^2 \mathbf{B}\|_{L^2}^2)\\
&\le C\left(\|\xi_1\|_{L^{\vec{p}}}^q+\|\xi_2\|_{L^{\vec{k}}}^l+\|\sigma_1\|_{L^\infty}^2+\|\sigma_2\|_{L^\infty}^2\right)(\|\nabla \mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2)\\
&= C\frac{\|\xi_1\|_{L^{\vec{p}}}^q+\|\xi_2\|_{L^{\vec{k}}}^l+\|\sigma_1\|_{L^\infty}^2+\|\sigma_2\|_{L^\infty}^2}{1+\ln(e+\|\mathbf{u}_h\|_{L^{\vec{s}}}+\|\mathbf{B}_h\|_{L^{\vec{r}}})}(1+\ln(e+\|\mathbf{u}_h\|_{L^{\vec{s}}}+\|\mathbf{B}_h\|_{L^{\vec{r}}}))(\|\nabla \mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2)\\
&\le C\frac{\|\xi_1\|_{L^{\vec{p}}}^q+\|\xi_2\|_{L^{\vec{k}}}^l+\|\sigma_1\|_{L^\infty}^2+\|\sigma_2\|_{L^\infty}^2}{1+\ln(e+\|\mathbf{u}_h\|_{L^{\vec{s}}}+\|\mathbf{B}_h\|_{L^{\vec{r}}})}\\
&\qquad\times\left(1+\ln\left(e+\|\mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{u}\|_{L^2}^2+\|\mathbf{B}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2+C\right)\right)(\|\nabla \mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2).
\end{aligned}
\tag{3.13}
$$

By (3.1), (3.2) and (3.13), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(1+\ln\left(e+\|\mathbf{u}\|_{L^2}^2+\|\mathbf{B}\|_{L^2}^2+\|\nabla \mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2+C\right)\right)+C(\|\nabla^2 \mathbf{u}\|_{L^2}^2+\|\nabla^2 \mathbf{B}\|_{L^2}^2)\\
\le& C\frac{\|\xi_1\|_{L^{\vec{p}}}^q+\|\xi_2\|_{L^{\vec{k}}}^l+\|\sigma_1\|_{L^\infty}^2+\|\sigma_2\|_{L^\infty}^2}{1+\ln(e+\|\mathbf{u}_h\|_{L^{\vec{s}}}+\|\mathbf{B}_h\|_{L^{\vec{r}}})}\left(1+\ln\left(e+\|\mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{u}\|_{L^2}^2+\|\mathbf{B}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2+C\right)\right).
\end{aligned}
$$

Applying Gronwall's inequality, we get

$$
\begin{aligned}
&\sup_{0\le t\le T}\ln\left(e+\|\mathbf{u}\|_{L^2}^2+\|\mathbf{B}\|_{L^2}^2+\|\nabla \mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{B}\|_{L^2}^2+C\right)\\
&\le\left(1+\ln\left(e+\|\mathbf{u}_0\|_{L^2}^2+\|\mathbf{B}_0\|_{L^2}^2+\|\nabla \mathbf{u}_0\|_{L^2}^2+\|\nabla \mathbf{B}_0\|_{L^2}^2+C\right)\right)\\
&\quad\times\exp\left\{C\int_0^T\frac{\|\xi_1\|_{L^{\vec{p}}}^q+\|\xi_2\|_{L^{\vec{k}}}^l+\|\sigma_1\|_{L^\infty}^2+\|\sigma_2\|_{L^\infty}^2}{1+\ln(e+\|\mathbf{u}_h\|_{L^{\vec{s}}}+\|\mathbf{B}_h\|_{L^{\vec{r}}})}\,dt\right\}.
\end{aligned}
$$

This gives

$$
\mathbf{u},\mathbf{B}\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3)).
$$

Now, we can prove Theorem 2.2. Since $\mathbf{u}_0,\mathbf{B}_0\in H^1(\mathbb{R}^3)$ and $\operatorname{div}\mathbf{u}_0=\operatorname{div}\mathbf{B}_0=0$, the weak solution $(\mathbf{u},\mathbf{B})$ is strong and unique on $[0,T_1]$ for some $T_1<T$. The above *a priori* esitmate ensures that this strong solution can be extended beyond T_1, which finally implies that $(\mathbf{u},\mathbf{B})$ is strong on $[0,T]$ and thus is regular up to T. This completes the proof of Theorem 1.1.

