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Infinite series about harmonic numbers inspired by Ramanujan–like formulae

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Abstract: By employing the coefficient extraction method from hypergeometric series, we shall establish numerous closed form evaluations for infinite series containing central binomial coefficients and harmonic numbers, including several conjectured ones made by Z.-W. Sun.

Keywords: harmonic number; central binomial coefficient; hypergeometric series

1. Introduction

There exist numerous infinite series representations for π and related mathematical constants in the literature (cf. [1–8]). About one century ago, Ramanujan [9] discovered 17 important series involving π. One of them can be reproduced as follows (cf. [10], Example 10):

$$\sum_{n=0}^{\infty}\Big(\frac{1}{4}\Big)^n\frac{(\frac{1}{2})_n^3}{(n!)^3}\{1+6n\}=\frac{4}{\pi}.$$

There is an elegant counterpart series due to Guillera [11] (cf. [5], Example 31)

$$\sum_{n=0}^{\infty}\Big(\frac{1}{4}\Big)^n\frac{(n!)^3}{(\frac{3}{2})_n^3}\{2+3n\}=\frac{\pi^2}{4}.$$

These two identities and further similar ones have been proved uniformly by means of the hypergeometric series approach (cf. [5, 6, 10, 12, 13]). The same approach suggests that we can go further to examine analogous series involving harmonic numbers, that become quite active topics recently (cf. [4, 7, 8, 14–16]). In this paper, several challenging series involving central binomial coefficients and harmonic numbers will be evaluated in closed form. They will be divided into four classes with eight sample series being highlighted in advance as follows:

$$\text{Eq (14)}\qquad \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\{(1+6n)\mathbf{O}_n\}=\frac{4\ln 2}{3\pi},$$

$$\text{Eq (15)} \quad \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n (n!)^3} \Big\{ 2 - (1+6n)\mathbf{H}_n \Big\} = \frac{8\ln 2}{\pi},$$

$$\text{Eq (31)} \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1-(2n+1)^2 \mathbf{O}_n^{\langle 2 \rangle}}{16^n (2n+1)^5} = \frac{253\pi^5}{77760},$$

$$\text{Eq (32)} \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2+3(2n+1)^4 (\mathbf{O}_n^{\langle 2 \rangle})^2}{16^n (2n+1)^5} = \frac{1091\pi^5}{155520},$$

$$\text{Eq (62)} \quad \sum_{n=1}^{\infty} \frac{16 - 12n\mathbf{H}_n + 27n^2 \mathbf{H}_n^2}{n^6 \binom{2n}{n}} = \frac{631\pi^6 + 340200\zeta(3)^2}{68040},$$

$$\text{Eq (65)} \quad \sum_{n=1}^{\infty} \frac{4 - 45n^3 \mathbf{H}_n^{\langle 3 \rangle} + 123n^5 \mathbf{H}_n^{\langle 5 \rangle}}{n^7 \binom{2n}{n}} = \frac{62\pi^2 \zeta(5) - 75\zeta(7)}{12},$$

$$\text{Eq (75)} \quad \sum_{n=1}^{\infty} \frac{(n!)^3}{(\frac{3}{2})_n^3} \Big\{ \frac{(1+2n)(3+8n)}{4^n \times n^4} - \frac{2(1+6n+6n^2)\mathbf{H}_n^{\langle 2 \rangle}}{4^n \times n^2} \Big\} = \frac{\pi^4}{24} - 2\pi^2 + 16,$$

$$\text{Eq (78)} \quad \sum_{n=1}^{\infty} \frac{(n!)^3}{(\frac{3}{2})_n^3} \Big\{ \frac{1+6n+10n^2}{4^n \times n^4 (1+2n)} + \frac{2(1+2n)(1+3n)\mathbf{O}_n^{\langle 2 \rangle}}{4^n \times n^3} \Big\} = \frac{\pi^4}{8} - \pi^2.$$

Furthermore, the hypergeometric series approach introduced in this paper enables the authors to confirm together the following remarkable identities conjectured by Sun [17–19]:

$$\text{Eq (25):} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n} \mathbf{O}_{n+1}^{\langle 3 \rangle}}{16^n (2n+1)} = \frac{5\pi}{18} \zeta(3).$$

$$\text{Eq (49):} \quad \sum_{n=1}^{\infty} \frac{n^3 \mathbf{H}_n^{\langle 3 \rangle}}{n^5 \binom{2n}{n}} = \frac{\pi^2 \zeta(3) + 3\zeta(5)}{27}.$$

$$\text{Eq (30):} \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2+3(2n+1)^4 \mathbf{O}_n^{\langle 4 \rangle}}{16^n (2n+1)^5} = \frac{121\pi^5}{17280}.$$

$$\text{Eq (68):} \quad \sum_{n=0}^{\infty} \frac{(n!)^3}{4^n (\frac{3}{2})_n^3} \Big\{ (2+3n)(\mathbf{O}_{n+1}^{\langle 2 \rangle} - \mathbf{H}_n^{\langle 2 \rangle}) \Big\} = \frac{\pi^4}{48}.$$

$$\text{Eq (69):} \quad \sum_{n=0}^{\infty} \frac{(n!)^3}{4^n (\frac{3}{2})_n^3} \Big\{ (2+3n)(\mathbf{O}_{n+1}^{\langle 3 \rangle} + \mathbf{H}_n^{\langle 3 \rangle}) \Big\} = \frac{\pi^2}{4} \zeta(3).$$

$$\text{Eq (34):} \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{33(2n+1)^3 \mathbf{O}_n^{\langle 3 \rangle} + 41}{16^n (2n+1)^6} = \frac{245\pi^3}{216} \zeta(3) - \frac{49\pi}{144} \zeta(5).$$

$$\text{Eq (35):} \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{33(2n+1)^5 \mathbf{O}_n^{\langle 5 \rangle} + 37}{16^n (2n+1)^6} = \frac{35\pi^3}{288} \zeta(3) + \frac{1003\pi}{96} \zeta(5).$$

In order to facilitate the subsequent presentation, we briefly review basic facts about harmonic numbers and the Γ-function as well as the "coefficient extraction".

1.1. Harmonic numbers

For $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, these numbers are defined by

$$\mathbf{H}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{1}{(x+k)^\lambda}, \quad \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(x+k)^\lambda};$$
$$\mathbf{O}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{1}{(x+2k)^\lambda}, \quad \bar{\mathbf{O}}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(x+2k)^\lambda}.$$

When $\lambda = 1$ and/or $x = 1$, they will be suppressed from these notations. We record also the following simple, but useful relations:

$$\mathbf{H}_{2n}^{\langle\lambda\rangle} = \mathbf{O}_n^{\langle\lambda\rangle} + 2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle}, \qquad \mathbf{H}_n^{\langle\lambda\rangle}(\tfrac{1}{2}) = 2^\lambda \mathbf{O}_n^{\langle\lambda\rangle};$$
$$\bar{\mathbf{H}}_{2n}^{\langle\lambda\rangle} = \mathbf{O}_n^{\langle\lambda\rangle} - 2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle}, \qquad \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(\tfrac{1}{2}) = 2^\lambda \bar{\mathbf{O}}_n^{\langle\lambda\rangle}.$$

1.2. The gamma function

It is defined by the Euler integral

$$\Gamma(x) = \int_0^\infty \tau^{x-1} e^{-\tau} d\tau \quad \text{for} \quad \Re(x) > 0.$$

The logarithmic differentiation of the Γ-function results in the digamma function (cf. Rainville [20],§9)

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \frac{z-1}{(n+1)(n+z)}$$

with the Euler–Mascheroni constant being given by

$$\gamma = \lim_{n\to\infty} (\mathbf{H}_n - \ln n).$$

There are power series expansions of the Γ-function [14]

$$\Gamma(1-x) = \exp\Big\{ \sum_{k\geq 1} \frac{\sigma_k}{k} x^k \Big\},$$
$$\Gamma(\tfrac{1}{2}-x) = \sqrt{\pi} \exp\Big\{ \sum_{k\geq 1} \frac{\tau_k}{k} x^k \Big\};$$

where σ_k and τ_k are defined respectively by

$$\sigma_1 = \gamma \quad \text{and} \quad \sigma_m = \zeta(m) \quad \text{for} \quad m \geq 2;$$
$$\tau_1 = \gamma + 2\ln 2 \quad \text{and} \quad \tau_m = (2^m - 1)\zeta(m) \quad \text{for} \quad m \geq 2;$$

with the usual Riemann zeta function $\zeta(x)$ being given by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad \text{for} \quad \Re(x) > 1.$$

1.3. Coefficient extraction

For $n \in \mathbb{N}_0$ and an indeterminate x, the shifted factorials are usually defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

