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Research article

Induced 3-Hom-Lie superalgebras

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Abstract: We construct 3-Hom-Lie superalgebras on a commutative Hom-superalgebra by means of involution and even degree derivation. We construct a representation of induced 3-Hom-Lie superalgebras by means of supertrace.

Keywords: induced 3-Hom-Lie superalgebra; derivation; induced representation; involution

1. Introduction

Ternary Lie algebras appeared first in Nambu's generalization of Hamiltonian mechanics [1] which use a generalization of Poisson algebras with a ternary bracket. The structure of n-Lie algebra was studied by Filippov [2]. The theory of cohomology for first-class n-Lie superalgebras can be found in [3]. In [4], the structure and cohomology of 3-Lie algebras induced by Lie algebras have been investigated. The reference [5] constructed super 3-Lie algebras by super Lie algebras. In [6], generalizations of n-ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-ary Nambu-Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type including n-ary totally associative and n-ary partially associative algebras. In [7], a method was demonstrated of how to construct ternary multiplications from the binary multiplication of a Hom-Lie algebra, a linear twisting map, and a trace function satisfying certain compatibility conditions; and it was shown that this method can be used to construct ternary Hom-Nambu-Lie algebras from Hom-Lie algebras. This construction was generalized to n-Lie algebras and n-Hom-Nambu-Lie algebras in [8]. The reference [9] constructed $(n+1)$-Hom-Lie algebras by n-Hom-Lie algebras. A method of constructing induced ternary Lie brackets were proposed in [10]. This method in a more general form can be described as follows: Given a Lie algebra $\mathfrak{g}$ and a generalized trace $\tau : \mathfrak{g} \to C$ on it, one can define the induced ternary Lie bracket by the formula

$$[x, y, z] \quad = \quad \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y], \tag{1.1}$$

where $x, y \in \mathfrak{g}$ and $[\cdot, \cdot, \cdot]$ is a Lie bracket of $\mathfrak{g}$. The structure of induced n-Lie algebras with n-Lie brackets constructed by means of a generalized trace, their cohomologies and Hom-generalizations were studied in the papers [4,8,9]. An extension of the method of constructing induced 3-Lie-algebras to 3-Lie-superalgebras by means of a generalized supertrace and possible application of this method in BRST-formalism of quantum field theory was proposed in the papers [11,12]. Later the method of constructing 3-Lie-superalgebras by means of supertrace proposed in [11,12] was extended to ternary Hom-Lie superalgebras by By me and my advisor in [13].

The reference [14] constructed 3-Lie superalgebras induced by means of derivation and involution, and it also constructed induced representation of induced 3-Lie superalgebras by means of supertrace. The paper generalizes the reference [14] to the case of 3-Hom-Lie superalgebras.

In Section 2, we construct induced 3-Hom-Lie superalgebras, whose ternary graded Hom-Lie brackets have the structure similar to (1.1). First of all, we propose two identities, which give sufficient and necessary conditions for graded skew-symmetric Hom-ternary bracket to satisfy the graded Filippov-Jacobi identity, in other words, to determine a 3-Hom-Lie superalgebra. Next we construct binary graded Hom-Lie brackets and then ternary graded Hom-Lie brackets on a commutative Hom-superalgebra with involution, where the structure of a ternary graded Hom-Lie bracket is similar to (1.1) and it is constructed by means of even degree derivation and involution. In Section 3, we construct induced representation of induced 3-Hom-Lie superalgebras by means of supertrace and prove that this is a representation of induced 3-Hom-Lie superalgebra.

2. 3-Hom-Lie superalgebras induced by means of derivation and involution

Definition 2.1. [15] *A Hom-superalgebra is a triple $(\mathfrak{g}, \mu, \alpha)$ in which $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space, $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is an even linear map, $\alpha : \mathfrak{g} \to \mathfrak{g}$ is an even linear map such that $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$.*

Definition 2.2. [16] *A 3-Hom-Lie superalgebra is a triple $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha_1, \alpha_2)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, an even trilinear map (bracket) $[\cdot, \cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and an algebra endomorphism $\alpha_i : \mathfrak{g} \to \mathfrak{g} (i = 1, 2)$ is an even linear map. If it satisfies the following conditions:*

$$\begin{aligned}
&|[x_1, x_2, x_3]| = |x_1| + |x_2| + |x_3|;\\
&[x_1, x_2, x_3] = -(-1)^{|x_1||x_2|}[x_2, x_1, x_3],\ [x_1, x_2, x_3] = -(-1)^{|x_2||x_3|}[x_1, x_3, x_2];\\
&[\alpha_1(x), \alpha_2(y), [z, u, v]] = [[x, y, z], \alpha_1(u), \alpha_2(v)]\\
&\ + (-1)^{|z|(|x|+|y|)}[\alpha_1(z), [x, y, u], \alpha_2(v)] + (-1)^{(|z|+|u|)(|x|+|y|)}[\alpha_1(z), \alpha_2(u), [x, y, v]].
\end{aligned}$$

where $|x|$ is the $\mathbb{Z}_2$-degree of the homogeneous element x in $\mathfrak{g}$.

Suppose that $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha_1, \alpha_2)$ is a 3-Hom-Lie superalgebra, if $\alpha_1 = \alpha_2 = \alpha$, it is satisfied

$$\alpha[x_1, x_2, x_3] = [\alpha(x_1), \alpha(x_2), \alpha(x_3)], \forall x_1, x_2, x_3 \in \mathfrak{g}.$$

Then $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\rho, \alpha_1, \alpha_2)$ is called as multiplicative.

In this section we consider a commutative Hom-superalgebra $(\mathfrak{g}, \mu, \alpha)$ in which $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ endowed with an involution $x \in \mathfrak{g} \mapsto x^\star \in \mathfrak{g}$ and an even degree derivation $\delta : \mathfrak{g} \to \mathfrak{g}$. By involution we mean an even degree linear mapping (even degree means that it preserves grading of any homogeneous element), which satisfies $(x^\star)^\star = x, x \in \mathfrak{g}$. By derivation of degree m, where m is an integer either $\bar{0}$ or $\bar{1}$, we mean linear mapping $\delta : \mathfrak{g} \to \mathfrak{g}$, which satisfies the graded Leibniz rule $\delta(uv) = \delta(u)\alpha(v) + (-1)^{m\widehat{uv}}\alpha(u)\delta(v)$, where $\widehat{x}, \widehat{y}$ are gradings of homogeneous elements x, y, respectively. Using an involution and derivation we construct three graded Hom-Lie brackets on Hom-superalgebra $\mathfrak{g}$. Furthermore, by considering a generalized supertrace we apply methods described in [5] and yield induced 3-Lie Hom-superalgebras whose bracket is defined using involution, derivation and both of them, together with generalized supertrace.

First of all, we start by proposing equivalent form to the graded Filippov-Jacobi identity. To simplify the equations, we will use the notation $\widehat{xy} = \widehat{x} + \widehat{y}$.

Proposition 2.3. *Assume $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a super vector space and let*

$$[\cdot, \cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{2.1}$$

be a skew-symmetric multilinear map, such that $\widehat{[x, y, z]} = \widehat{xyz}$. $\alpha_1 : \mathfrak{g} \to \mathfrak{g}$ and $\alpha_2 : \mathfrak{g} \to \mathfrak{g}$ are algebra endomorphisms. Then $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha_1, \alpha_2)$ is a 3-Lie Hom-superalgebra if and only if the equalities

$$\begin{aligned} [[x, y, z], \alpha_1(u), \alpha_2(v)] = {} & (-1)^{\widehat{u\widehat{xyz}}+\widehat{xyz}}[[u, y, z], \alpha_1(x), \alpha_2(v)] \\ & + (-1)^{\widehat{u\widehat{yz}}+\widehat{yz}}[[x, u, z], \alpha_1(y), \alpha_2(v)] + (-1)^{\widehat{uz}}[[x, y, u], \alpha_1(z), \alpha_2(v)] \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} & [[x, y, z], \alpha_1(u), \alpha_2(v)] + (-1)^{\widehat{xy}\widehat{zuv}+\widehat{zuv}}[[u, v, z], \alpha_1(x), \alpha_2(y)] \\ & - (-1)^{\widehat{yu}+\widehat{yz}+\widehat{zu}}[[x, u, z], \alpha_1(y), \alpha_2(v)] - (-1)^{\widehat{v\widehat{yzu}}+\widehat{x\widehat{yzu}}+\widehat{xv}}[[v, y, z], \alpha_1(u), \alpha_2(x)] \\ & - (-1)^{\widehat{yv}\widehat{zu}+\widehat{vy}}[[x, v, z], \alpha_1(u), \alpha_2(y)] - (-1)^{\widehat{xu}\widehat{yz}+\widehat{xu}}[[u, y, z], \alpha_1(x), \alpha_2(v)] = 0 \end{aligned} \tag{2.3}$$

hold for bracket $[\cdot, \cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Proof. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a super vector space and assume that $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha_1, \alpha_2)$ is a 3-Lie Hom-superalgebra. If this is the case, then the Filippov-Jacobi identity must hold:

$$\begin{aligned} [\alpha_1(x), \alpha_2(y), [z, u, v]] = {} & [[x, y, z], \alpha_1(u), \alpha_2(v)] + (-1)^{\widehat{xy}\widehat{z}}[\alpha_1(z), [x, y, u], \alpha_2(v)] \\ & + (-1)^{\widehat{xy}\widehat{zu}}[\alpha_1(z), \alpha_2(u), [x, y, v]]. \end{aligned} \tag{2.4}$$