4. Proof of Theorem 2.3

Let us introduce Elsässer's variables $\mathbf{W}^+$ and $\mathbf{W}^-$:

$$
\mathbf{W}^+=\mathbf{u}+\mathbf{B},\quad \mathbf{W}^-=\mathbf{u}-\mathbf{B},\quad \mathbf{W}^+(0)=\mathbf{u}_0+\mathbf{B}_0,\quad \mathbf{W}^-(0)=\mathbf{u}_0-\mathbf{B}_0
$$

and $\nabla P=\nabla\left(p+\frac{1}{2}|\mathbf{B}|^2\right)$. Then, $\mathbf{W}^+,\mathbf{W}^-$ satisfies

$$
\begin{cases}
\partial_t\mathbf{W}^+-\Delta\mathbf{W}^++(\mathbf{W}^-\cdot\nabla)\mathbf{W}^++\nabla P=0,\\
\partial_t\mathbf{W}^--\Delta\mathbf{W}^-+(\mathbf{W}^+\cdot\nabla)\mathbf{W}^-+\nabla P=0.
\end{cases}
\tag{4.1}
$$

Multiplying the first equation of (4.1) by $|\mathbf{W}^+|^2\mathbf{W}^+$ and the second equation of (4.1) by $|\mathbf{W}^-|^2\mathbf{W}^-$, integrating by parts and using the divergence free property of $\mathbf{W}^+$ and $\mathbf{W}^-$, we conclude that

$$\frac{1}{4}\frac{d}{dt}\|\mathbf{W}^+\|_{L^4}^4 + \||\mathbf{W}^+||\nabla\mathbf{W}^+|\|_{L^2}^2 + \frac{1}{2}\left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}^2 = -\int_{\mathbb{R}^3}\nabla P\cdot|\mathbf{W}^+|^2\mathbf{W}^+\,dx, \tag{4.2}$$

$$\frac{1}{4}\frac{d}{dt}\|\mathbf{W}^-\|_{L^4}^4 + \||\mathbf{W}^-||\nabla\mathbf{W}^-|\|_{L^2}^2 + \frac{1}{2}\left\|\nabla|\mathbf{W}^-|^2\right\|_{L^2}^2 = -\int_{\mathbb{R}^3}\nabla P\cdot|\mathbf{W}^-|^2\mathbf{W}^-\,dx. \tag{4.3}$$

Combine (4.2) and (4.3) and the vector identity

$$-\frac{\mathbf{a}}{|\mathbf{a}|}\cdot\nabla|\mathbf{a}| = |\mathbf{a}|\,\mathrm{div}\left(\frac{\mathbf{a}}{|\mathbf{a}|}\right)$$

for $\mathbf{a} = \mathbf{W}^\pm$, we have

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4) + \||\mathbf{W}^+||\nabla\mathbf{W}^+|\|_{L^2}^2 + \||\mathbf{W}^-||\nabla\mathbf{W}^-|\|_{L^2}^2\\
&+\frac{1}{2}(\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + \|\nabla|\mathbf{W}^-|^2\|_{L^2}^2)\\
=&-\int_{\mathbb{R}^3}\nabla P\cdot(|\mathbf{W}^+|^2\mathbf{W}^+ + |\mathbf{W}^-|^2\mathbf{W}^-)\,dx = \int_{\mathbb{R}^3}P(\mathbf{W}^+\nabla|\mathbf{W}^+|^2 + \mathbf{W}^-\nabla|\mathbf{W}^-|^2)\,dx\\
\le&C\int_{\mathbb{R}^3}|P\mathbf{W}^+||\mathbf{W}^+||\nabla|\mathbf{W}^+||\,dx + C\int_{\mathbb{R}^3}|P\mathbf{W}^-||\mathbf{W}^-||\nabla|\mathbf{W}^-||\,dx\\
\le&C\int_{\mathbb{R}^3}|P\mathbf{W}^+||\mathbf{W}^+|\left|\left(\frac{\mathbf{W}^+}{|\mathbf{W}^+|}|\nabla|\mathbf{W}^+|\right)\right|\,dx + C\int_{\mathbb{R}^3}|P\mathbf{W}^-||\mathbf{W}^-|\left|\left(\frac{\mathbf{W}^-}{|\mathbf{W}^-|}\nabla|\mathbf{W}^-|\right)\right|\,dx\\
=&C\int_{\mathbb{R}^3}|P\mathbf{W}^+||\mathbf{W}^+||\mathbf{W}^+|\left|\mathrm{div}\left(\frac{\mathbf{W}^+}{|\mathbf{W}^+|}\right)\right|\,dx + C\int_{\mathbb{R}^3}|P\mathbf{W}^-||\mathbf{W}^-||\mathbf{W}^-|\left|\mathrm{div}\left(\frac{\mathbf{W}^-}{|\mathbf{W}^-|}\right)\right|\,dx\\
=&K_1 + K_2.
\end{aligned}\tag{4.4}$$