Let $[x^m]\phi(x)$ stand for the coefficient of x^m in the formal power series $\phi(x)$. We shall frequently use the following relations:

$$\begin{aligned}
&[x]\frac{(1+x)_n}{n!} = \mathbf{H}_n, && [x^2]\frac{(1+x)_n}{n!} = \frac{\mathbf{H}_n^2 - \mathbf{H}_n^{\langle 2\rangle}}{2},\\
&[x]\frac{n!}{(1-x)_n} = \mathbf{H}_n, && [x^2]\frac{n!}{(1-x)_n} = \frac{\mathbf{H}_n^2 + \mathbf{H}_n^{\langle 2\rangle}}{2},\\
&[y]\frac{(\frac{1}{2}+y)_n}{(\frac{1}{2})_n} = 2\mathbf{O}_n, && [y^2]\frac{(\frac{1}{2}+y)_n}{(\frac{1}{2})_n} = 2(\mathbf{O}_n^2 - \mathbf{O}_n^{\langle 2\rangle}),\\
&[y]\frac{(\frac{1}{2})_n}{(\frac{1}{2}-y)_n} = 2\mathbf{O}_n, && [y^2]\frac{(\frac{1}{2})_n}{(\frac{1}{2}-y)_n} = 2(\mathbf{O}_n^2 + \mathbf{O}_n^{\langle 2\rangle}).
\end{aligned}$$

By means of the generating function method, it is not difficult to show that (cf. Chen–Chu [21] and Chu [4, 5]) in general there hold the relations:

$$[x^m]\frac{(\lambda-x)_n}{(\lambda)_n} = \Omega_m(-\mathrm{hh}_k) \quad \text{and} \quad [x^m]\frac{(\lambda)_n}{(\lambda-x)_n} = \Omega_m(\mathrm{hh}_k).$$

Here "hh_k" stands for the harmonic number $\mathrm{hh}_k := \mathbf{H}_n^{\langle k\rangle}(\lambda)$ of order k, and the Bell polynomials (cf. [22], §3.3) are expressed explicitly as

$$\Omega_m(\pm\mathrm{hh}_k) = \sum_{\sigma(m)} \prod_{k=1}^{m} \frac{\{\pm \mathbf{H}_n^{\langle k\rangle}(\lambda)\}^{\ell_k}}{\ell_k!\, k^{\ell_k}},$$

where the sum runs over $\sigma(m)$, the set of m-partitions represented by m-tuples of $(\ell_1, \ell_2, \cdots, \ell_m) \in \mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^{m} k\ell_k = m$.

The aim of this paper is to find exact evaluations, in closed form, for infinite series of convergence rate "$\frac{1}{4}$" that contain both central binomial coefficients and harmonic numbers. This will mainly be fulfilled by extracting coefficients from hypergeometric seires, which has been shown, by the second author and his collaborators, powerful in dealing with both Ramanujan–like series [5, 6, 10, 12, 13] and harmonic number identities [4, 7, 8, 14–16]. Numerous elegant summation formulae are established, including several conjectured ones made by Sun [17–19].

As preliminaries, we shall illustrate how to derive infinite series identities by the hypergeometric series approach in the next section, where four summation theorems of hypergeometric series are recorded for subsequent applications. Then the remaining four sections will be devoted to infinite series identities involving harmonic numbers classified accoring to the positions of the central binomial coefficients.

In order to assure accuracy of computations, numerical tests for all the equations have been made by appropriately devised *Mathematica* commands.

2. Hypergeometric series approach

Following Bailey [23], the hypergeometric series reads as

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} \middle| z\right] = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k}. $$

For the sake of brevity, the Γ-function quotient will be abbreviated to

$$ \Gamma\left[\begin{matrix} \alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C \end{matrix}\right] = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}. $$

There exist numerous summation theorems for hypergeometric series (see Bailey [23] and Brychkov [24]) in the mathematical literature. The following formula can be found in Karlsson [25] (which is, in fact, a reduced case of Lemma 1):

$$ \begin{aligned} \mathcal{W}(a, b; x) &= {}_3F_2\left[\begin{matrix} 2ax, 2bx, 1-2bx \\ 1+ax-bx, \frac{1}{2}+ax+bx \end{matrix} \middle| \frac{1}{4}\right] \\ &= \Gamma\left[\begin{matrix} 1+\frac{ax}{3}, \frac{1}{2}+\frac{ax}{3}, 1+ax-bx, \frac{1}{2}+ax+bx \\ 1+ax, \frac{1}{2}+ax, 1+\frac{ax}{3}-bx, \frac{1}{2}+\frac{ax}{3}+bx \end{matrix}\right]. \end{aligned} $$

As a warm up, we are going to take this formula as an example to illustrate how to derive infinite series identities involving harmonic numbers. The two expressions of $\mathcal{W}(a, b; x)$ in terms of hypergeometric ${}_3F_2$-series and in the quotient of Γ-function are analytic in x in the neighborhood of $x = 0$. Therefore they can be expanded into Maclaurin series

$$ \mathcal{W}(a, b; x) = \sum_{m=0}^{\infty} x^m \times \mathrm{W}_m(a, b) \Longleftrightarrow \mathrm{W}_m(a, b) := [x^m]\mathcal{W}(a, b; x). $$

By manipulating these coefficients $\mathrm{W}_m(a, b)$, we can establish the following identities. More identities containing the central binomial coefficients and/or harmonic numbers can be found in [26–33].

- Coefficient $\mathrm{W}_2(a, b)$: See also Elsner [28] and Zucker [33]

$$ \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}. $$

- Coefficient $\mathrm{W}_3(a, b)$

$$ \sum_{n=1}^{\infty} \frac{1 + n\bar{\mathbf{H}}_{2n}}{n^3\binom{2n}{n}} = \frac{2}{3}\zeta(3). $$

- Coefficient $\mathrm{W}_4(1, -1)$: see (Sun [19], Equation over Theorem 1.1)

$$ \sum_{n=1}^{\infty} \frac{n^2\mathbf{H}_n^{\langle 2\rangle} - 1}{n^4\binom{2n}{n}} = \frac{\pi^4}{1944}. $$

- Furthermore, by comparing the coefficients of $[a^3b]\mathrm{W}_4(a,b)$ and $[ab^3]\mathrm{W}_4(a,b)$, we obtain two equations

$$[ab^3]\mathrm{W}_4(a,b) = \sum_{n=1}^{\infty} \frac{16 + 8n^2\bar{\mathbf{H}}_{2n}^2 + 16n\bar{\mathbf{H}}_{2n} - 14n^2\mathbf{H}_n^{\langle 2\rangle} + 8n^2\mathbf{O}_n^{\langle 2\rangle}}{n^4\binom{2n}{n}},$$
$$[a^3b]\mathrm{W}_4(a,b) = \sum_{n=1}^{\infty} \frac{16 + 8n^2\bar{\mathbf{H}}_{2n}^2 + 16n\bar{\mathbf{H}}_{2n} - 6n^2\mathbf{H}_n^{\langle 2\rangle} + 8n^2\mathbf{O}_n^{\langle 2\rangle}}{n^4\binom{2n}{n}}.$$

Resolving this system of equations for the sum about $n^2\mathbf{H}_n^{\langle 2\rangle}$ (subtracting the first equation from the second), we have the expression

$$\sum_{n=1}^{\infty} \frac{\mathbf{H}_n^{\langle 2\rangle}}{n^2\binom{2n}{n}} = \frac{[ab^3]\mathrm{W}_4(a,b) - [a^3b]\mathrm{W}_4(a,b)}{8}.$$

Alternatively, the above two coefficients on the right hand side can also be computed from the Γ-function quotient

$$[ab^3]\mathrm{W}_4(a,b) = \frac{182\pi^4}{1215} \quad \text{and} \quad [a^3b]\mathrm{W}_4(a,b) = \frac{14\pi^4}{135}.$$

Hence, we find the following closed formula

$$\sum_{n=1}^{\infty} \frac{\mathbf{H}_n^{\langle 2\rangle}}{n^2\binom{2n}{n}} = \frac{91\pi^4}{4860} - \frac{7\pi^4}{540} = \frac{7\pi^4}{1215}.$$

As a bonus, we recover, by combining the two series for $\mathbf{H}_n^{\langle 2\rangle}$, the well–known identity of Comtet (cf. [22], page 89) below

$$\sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}} = \frac{17\pi^4}{3240}.$$

This example demonstrates that the hypergeometric series approach is indeed powerful. For a given hypergeometric series formula, the above procedure can be summarized as follows:

- Reformulate the equality by identifying a variable “x” and eventual parameters $\{a,b,c\}$ so that both sides of the resulting equality are analytic in x at $x = 0$.
- Find infinite series identities by extracting and then equating the coefficients $\mathrm{W}_m(a,b,c)$ for small integer values of m across the equation.
- Find infinite series identities by computing the coefficients $\mathrm{W}_m(a,b,c)$ for particular values of parameters $\{a,b,c\}$.
- Find infinite series identities by determining the coefficients of specific monomials “$a^ib^jc^k$” in $\mathrm{W}_m(a,b,c)$.
- Furthermore, find infinite series identities by constructing and then resolving a linear system formed by linear equations characterized by coefficients of monomials “$a^ib^jc^k$” in $\mathrm{W}_m(a,b,c)$.