To show that identity (2.2) holds for bracket (2.1), apply Filippov-Jacobi identity (2.4) recursively to itself on the right-most bracket. This yields us the following result:

$$\begin{aligned} [\alpha_1(x), \alpha_2(y), [z, u, v]] = {} & [[x, y, z], \alpha_1(u), \alpha_2(v)] + (-1)^{\widehat{xy}\widehat{z}}[\alpha_1(z), [x, y, u], \alpha_2(v)] \\ & + (-1)^{\widehat{xy}\widehat{zu}}([[z, u, x], \alpha_1(y), \alpha_2(v)] + (-1)^{\widehat{zu}\widehat{x}}[\alpha_1(x), [z, u, y], \alpha_2(v)] \\ & + (-1)^{\widehat{zu}\widehat{xy}}[\alpha_1(x), \alpha_2(y), [z, u, v]]) \\ = {} & [[x, y, z], \alpha_1(u), \alpha_2(v)] + (-1)^{\widehat{xy}\widehat{z}}[\alpha_1(z), [x, y, u], \alpha_2(v)] \\ & + (-1)^{\widehat{xy}\widehat{zu}}[[z, u, x], \alpha_1(y), \alpha_2(v)] + (-1)^{\widehat{xy}\widehat{zu}+\widehat{zu}\widehat{x}} \end{aligned}$$

$$[\alpha_1(x),[z,u,y],\alpha_2(v)]+(-1)^{\widehat{xy}\widehat{zu}+\widehat{zu}\widehat{xy}}[\alpha_1(x),\alpha_2(y),[z,u,v]],$$

which gives

$$\begin{aligned}[[x,y,z],\alpha_1(u),\alpha_2(v)]=&-(-1)^{\widehat{\widehat{xy}z}}[\alpha_1(z),[x,y,u],\alpha_2(v)]-(-1)^{\widehat{xy}\widehat{zu}}[[z,u,x],\alpha_1(y),\alpha_2(v)]\\&-(-1)^{\widehat{\widehat{zu}y}}[\alpha_1(x),[z,u,y],\alpha_2(v)].\end{aligned}\tag{2.5}$$

We need to show that the right hand sides of (2.2) and (2.5) coincide. It is indeed the case:

$$\begin{aligned}-(-1)^{\widehat{\widehat{zu}y}}[\alpha_1(x),[z,u,y],\alpha_2(v)]&=-(-1)^3(-1)^{\widehat{\widehat{zu}y}+\widehat{x\widehat{zu}y}+\widehat{uz}+\widehat{zy}}[[u,y,z],\alpha_1(x),\alpha_2(v)]\\&=(-1)^{\widehat{u\widehat{xy}z}+\widehat{\widehat{xy}z}}[[u,y,z],\alpha_1(x),\alpha_2(v)],\\-(-1)^{\widehat{xy}\widehat{zu}}[[z,u,x],\alpha_1(y),\alpha_2(v)]&=-(-1)^3(-1)^{\widehat{xy}\widehat{zu}+\widehat{zu}+\widehat{zx}+\widehat{xu}}[[x,u,z],\alpha_1(y),\alpha_2(v)]\\&=(-1)^{\widehat{\widehat{uy}z}+\widehat{yz}}[[x,u,z],\alpha_1(y),\alpha_2(v)],\\-(-1)^{\widehat{\widehat{xy}z}}[\alpha_1(z),[x,y,u],\alpha_2(v)]&=-(-1)(-1)^{\widehat{\widehat{xy}z}+\widehat{z\widehat{xy}u}}[[x,y,u],\alpha_1(z),\alpha_2(v)]\\&=(-1)^{\widehat{uz}}[[x,y,u],\alpha_1(z),\alpha_2(v)].\end{aligned}$$

From those equalities we can deduce that if $(\mathfrak{g},[\cdot,\cdot,\cdot],\alpha_1,\alpha_2)$ is indeed a 3-Lie Hom-superalgebra, then identity (2.2) holds for bracket $[\cdot,\cdot,\cdot]$.

In order to prove the identity (2.3), we can apply Filippov-Jacobi identity recursively to itself once again, but this time to both the right-most and middle brackets on the right hand side of identity (2.4).

$$\begin{aligned}&[\alpha_1(x),\alpha_2(y),[z,u,v]]\\=&[[x,y,z],\alpha_1(u),\alpha_2(v)]-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}}[\alpha_1(z),\alpha_2(v),[x,y,u]]\\&+(-1)^{\widehat{xy}\widehat{zu}}[\alpha_1(z),\alpha_2(u),[x,y,v]]\\=&[[x,y,z],\alpha_1(u),\alpha_2(v)]-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}}([[z,v,x],\alpha_1(y),\alpha_2(u)]\\&+(-1)^{\widehat{\widehat{zv}x}}[\alpha_1(x),[z,v,y],\alpha_2(u)]+(-1)^{\widehat{zv}\widehat{xy}}[\alpha_1(x),\alpha_2(y),[z,v,u]])\\&+(-1)^{\widehat{xy}\widehat{zu}}([[z,u,x],\alpha_1(y),\alpha_2(v)]+(-1)^{\widehat{\widehat{zu}x}}[\alpha_1(x),[z,u,y],\alpha_2(v)]\\&+(-1)^{\widehat{zu}\widehat{xy}}[\alpha_1(x),\alpha_2(y),[z,u,v]])\\=&[[x,y,z],\alpha_1(u),\alpha_2(v)]-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}}([[z,v,x],\alpha_1(y),\alpha_2(u)]\\&-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}+\widehat{\widehat{zv}x}}[\alpha_1(x),[z,v,y],\alpha_2(u)]-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}+\widehat{zv}\widehat{xy}}[\alpha_1(x),\alpha_2(y),[z,v,u]]\\&+(-1)^{\widehat{xy}\widehat{zu}}[[z,u,x],\alpha_1(y),\alpha_2(v)]+(-1)^{\widehat{xy}\widehat{zu}+\widehat{\widehat{zu}x}}[\alpha_1(x),[z,u,y],\alpha_2(v)]\\&+(-1)^{\widehat{xy}\widehat{zu}+\widehat{zu}\widehat{xy}}[\alpha_1(x),\alpha_2(y),[z,u,v]].\end{aligned}$$

Reordering the summands in the last equation results in

$$\begin{aligned}&[[x,y,z],\alpha_1(u),\alpha_2(v)]-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}+\widehat{zv}\widehat{xy}}[\alpha_1(x),\alpha_2(y),[z,v,u]]\\&+(-1)^{\widehat{xy}\widehat{zu}}[[z,u,x],\alpha_1(y),\alpha_2(v)]-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}+\widehat{\widehat{zv}x}}[\alpha_1(x),[z,v,y],\alpha_2(u)]\\&-(-1)^{\widehat{\widehat{xy}z}+\widehat{v\widehat{xy}u}}[[z,v,x],\alpha_1(y),\alpha_2(u)]+(-1)^{\widehat{xy}\widehat{zu}+\widehat{\widehat{zu}x}}[\alpha_1(x),[z,u,y],\alpha_2(v)]=0,\end{aligned}$$

which is exactly (2.3) once the elements are ordered accordingly within the brackets:

$$
\begin{aligned}
-(-1)^{\widehat{xyz}+\widehat{vxyu}+\widehat{zv}\widehat{xy}}[\alpha_1(x),\alpha_2(y),[z,v,u]] &= +\,(-1)^{\widehat{xy}\widehat{zuv}+\widehat{zuv}}[[u,v,z],\alpha_1(x),\alpha_2(y)]\\
+(-1)^{\widehat{xy}\widehat{zu}}[[z,u,x],\alpha_1(y),\alpha_2(v)] &= -\,(-1)^{\widehat{yu}+\widehat{yz}+\widehat{zu}}[[x,u,z],\alpha_1(y),\alpha_2(v)]\\
-(-1)^{\widehat{xyz}+\widehat{vxyu}+\widehat{zvx}}[\alpha_1(x),[z,v,y],\alpha_2(u)] &= -\,(-1)^{\widehat{vyzu}+\widehat{xyzu}+\widehat{xv}}[[v,y,z],\alpha_1(u),\alpha_2(x)]\\
-(-1)^{\widehat{xyz}+\widehat{vxyu}}[[z,v,x],\alpha_1(y),\alpha_2(u)] &= -\,(-1)^{\widehat{yv}\widehat{zu}+\widehat{vy}}[[x,v,z],\alpha_1(u),\alpha_2(y)]\\
+(-1)^{\widehat{xy}\widehat{zu}+\widehat{zu}\widehat{x}}[\alpha_1(x),[z,u,y],\alpha_2(v)] &= -\,(-1)^{\widehat{xu}\widehat{yz}+\widehat{xu}}[[u,y,z],\alpha_1(x),\alpha_2(v)].
\end{aligned}
$$