To estimate K_1 and K_2, let us first establish an estimate of the pressure P. Taking the divergence operator $\nabla\cdot$ on both sides of the first equation of (4.1), it follows that

$$-\Delta P = \mathrm{div}((\mathbf{W}^-\cdot\nabla)\mathbf{W}^+) = \partial_j(W_i^-\partial_iW_j^+) = \partial_j\partial_i(W_i^-W_j^+).$$

Here we used the divergence free property of $\mathbf{W}^-$ and the summation symbol $\sum_{i,j=1}^3$ was omitted for convenience. As a consequence

$$P = \mathcal{R}_i\mathcal{R}_j\left(W_i^+W_j^-\right),$$

where $\mathcal{R}_i$ represents the classical three-dimensional Riesz transformations. By using the boundedness of Riesz transformations in $L^r(1 < r < \infty)$ space, we get

$$\|P\|_{L^r} \le C\|\mathbf{W}^+\|_{L^{2r}}\|\mathbf{W}^-\|_{L^{2r}}. \tag{4.5}$$

Using Hölder's inequality, Poincaré's inequality and (4.5), we obtain

$$\begin{aligned}
\|P\mathbf{W}^+\|_{L^2}^2 =& \int_{\mathbb{R}^3}|P|^2|\mathbf{W}^+|^2\,dx\\
\le&C\|P\|_{L^2}\|P\|_{L^3}\left\||\mathbf{W}^+|^2\right\|_{L^6}\\
\le&C\|\mathbf{W}^+\|_{L^4}\|\mathbf{W}^-\|_{L^4}\|\mathbf{W}^+\|_{L^6}\|\mathbf{W}^-\|_{L^6}\left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}\\
\le&C\|\mathbf{W}^+\|_{L^4}\|\mathbf{W}^-\|_{L^4}\|\nabla\mathbf{W}^+\|_{L^2}\|\nabla\mathbf{W}^-\|_{L^2}\left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}.
\end{aligned}\tag{4.6}$$

Similarly, we can get

$$\|P\mathbf{W}^-\|_{L^2}^2 \leq C\|\mathbf{W}^+\|_{L^4}\|\mathbf{W}^-\|_{L^4}\|\nabla\mathbf{W}^+\|_{L^2}\|\nabla\mathbf{W}^-\|_{L^2}\left\|\nabla|\mathbf{W}^-|^2\right\|_{L^2}. \tag{4.7}$$

Now, we are in position of estimating K_1 and K_2. For K_1, recall the decompostion of $div\left(\frac{W^+}{|W^+|}\right)$ and Lemma 2.4, we have

$$\begin{aligned}
K_1 &\leq C\|f_1\|_{L^{\vec{p}}}\||\mathbf{W}^+|^2\|_{L_1^{\frac{2p_1}{p_1-2}}L_2^{\frac{2p_2}{p_2-2}}L_3^{\frac{2p_3}{p_3-2}}}\|P\mathbf{W}^+\|_{L^2} + C\|g_1\|_{L^\infty}\left\||\mathbf{W}^+|^2\right\|_{L^2}\|P\mathbf{W}^+\|_{L^2} \\
&\leq C\|f_1\|_{L^{\vec{p}}}\left\||\mathbf{W}^+|^2\right\|_{L^2}^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}\left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}\|P\mathbf{W}^+\|_{L^2} \\
&\quad + C\|g_1\|_{L^\infty}\left\||\mathbf{W}^+|^2\right\|_{L^2}\|P\mathbf{W}^+\|_{L^2} \\
&= K_1' + K_1''.
\end{aligned}$$

