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*Research article*

# The sixth power mean of one kind generalized two-term exponential sums and their asymptotic properties

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**Abstract:** The main aim of this article is using the elementary method and the number of the solutions of some congruence equations modulo an odd prime $p$, to study the calculating problem of the sixth power mean of one kind generalized two-term exponential sums, and give a sharp asymptotic formula for it.

**Keywords:** the generalized two-term exponential sums; sixth power mean; elementary method; asymptotic formula

## 1. Introduction

As usual, we let $p$ be an odd prime, $\chi$ denotes a Dirichlet character modulo $p$. For any integers $k > h \geq 1$ and integers $m$ and $n$, the generalized two-term exponential sum $S(m, n, k, h, \chi; p)$ is defined as follows:

$$S(m,n,k,h,\chi;p) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k + na^h}{p}\right),$$

where $e(y) = e^{2\pi i y}$ and $i$ is the imaginary unit.

In analytic number theory, these sums play important roles, many classical number theory problems are closely related to it, such as the prime distribution and Waring's problems etc. And because of that, many number theorists and scholars had studied the various properties of $S(m, n, k, h, \chi; p)$, and obtained a series of meaningful research results. For example, R. Duan and W. P. Zhang [1] proved that for any prime $p$ with $3 \nmid (p-1)$, and any Dirichlet character $\lambda$ mod $p$, one has the identities

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 3p^3 - 8p^2 & \text{if } \lambda = \left(\frac{*}{p}\right), \\ 2p^3 - 7p^2 & \text{if } \lambda \neq \chi_0, \lambda \neq \left(\frac{*}{p}\right), \\ 2p^3 - 3p^2 - 3p - 1 & \text{if } \lambda = \chi_0, \end{cases}$$

where $\left(\frac{*}{p}\right)$ is the Legendre symbol and $\chi_0$ is the principal character modulo $p$.

L. Chen and X. Wang [2] used the elementary method to prove the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = \begin{cases} 2p^2(p-2), & \text{if } p\equiv 7 \bmod 12,\\ 2p^3, & \text{if } p\equiv 11 \bmod 12,\\ 2p(p^2-10p-2\alpha^2), & \text{if } p\equiv 1 \bmod 24,\\ 2p(p^2-4p-2\alpha^2), & \text{if } p\equiv 5 \bmod 24,\\ 2p(p^2-6p-2\alpha^2), & \text{if } p\equiv 13 \bmod 24,\\ 2p(p^2-8p-2\alpha^2), & \text{if } p\equiv 17 \bmod 24,\end{cases}$$

where the character sum $\alpha=\alpha(p)=\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a^3+a}{p}\right)$ is an integer satisfying the identity (see [3, Theorem 4–11]):

$$p=\alpha^2+\beta^2=\left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a^3+a}{p}\right)\right)^2+\left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a^3+ra}{p}\right)\right)^2,$$

and $r$ denotes any quadratic non-residue modulo $p$.

Recently, W. P. Zhang and Y. Y. Meng [4] studied the sixth power mean of $S(m,n,3,1,\chi_0;p)$, and proved that for any odd prime $p$ and integer $n$ with $(n,p)=1$, one has the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^6 = \begin{cases} 5p^3(p-1) & \text{if } p\equiv 5 \bmod 6;\\ p^2\left(5p^2-23p-d^2\right) & \text{if } p\equiv 1 \bmod 6,\end{cases}$$

where $4p=d^2+27\cdot b^2$, and $d$ is uniquely determined by $d\equiv 1 \bmod 3$ and $b>0$.

In addition, X. Y. Liu and W. P. Zhang [5] studied the calculating problem of the sixth power mean of the generalized two-term exponential sums

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right|^6, \tag{1.1}$$

and proved that for any odd prime $p$ with $3\nmid(p-1)$, one has the identity

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right|^6 = p(p-1)\left(6p^3-28p^2+39p+5\right).$$

Some papers related to the exponential sums and the generalized exponential sums can also be found in [6–10], to save space, we will not list them all here.