This completes the proof of necessity. Sufficiency can be shown analogously. □

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Hom-superalgebra. We say that superalgebra $\mathfrak{g}$ is commutative Hom-superalgebra if for any two homogeneous elements $u, v \in \mathfrak{g}$ it holds that $uv = (-1)^{\widehat{uv}}$. Let $m \in \mathbb{Z}_2$. linear mapping $\delta : \mathfrak{g} \to \mathfrak{g}$ is said to be a degree m derivation of a superalgebra $\mathfrak{g}$ if $\widehat{\delta(u)} = \widehat{u} + m$, for all $u \in \mathfrak{g}$, and it satisfies the graded leibniz rule

$$\delta(uv) = \delta(u)\alpha(v) + (-1)^{m\widehat{uv}}\alpha(u)\delta(v).$$

In case $m = 0$, we call it even degree derivation, and otherwise, for $m = 1$, we call it odd degree derivation. We denote the degree of δ as $\widehat{\delta}$. Consequently, if δ is an even degree derivation of Hom-superalgebra $\mathfrak{g}$, then $\widehat{\delta(u)} = \widehat{u}$. Or, in other words, derivation δ does not change the degree of homogeneous element $u \in \mathfrak{g}$. Furthermore, in case of even δ the Leibniz rule for any $u, v \in \mathfrak{g}$ simplifies to

$$\delta(uv) = \delta(u)\alpha(v) + \alpha(u)\delta(v).$$

Mapping $(\cdot)^\star : \mathfrak{g} \to \mathfrak{g}, u \mapsto u^\star$ is said to be an involution of a Hom-superalgebra $\mathfrak{g}$ if it satisfies the following conditions:

(1) involution is an even degree mapping of a Hom-superalgebra $\mathfrak{g}$, $\mathfrak{g}_i^\star \subset \mathfrak{g}_i, i \in \mathbb{Z}_2, \widehat{u^\star} = \widehat{u}$;

(2) it is linear, $(\lambda u + \mu v)^\star = \lambda u^\star + \mu v^\star$;

(3) $(\cdot)^\star : \mathfrak{g} \to \mathfrak{g}$ is its own inverse, $(u^\star)^\star = u$;

(4) $(uv)^\star = (-1)^{\widehat{uv}} v^\star u^\star$;

(5) $(\alpha(u))^\star = \alpha(u^\star)$.

In the case of commutative Hom-superalgebra the condition (4) takes on the form $(uv)^\star = u^\star v^\star$.

Making use of involution and even degree derivation we can construct graded Hom-Lie brackets on a Hom-superalgebra $\mathfrak{g}$. To achieve that, let us define

$$
\begin{aligned}
[u,v]_\star &= u^\star\alpha(v) - (-1)^{\widehat{uv}} v^\star\alpha(u); &(2.6)\\
[u,v]_\delta &= \alpha(u)\delta(v) - (-1)^{\widehat{uv}}\alpha(v)\delta(u); &(2.7)\\
[u,v]_{\star,\delta} &= (\alpha(u) - \alpha(u^\star))\delta(v) - (-1)^{\widehat{uv}}(\alpha(v) - \alpha(v^\star))\delta(u). &(2.8)
\end{aligned}
$$

Proposition 2.4. *Brackets (2.6) and (2.7) are graded Hom-Lie brackets. If involution* $(\cdot)^\star : \mathfrak{g} \to \mathfrak{g}$ *and an even degree derivation* $\delta : \mathfrak{g} \to \mathfrak{g}$ *satisfy the condition*

$$(\delta(u))^\star = -\delta(u^\star),$$

then bracket (2.8) is also a graded Hom-Lie bracket.

Proof. Proving the proposition is similar for all three brackets (2.6)–(2.8). Let us only observe (2.8), which is the most involved. To assure linearity, pick coefficients $\lambda, \mu \in \mathbb{K}$ and homogeneous elements $u, v, w \in \mathfrak{g}$ such that $\widehat{v} = \widehat{w}$. Note that $\widehat{\omega} = \widehat{\lambda v + \mu w} = \widehat{v} = \widehat{w}$ as both v and w are homogeneous, and calculate

$$\begin{aligned}
[u, \lambda v + \mu w]_{\star,\delta} =&(\alpha(u) - \alpha(u^\star))\delta(\lambda v + \mu w) - (-1)^{\widehat{uw}}(\alpha(\lambda v + \mu w) - \alpha(\lambda v + \mu w)^\star)\delta(u)\\
=&\lambda(\alpha(u) - \alpha(u^\star))\delta(v) + \mu(\alpha(u) - \alpha(u^\star))\delta(w)\\
&- (-1)^{\widehat{uw}}(\lambda(\alpha(v) - \alpha(v^\star)) + \mu(\alpha(w) - \alpha(w^\star)))\delta(u)\\
=&\lambda(\alpha(u) - \alpha(u^\star))\delta(v) + \mu(\alpha(u) - \alpha(u^\star))\delta(w)\\
&- (-1)^{\widehat{uw}}\lambda(\alpha(v) - \alpha(v^\star))\delta(u) - (-1)^{\widehat{uw}}\mu(\alpha(w) - \alpha(w^\star))\delta(u)\\
=&\lambda\{(\alpha(u) - \alpha(u^\star))\delta(v) - (-1)^{\widehat{uw}}(\alpha(v) - \alpha(v^\star))\delta(u)\}\\
&+ \mu\{(\alpha(u) - \alpha(u^\star))\delta(w) - (-1)^{\widehat{uw}}(\alpha(w) - \alpha(w^\star))\delta(u)\}\\
=&\lambda[u, v]_{\star,\delta} + \mu[u, w]_{\star,\delta}.
\end{aligned}$$

Antisymmetry is a result of direct computation

$$\begin{aligned}
[u, v]_{\star,\delta} =&(\alpha(u) - \alpha(u^\star))\delta(v) - (-1)^{\widehat{uv}}(\alpha(v) - \alpha(v)^\star)\delta(u)\\
=& - (-1)^{\widehat{uv}}\{(\alpha(v) - \alpha(v^\star))\delta(u) - (-1)^{\widehat{uv}}(\alpha(u) - \alpha(u)^\star)\delta(v)\}\\
=& - (-1)^{\widehat{uv}}[v, u]_{\star,\delta}.
\end{aligned}$$

In order to complete the proof, we still need to show that bracket defined by (2.8) satisfies the Hom-Jacobi identity. To achieve that, first observe that due to the commutativity of $\mathfrak{g}$ we can write bracket $[\cdot, \cdot]_{\star,\delta}$ as

$$[u, v]_{\star,\delta} = \alpha(u)\delta(v) - \alpha(u^\star)\delta(v) - \delta(u)\alpha(v) + \delta(u)\alpha(v^\star).$$

Furthermore, if δ is even and $(\delta(u))^\star = -\delta(u^\star)$, then we can write

$$\begin{aligned}
[\alpha(u), \alpha(v)\delta(w)]_{\star,\delta} =&\alpha^2(u)\delta(\alpha(v))\alpha\delta(w) + \alpha^2(u)\alpha^2(v)\delta^2(w) - \alpha^2(u^\star)\delta(\alpha(v))\alpha\delta(w)\\
&- \alpha^2(u^\star)\alpha^2(v)\delta^2(w) - \delta\alpha(u)\alpha^2(v)\alpha\delta(w) - \delta\alpha(u)\alpha^2(v^\star)\alpha\delta(w^\star).
\end{aligned}$$

$$\begin{aligned}
[\alpha(u), \delta(v)\alpha(w)]_{\star,\delta} =&\alpha^2(u)\delta^2(v)\alpha^2(w) + \alpha^2(u)\alpha\delta(v)\delta\alpha(w) - \alpha^2(u^\star)\delta^2(v)\alpha^2(w)\\
&- \alpha^2(u^\star)\alpha\delta(v)\delta\alpha(w) - \delta\alpha(u)\alpha\delta(v)\alpha^2(w) - \delta\alpha(u)\alpha\delta(v^\star)\alpha^2(w^\star).
\end{aligned}$$