Let's estimate K_1' first, by (4.6) we have

$$\begin{aligned}
K_1' \leq& C\|f_1\|_{L^{\vec{p}}}\left\||\mathbf{W}^+|^2\right\|_{L^2}^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}\left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}} \\
&\times\|\mathbf{W}^+\|_{L^4}^{\frac{1}{2}}\|\mathbf{W}^-\|_{L^4}^{\frac{1}{2}}\|\nabla\mathbf{W}^+\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{W}^-\|_{L^2}^{\frac{1}{2}}\left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}^{\frac{1}{2}} \\
\leq& C\|f_1\|_{L^{\vec{p}}}\left\|\mathbf{W}^+\right\|_{L^4}^{\frac{5}{2}-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}\left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}^{\frac{1}{2}+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}\|\mathbf{W}^-\|_{L^4}^{\frac{1}{2}}\|\nabla\mathbf{W}^+\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{W}^-\|_{L^2}^{\frac{1}{2}} \\
\leq& \varepsilon\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + C\|f_1\|_{L^{\vec{p}}}^{\frac{4}{3-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}}\|\mathbf{W}^+\|_{L^4}^{\frac{4(\frac{5}{2}-2(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}))}{3-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}}\|\mathbf{W}^-\|_{L^4}^{\frac{2}{3-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}} \\
&\times\|\nabla\mathbf{W}^+\|_{L^2}^{\frac{2}{3-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}}\|\nabla\mathbf{W}^-\|_{L^2}^{\frac{2}{3-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}}.
\end{aligned} \tag{4.8}$$

Let

$$\alpha = \frac{4(\frac{5}{2}-2(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}))}{3-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)} \geq 3, \quad \beta = \frac{2}{3-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)} \in \left(\frac{2}{3},1\right).$$

It is easily checked that $\alpha+\beta=4$, therefore, by Young's inequality we have

$$\begin{aligned}
\|\mathbf{W}^+\|_{L^4}^{\alpha}\|\mathbf{W}^-\|_{L^4}^{\beta} &\leq \frac{\alpha}{\alpha+\beta}\|\mathbf{W}^+\|_{L^4}^{\alpha\cdot\frac{\alpha+\beta}{\alpha}} + \frac{\beta}{\alpha+\beta}\|\mathbf{W}^-\|_{L^4}^{\beta\cdot\frac{\alpha+\beta}{\beta}} \\
&\leq \|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4,
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
\|f_1\|_{L^{\vec{p}}}^{2\beta}\|\nabla\mathbf{W}^+\|_{L^2}^{\beta}\|\nabla\mathbf{W}^-\|_{L^2}^{\beta} &\leq (1-\beta)\|f_1\|_{L^{\vec{p}}}^{\frac{2\beta}{1-\beta}} + \beta(\|\nabla\mathbf{W}^+\|_{L^2}^{\beta}\|\nabla\mathbf{W}^{-\beta}\|_{L^2})^{\frac{1}{\beta}} \\
&\leq C(\|f_1\|_{L^{\vec{p}}}^{q} + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^-\|_{L^2}^2),
\end{aligned} \tag{4.10}$$

where $q = \frac{2\beta}{1-\beta} = \frac{4}{1-2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}$.

Concluding (4.8)–(4.10), we get the estimates for K_1':

$$K_1' \leq \varepsilon\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + C(\|f_1\|_{L^{\vec{p}}}^{q} + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^-\|_{L^2}^2)(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4). \tag{4.11}$$