Of course, the results in [5] are very neat and beautiful, the only drawback is that they do not talk about the case $3\mid(p-1)$. Then for prime $p$ with $p\equiv 1 \bmod 3$, what is going to happen? This seems to be an open problem. In this paper, we will use the elementary and analytic methods, and the number of the solutions of some congruence equations to study this problem, and prove the following conclusion:

**Theorem.** For any prime $p$ with $p \equiv 1 \bmod 3$, we have the asymptotic formula

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3 + a}{p}\right) \right|^6 = 6p^5 + O\left(p^4\right).$$

Combining our theorem and the result in [5] we can deduce the following:

**Corollary.** For any odd prime $p$, we have the asymptotic formula

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3 + a}{p}\right) \right|^6 = 6p^5 + O\left(p^4\right).$$

**Some notes:** For prime $p$ with $p \equiv 1 \bmod 3$, we can only get an asymptotic formula for (1.1), but can not get an exact calculating formula. The reason is that we can not get an exact value in Lemma 6.

Whether there exists an exact calculating formula for (1.1) with $p \equiv 1 \bmod 3$ is an open problem. It remains to be further studied.

## 2. Several lemmas

In this section, we decompose the proof of the theorem into six simple lemmas. Of course, the proofs of these lemmas need some knowledge of elementary or analytic number theory, all these can be found in [3, 11, 12], we will not repeat them here. First we have the following:

**Lemma 1.** Let $p$ be an odd prime with $p \equiv 1 \bmod 3$, then we have the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{\substack{d=1 \\ a+b+c \equiv d+e+1 \bmod p \\ abc \equiv de \bmod p}}^{p-1} \sum_{e=1}^{p-1} 1 = p^3 - 3p^2 + 5p - 5.$$

**Proof.** For any integer $a$, from the properties of the character sums

$$\sum_{\chi \bmod p} \chi(a) = \begin{cases} p-1 & \text{if } a \equiv 1 \bmod p; \\ 0 & \text{otherwise,} \end{cases}$$

and the properties of the classical Gauss sums modulo $p$

$$\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma}{p}\right) = \overline{\chi}(m) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right) = \overline{\chi}(m) \tau(\chi),$$

we have

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left( \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma}{p}\right) \right)^3 \left( \sum_{a=1}^{p-1} \overline{\chi}(a) e\left(\frac{-ma}{p}\right) \right)^3$$

$$= p(p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{\substack{d=1 \\ a+b+c \equiv d+e+f \bmod p \\ abc \equiv def \bmod p}}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} 1 = p(p-1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{\substack{d=1 \\ af+bf+cf \equiv df+ef+f \bmod p \\ abcf^3 \equiv def^3 \bmod p}}^{p-1} \sum_{e=1}^{p-1} \sum_{f=1}^{p-1} 1$$

$$= p(p-1)^2 \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1. \tag{2.1}$$

On the other hand, we also have

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left( \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma}{p}\right) \right)^3 \left( \sum_{a=1}^{p-1} \overline{\chi}(a) e\left(\frac{-ma}{p}\right) \right)^3$$
$$= (p-1)^6 + \sum_{\chi \bmod p} \sum_{m=1}^{p-1} \tau^3(\chi) \overline{\tau(\chi)}^3$$
$$= (p-1)^6 + (p-1) + (p-1) \sum_{\chi \neq \chi_0} \tau^3(\chi) \overline{\tau(\chi)}^3$$
$$= (p-1)^6 + (p-1)\left[1 + (p-2)p^3\right] = p(p-1)^2 \left(p^3 - 3p^2 + 5p - 5\right), \tag{2.2}$$

where $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums, $\chi_0$ denotes the principal character modulo $p$, $\overline{\chi}(a)$ and $\overline{\tau(\chi)}$ denote the complex conjugate of $\chi(a)$ and $\tau(\chi)$.

Combining (2.1) and (2.2) we have the identity

$$\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} 1 = p^3 - 3p^2 + 5p - 5.$$

This proves Lemma 1.