Using the results above we can now calculate $[\alpha(u), [v, w]_{\star,\delta}]_{\star,\delta}$:

$$
\begin{aligned}
[\alpha(u), [v, w]_{\star,\delta}]_{\star,\delta} =& [\alpha(u), \alpha(v)\delta(w)]_{\star,\delta} - [\alpha(u), \alpha(v^\star)\delta(w)]_{\star,\delta} \\
& - [\alpha(u), \delta(v)\alpha(w)]_{\star,\delta} + [\alpha(u), \delta(v)\alpha(w^\star)]_{\star,\delta} \\
=& \alpha^2(u)\delta(\alpha(v))\alpha\delta(w) + \alpha^2(u)\alpha^2(v)\delta^2(w) - \alpha^2(u^\star)\delta(\alpha(v))\alpha\delta(w) \\
& - \alpha^2(u^\star)\alpha^2(v)\delta^2(w) - \delta\alpha(u)\alpha^2(v)\alpha\delta(w) - \delta\alpha(u)\alpha^2(v^\star)\alpha\delta(w^\star) \\
& - \alpha^2(u)\delta(\alpha(v^\star))\alpha\delta(w) - \alpha^2(u)\alpha^2(v^\star)\delta^2(w) + \alpha^2(u^\star)\delta(\alpha(v^\star))\alpha\delta(w) \\
& + \alpha^2(u^\star)\alpha^2(v^\star)\delta^2(w) + \delta\alpha(u)\alpha^2(v^\star)\alpha\delta(w) + \delta\alpha(u)\alpha^2(v)\alpha\delta(w^\star) \\
& - \alpha^2(u)\delta^2(v)\alpha^2(w) - \alpha^2(u)\alpha\delta(v)\delta\alpha(w) + \alpha^2(u^\star)\delta^2(v)\alpha^2(w) \\
& + \alpha^2(u^\star)\alpha\delta(v)\delta\alpha(w) + \delta\alpha(u)\alpha\delta(v)\alpha^2(w) + \delta\alpha(u)\alpha\delta(v^\star)\alpha^2(w^\star) \\
& + \alpha^2(u)\delta^2(v)\alpha^2(w^\star) + \alpha^2(u)\alpha\delta(v)\delta\alpha(w^\star) - \alpha^2(u^\star)\delta^2(v)\alpha^2(w^\star) \\
& - \alpha^2(u^\star)\alpha\delta(v)\delta\alpha(w^\star) - \delta\alpha(u)\alpha\delta(v)\alpha^2(w^\star) - \delta\alpha(u)\alpha\delta(v^\star)\alpha^2(w).
\end{aligned}
\tag{2.9}
$$

As a next step we can apply (2.9) also to $[\alpha(v), [w, u]_{\star,\delta}]_{\star,\delta}$ and $[\alpha(w), [u, v]_{\star,\delta}]_{\star,\delta}$, yielding all elements on the left hand side of Jacobi identity.

$$
[\alpha(u), [v, w]_{\star,\delta}]_{\star,\delta} + (-1)^{\widehat{uvw}}[\alpha(v), [w, u]_{\star,\delta}]_{\star,\delta} + (-1)^{\widehat{wuv}}[\alpha(w), [u, v]_{\star,\delta}]_{\star,\delta} =
$$

$$
\left.
\begin{array}{c}
+\alpha^2(u)\delta(\alpha(v))\alpha\delta(w) + \alpha^2(u)\alpha^2(v)\delta^2(w) \\
-\alpha^2(u^\star)\delta(\alpha(v))\alpha\delta(w) - \alpha^2(u^\star)\alpha^2(v)\delta^2(w) \\
-\delta\alpha(u)\alpha^2(v)\alpha\delta(w) - \delta\alpha(u)\alpha^2(v^\star)\alpha\delta(w^\star) \\
-\alpha^2(u)\delta(\alpha(v^\star))\alpha\delta(w) - \alpha^2(u)\alpha^2(v^\star)\delta^2(w) \\
+\alpha^2(u^\star)\delta(\alpha(v^\star))\alpha\delta(w) + \alpha^2(u^\star)\alpha^2(v^\star)\delta^2(w) \\
+\delta\alpha(u)\alpha^2(v^\star)\alpha\delta(w) + \delta\alpha(u)\alpha^2(v)\alpha\delta(w^\star) \\
-\alpha^2(u)\delta^2(v)\alpha^2(w) - \alpha^2(u)\alpha\delta(v)\delta\alpha(w) \\
+\alpha^2(u^\star)\delta^2(v)\alpha^2(w) + \alpha^2(u^\star)\alpha\delta(v)\delta\alpha(w) \\
+\delta\alpha(u)\alpha\delta(v)\alpha^2(w) + \delta\alpha(u)\alpha\delta(v^\star)\alpha^2(w^\star) \\
+\alpha^2(u)\delta^2(v)\alpha^2(w^\star) + \alpha^2(u)\alpha\delta(v)\delta\alpha(w^\star) \\
-\alpha^2(u^\star)\delta^2(v)\alpha^2(w^\star) - \alpha^2(u^\star)\alpha\delta(v)\delta\alpha(w^\star) \\
-\delta\alpha(u)\alpha\delta(v)\alpha^2(w^\star) - \delta\alpha(u)\alpha\delta(v^\star)\alpha^2(w)
\end{array}
\right\} [\alpha(u), [v, w]_{\star,\delta}]_{\star,\delta}
$$

$$
\left.
\begin{array}{c}
+\alpha\delta(u)\alpha^2(v)\delta(\alpha(w)) + \delta^2(u)\alpha^2(v)\alpha^2(w) \\
-\alpha\delta(u)\alpha^2(v^\star)\delta(\alpha(w)) - \delta^2(u)\alpha^2(v^\star)\alpha^2(w) \\
-\alpha\delta(u)\delta\alpha(v)\alpha^2(w) - \alpha\delta(u^\star)\delta\alpha(v)\alpha^2(w^\star) \\
-\alpha\delta(u)\alpha^2(v)\delta(\alpha(w^\star)) - \delta^2(u)\alpha^2(v)\alpha^2(w^\star) \\
+\alpha\delta(u)\alpha^2(v^\star)\delta(\alpha(w^\star)) + \delta^2(u)\alpha^2(v^\star)\alpha^2(w^\star) \\
+\alpha\delta(u)\delta\alpha(v)\alpha^2(w^\star) + \alpha\delta(u^\star)\delta\alpha(v)\alpha^2(w) \\
-\alpha^2(u)\alpha^2(v)\delta^2(w) - \delta\alpha(u)\alpha^2(v)\alpha\delta(w) \\
+\alpha^2(u)\alpha^2(v^\star)\delta^2(w) + \delta\alpha(u)\alpha^2(v^\star)\alpha\delta(w) \\
+\alpha^2(u)\delta\alpha(v)\alpha\delta(w) + \alpha^2(u^\star)\delta\alpha(v)\alpha\delta(w^\star) \\
+\alpha^2(u^\star)\alpha^2(v)\delta^2(w) + \delta\alpha(u^\star)\alpha^2(v)\alpha\delta(w) \\
-\alpha^2(u^\star)\alpha^2(v^\star)\delta^2(w) - \delta\alpha(u^\star)\alpha^2(v^\star)\alpha\delta(w) \\
-\alpha^2(u^\star)\delta\alpha(v)\alpha\delta(w) - \alpha^2(u)\delta\alpha(v)\alpha\delta(w^\star)
\end{array}
\right\} (-1)^{\widehat{uvw}}[\alpha(v), [w, u]_{\star,\delta}]_{\star,\delta}
$$

$$\left.\begin{array}{c}
+\delta(\alpha(u))\alpha\delta(v)\alpha^2(w)+\alpha^2(u)\delta^2(v)\alpha^2(w)\\
-\delta\alpha(u)\alpha\delta(v)\alpha^2(w^\star)-\alpha^2(u)\delta^2(v)\alpha^2(w^\star)\\
-\alpha^2(u)\alpha\delta(v)\delta\alpha(w)-\alpha^2(u^\star)\alpha\delta(v^\star)\delta\alpha(w)\\
-\delta(\alpha(u^\star))\alpha\delta(v)\alpha^2(w)-\alpha^2(u^\star)\delta^2(v)\alpha^2(w)\\
+\delta(\alpha(u^\star))\alpha\delta(v)\alpha^2(w^\star)+\alpha^2(u^\star)\delta^2(v)\alpha^2(w^\star)\\
+\alpha^2(u^\star)\alpha\delta(v)\delta\alpha(w)+\alpha^2(u)\alpha\delta(v^\star)\delta\alpha(w)\\
-\delta^2(u)\alpha^2(v)\alpha^2(w)-\alpha\delta(u)\delta\alpha(v)\alpha^2(w)+\\
\delta^2(u)\alpha^2(v)\alpha^2(w^\star)+\alpha\delta(u)\delta\alpha(v)\alpha^2(w^\star)\\
+\alpha\delta(u)\alpha^2(v)\delta\alpha(w)+\alpha\delta(u^\star)\alpha^2(v^\star)\delta\alpha(w)\\
+\delta^2(u)\alpha^2(v^\star)\alpha^2(w)+\alpha\delta(u)\delta\alpha(v^\star)\alpha^2(w)\\
-\delta^2(u)\alpha^2(v^\star)\alpha^2(w^\star)-\alpha\delta(u)\delta\alpha(v^\star)\alpha^2(w^\star)\\
-\alpha\delta(u)\alpha^2(v^\star)\delta\alpha(w)-\alpha\delta(u^\star)\alpha^2(v)\delta\alpha(w)
\end{array}\right\}(-1)^{\widehat{wuv}}[\alpha(w),[u,v]_{\star,\delta}]_{\star,\delta}$$

=0.