The estimate of K_1'' can be obtained in a similar way:

$$
\begin{aligned}
K_1'' &\le C\|g_1\|_{L^\infty} \left\||\mathbf{W}^+|^2\right\|_{L^2} \|\mathbf{W}^+\|_{L^4}^{\frac{1}{2}}\|\mathbf{W}^-\|_{L^4}^{\frac{1}{2}}\|\nabla\mathbf{W}^+\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{W}^-\|_{L^2}^{\frac{1}{2}} \left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}^{\frac{1}{2}} \\
&\le C\|g_1\|_{L^\infty}\|\mathbf{W}^+\|_{L^4}^{\frac{5}{2}}\|\mathbf{W}^-\|_{L^4}^{\frac{1}{2}}\|\nabla\mathbf{W}^+\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{W}^-\|_{L^2}^{\frac{1}{2}} \left\|\nabla|\mathbf{W}^+|^2\right\|_{L^2}^{\frac{1}{2}} \\
&\le \varepsilon\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + C\|g_1\|_{L^\infty}^{\frac{4}{3}}\|\mathbf{W}^+\|_{L^4}^{\frac{10}{3}}\|\mathbf{W}^-\|_{L^4}^{\frac{2}{3}}\|\nabla\mathbf{W}^+\|_{L^2}^{\frac{2}{3}}\|\nabla\mathbf{W}^-\|_{L^2}^{\frac{2}{3}} \\
&\le \varepsilon\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + C(\|g_1\|_{L^\infty}^4 + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^-\|_{L^2}^2)(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4).
\end{aligned}
\tag{4.12}
$$

Thus, we get

$$
\begin{aligned}
K_1 &\le 2\varepsilon\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + C(\|f_1\|_{L^{\tilde{p}}}^q + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^-\|_{L^2}^2)(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4) \\
&\quad + C(\|g_1\|_{L^\infty}^4 + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^-\|_{L^2}^2)(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4).
\end{aligned}
\tag{4.13}
$$

Following a similar argument, we get the estimate for K_2:

$$
\begin{aligned}
K_2 &\le 2\varepsilon\|\nabla|\mathbf{W}^-|^2\|_{L^2}^2 + C(\|f_2\|_{L^{\tilde{r}}}^s + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^-\|_{L^2}^2)(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4) \\
&\quad + C(\|g_2\|_{L^\infty}^4 + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^-\|_{L^2}^2)(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4).
\end{aligned}
\tag{4.14}
$$

Substitute (4.13) and (4.14) into (4.4), we have

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4) + \||\mathbf{W}^+||\nabla\mathbf{W}^+|\|_{L^2}^2 + \||\mathbf{W}^-||\nabla\mathbf{W}^-|\|_{L^2}^2 \\
&\quad + \frac{1}{2}(\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + \|\nabla|\mathbf{W}^-|^2\|_{L^2}^2) \\
\le& C(\|f_1\|_{L^{\tilde{p}}}^q + \|f_2\|_{L^{\tilde{r}}}^s + \sum_{i=1}^2 \|g_i\|_{L^\infty}^4 + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^+\|_{L^2}^2)(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4) \\
&\quad + 2\varepsilon(\|\nabla|\mathbf{W}^+|^2\|_{L^2}^2 + \|\nabla|\mathbf{W}^-|^2\|_{L^2}^2).
\end{aligned}
$$

Choosing ε small and applying Gronwall's inequality, we see that

$$
\begin{aligned}
&\sup_{0\le t\le T}(\|\mathbf{W}^+\|_{L^4}^4 + \|\mathbf{W}^-\|_{L^4}^4) \\
\le& (\|\mathbf{W}^+(0)\|_{L^4}^4 + \|\mathbf{W}^-(0)\|_{L^4}^4) \\
&\times \exp\left\{C\int_0^T (\|f_1\|_{L^{\tilde{p}}}^q + \|f_2\|_{L^{\tilde{r}}}^s + \sum_{i=1}^2 \|g_i\|_{L^\infty}^4 + \|\nabla\mathbf{W}^+\|_{L^2}^2 + \|\nabla\mathbf{W}^+\|_{L^2}^2)\,dt\right\},
\end{aligned}
$$

which implies that

$$
\mathbf{u}, \mathbf{B} \in L^\infty\left(0, T; L^4\left(\mathbb{R}^3\right)\right) \subset L^8\left(0, T; L^4\left(\mathbb{R}^3\right)\right),
$$

combining this with the Serrin-type regularity criteria for MHD system (see for example [4, Theorem 4]), we complete the proof of Theorem 2.3.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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