By carrying out this procedure, we shall prove, in the rest of the paper, numerous interesting infinite series identities, including several conjectured ones mainly made by Sun [17–19] by examining the following four lemmas of hypergeometric series with the same convergence rate "$\frac{1}{4}$".

The first formula below was among a list of conjectured identities made by Gosper (1977), which was first confirmed by Gessel and Stanton (cf. [34], Eq 1.7)as a limiting case of a terminating ${}_7F_6$-series.

Lemma 1 (Gessel and Stanton [34]: see also Chu [36])**.**

$$\Gamma\left[\begin{matrix}1+d,\frac{1}{2}+d,1+a-b,\frac{1}{2}+a+b\\1+a,\frac{1}{2}+a,1-b+d,\frac{1}{2}+b+d\end{matrix}\right]={}_5F_4\left[\begin{matrix}2a,\ 1+\frac{2a}{3},\ 2b,\ 1-2b,\ a-d\\\frac{2a}{3},1+a-b,\frac{1}{2}+a+b,1+2d\end{matrix}\Big|\frac{1}{4}\right].$$

The next formula was also conjectured by Gosper (1977), that was shown by Chu (cf. [36], Eq 5.1e) as a limiting result from another terminating ${}_7F_6$-series.

Lemma 2 (Gosper (1977): see also Chu [36])**.**

$$\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2}+b+d,1+a-b,1+a-d\\\frac{1}{2}+b,\frac{1}{2}+d,1+a,1+a-b-d\end{matrix}\right]={}_5F_4\left[\begin{matrix}a,1+\frac{2a}{3},2b,2d,1+2a-2b-2d\\\frac{2a}{3},1+a-b,1+a-d,\frac{1}{2}+b+d\end{matrix}\Big|\frac{1}{4}\right].$$

The third summation formula is due to the second author, who discovered it by making use of the inverse series relations found by Gould and Hsu [35].

Lemma 3 (Chu [6], Theorem 2.2)**.** *For three complex parameters $\{a,b,d\}$ subject to $a+d,1+b-d\notin\mathbb{Z}\backslash\mathbb{N}$, define the the factorial quotient* U_n *by*

$$\mathrm{U}_n(a,b,d)=\frac{(d)_n(1-d)_n(a-b+d)_n(1+b-a-d)_n}{(a+d)_n(1+b-d)_n(2n+1)!}$$

and the polynomial $\mathcal{P}_n$ by

$$\mathcal{P}_n(a,b,d)=(1+2n)(b-d+n)+(d+n)(a-b+d+n).$$

Then there holds the transformation formula

$$\Phi(a,b,d):=\Gamma\left[\begin{matrix}a+d,1+b-d\\a,\ b\end{matrix}\right]=\sum_{n=0}^{\infty}\mathrm{U}_n(a,b,d)\mathcal{P}_n(a,b,d).$$

Finally, we need a transformation formula which can be proved by means of the modified Abel lemma on summation by parts.

Lemma 4 (Chu [12],Theorem 2.7)**.** *For four complex parameters $\{a,b,c,d\}$ subject to $\Re(c+d-a-b)>1$ and $c,d\notin\mathbb{Z}\backslash\mathbb{N}$, define the the factorial quotient* V_n *by*

$$\mathrm{V}_n(a,b,c,d)=\frac{(c-a)_n(c-b)_n(d-a)_n(d-b)_n}{(c)_n(d)_n(c+d-a-b)_{2n+1}}$$

and the quadratic polynomial $\mathcal{Q}_n$ by

$$\mathcal{Q}_n(a,b,c,d)=(c-1+n)(c+d-a-b+2n)+(d-a+n)(d-b+n).$$

Then there holds the transformation formula

$$\Psi(a,b,c,d):=(c+d-a-b-1)\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k(d)_k}=\sum_{n=0}^{\infty}\mathrm{V}_n(a,b,c,d)\mathcal{Q}_n(a,b,c,d).$$

3. Series containing $\binom{2n}{n}^3$ in numerators

By making use of Lemmas 1, 3, and 4, we shall evaluate several infinite series containing the cubic central binomial coefficient $\binom{2n}{n}^3$ in numerators.

3.1. Series from Lemma 1

Under the parameter replacements

$$a \to \frac{1}{4} + ax,\ b \to \frac{1}{4} + bx,\ d \to -\frac{1}{4} + dx$$

the equality in Lemma 1 can be restated as in the proposition below.

Proposition 5.

$$\begin{aligned}\mathcal{A}(a,b,d;x) &= 4^{1+ax-dx}\Gamma\left[\begin{matrix}\frac{1}{2}+2dx, 1+ax+bx, 1+ax-bx\\ \frac{1}{2}+2ax, \frac{1}{2}+bx+dx, \frac{1}{2}-bx+dx\end{matrix}\right]\\ &= \sum_{n=0}^{\infty}(1+4ax+6n)\frac{(\frac{1}{2}+2ax)_n(\frac{1}{2}+2bx)_n(\frac{1}{2}-2bx)_n(\frac{1}{2}+ax-dx)_n}{4^n n!(1+ax+bx)_n(1+ax-bx)_n(\frac{1}{2}+2dx)_n}.\end{aligned}$$

Both sides of the above equality are analytic functions of x in the neighborhood of $x = 0$ and can be expanded into power series in x as follows:

$$\mathcal{A}(a,b,d;x) = \sum_{m=0}^{\infty} x^m \mathrm{A}_m(a,b,d).$$

By computing the initial coefficients across the equation in Proposition 5, and then equating the resulting expressions, we can derive a number of infinite series identities. The first coefficient $\mathrm{A}_0(a,b,d)$ recovers Ramanujan's identity as anticipated in the introduction. Further elegant ones are recorded below as examples.

- Coefficient $\mathrm{A}_2(0,0,1)$

$$\frac{4\ln^2 2+\pi^2}{3\pi} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{3\mathbf{O}_n^2+\mathbf{O}_n^{\langle 2\rangle}\Big\}. \tag{1}$$

- Coefficient $\mathrm{A}_2(0,1,0)$: Conjectured by Guo–Lian [37] and proved in (cf. [38], Theorem 1.2)

$$\frac{4\pi}{3} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{16\mathbf{O}_n^{\langle 2\rangle}-\mathbf{H}_n^{\langle 2\rangle}\Big\}. \tag{2}$$

We shall succeed in refining the above two identities in (10), (11) and (16).

- Coefficient $[b^2d]\mathrm{A}_3(a,b,d)$

$$\frac{4}{9\pi}\{21\zeta(3)+\pi^2\ln 2\} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{\mathbf{O}_n(16\mathbf{O}_n^{\langle 2\rangle}-\mathbf{H}_n^{\langle 2\rangle})\Big\}. \tag{3}$$

- Coefficients $\mathrm{A}_3(0,0,1) \rightleftharpoons [d^3]\mathrm{A}_3(a,b,d)$

$$\frac{42\zeta(3)+4\ln^3 2+3\pi^2\ln 2}{9\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{2\mathbf{O}_n^{\langle 3\rangle}+3\mathbf{O}_n^3+3\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle}\Big\}. \tag{4}$$

- Coefficients $\mathrm{A}_4(0,1,0) \rightleftharpoons [b^4]\mathrm{A}_4(a,b,d)$

$$\frac{8\pi^3}{45}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{256\mathbf{O}_n^{\langle 4\rangle}-\mathbf{H}_n^{\langle 4\rangle}-(\mathbf{H}_n^{\langle 2\rangle}-16\mathbf{O}_n^{\langle 2\rangle})^2\Big\}. \tag{5}$$

- Coefficient $[b^2d^2]\mathrm{A}_4(a,b,d)$

$$\frac{4}{9\pi}\{\pi^4+\pi^2\ln^2 2+42\ln 2\zeta(3)\}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{(3\mathbf{O}_n^2+\mathbf{O}_n^{\langle 2\rangle})(16\mathbf{O}_n^{\langle 2\rangle}-\mathbf{H}_n^{\langle 2\rangle})\Big\}. \tag{6}$$