This means that Hom-Jacobi identity indeed holds, and $[\cdot,\cdot]_{\star,\delta}$ is a graded Hom-Lie bracket. □

What the proposition tells us is that graded Hom-Lie brackets (2.6), (2.7) naturally define Hom-Lie superalgebra structures $(\mathfrak{g},[u,v]_\delta,\alpha)$ and $(\mathfrak{g},[u,v]_\star,\beta)$ on commutative Hom-superalgebra $\mathfrak{g}$. In case we further assume that the derivation δ is even, and together with involution it satisfies the condition $(\delta(u))^\star=-\delta(u^\star)$, then graded Hom-Lie bracket (2.8) defines another Hom-Lie superalgebra structure $(\mathfrak{g},[u,v]_{\star,\delta},\gamma)$ on $\mathfrak{g}$.

Next let us consider the generalized supertraces on those Hom-Lie superalgebras:

$$\xi:\quad (\mathfrak{g},[u,v]_\star)\to\mathbb{K}, \tag{2.10}$$
$$\eta:\quad (\mathfrak{g},[u,v]_\delta)\to\mathbb{K}, \tag{2.11}$$
$$\chi:\quad (\mathfrak{g},[u,v]_{\star,\delta})\to\mathbb{K}. \tag{2.12}$$

With the help of those generalized supertraces we can induce ternary Hom-Lie superalgebras out of the binary Lie superalgebras, by following the construction described in [13].

Theorem 2.5. *Let $(\mathfrak{g},\mu,\alpha)$ in which $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ be a commutative Hom-superalgebra, and let $(\cdot)^\star:\mathfrak{g}\to\mathfrak{g}$ and $\delta:\mathfrak{g}\to\mathfrak{g}$ be involution and derivation of $\mathfrak{g}$, respectively. If (2.10) and (2.11) are supertraces, then graded ternary brackets $[\cdot,\cdot,\cdot]_\star:\mathfrak{g}^{\oplus 3}\to\mathfrak{g}$ and $[\cdot,\cdot,\cdot]_\delta:\mathfrak{g}^{\oplus 3}\to\mathfrak{g}$, defined by*

$$[u,v,w]_\star \;=\; \xi(u)[v,w]_\star+(-1)^{\widehat{uvw}}\xi(v)[w,u]_\star+(-1)^{\widehat{wuv}}\xi(w)[u,v]_\star, \tag{2.13}$$
$$[u,v,w]_\delta \;=\; \eta(u)[v,w]_\delta+(-1)^{\widehat{uvw}}\eta(v)[w,u]_\delta+(-1)^{\widehat{wuv}}\eta(w)[u,v]_\delta \tag{2.14}$$

are graded ternary Hom-Lie brackets. If δ is even, $(\delta(u))^\star=-\delta(u^\star)$ and (2.12) is a supertrace, then graded Hom-ternary bracket $[\cdot,\cdot,\cdot]_\star:\mathfrak{g}^{\oplus 3}\to\mathfrak{g}$, defined as

$$[u,v,w]_{\star,\delta} \;=\; \chi(u)[v,w]_{\star,\delta}+(-1)^{\widehat{uvw}}\chi(v)[w,u]_{\star,\delta}+(-1)^{\widehat{wuv}}\chi(w)[u,v]_{\star,\delta} \tag{2.15}$$

is ternary Hom-Lie bracket.

Consequences of the theorem are that each and every one of $(\mathfrak{g},[\cdot,\cdot,\cdot]_\delta,\alpha)$, $(\mathfrak{g},[\cdot,\cdot,\cdot]_\star,\beta)$ and $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\star,\delta},\gamma)$ are all ternary Hom-Lie superalgebras. Of course granted that for the latter the derivation and involution satisfy the required condition.

3. Induced representations of Induced 3-Hom-Lie superalgebras

In this section we consider 3-Hom-Lie superalgebras. It was shown [7] that the method of constructing an induced 3-Hom-Lie algebras with the help of a generalized trace can be extended to the case of Hom-Lie superalgebras by means of a concept of a generalized supertrace. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ in which $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Hom-Lie superalgebra. Then by generalized supertrace we mean a linear function $S\tau : \mathfrak{g} \to C$ such that it vanishes on graded Hom-Lie brachet of $\mathfrak{g}$, i.e., $S\tau([x,y]) = 0$, and it also vanishes when restricted to $\mathfrak{g}_{\bar{1}}$, i.e., $S\tau|_{\mathfrak{g}_{\bar{1}}} \equiv 0$.

Let $(\eta, [\cdot,\cdot,\cdot], \alpha)$ in which $\eta = \eta_{\bar{0}} \oplus \eta_{\bar{1}}$ be a 3-Hom-Lie superalgebra, $(V, [\cdot,\cdot], \alpha)$ in which $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space and End(V) be the super vector space of endomorphisms of V. The graded commutator of two endomorphisms $A, B \in$ End(V) of a super vector space V, defined by formula $[A,B] = AB - (-1)^{\widehat{A}\widehat{B}}BA$, where A, B are homogeneous endomorphisms and $\widehat{A}, \widehat{B}$ are their gradings, determines the structure of the Lie superalgebra on End(V) and we denote this Hom-Lie superalgebra by $\mathfrak{sgl}(V)$. There is a canonical structure of a super vector space on the tensor product $\eta \otimes \eta$, which is defined as follows:

$$\eta \otimes \eta = (\eta \otimes \eta)_{\bar{0}} \oplus (\eta \otimes \eta)_{\bar{1}},$$

where $(\eta \otimes \eta)_{\bar{0}} = (\eta_{\bar{0}} \otimes \eta_{\bar{0}}) \oplus (\eta_{\bar{1}} \otimes \eta_{\bar{1}})$ and $(\eta \otimes \eta)_{\bar{1}} = (\eta_{\bar{0}} \otimes \eta_{\bar{1}}) \oplus (\eta_{\bar{1}} \otimes \eta_{\bar{0}})$. In [5], the supertrace of an endomorphism $a : V \to V$ be defined by

$$\mathrm{str}(a) = \begin{cases} Tr(a|_{V_0}) - Tr(a|_{V_1}), & \text{if } a \text{ is even;} \\ 0, & \text{if } a \text{ is odd.} \end{cases}$$

Definition 3.1. *A representation of a multiplicative* 3*-Hom-Lie superalgebra* $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ *on a vector space* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *with respect to a linear autmorphism* $\beta(\beta \in \mathrm{GL}(V))$ *is a even linear map* $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{sgl}(V)$ *such that for all* $x, y, z, u, v \in \mathfrak{g}$. *We have*
(1) $\rho(x,y) = -(-1)^{\widehat{x}\widehat{y}}\rho(y,x)$;
(2) $\rho(\alpha(x), \alpha(y)) \circ \beta = \beta \circ \rho(x,y)$;
(3) $\rho(\alpha(x), \alpha(y)) \circ \rho(u,v) - (-1)^{\widehat{xy}\widehat{uv}}\rho(\alpha(u), \alpha(v)) \circ \rho(x,y) = \rho([x,y,u], \alpha(v)) \circ \beta + (-1)^{\widehat{u}\widehat{xy}}$
$\rho(\alpha(u), [x,y,v]) \circ \beta$;
(4) $\rho([x,y,z], \alpha(u)) \circ \beta = \rho(\alpha(x), \alpha(y)) \circ \rho(z,u) + (-1)^{\widehat{x}\widehat{yz}}\rho(\alpha(y), \alpha(z)) \circ \rho(x,u) + (-1)^{\widehat{z}\widehat{xy}}$
$\rho(\alpha(z), \alpha(x)) \circ \rho(y,u)$. *We will this representation of 3-Hom-Lie superalgebra* $\mathfrak{g}$ *in a super vector space* V *by* $(\mathfrak{g}, \rho, V, \beta)$.