**Lemma 2.** Let $p$ be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order character $\lambda$ modulo $p$, we have the identity

$$\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a)\overline{\lambda}(d) = \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \overline{\lambda}(a)\lambda(d)$$
$$= \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) = \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \overline{\lambda}(a) = p^2 - 2p - 1.$$

**Proof.** First from the properties of the reduced residue system modulo $p$ we have

$$\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) = \sum_{\substack{a=1 \\ a-1+d(b-1)+e(c-1)\equiv 0 \bmod p \\ abc\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a). \tag{2.3}$$

For any integer $b$ with $(b, p) = 1$, let $\overline{b}$ denotes the multiplicative inverse of $b$ modulo $p$. If $a = b = c = 1$, then the congruence equations $a - 1 + d(b-1) + e(c-1) \equiv 0 \bmod p$ and $abc \equiv 1 \bmod p$ have $(p-1)^2$ solutions and $\lambda(1) = 1$.

If $a = 1$, $b \neq 1$ and $c = \overline{b}$, then the congruence equations $a - 1 + d(b-1) + e(c-1) \equiv 0 \bmod p$ and $abc \equiv 1 \bmod p$ have $(p-1)(p-2)$ solutions.

Similarly, if $b = 1$, $a \neq 1$ and $c = \overline{a}$, then we have

$$\sum_{\substack{a=2 \\ a-1+e(\overline{a}-1)\equiv 0 \bmod p}}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) = (p-1)\sum_{a=2}^{p-1} \lambda(a) = -(p-1). \tag{2.4}$$

If $c = 1$, $a \neq 1$ and $b = \overline{a}$, then we also have

$$\sum_{\substack{a=2 \\ a-1+d(\overline{a}-1)\equiv 0 \bmod p}}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) = (p-1)\sum_{a=2}^{p-1} \lambda(a) = -(p-1). \tag{2.5}$$

If $a \neq 1$, $b \neq 1$, $c \neq 1$ and $abc \equiv 1 \bmod p$, then we have

$$\begin{aligned}
\sum_{\substack{a=2 \\ a-1+d(b-1)+e(c-1)\equiv 0 \bmod p \\ abc\equiv 1 \bmod p}}^{p-1} \sum_{b=2}^{p-1} \sum_{c=2}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) &= \sum_{\substack{a=2 \\ 1+d+e\equiv 0 \bmod p \\ abc\equiv 1 \bmod p}}^{p-1} \sum_{b=2}^{p-1} \sum_{c=2}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) \\
&= (p-2)\left[(p-1)\sum_{a=1}^{p-1} \lambda(a) - (p-2) - 2\sum_{a=2}^{p-1} \lambda(a) - 1\right] \\
&= (p-2)(3-p).
\end{aligned} \tag{2.6}$$

Combining (2.3)–(2.6) we have the identity

$$\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) = p^2 - 2p - 1. \tag{2.7}$$

From the properties of the reduced residue system modulo $p$ and (2.7) we also have

$$\begin{aligned}
\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a)\overline{\lambda}(d) &= \sum_{\substack{a=1 \\ ad+bd+cd\equiv d+ed+1 \bmod p \\ abcd^3\equiv d^2 e \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(ad)\overline{\lambda}(d) \\
= \sum_{\substack{a=1 \\ a+b+c\equiv 1+e+\overline{d} \bmod p \\ abc\equiv \overline{d}e \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) &= \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(a) = p^2 - 2p - 1.
\end{aligned} \tag{2.8}$$

Now Lemma 2 follows from (2.7) and (2.8).