- Coefficients $\mathrm{A}_4(0,0,1) \rightleftharpoons [d^4]\mathrm{A}_4(a,b,d)$

$$\begin{aligned}&\frac{672\zeta(3)\ln 2+17\pi^4+16\ln^4 2+24\pi^2\ln^2 2}{36\pi}\\&=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{10\mathbf{O}_n^{\langle 4\rangle}-18\mathbf{O}_n^4+3(\mathbf{O}_n^{\langle 2\rangle}+3\mathbf{O}_n^2)^2+24\mathbf{O}_n\mathbf{O}_n^{\langle 3\rangle}\Big\}.\end{aligned} \tag{7}$$

- Coefficient $[b^4d]\mathrm{A}_5(a,b,d)$

$$\begin{aligned}&\frac{8\{105\pi^2\zeta(3)-1395\zeta(5)-\pi^4\ln 2\}}{135\pi}\\&=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{\mathbf{H}_n^{\langle 4\rangle}\mathbf{O}_n-256\mathbf{O}_n\mathbf{O}_n^{\langle 4\rangle}+\mathbf{O}_n(\mathbf{H}_n^{\langle 2\rangle}-16\mathbf{O}_n^{\langle 2\rangle})^2\Big\}.\end{aligned} \tag{8}$$

- Coefficient $[b^2d^3]\mathrm{A}_5(a,b,d)$

$$\begin{aligned}&\frac{1116\zeta(5)+105\pi^2\zeta(3)+252\zeta(3)\ln^2 2+4\pi^2\ln^3 2+12\pi^4\ln 2}{27\pi}\\&=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3(1+6n)}{4^n(n!)^3}\Big\{(16\mathbf{O}_n^{\langle 2\rangle}-\mathbf{H}_n^{\langle 2\rangle})(2\mathbf{O}_n^{\langle 3\rangle}+3\mathbf{O}_n^3+3\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle})\Big\}.\end{aligned} \tag{9}$$

3.2. Series from Lemma 3

Under the parameter replacements

$$a\to\frac{1}{2}+ax,\ b\to\frac{1}{2}+bx,\ d\to\frac{1}{2}+dx$$

the equality in Lemma 3 can be restated as in the proposition below.

Proposition 6.

$$\begin{aligned} C(a,b,d;x) &= \Gamma\left[\begin{matrix} 1+ax+dx, 1+bx-dx \\ \frac{1}{2}+ax, \frac{1}{2}+bx \end{matrix}\right] \\ &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+dx)_n(\frac{1}{2}-dx)_n(\frac{1}{2}+ax-bx+dx)_n(\frac{1}{2}-ax+bx-dx)_n}{(1+ax+dx)_n(1+bx-dx)_n(2n+1)!} \\ &\times \Big\{(1+2n)(n+bx-dx)+(\tfrac{1}{2}+n+dx)(\tfrac{1}{2}+n+ax-bx+dx)\Big\}. \end{aligned}$$

Expanding both sides of the above equation into power series in x

$$C(a,b,d;x) = \sum_{m=0}^{\infty} x^m \mathrm{C}_m(a,b,d)$$

and then comparing further the coefficients of monomials $a^i b^j d^k$ (subject to $i+j+k=2$) in $\mathrm{C}_2(a,b,d)$, we construct a system of linear equations. By resolving this system, we can derive the three identities below. This procedure will be denominated as "Resolving linear system" formed by $\mathrm{C}_2(a,b,d)$.

- Resolving Linear system $\mathrm{C}_2(a,b,d)$

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3} \frac{1+2(1+2n)(1+6n)\mathbf{O}_n^{(2)}}{2n+1}. \tag{10}$$

- Resolving Linear system $\mathrm{C}_2(a,b,d)$

$$\frac{8\pi}{3} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3} \frac{8+(1+2n)(1+6n)\mathbf{H}_n^{(2)}}{2n+1}. \tag{11}$$

- Resolving Linear system $\mathrm{C}_2(a,b,d)$

$$\frac{2\pi^2-16\ln^2 2}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3} \Big\{\frac{4}{2n+1} + 4\mathbf{H}_n - (1+6n)\mathbf{H}_n^2\Big\}. \tag{12}$$

- Resolving Linear system $\mathrm{C}_4(a,b,d)$

$$\frac{\pi^3}{96} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3} \Big\{\frac{2\mathbf{O}_n^{(2)}}{2n+1} - (1+6n)\big[\mathbf{O}_n^{(4)} - 2(\mathbf{O}_n^{(2)})^2\big]\Big\}. \tag{13}$$

3.3. Series from Lemma 4

Under the parameter replacements

$$a \to \frac{1}{2}+ax,\ b \to \frac{1}{2}+bx,\ c \to 1,\ d \to 1+dx$$

the sum with respect to k in Lemma 4 can be evaluated by the Gauss summation theorem (cf. Bailey [23],§1.3). This leads us to the summation formula as in the proposition below.

Proposition 7.

$$\begin{aligned}\mathcal{D}(a,b,d;x) &= \Gamma\begin{bmatrix} 1+dx, 1-ax-bx+dx \\ \frac{1}{2}-ax+dx, \frac{1}{2}-bx+dx \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-ax)_n(\frac{1}{2}-bx)_n(\frac{1}{2}-ax+dx)_n(\frac{1}{2}-bx+dx)_n}{n!(1+dx)_n(1-ax-bx+dx)_{2n+1}} \\ &\times \Big\{(\tfrac{1}{2}+n-ax+dx)(\tfrac{1}{2}+n-bx+dx)+n(1+2n-ax-bx+dx)\Big\}.\end{aligned}$$

Expanding both sides of the above equation into power series in x

$$\mathcal{D}(a,b,d;x) = \sum_{m=0}^{\infty} x^m \mathrm{D}_m(a,b,d)$$

we can show by the initial coefficients the following infinite series identities, where the first two identities are equivalent to those conjectured by Sun (cf. [19], Eqs 3.52 & 3.53) in view of Sun [19], Remark 3.14.

- Coefficient $\mathrm{D}_1(1,2,1)$

$$\frac{4\ln 2}{3\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\{(1+6n)\mathbf{O}_n\}. \tag{14}$$

- Coefficient $\mathrm{D}_1(1,0,1)$

$$\frac{8\ln 2}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\{2-(1+6n)\mathbf{H}_n\}. \tag{15}$$

- Resolving Linear system $\mathrm{D}_2(a,b,d)$

$$\frac{16\ln^2 2+\pi^2}{6\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\Big\{6(1+6n)\mathbf{O}_n^2 - \frac{1}{1+2n}\Big\}. \tag{16}$$

- Resolving Linear system $\mathrm{D}_2(a,b,d)$

$$\frac{8\ln^2 2}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\Big\{\frac{2}{1+2n} + 6\mathbf{O}_n - 3(1+6n)\mathbf{O}_n\mathbf{H}_n\Big\}. \tag{17}$$

- Resolving Linear system $\mathrm{D}_3(a,b,d)$

$$\frac{7\zeta(3)}{\pi} - \pi\ln 2 = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\Big\{\frac{\mathbf{H}_n}{2n+1} - 4\mathbf{O}_n^{\langle 2\rangle} + 2(1+6n)\mathbf{H}_n\mathbf{O}_n^{\langle 2\rangle}\Big\}. \tag{18}$$

- Resolving combined linear system $\mathrm{C}_3(a,b,d)$ & $\mathrm{D}_3(a,b,d)$

$$\frac{96\zeta(3)-16\pi^2\ln 2}{3\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\Big\{\frac{8\mathbf{H}_n}{2n+1} - 2\mathbf{H}_n^{\langle 2\rangle} + (1+6n)(\mathbf{H}_n^{\langle 3\rangle} + \mathbf{H}_n\mathbf{H}_n^{\langle 2\rangle})\Big\}. \tag{19}$$

- Resolving combined linear system $C_4(a,b,d)$ & $D_4(a,b,d)$

$$\frac{52\pi^3}{45}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\Big\{\frac{16\mathbf{H}_n^{\langle 2\rangle}}{2n+1}+(1+6n)[\mathbf{H}_n^{\langle 4\rangle}+(\mathbf{H}_n^{\langle 2\rangle})^2+128\mathbf{O}_n^{\langle 4\rangle}]\Big\}. \tag{20}$$

- Resolving combined linear system $C_4(a,b,d)$ & $D_4(a,b,d)$

$$\frac{\pi^3}{6}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\Big\{\frac{\mathbf{H}_n^{\langle 2\rangle}+16\mathbf{O}_n^{\langle 2\rangle}}{2n+1}+2(1+6n)(8\mathbf{O}_n^{\langle 4\rangle}+\mathbf{H}_n^{\langle 2\rangle}\mathbf{O}_n^{\langle 2\rangle})\Big\}. \tag{21}$$

- Resolving combined linear system $C_5(a,b,d)$ & $D_5(a,b,d)$

$$\frac{42\pi^2\zeta(3)-372\zeta(5)-\pi^4\ln 2}{48\pi}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3}{4^n(n!)^3}\Big\{\frac{2\mathbf{H}_n\mathbf{O}_n^{\langle 2\rangle}}{2n+1}+[2-(1+6n)\mathbf{H}_n][\mathbf{O}_n^{\langle 4\rangle}-2(\mathbf{O}_n^{\langle 2\rangle})^2]\Big\}. \tag{22}$$

The last four identities are derived by resolving the combined linear system formed by equations from both $C_5(a,b,d)$ and $D_5(a,b,d)$.