An evident example of a representation of 3-Hom-Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ is an adjoint representation. Fix two elements x, y of a 3-Hom-Lie superalgebra $\mathfrak{g}$ and for any $u \in \mathfrak{g}$, define $\mathrm{ad}_{(x,y)}u = [x,y,u]$. Thus $\mathrm{ad} : \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, and $(\mathfrak{g}, \mathrm{ad}, \mathfrak{g}, \alpha)$ is a representation of multiplicative 3-Hom-Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ on $\mathfrak{g}$ with respect to α. In fact, conditions (1) and (2) of Definition 3.1 immediately follow from the properties of graded ternary Lie bracket. In order to prove condition (3) of Definition 3.1, we calculate the graded commutator of two linear operators $\mathrm{ad}_{(x,y)}$ and $\mathrm{ad}_{(u,v)}$ by means of graded Filippov-Jacobi identity. Then we have

$$\begin{aligned}
&(\mathrm{ad}_{(\alpha(x),\alpha(y))}\mathrm{ad}_{(u,v)} - (-1)^{\widehat{xy}\widehat{uv}}\mathrm{ad}_{(\alpha(u),\alpha(v))}\mathrm{ad}_{(x,y)})z \\
&= [\alpha(x), \alpha(y)), [u,v,z]] - (-1)^{\widehat{xy}\widehat{uv}}[\alpha(u), \alpha(v), [x,y,z]] \\
&= [[x,y,u], \alpha(v), \alpha(z)] + (-1)^{\widehat{u}\widehat{xy}}[\alpha(u), [x,y,v], \alpha(z)]
\end{aligned}$$

$$
\begin{aligned}
&+ (-1)^{\widehat{uv}\widehat{xy}}[\alpha(u), \alpha(v), [x, y, z]] - (-1)^{\widehat{xy}\widehat{uv}}[\alpha(u), \alpha(v), [x, y, z]]\\
&= [[x, y, u], \alpha(v), \alpha(z)] + (-1)^{\widehat{uxy}}[\alpha(u), [x, y, v], \alpha(z)]\\
&= \mathrm{ad}_{([x,y,u],\alpha(v))}\alpha(z) + (-1)^{\widehat{uxy}}\mathrm{ad}_{(\alpha(u),[x,y,v])}\alpha(z).
\end{aligned}
$$

The last property of Definition 3.1 can be checked as follows:

$$
\mathrm{ad}_{([x,y,z],\alpha(u))}\alpha(v) = [[x, y, z], \alpha(u), \alpha(v)] = (-1)^{\widehat{uv}\widehat{xyz}}[\alpha(u), \alpha(v), [x, y, z]]. \tag{3.1}
$$

Now making use of graded Filippov-Jacobi identity, we obtain

$$
\begin{aligned}
[\alpha(u), \alpha(v), [x, y, z]] =&[[u, v, x], \alpha(y), \alpha(z)] + (-1)^{\widehat{xuv}}[\alpha(x), [u, v, y], \alpha(z)]\\
&+ (-1)^{\widehat{uv}\widehat{xy}}[\alpha(x), \alpha(y), [u, v, z]].
\end{aligned}
$$

Now we should substitute this expression into the right hand side of formula (3.1), but first we will calculate the sign of each term in resulting expression. In the term $[\alpha(x), \alpha(y), [u, v, z]]$, we will do the following permutation of the arguments $[\alpha(x), \alpha(y), [z, u, v]]$, which will entail multiplicaion by $(-1)^{\widehat{zuv}}$. Therefore, the coefficient of this term will be (-1) in power

$$
\widehat{uv}\widehat{xyz} + \widehat{uv}\widehat{xy} + \widehat{zuv} = \widehat{uv}\widehat{xyz} + \widehat{uv}\widehat{xyz} = 0.
$$

Analogously, we permute the arguments of the double bracket $[[u, v, x], \alpha(y), \alpha(z)]$ as follows $[\alpha(y), \alpha(z), [x, u, v]]$ and this entails the appearance of the factor (-1) in power $\widehat{yz}\widehat{uvx} + \widehat{xuv}$. All together it gives the following sign

$$
\widehat{uv}\widehat{xyz} + \widehat{yz}\widehat{uvx} + \widehat{xuv} = \widehat{uv}\widehat{yz} + \widehat{yz}\widehat{uvx} = \widehat{yz}\widehat{x}.
$$

Similarly, we permute the arguments of the double bracket $[\alpha(x), [u, v, y], \alpha(z)]$ to cast it into the form $[\alpha(z), \alpha(x), [y, u, v]]$, then calculate the sign, which turns out be $\widehat{zxy}$. Hence, we get

$$
\begin{aligned}
\mathrm{ad}_{([x,y,z],\alpha(u))} \circ \alpha =&\mathrm{ad}_{(\alpha(x),\alpha(y))}\mathrm{ad}_{(z,u)} + (-1)^{\widehat{xyz}}\mathrm{ad}_{(\alpha(y),\alpha(z))}\mathrm{ad}_{(x,u)}\\
&+ (-1)^{\widehat{zxy}}\mathrm{ad}_{(\alpha(z),\alpha(x))}\mathrm{ad}_{(y,u)}.
\end{aligned}
$$

Now we assume that $\eta = \eta_{\bar 0} \oplus \eta_{\bar 1}$ is a 3-Hom-Lie superalgebra, $V = V_{\bar 0} \oplus V_{\bar 1}$ is a super vector space and $\rho : \eta \otimes \eta \to \mathrm{End}(V)$ is a graded skew-symmetric mapping. Consider the direct sum $\eta \oplus V$. We equip it with a structure of super vector space if we associate grade $\bar 0$ to elements $x + v$(elements of even grade), where $x \in \eta_{\bar 0}, v \in V_{\bar 0}$, and grade $\bar 1$ to elements $x + v$(elements of odd grade), where $x \in \eta_{\bar 1}, v \in V_{\bar 1}$. Then $\eta \oplus V = (\eta \oplus V)_{\bar 0} \oplus (\eta \oplus V)_{\bar 1}$, where $(\eta \oplus V)_{\bar 0} = \eta_{\bar 0} \oplus V_{\bar 0}$ and $(\eta \oplus V)_{\bar 1} = \eta_{\bar 1} \oplus V_{\bar 1}$. In analogy with representations of 3-Lie algebra [8], we define the ternary bracket on the super vector space $\eta \oplus V$ as follows:

$$
\begin{aligned}
[x_1 + v_1, x_2 + v_2, x_3 + v_3] =&[x_1, x_2, x_3] + \rho(x_1, x_2)v_3 + (-1)^{\widehat{x_1}\widehat{x_2 x_3}}\rho(x_2, x_3)v_1\\
&+ (-1)^{\widehat{x_3}\widehat{x_1 x_2}}\rho(x_3, x_1)v_2.
\end{aligned} \tag{3.2}
$$

It is easy to show that this ternary bracket is a graded ternary bracket. Indeed, if we assume that all arguments of this ternary bracket are homogenous elements $\eta \oplus V$, then the grading of $x_i + v_i$ is equal to the grading of x_i(or v_i). Thus it is sufficient to show that the grading of ternary bracket (3.2) is $\widehat{x_1}+\widehat{x_2}+\widehat{x_3}$. But this is true, because the grading of the first term $[x_1, x_2, x_3]$ is $\widehat{x_1}+\widehat{x_2}+\widehat{x_3}$ and the grading of the each term of the form $\rho(x_i, x_j)v_k$, where i, j, k is a cyclic permutation of $1, 2, 3$, is the same integer, because

$$\widehat{x_i}+\widehat{x_j}+\widehat{v_k}=\widehat{x_i}+\widehat{x_j}+\widehat{x_k}=\widehat{x_1}+\widehat{x_2}+\widehat{x_3}.$$

The fact that this ternary bracket has the correct graded symmetries is checked on the permutation of the first two arguments x_1+v_1, x_2+v_2. Making use of the graded symmetry properties of a graded ternary Hom-Lie bracket in η and the property (2) of Definition 3.1, we get

$$\begin{aligned}
[x_2+v_2, x_1+v_1, x_3+v_3] =&[x_2, x_1, x_3]+\rho(x_2, x_1)v_3+(-1)^{\widehat{x_2}\widehat{x_1x_3}}\rho(x_1, x_3)v_2\\
&+(-1)^{\widehat{x_3}\widehat{x_1x_2}}\rho(x_3, x_2)v_1\\
=&-(-1)^{\widehat{x_1}\widehat{x_2}}[x_1, x_2, x_3]-(-1)^{\widehat{x_1}\widehat{x_2}}\rho(x_1, x_2)v_3\\
&-(-1)^{\widehat{x_1}\widehat{x_3}}\rho(x_2, x_3)v_1-(-1)^{\widehat{x_1}\widehat{x_2}+\widehat{x_3}\widehat{x_1x_2}}\rho(x_3, x_1)v_2\\
=&-(-1)^{\widehat{x_1}\widehat{x_2}}[x_1+v_1, x_2+v_2, x_3+v_3].
\end{aligned}$$

We will need the following theorem or the induced representation, which will be discussed later in this paper.