4. Series containing $\binom{2n}{n}$ in numerators

By means of Lemmas 1 and 4, we shall evaluate, in this section, infinite series involving the central binomial coefficient $\binom{2n}{n}$ in numerators, including four challenging ones conjectured by Sun [18].

4.1. Series from Lemma 1

Under the parameter settings

$$a\to\frac{3}{4}+ax,\ b\to\frac{1}{4}+bx,\ d\to\frac{1}{4}+dx$$

we can reformulate the equality in Lemma 1 as in the proposition below.

Proposition 8.

$$\begin{aligned}\mathcal{A}(a,b,d;x)&=4^{1+ax-dx}\Gamma\begin{bmatrix}\frac{1}{2}+ax+bx,\frac{1}{2}+ax-bx,\frac{1}{2}+2dx\\ \frac{1}{2}+2ax,1+bx+dx,1-bx+dx\end{bmatrix}\\&=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}+2ax)_{n+1}(\frac{1}{2}+2bx)_n(\frac{1}{2}-2bx)_n(\frac{1}{2}+ax-dx)_n}{4^n n!(\frac{1}{2}+ax+bx)_{n+1}(\frac{1}{2}+ax-bx)_{n+1}(\frac{1}{2}+2dx)_{n+1}}\Big\{3+4ax+6n\Big\}.\end{aligned}$$

By comparing the Maclaurin series coefficients across the equation

$$\mathcal{A}(a,b,d;x)=\sum_{m=0}^{\infty}x^m\mathrm{A}_m(a,b,d)$$

we establish several interesting infinite series identities as follows. Among them, (25), (30), (34) and (35) were first conjectured by Sun [41] and subsequently confirmed by Albinger [39].

- Coefficient $A_0(a,b,d)$: see (Sun [19], Equation below (1.2))

$$\frac{\pi}{3}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)}. \tag{23}$$

- Coefficient $A_2(3,\sqrt{-3},1)$: Zucker [33]

$$\frac{7\pi^3}{216}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)^3}. \tag{24}$$

- Coefficient $A_3(3,2,1)$

$$\frac{5\pi}{18}\zeta(3)=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}\mathbf{O}_{n+1}^{\langle 3\rangle}}{16^n(2n+1)}. \tag{25}$$

- Resolving linear system $A_2(a,b,d)$

$$\pi\ln^2 2-\frac{\pi^3}{24}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{4(2n+1)\mathbf{O}_n+3(2n+1)^2\mathbf{O}_n^2}{16^n(2n+1)^3}. \tag{26}$$

- Resolving linear system $A_3(a,b,d)$

$$\frac{5\pi\zeta(3)+7\pi^3\ln 2}{24}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{7+9(2n+1)\mathbf{O}_n}{16^n(2n+1)^4}. \tag{27}$$

- Resolving linear system $A_3(a,b,d)$

$$\frac{11\pi\zeta(3)+\pi^3\ln 2}{24}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1+18(2n+1)^2\mathbf{O}_n^{\langle 2\rangle}+27(2n+1)^3\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle}}{16^n(2n+1)^4}. \tag{28}$$

- Resolving linear system $A_3(a,b,d)$

$$\frac{3\pi^3\ln 2-24\pi\ln^3 2+7\pi\zeta(3)}{72}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1-2(2n+1)^2\mathbf{O}_n^2-(2n+1)^3\mathbf{O}_n^3}{16^n(2n+1)^4}. \tag{29}$$

- Coefficient $A_4(3,\sqrt{-3},1)$

$$\frac{121\pi^5}{17280}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2+3(2n+1)^4\mathbf{O}_n^{\langle 4\rangle}}{16^n(2n+1)^5}. \tag{30}$$

- Resolving linear system $A_4(a,b,d)$

$$\frac{253\pi^5}{77760}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1-(2n+1)^2\mathbf{O}_n^{\langle 2\rangle}}{16^n(2n+1)^5}. \tag{31}$$

- Resolving linear system $A_4(a,b,d)$

$$\frac{1091\pi^5}{155520}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2+3(2n+1)^4(\mathbf{O}_n^{\langle 2\rangle})^2}{16^n(2n+1)^5}. \tag{32}$$

- Resolving linear system $A_4(a,b,d)$

$$\begin{aligned}&\frac{\pi}{8640}\Big\{10800\zeta(3)\ln 2+421\pi^4+7560\pi^2\ln^2 2\Big\}\\&=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{31+42(2n+1)\mathbf{O}_n+27(2n+1)^2\mathbf{O}_n^2}{16^n(2n+1)^5}.\end{aligned} \tag{33}$$

- Resolving linear system $A_5(a,b,d)$

$$\frac{245\pi^3}{216}\zeta(3)-\frac{49\pi}{144}\zeta(5)=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{33(2n+1)^3\mathbf{O}_n^{\langle 3\rangle}+41}{16^n(2n+1)^6}. \tag{34}$$

- Resolving linear system $A_5(a,b,d)$

$$\frac{35\pi^3}{288}\zeta(3)+\frac{1003\pi}{96}\zeta(5)=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{33(2n+1)^5\mathbf{O}_n^{\langle 5\rangle}+37}{16^n(2n+1)^6}. \tag{35}$$

- Resolving linear system $A_5(a,b,d)$

$$\frac{425\pi^3}{2592}\zeta(3)-\frac{125\pi}{864}\zeta(5)=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{4+33(2n+1)^2\mathbf{O}_n^{\langle 2\rangle}+33(2n+1)^5\mathbf{O}_n^{\langle 2\rangle}\mathbf{O}_n^{\langle 3\rangle}}{16^n(2n+1)^6}. \tag{36}$$

4.2. Series from Lemma 4

Under the parameter replacements

$$a\to 1+ax,\ b\to 1+bx,\ c\to\frac{3}{2}+cx,\ d\to\frac{3}{2}+d$$

the equality in Lemma 4 can be reformulated as

$$\begin{aligned}x(c+d-a-b)\sum_{k=0}^{\infty}\frac{(1+ax)_k(1+bx)_k}{(\frac{3}{2}+cx)_k(\frac{3}{2}+dx)_k}&=\sum_{n=0}^{\infty}\begin{Bmatrix}(\frac{1}{2}+cx+n)(1-ax-bx+cx+dx+2n)\\+(\frac{1}{2}-ax+dx+n)(\frac{1}{2}-bx+dx+n)\end{Bmatrix}\\&\times\frac{(\frac{1}{2}-ax+cx)_n(\frac{1}{2}-bx+cx)_n(\frac{1}{2}-ax+dx)_n(\frac{1}{2}-bx+dx)_n}{(\frac{3}{2}+cx)_n(\frac{3}{2}+dx)_n(1-ax-bx+cx+dx)_{2n+1}}.\end{aligned}$$

By applying Thomae's transformation (cf. Bailey [23], §3.2), we can further manipulate the series on the left hand side

$$x(c+d-a-b)\sum_{k=0}^{\infty}\frac{(1+ax)_k(1+bx)_k}{(\frac{3}{2}+cx)_k(\frac{3}{2}+dx)_k}=(c+d-a-b)\,{}_3F_2\left[\begin{matrix}1,\ 1+ax,1+bx\\ \frac{3}{2}+cx,\frac{3}{2}+dx\end{matrix}\middle|1\right]$$

$$= {}_3F_2\left[\begin{matrix} cx+dx-ax-bx, \frac{1}{2}+cx, \frac{1}{2}+dx \\ 1-ax+cx+dx, 1-bx+cx+dx \end{matrix} \middle| 1\right] \times \Gamma\left[\begin{matrix} 1-ax-bx+cx+dx, \frac{3}{2}+cx, \frac{3}{2}+dx \\ 1-ax+cx+dx, 1-bx+cx+dx \end{matrix}\right].$$

Now letting $c \to a+b-d$ and then $d \to b-d$, we derive, from the corresponding limiting case, the following summation formula.

Proposition 9.

$$\mathcal{D}(a,b,d;x) = \Gamma\left[\begin{matrix} \frac{1}{2}+ax+dx, \frac{1}{2}+bx-dx \\ 1+ax, 1+bx \end{matrix}\right] = \sum_{n=0}^{\infty} \left\{\begin{matrix} (1+2n)(\frac{1}{2}+ax+dx+n) \\ +(\frac{1}{2}-ax+bx-dx+n)(\frac{1}{2}-dx+n) \end{matrix}\right\}$$
$$\times \frac{(\frac{1}{2}+ax-bx+dx)_n(\frac{1}{2}-ax+bx-dx)_n(\frac{1}{2}+dx)_n(\frac{1}{2}-dx)_n}{(2n+1)!(\frac{1}{2}+ax+dx)_{n+1}(\frac{1}{2}+bx-dx)_{n+1}}.$$

By comparing the Maclaurin series coefficients across the equation

$$\mathcal{D}(a,b,d;x) = \sum_{m=0}^{\infty} x^m \mathrm{D}_m(a,b,d)$$

we establish several interesting infinite series identities as follows.

- Coefficient $\mathrm{D}_1(a,b,d)$

$$\pi \ln 2 = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2+3(2n+1)\mathbf{O}_n}{16^n(2n+1)^2}. \tag{37}$$

- Coefficient $\mathrm{D}_2(1,-1,-\frac{1}{2})$: (see Sun [19], Eq 1.3)

$$\frac{\pi^3}{648} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}\mathbf{O}_n^{\langle 2\rangle}}{16^n(2n+1)}. \tag{38}$$

To reduce lengthy expressions in the next few series, we adopt the notation below

$$\mathcal{O}_n^{\langle m\rangle} = (2n+1)^m \mathbf{O}_n^{\langle m\rangle} \quad \text{and} \quad \mathcal{O}_n^m = (2n+1)^m \mathbf{O}_n^m.$$

- Resolving combined linear system $\mathrm{A}_5(a,b,d)$ & $\mathrm{D}_5(a,b,d)$

$$\frac{4631\pi^5 \ln 2 + 27720\pi^3 \ln^3 2 - 3560\pi^3\zeta(3) + 59400\pi\zeta(3)\ln^2 2 + 32220\pi\zeta(5)}{8640}$$
$$= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{155+341\mathcal{O}_n+231\mathcal{O}_n^2+99\mathcal{O}_n^3}{16^n(2n+1)^6}. \tag{39}$$

- Resolving combined linear system $\mathrm{D}_5(a,b,d)$ & $\mathrm{D}_5(a,b,d)$

$$\frac{2783\pi^5 \ln 2 - 5470\pi^3\zeta(3) + 11640\pi\zeta(5)}{2880}$$
$$= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{149+297\mathcal{O}_n-231\mathcal{O}_n^{\langle 2\rangle}-297\mathcal{O}_n\mathcal{O}_n^{\langle 2\rangle}}{16^n(2n+1)^6}. \tag{40}$$

- Resolving combined linear system $D_5(a,b,d)$ & $D_5(a,b,d)$

$$\frac{1331\pi^5\ln 2-2450\pi^3\zeta(3)+4860\pi\zeta(5)}{17280}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{10+22O_n+22O_n^{\langle 4\rangle}+33O_nO_n^{\langle 4\rangle}}{16^n(2n+1)^6}. \tag{41}$$

- Resolving combined linear system $D_5(a,b,d)$ & $D_5(a,b,d)$

$$\frac{12001\pi^5\ln 2-17510\pi^3\zeta(3)+18420\pi\zeta(5)}{5760}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{284+594O_n+66O_n^{\langle 2\rangle}+594(O_n^{\langle 2\rangle})^2+891O_n(O_n^{\langle 2\rangle})^2}{16^n(2n+1)^6}. \tag{42}$$

- Resolving combined linear system $D_5(a,b,d)$ & $D_5(a,b,d)$

$$\frac{\pi}{3456}\left\{\begin{matrix}16753\pi^4\ln 2-12670\pi^2\zeta(3)+332640\zeta(3)\ln^2 2\\ +130356\zeta(5)+47520\pi^2\ln^3 2-114048\ln^5 2\end{matrix}\right\}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1324+2530O_n+990O_n^2-330O_n^4-99O_n^5}{16^n(2n+1)^6}. \tag{43}$$

- Resolving combined linear system $D_5(a,b,d)$ & $D_5(a,b,d)$

$$\frac{283\pi^5\ln 2-1600\pi^3\zeta(3)+21600\pi\zeta(3)\ln^2 2+270\pi\zeta(5)}{8640}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{4+22O_n+9O_n^2+12O_nO_n^{\langle 3\rangle}+91O_n^2O_n^{\langle 3\rangle}}{16^n(2n+1)^6}. \tag{44}$$

5. Series containing $\binom{2n}{n}$ in denominators

According to Lemmas 1 and 2, we shall examine, in this section, infinite series containing the central binomial coefficient $\binom{2n}{n}$ in denominators.

5.1. Series from Lemma 1

By specifying the parameters in Lemma 1

$$a\to ax,\ b\to bx,\ d\to dx$$

we deduce the equality as in the following proposition.

Proposition 10.

$$\mathcal{A}(a,b,d;x)=\Gamma\left[\begin{matrix}1+ax-bx,\frac{1}{2}+ax+bx,1+dx,\frac{1}{2}+dx\\ 1+ax,\frac{1}{2}+ax,\frac{1}{2}+bx+dx,1-bx+dx\end{matrix}\right]=\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\left[\begin{matrix}2ax,2bx,1-2bx,ax-dx\\ 1,1+ax-bx,\frac{1}{2}+ax+bx,1+2dx\end{matrix}\right]_n\frac{2ax+3n}{2ax}.$$

By expanding both sides of the above equation into power series in x

$$\mathcal{A}(a,b,d;x) = \sum_{m=0}^{\infty} x^m \mathrm{A}_m(a,b,d)$$

we can derive the following remarkable infinite series identities. Among them, Sun [41] first conjectured and Albinger [39] subsequently confirmed the first two (45) and (46), as well as

$$\frac{2}{3}\zeta(3) = \sum_{n=1}^{\infty} \frac{1+n\bar{\mathbf{H}}_{2n}}{n^3\binom{2n}{n}} = \sum_{n=1}^{\infty} \frac{1+n(\mathbf{H}_{2n}-\mathbf{H}_n)}{n^3\binom{2n}{n}},$$

where this last identity is just a linear combination of (45) and (46).

- Coefficient $\mathrm{A}_3(1,-1,1)$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{3n\mathbf{H}_n - 1}{n^3\binom{2n}{n}}. \tag{45}$$

- Coefficient $\mathrm{A}_3(1,1,1)$

$$5\zeta(3) = \sum_{n=1}^{\infty} \frac{6n\mathbf{O}_n + 5}{n^3\binom{2n}{n}}. \tag{46}$$

- Resolving linear system $\mathrm{A}_5(a,b,d)$: (cf. Chu [7], Proposition 3.1)

$$\frac{5\pi^2\zeta(3)+6\zeta(5)}{18} = \sum_{n=1}^{\infty} \frac{9n\mathbf{H}_n - 2}{n^5\binom{2n}{n}}. \tag{47}$$

- Resolving linear system $\mathrm{A}_5(a,b,d)$

$$\frac{5\pi^2\zeta(3)+402\zeta(5)}{36} = \sum_{n=1}^{\infty} \frac{9n\mathbf{O}_n + 17}{n^5\binom{2n}{n}}. \tag{48}$$

- Resolving linear system $\mathrm{A}_5(a,b,d)$: This was conjectured by Sun [18] and confirmed subsequently by Ablinger (cf. [39](108)) and Chu [4,7].

$$\frac{\pi^2\zeta(3)+3\zeta(5)}{27} = \sum_{n=1}^{\infty} \frac{n^3\mathbf{H}_n^{\langle 3\rangle}}{n^5\binom{2n}{n}}. \tag{49}$$

- Resolving linear system $\mathrm{A}_5(a,b,d)$

$$\frac{8\pi^2\zeta(3)-93\zeta(5)}{9} = \sum_{n=1}^{\infty} \frac{2+9n^2\mathbf{H}_n^2-9n^3\mathbf{H}_n^3}{n^5\binom{2n}{n}}. \tag{50}$$

- Resolving linear system $\mathrm{A}_5(a,b,d)$

$$\frac{13\pi^2\zeta(3)-33\zeta(5)}{9} = \sum_{n=1}^{\infty} \frac{4-9n^2\mathbf{H}_n^{\langle 2\rangle}+27n^3\mathbf{H}_n\mathbf{H}_n^{\langle 2\rangle}}{n^5\binom{2n}{n}}. \tag{51}$$

- Resolving linear system $A_5(a,b,d)$

$$\frac{22\pi^2\zeta(3)+246\zeta(5)}{9}=\sum_{n=1}^{\infty}\frac{4+45n^2\mathbf{H}_n^{\langle 2\rangle}+54n^3\mathbf{H}_n^{\langle 2\rangle}\mathbf{O}_n}{n^5\binom{2n}{n}}. \tag{52}$$