Theorem 3.2. *Let $\eta=\eta_{\bar{0}}\oplus\eta_{\bar{1}}$ be a 3-Hom-Lie superalgebra, $V=V_{\bar{0}}\oplus V_{\bar{1}}$ be a super vector space, $\rho:\eta\times\eta\to \mathrm{gl}(V)$ be a graded skew-symmetric bilinear mapping. Then (η, ρ, V) is a representation of 3-Hom-Lie superalgebras η in a super vector space V if and only if the direct sum of super vector space $\eta\oplus V$ equipped with the graded ternary bracket (3.2) is a 3-Hom-Lie superalgebra, or, in the words, the graded ternary bracket (3.2) satisfies the graded Filippov-Jacobi identity.*

Proof. First of all, we prove that if (η, ρ, V) is a representation of a 3-Hom-Lie superalgebra η, then the graded ternary bracket (3.2) defines the structure of 3-Hom-Lie superalgebra on the direct sum of super vector spaces $\eta\oplus V$. Since we have already proved that the ternary bracket (3.2) is a graded ternary bracket, the only thing we need to prove is that this bracket satisfies the graded Filippov-Jacobi identity. To this end, we introduce the following notations:

$$Y=y+v, Z=z+w, X_i=x_i+u_i,$$

where $i=1, 2, 3, y, z, x_i\in\eta$ and $v, w, u_i\in V$. We assume that all elements Y, Z, X_i are homogeneous with respect to super vector space structure of $\eta\oplus V$. Evidently the grading of Y is equal to $\widehat{y}$, grading of Z is $\widehat{z}$ and the grading of X_i is $\widehat{x_i}$. Now our aim is to prove the graded Filippov-Jacobi identity for graded ternary bracket (3.2), that is, we need to show that the following expression

$$\begin{aligned}
&[\alpha(Y), \alpha(Z), [X_1, X_2, X_3]]-[[Y, Z, X_1], \alpha(X_2), \alpha(X_3)]]\\
&-(-1)^{\widehat{x_1}\widehat{yz}}[\alpha(X_1), [Y, Z, X_2], \alpha(X_3)]-(-1)^{\widehat{x_1x_2}\widehat{yz}}[\alpha(X_1), \alpha(X_2), [Y, Z, X_3]]
\end{aligned}\tag{3.3}$$

is equal to zero. If we expand each double ternary bracket in this expression by means of (3.2), then the η-component of resulting expression is

$$\begin{aligned}
&[\alpha(y), \alpha(z), [x_1, x_2, x_3]]-[[y, z, x_1], \alpha(x_2), \alpha(x_3)]]\\
&-(-1)^{\widehat{x_1}\widehat{yz}}[\alpha(x_1), [y, z, x_2], \alpha(x_3)]-(-1)^{\widehat{x_1x_2}\widehat{yz}}[\alpha(x_1), \alpha(x_2), [y, z, x_3]]
\end{aligned}\tag{3.4}$$

and this is zero by virtue of the graded Filippov-Jacobi identity in a 3-Lie superalgebra η. The V-component of the resulting expression can be written in the form

$$\Psi_1(u_1) + \Psi_2(u_2) + \Psi_3(u_3) + \Psi_v(v) + \Psi_w(w), \tag{3.5}$$

where $\Psi_1, \Psi_2, \Psi_3, \Psi_v, \Psi_w \in \mathfrak{gl}(V)$ and

$$\begin{aligned}
\Psi_1 =&(-1)^{\widehat{x_1}\widehat{x_2 x_3}}([\rho(y,z),\rho(x_2,x_3)] - \rho([y,z,x_2],x_3) - (-1)^{\widehat{x_2}\widehat{yz}}\rho(x_2,[y,z,x_3])),\\
\Psi_2 =&(-1)^{\widehat{x_3}\widehat{x_1 x_2}}([\rho(y,z),\rho(x_3,x_1)] - \rho([y,z,x_3],x_1) - (-1)^{\widehat{x_3}\widehat{yz}}\rho(x_3,[y,z,x_1])),\\
\Psi_3 =&[\rho(y,z),\rho(x_1,x_2)] - \rho([y,z,x_1],x_2) - (-1)^{\widehat{x_1}\widehat{yz}}\rho(x_1,[y,z,x_2]),\\
\Psi_v =&(-1)^{\alpha}(\rho([x_1,x_2,x_3],z) - \rho(x_1,x_2)\rho(x_3,z) - (-1)^{\widehat{x_1}\widehat{x_2 x_3}}\rho(x_2,x_3)\rho(x_1,z)\\
&- (-1)^{\widehat{x_3}\widehat{x_1 x_2}}\rho(x_3,x_1)\rho(x_2,z)),\\
\Psi_w =&(-1)^{\beta}(\rho([x_1,x_2,x_3],y) - \rho(x_1,x_2)\rho(x_3,y) - (-1)^{\widehat{x_1}\widehat{x_2 x_3}}\rho(x_2,x_3)\rho(x_1,y)\\
&- (-1)^{\widehat{x_3}\widehat{x_1 x_2}}\rho(x_3,x_1)\rho(x_2,y)),
\end{aligned}$$

where $\alpha = \widehat{yz}[\widehat{x_1,x_2,x_3}] + \widehat{yz} + 1, \beta = \widehat{yz}[\widehat{x_1,x_2,x_3}]$. Expressions Ψ_1, Ψ_2, Ψ_3 vanish by virtue of condition (3) of Definition 3.1 and expressions Ψ_v, Ψ_w by virtue of condition (4). Hence, the V-component of expression (3.3) also vanishes and this means that the graded ternary bracket (3.2) is a graded ternary Lie bracket, i.e. it satisfies the graded Filippov-Jacobi identity.

Now we prove that if the graded ternary bracket (3.2) satisfies the graded Filippov-Jacobi identity, then $(\mathfrak{g}, \rho, V, \alpha)$ is a representation of 3-Hom-Lie superalgebra η. By other words, we assume that the expression (3.3) vanishes. Vanishing of the η-component of this expression gives us nothing, because it reduces to the graded Filippov-Jacobi identity in η, which already holds according to our assumption that η is a 3-Hom-Lie superalgebra. From the equality to zero of the V-component, it immediately follows that the expression (3.5), where u_1, u_2, u_3, v, w are arbitrary vectors of V, is equal to zero. Taking $u_2 = u_3 = v = w = 0$, we get $\Psi_1(u_1) = 0$ for any u_1, which means that $\Psi_1 = 0$. Hence, condition (3) of Definition 3.1 is satisfied. Analogously we can prove that condition (4) is also satisfied and $(\mathfrak{g}, \rho, V, \alpha)$ is a representation of 3-Hom-Lie superalgebra. □

Recall that if a Hom-Lie algebra is equipped with a generalized trace, then one can construct the induced ternary Hom-Lie algebra (Section 3). This method of constructing the induced ternary Lie algebras can be extended by means of a generalized supertrace to Hom-Lie superalgebras, as was shown in [13]. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Hom-Lie superalgebra and $S\tau$ be a generalized supertrace of this Hom-Lie superalgebra. It can be proved then [12] that the graded ternary bracket

$$[x,y,z] = \mathrm{S}\tau(x)[y,z] + (-1)^{\widehat{x}\widehat{yz}}\mathrm{S}\tau(y)[z,x] + (-1)^{\widehat{z}\widehat{xy}}\mathrm{S}\tau(z)[x,y],\ x,y,z \in \mathfrak{g}. \tag{3.6}$$

determines the 3-Lie superalgebra on the super vector space of a Lie superalgebra $\mathfrak{g}$. We will call this 3-Lie superalgebra constructed by means of a generalized supertrace induced 3-Lie superalgebra. Particularly, if we have a representation $\pi : \mathfrak{g} \to \mathfrak{sgl}(V)$ of a Lie superalgebra $\mathfrak{g}$, then we construct the induced 3-Lie superalgebra (3.6) by simply using the supertrace of matrices in $\mathfrak{sgl}(V)$, i.e., we define the ternary bracket as follows:

$$[x,y,z] = \mathrm{str}(\pi(x))[y,z] + (-1)^{\widehat{x}\widehat{yz}}\mathrm{str}(\pi(y))[z,x] + (-1)^{\widehat{z}\widehat{xy}}\mathrm{str}(\pi(z))[x,y],\ x,y,z \in \mathfrak{g}. \tag{3.7}$$

We will denote the induced 3-Lie superalgebra with graded ternary bracket (3.7) by $\mathrm{t}\mathfrak{g}_\pi$. We can also extend the method of constructing induced representations of induced 3-Hom-Lie algebras to induced 3-Hom-Lie superalgebras.