- Resolving linear system $A_6(a,b,d)$: Conjectured by Sun (cf. [41], Eq 4.6) and proved first by Albinger [39] and then by Chu (cf. [7], Proposition 3.4)

$$\frac{313\pi^6}{612360}=\sum_{n=1}^{\infty}\frac{2-n^2\mathbf{H}_n^{\langle 2\rangle}}{n^6\binom{2n}{n}}. \tag{53}$$

- Resolving linear system $A_6(a,b,d)$: Chu (cf. [7], Proposition 3.4)

$$\frac{163\pi^6}{136080}=\sum_{n=1}^{\infty}\frac{3n^4\mathbf{H}_n^{\langle 4\rangle}-1}{n^6\binom{2n}{n}}. \tag{54}$$

- Resolving linear system $A_6(a,b,d)$: Chu (cf. [7], Proposition 3.4)

$$\frac{65\pi^6}{34992}=\sum_{n=1}^{\infty}\frac{7-3n^4(\mathbf{H}_n^{\langle 2\rangle})^2}{n^6\binom{2n}{n}}. \tag{55}$$

5.2. Series from Lemma 2

By specifying the parameters in Lemma 2

$$a\to ax,\ b\to\frac{1}{2}+bx,\ d\to dx$$

we deduce the equality as in the following proposition.

Proposition 11.

$$\begin{aligned}\mathcal{B}(a,b,d;x)&=\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2}+ax-bx,1+ax-dx,1+bx+dx\\ 1+ax,1+bx,\frac{1}{2}+dx,\frac{1}{2}+ax-bx-dx\end{matrix}\right]\\ &=\sum_{n=0}^{\infty}\Big(\frac{1}{4}\Big)^n\left[\begin{matrix}ax,1+2bx,2dx,2ax-2bx-2dx\\ 1,1+ax-dx,1+bx+dx,\frac{1}{2}+ax-bx\end{matrix}\right]_n\frac{2ax+3n}{2ax}.\end{aligned}$$

By expanding both sides of the above equation into power series in x

$$\mathcal{B}(a,b,d;x)=\sum_{m=0}^{\infty}x^m\mathrm{B}_m(a,b,d)$$

we can derive the following significant infinite series identities. Among them, the initial identity (56) is equivalent to conjecture 10.53(i) by Sun [17], which was first proved by Chu (cf. [4], example 3.13) and recently by Xu and Zhao [44].

- Resolving linear system $B_4(a, b, d)$

$$\frac{13\pi^4}{1620} = \sum_{n=1}^{\infty} \frac{3n\mathbf{H}_n^2 - 2\mathbf{H}_n}{n^3\binom{2n}{n}}. \tag{56}$$

- Resolving linear system $B_4(a, b, d)$

$$\frac{\pi^4}{16} = \sum_{n=1}^{\infty} \frac{5\mathbf{O}_n + 3n\mathbf{O}_n^{\langle 2\rangle} + 3n\mathbf{O}_n^2}{n^3\binom{2n}{n}}. \tag{57}$$

- Resolving linear system $B_4(a, b, d)$

$$\frac{29\pi^4}{540} = \sum_{n=1}^{\infty} \frac{5\mathbf{H}_n - 2\mathbf{O}_n + 6n\mathbf{H}_n\mathbf{O}_n}{n^3\binom{2n}{n}}. \tag{58}$$

- Resolving linear system $B_4(a, b, d)$

$$\frac{61\pi^4}{4860} = \sum_{n=1}^{\infty} \frac{n\mathbf{O}_n^{\langle 2\rangle} + n\bar{\mathbf{H}}_{2n}^2 + 2\bar{\mathbf{H}}_{2n}}{n^3\binom{2n}{n}}. \tag{59}$$

- Resolving linear system $B_4(a, b, d)$

$$\frac{80\pi^4}{243} = \sum_{n=1}^{\infty} \frac{4\mathbf{O}_n^{\langle 2\rangle} + (5\mathbf{H}_n + 2\mathbf{O}_n)^2}{n^2\binom{2n}{n}}. \tag{60}$$

- Resolving linear system $B_5(a, b, d)$

$$\frac{130\pi^2\zeta(3) - 1896\zeta(5)}{9} = \sum_{n=1}^{\infty} \frac{58 + 9n\mathbf{H}_n(2 - 3n\mathbf{H}_n)(5 + 6n\mathbf{O}_n)}{n^5\binom{2n}{n}}. \tag{61}$$

- Resolving combined linear system $A_6(a, b, d)$ & $B_6(a, b, d)$

$$\frac{631\pi^6 + 340200\zeta(3)^2}{68040} = \sum_{n=1}^{\infty} \frac{16 - 12n\mathbf{H}_n + 27n^2\mathbf{H}_n^2}{n^6\binom{2n}{n}}. \tag{62}$$

- Resolving combined linear system $A_6(a, b, d)$ & $B_6(a, b, d)$

$$\frac{673\pi^6 + 136080\zeta(3)^2}{136080} = \sum_{n=1}^{\infty} \frac{5 - 3n^3\mathbf{H}_n^{\langle 3\rangle} + 9n^4\mathbf{H}_n\mathbf{H}_n^{\langle 3\rangle}}{n^6\binom{2n}{n}}. \tag{63}$$

- Resolving combined linear system $A_6(a, b, d)$ & $B_6(a, b, d)$

$$\frac{1277\pi^6 + 340200\zeta(3)^2}{68040} = \sum_{n=1}^{\infty} \frac{14 + 15n^3\mathbf{H}_n^{\langle 3\rangle} + 18n^4\mathbf{O}_n\mathbf{H}_n^{\langle 3\rangle}}{n^6\binom{2n}{n}}. \tag{64}$$

- Resolving combined linear system $A_7(a,b,d)$ & $B_7(a,b,d)$

$$\frac{62\pi^2\zeta(5)-75\zeta(7)}{12}=\sum_{n=1}^{\infty}\frac{4-45n^3\mathbf{H}_n^{\langle 3\rangle}+123n^5\mathbf{H}_n^{\langle 5\rangle}}{n^7\binom{2n}{n}}. \tag{65}$$

- Resolving combined linear system $A_7(a,b,d)$ & $B_7(a,b,d)$

$$\frac{533\pi^4\zeta(3)+465\pi^2\zeta(5)+153495\zeta(7)}{7290}=\sum_{n=1}^{\infty}\frac{291n^3\mathbf{H}_n^{\langle 3\rangle}-205n^5\mathbf{H}_n^{\langle 2\rangle}\mathbf{H}_n^{\langle 3\rangle}-4}{n^7\binom{2n}{n}}. \tag{66}$$

6. Series containing $\binom{2n}{n}^3$ in denominators

Finally, by applying Lemmas 2 and 3, we are going to evaluate, in closed form, a few infinite series involving the cubic central binomial coefficient $\binom{2n}{n}^3$ in denominators, including a couple of conjectured ones made recently by Sun [19].

6.1. Series from Lemma 2

By making the replacements in Lemma 2

$$a\to 1+ax,\ b\to\frac{1}{2}+bx,\ d\to\frac{1}{2}+dx$$

we obtain the equality as in the proposition below.

Proposition 12.

$$\mathcal{B}(a,b,d;x)=2\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2}+bx+dx,\frac{1}{2}+ax-bx,\frac{1}{2}+ax-dx\\ 1+ax,1+bx,1+dx,1+ax-bx-dx\end{matrix}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(1+ax)_n(1+2bx)_n(1+2dx)_n(1+2ax-2bx-2dx)_n}{n!(\frac{1}{2}+ax-bx)_{n+1}(\frac{1}{2}+ax-dx)_{n+1}(\frac{1}{2}+bx+dx)_{n+1}}\frac{2+2ax+3n}{4^n}.$$

In view of the power series expansion in x

$$\mathcal{B}(a,b,d;x)=\sum_{m=0}^{\infty}x^m B_m(a,b,d)$$

we recover from the first coefficient $B_0(a,b,d)$ Guilera's series for π^2 stated in the introduction. Further identities are highlighted as follows.

- Coefficient $B_1(1,0,0)$

$$\pi^2\ln 2=\sum_{n=0}^{\infty}\frac{(n!)^3}{4^n(\frac{3}{2})_n^3}\Big\{(2+3n)(4\mathbf{O}_{n+1}-3\mathbf{H}_n)-2\Big\}. \tag{67}$$

- Coefficients $\mathrm{B}_2(0,1,1) \rightleftharpoons [ab]\mathrm{B}_2(a,b,d)$: Conjectured by Sun (cf. [19], Eq 3.13) and confirmed in [40,43]

$$\frac{\pi^4}{48} = \sum_{n=0}^{\infty} \frac{(n!)^3}{4^n(\frac{3}{2})_n^3}\Big\{(2+3n)(\mathbf{O}_{n+1}^{\langle 2\rangle} - \mathbf{H}_n^{\langle 2\rangle})\Big\}. \tag{68}$$

- Coefficients $\mathrm{B}_3(0,1,1) \rightleftharpoons [bd^2]\mathrm{B}_3(a,b,d)$: Conjectured by Sun (cf. [19], Eq 3.18)

$$\frac{\pi^2}{4}\zeta(3) = \sum_{n=0}^{\infty} \frac{(n!)^3}{4^n(\frac{3}{2})_n^3}\Big\{(2+3n)(\mathbf{O}_{n+1}^{\langle 3\rangle} + \mathbf{H}_n^{\langle 3\rangle})\Big\}. \tag{69}$$

- Coefficients $\mathrm{B}_4(0,1,1) \rightleftharpoons [b^2d^2]\mathrm{B}_4(a,b,d)$

$$\frac{\pi^6}{240} = \sum_{n=0}^{\infty} \frac{(n!)^3(2+3n)}{4^n(\frac{3}{2})_n^3}\Big\{\mathbf{O}_{n+1}^{\langle 4\rangle} - \mathbf{H}_n^{\langle 4\rangle} + (\mathbf{O}_{n+1}^{\langle 2\rangle} - \mathbf{H}_n^{\langle 2\rangle})^2\Big\}. \tag{70}$$

- Coefficient $\mathrm{B}_5(0,1,1)$

$$\frac{\pi^4}{48}\zeta(3) + \frac{\pi^2}{4}\zeta(5) = \sum_{n=0}^{\infty} \frac{(n!)^3(2+3n)}{4^n(\frac{3}{2})_n^3}\Big\{\mathbf{O}_{n+1}^{\langle 5\rangle} + \mathbf{H}_n^{\langle 5\rangle} + (\mathbf{O}_{n+1}^{\langle 2\rangle} - \mathbf{H}_n^{\langle 2\rangle})(\mathbf{O}_{n+1}^{\langle 3\rangle} + \mathbf{H}_n^{\langle 3\rangle})\Big\}. \tag{71}$$

6.2. Series from Lemma 3

By making the replacements in Lemma 3

$$a \to \frac{3}{2} + ax, \; b \to \frac{1}{2} + bx, \; d \to dx$$

we obtain the equality as in the proposition below.

Proposition 13.

$$\begin{aligned} C(a,b,d;x) &= \Gamma\begin{bmatrix} \frac{1}{2}+ax+dx, \frac{1}{2}+bx-dx \\ \frac{3}{2}+ax, \frac{1}{2}+bx \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{(dx)_n(1-dx)_n(1+ax-bx+dx)_n(bx-ax-dx)_n}{(\frac{1}{2}+ax+dx)_{n+1}(\frac{1}{2}+bx-dx)_{n+1}(2n+1)!} \\ &\times \Big\{(1+2n)(\tfrac{1}{2}+n+bx-dx) + (n+dx)(1+n+ax-bx+dx)\Big\}. \end{aligned}$$

In view of the power series expansion in x

$$C(a,b,d;x) = \sum_{m=0}^{\infty} x^m \mathrm{C}_m(a,b,d)$$

we can prove the following curious infinite series identities.

- Coefficient $\mathrm{C}_2(a,b,d)$

$$6 - \frac{\pi^2}{2} = \sum_{n=1}^{\infty} \frac{(n!)^3}{(\frac{3}{2})_n^3}\frac{1+6n+6n^2}{4^n \times n^2}. \tag{72}$$

- Coefficient $C_3(1,-1,-1)$: Sun (cf. [42], Eq 1.77)

$$\frac{\pi^2}{2}-4=\sum_{n=1}^{\infty}\frac{(n!)^3}{(\frac{3}{2})_n^3}\frac{(1+2n)(1+3n)}{4^n\times n^3}. \tag{73}$$

- Resolving linear system $C_4(a,b,d)$

$$\pi^2+7\zeta(3)-16=\sum_{n=1}^{\infty}\frac{(n!)^3}{(\frac{3}{2})_n^3}\frac{(1+4n)+2(1+2n)(1+3n)\mathbf{O}_n}{4^n\times n^3}. \tag{74}$$

- Coefficient $C_4(1,-1,-1)$

$$\frac{\pi^4-48\pi^2+384}{24}=\sum_{n=1}^{\infty}\frac{(n!)^3}{(\frac{3}{2})_n^3}\left\{\frac{(1+2n)(3+8n)}{4^n\times n^4}-\frac{2(1+6n+6n^2)\mathbf{H}_n^{\langle 2\rangle}}{4^n\times n^2}\right\}. \tag{75}$$

- Coefficient $C_5(1,-1,-1)$

$$\frac{\pi^4-48\pi^2+384}{-24}=\sum_{n=1}^{\infty}\frac{(n!)^3}{(\frac{3}{2})_n^3}\left\{\frac{(1+2n)(2+5n)}{4^n\times n^5}-\frac{2(1+2n)(1+3n)\mathbf{H}_n^{\langle 2\rangle}}{4^n\times n^3}\right\}. \tag{76}$$

- Coefficient $C_4(-1,1,1+\sqrt{-1})$

$$\frac{\pi^4-8\pi^2-32}{-8}=\sum_{n=1}^{\infty}\frac{(n!)^3}{(\frac{3}{2})_n^3}\left\{\frac{2(1+6n+6n^2)\mathbf{O}_n^{\langle 2\rangle}}{4^n\times n^2}-\frac{1+8n+22n^2+20n^3+4n^4}{4^n\times n^4(1+2n)^2}\right\}. \tag{77}$$

- Coefficient $C_5(-1,1,1+\sqrt{-1})$

$$\frac{\pi^4-8\pi^2}{8}=\sum_{n=1}^{\infty}\frac{(n!)^3}{(\frac{3}{2})_n^3}\left\{\frac{1+6n+10n^2}{4^n\times n^4(1+2n)}+\frac{2(1+2n)(1+3n)\mathbf{O}_n^{\langle 2\rangle}}{4^n\times n^3}\right\}. \tag{78}$$

- Coefficient $C_6(1,-1,-1)$

$$\begin{aligned}\frac{\pi^6-120\pi^4+5760\pi^2-46080}{-720}&=\sum_{n=1}^{\infty}\frac{(n!)^3(1+2n)}{(\frac{3}{2})_n^3}\\ &\times\left\{\frac{5+12n}{4^n\times n^6}-\frac{2(3+8n)\mathbf{H}_n^{\langle 2\rangle}}{4^n\times n^4}+\frac{(1+6n+6n^2)[2(\mathbf{H}_n^{\langle 2\rangle})^2-\mathbf{H}_n^{\langle 4\rangle}]}{4^n\times n^2(1+2n)}\right\}.\end{aligned} \tag{79}$$

- Coefficient $C_7(1,-1,-1)$

$$\begin{aligned}\frac{\pi^6-120\pi^4+5760\pi^2-46080}{720}&=\sum_{n=1}^{\infty}\frac{(n!)^3(1+2n)}{(\frac{3}{2})_n^3}\\ &\times\left\{\frac{3+7n}{4^n\times n^7}-\frac{2(2+5n)\mathbf{H}_n^{\langle 2\rangle}}{4^n\times n^5}+\frac{(1+3n)[2(\mathbf{H}_n^{\langle 2\rangle})^2-\mathbf{H}_n^{\langle 4\rangle}]}{4^n\times n^3}\right\}.\end{aligned} \tag{80}$$

There is a curious phenomenon among the just displayed series.

$$\begin{aligned}2 &= \text{Equation}(72) + \text{Equation}(73),\\ 0 &= \text{Equation}(75) + \text{Equation}(76),\\ 4 &= \text{Equation}(77) + \text{Equation}(78),\\ 0 &= \text{Equation}(79) + \text{Equation}(80).\end{aligned}$$

However, these equalities are not immediate from the related series involved.

7. Conclusions

By employing the hypergeometric series approach, we have exhibited numerous striking infinite series identities, including several difficult ones conjectured by Sun [17–19]. However, our list is far from exhaustive. For instance, under the parameter setting

$$a \to \frac{1}{4} + ax,\ b \to \frac{1}{4} + bx,\ d \to \frac{1}{4} + dx,$$

Lemma 2 would lead to several unusual series represented by

$$\frac{\sqrt{2}\Gamma^2(\frac{1}{4})}{\pi^{5/2}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{4})_n}{4^n(n!)^4}\{1 + 6n\}.$$

The interested reader is enthusiastically encouraged to make further explorations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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