Lemma 3.3. *Let* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *be a Hom-Lie superalgebra,* $\pi : \mathfrak{g} \to \mathfrak{sgl}(V)$ *be a representation of this Hom-Lie superalgebra. If we equip the super vector space* $\mathfrak{g} \oplus V$ *with the graded skew-symmetric bracket*

$$[[x + v, y + w]] = [x, y] + \pi(x).w - (-1)^{\widehat{xy}}\pi(y).v, \tag{3.8}$$

where $x, y \in \mathfrak{g}, v, w \in V$ *and* $[x, y]$ *is a Hom-Lie bracket in* $\mathfrak{g}$, *then the direct sum of two super vector spaces* $\mathfrak{g} \oplus V$ *becomes a Hom-Lie superalgebra, i.e. the graded skew-symmetric bracket (3.8) satisfies the graded Jacobi-identity.*

Proof. The proof of this lemma is simply to verify the graded Jacobi-identity for the bracket (3.8). In order to simplify notations, we will denote $\mu_1 = \widehat{x_1}\widehat{x_2 x_3}, \mu_2 = \widehat{x_2}\widehat{x_1 x_3}, \mu_3 = \widehat{x_3}\widehat{x_1 x_2}$ and $\nu = \widehat{x_1}\widehat{x_2} + \widehat{x_2}\widehat{x_3} + \widehat{x_1}\widehat{x_3}$. Then the first term of the graded Jacobi-identity can be expanded as follows:

$$\begin{aligned}[][[[[x_1 + v_1, x_2 + v_2]],\alpha(x_3) + \alpha(v_3)]] = {} & [[x_1, x_2], \alpha(v_3)] + \pi([x_1, x_2])\alpha(v_3)\\ & - (-1)^{\mu_3}\pi(\alpha(x_3))\alpha(x_1).v_2 + (-1)^{\nu}\pi(\alpha(x_3))\alpha(x_2).v_1.\end{aligned}$$

The second term of the identity gives

$$\begin{aligned}(-1)^{\mu_1}[[[[x_2 + v_2, x_3 + v_3]],\alpha(x_1) + \alpha(v_1)]] = {} & (-1)^{\mu_1}[[x_2, x_3], \alpha(x_1)] + (-1)^{\mu_1}\pi([x_2, x_3])\alpha(v_1)\\ & - \pi(\alpha(x_1))\pi(x_2).v_3 + (-1)^{\widehat{x_2}\widehat{x_3}}\pi(\alpha(x_1))\pi(x_3).v_2.\end{aligned}$$

The third one yields the expression

$$\begin{aligned}(-1)^{\mu_3}[[[[x_3 + v_3, x_1 + v_1]],\alpha(x_2) + \alpha(v_2)]] = {} & (-1)^{\mu_3}[[x_3, x_1], \alpha(x_2)] + (-1)^{\mu_3}\pi([x_3, x_1])\alpha(v_2)\\ & - (-1)^{\mu_1}\pi(\alpha(x_2))\pi(x_3).v_1 + (-1)^{\widehat{x_1}\widehat{x_2}}\pi(\alpha(x_2))\pi(x_1).v_3.\end{aligned}$$

If we now take the sum of the left hand sides of these relations, we get the left hand side of the graded Jacobi identity for bracket (3.8). The sum of the right-hand sides of these relations gives zero. Indeed, terms underlined by a solid line add up to zero, because the graded Jacobi identity holds in the Hom-Lie superalgebra $\mathfrak{g}$. The terms underlined with dashed lines or not underlined at all also add up to zero due to the fact that the terms in each group simply cancel each other. □

We will prove the following theorem by means of Theorem 3.2 and Lemma 3.3.

Theorem 3.4. *Let* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *be a Hom-Lie superalgebra and* $\pi : \mathfrak{g} \to \mathfrak{sgl}(V)$ *be a representation of* $\mathfrak{g}$. *If*

$$\mathrm{Tr}(\pi(\alpha(v))) = \mathrm{Tr}(\pi(v))$$

for arbitrary $v \in \mathfrak{g}$, *then mapping* $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{sgl}(V)$, *defined by the formula*

$$\rho(x, y) = \mathrm{str}(\pi(x))\pi(y) - (-1)^{\widehat{xy}}\mathrm{str}(\pi(y))\pi(x), \tag{3.9}$$

where $x, y \in \mathfrak{g}$, *is a representation of induced 3-Hom-Lie superalgebra* $\mathrm{t}\mathfrak{g}_\pi$.

Proof. According to Theorem 3.2, if we show that the graded ternary bracket (3.2), where the first term at the right hand side of (3.2) is the graded ternary bracket (3.7) and ρ is (3.9), determines 3-Hom-Lie superalgebra on the direct sum $\mathfrak{g} \oplus V$, then we prove that (3.9) is a representation of induced 3-Hom-Lie superalgebra $\mathfrak{tg}_\pi$. Substituting (3.7) and (3.9) into (3.2), we find

$$\begin{aligned}[x_1 + v_1, x_2 + v_2, x_3 + v_3] &= \mathrm{str}\pi(x_1)[[x_2 + v_2, x_3 + v_3]] \\ &+ (-1)^{\widehat{x_1}\widehat{x_2 x_3}}\mathrm{str}\pi(x_2)[[x_3 + v_3, x_1 + v_1]] + (-1)^{\widehat{x_3}\widehat{x_2 x_1}}\mathrm{str}\pi(x_3)[[x_1 + v_1, x_2 + v_2]],\end{aligned} \tag{3.10}$$

According to Lemma 3.3, the bracket [[x,y]] determines the structure of Hom-Lie superalgebra on $\mathfrak{g} \oplus V$. Thus, the graded ternary bracket (3.10) has the form of a graded ternary bracket for an induced 3-Hom-Lie superalgebra constructed with the help of a graded Hom-Lie bracket and the super trace. Hence, the graded ternary bracket (3.10) determines the induced 3-Hom-Lie superalgebra on $\mathfrak{g} \oplus V$ and therefore, (3.9) is a representation of 3-Hom-Lie superalgebra. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. Y. Nambu, Generalized Hamiltonian dynamics, *Phys. Rev. D*, **7** (1973), 2405–2412. https://doi.org/10.1103/PhysRevD.7.2405
2. V. T. Filippov, *n*-Lie algebras, *Sib. Math. J.*, **26** (1985), 879–891. https://doi.org/10.1007/BF00969110
3. Y. Ma, L. Y. Chen, On the cohomology and extensions of first-class *n*-Lie superalgebras, *Commun. Algebra*, **42** (2014), 4578–4599. https://doi.org/10.1080/00927872.2013.822877
4. J. Arnlind, A. Kitouni, A. Makhlouf, S. Silvestrov, Structure and cohomology of 3-Lie algebras induced by Lie algebras, *Algebra Geom. Math. Phys.*, **85** (2014), 123–144. https://doi.org/10.1007/978-3-642-55361-5_9
5. V. Abramov, 3-Lie superalgebras induced by Lie superalgebras, *Axioms*, **8** (2019), 1–8. https://doi.org/10.3390/axioms8010021
6. H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of *n*-ary Nambu algebras and beyond, *J. Math. Phys.*, **50** (2009), 083501. https://doi.org/10.1063/1.3167801

7. J. Arnlind, A. Makhlouf, S. Silvestrov, Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras, *J. Math. Phys.*, **51** (2010), 11. https://doi.org/10.1063/1.3359004
8. J. Arnlind, A. Makhlouf, S. Silvestrov, Construction of n-Lie algebras and n-ary Hom-Nambu-Lie algebras, *J. Math. Phys.*, **52** (2011), 13. https://doi.org/10.1063/1.3653197
9. A. Kitouni, A. Makhlouf, S. Silvestrov, On $(n + 1)$-Hom-Lie algebras induced by n-Hom-Lie algebras, *Georgian Math. J.*, **23** (2016), 75–95. https://doi.org/10.1515/gmj-2015-0063
10. H. Awata, M. Li, D. Minic, T. Yaneya, On the quantization of Nambu brackets, *JHEP02*, **013** (2001), 1–16. https://doi.org/10.1088/1126-6708/2001/02/013
11. V. Abramov, Super 3-Lie algebras induced by super Lie algebras, *Adv. Appl. Clifford Algebras*, **27** (2017), 9–16. https://doi.org/10.1007/s00006-015-0604-3
12. V. Abramov, Matrix 3-Lie superalgebras and BRST supersymmetry, *Int. J. Geom. Methods Mod. Phys.*, **14** (2017), 1750160. https://doi.org/10.1142/S0219887817501602
13. B. L. Guan, L. Y. Chen, B. Sun, 3-ary Hom-Lie superalgebras induced by Hom-Lie superalgebras, *Adv. Appl. Clifford Algebras*, **27** (2017), 3063–3082. https://doi.org/10.1007/s00006-017-0801-3
14. V. Abramov, P. Lätt, Induced 3-Lie algebras, superalgebras and induced representations, *P. Est. Acda. Sci.*, **69** (2020), 116–133. https://doi.org/10.3176/proc.2020.2.05
15. E. K. Abdaoui, F. Ammar, A. Makhlouf, Hom-Alternative, Hom-Malcev and Hom-Jordan superalgebras, *Bull. Malays. Math. Sci. Soc.*, **40** (2017), 439–472. https://doi.org/10.1007/s40840-016-0323-5
16. B. L. Guan, *Representations for Hom-n-Lie superalgebras and structures for generalized restricted Lie algebras*, Ph.D thesis, Northeast Normal University, 2014.