**Lemma 3.** Let $p$ be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order character $\lambda$ modulo $p$, we have the identity

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\lambda(d) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\overline{\lambda}(d)$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\lambda(a)\overline{\lambda}(b) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\lambda(b)\overline{\lambda}(a) = -(p-1).$$

**Proof.** First from the properties of the reduced residue system we have

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\lambda(d) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ d(a-1)+e(b-1)+c-1\equiv 0 \bmod p\\ abc\equiv 1 \bmod p}}^{p-1}\lambda(d). \tag{2.9}$$

If $a = b = c = 1$, then from (2.9) we have

$$\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(d) = 0. \tag{2.10}$$

If $a = 1$, $b \neq 1$ and $c = \overline{b}$, then from (2.9) we have

$$\sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ e(b-1)+c-1\equiv 0 \bmod p\\ bc\equiv 1 \bmod p}}^{p-1}\lambda(d) = \sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\sum_{\substack{e=1\\ e(b-1)+c-1\equiv 0 \bmod p\\ bc\equiv 1 \bmod p}}^{p-1}\sum_{d=1}^{p-1}\lambda(d) = 0. \tag{2.11}$$

Similarly, if $b = 1$, $a \neq 1$ and $c = \overline{a}$, then from (2.9) we have

$$\sum_{a=2}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ d(a-1)+\overline{a}-1\equiv 0 \bmod p}}^{p-1}\lambda(d) = (p-1)\sum_{a=2}^{p-1}\lambda(d) = -(p-1). \tag{2.12}$$

If $c = 1$, $a \neq 1$ and $b = \overline{a}$, then we also have

$$\sum_{a=2}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ d(a-1)+e(\overline{a}-1)\equiv 0 \bmod p}}^{p-1}\lambda(d) = \sum_{a=2}^{p-1}\sum_{e=1}^{p-1}\lambda\left(\overline{a}\right)\lambda(e) = 0. \tag{2.13}$$

If $a \neq 1$, $b \neq 1$, $c \neq 1$ and $abc \equiv 1 \bmod p$, then note that $\lambda\left(\overline{a}\right) = \overline{\lambda}(a)$, from (2.9) we have the identity

$$\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ d(a-1)+e(b-1)+(c-1)\equiv 0 \bmod p\\ abc\equiv 1 \bmod p}}^{p-1}\lambda(d) = \sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ d+e+c-1\equiv 0 \bmod p\\ abc\equiv 1 \bmod p}}^{p-1}\overline{\lambda}(a-1)\lambda(d)$$

$$
\begin{aligned}
&= \sum_{\substack{a=2\\ abc\equiv 1 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\sum_{e=1}^{p-1}\overline{\lambda}(a-1)\lambda(1-c-e) = -\sum_{\substack{a=1\\ abc\equiv 1 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\sum_{c=1}^{p-1}\overline{\lambda}(a-1)\lambda(c-1)\\
&= -\sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\overline{\lambda}(a-1)\lambda(c-1) + \sum_{a=1}^{p-1}\overline{\lambda}(a-1)\lambda\left(\overline{a}-1\right)\\
&= -1 - \sum_{a=2}^{p-1}\lambda\left(\overline{a}\right) = 0. \qquad (2.14)
\end{aligned}
$$

Combining (2.9)–(2.14) we have the identity

$$
\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(d) = -(p-1). \qquad (2.15)
$$

From the properties of the reduced residue system modulo $p$ and (2.15) we have

$$
\begin{aligned}
&\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(a)\overline{\lambda}(b) = \sum_{\substack{a=1\\ ab+b+cb\equiv db+eb+1 \bmod p\\ ab^3c\equiv dbeb \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(ab)\overline{\lambda}(b)\\
&= \sum_{\substack{a=1\\ a+1+c\equiv d+e+\overline{b} \bmod p\\ ac\equiv de\overline{b} \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(a) = \sum_{\substack{a=1\\ a+1+c\equiv d+e+b \bmod p\\ ac\equiv deb \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(a) = -(p-1). \qquad (2.16)
\end{aligned}
$$

Now Lemma 3 follows from (2.15) and (2.16).

**Lemma 4.** Let $p$ be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order character $\lambda$ modulo $p$, we have the identities

$$
\left|\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(ab)\right| = \left|\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\overline{\lambda}(c)\lambda(de)\right| \le p^2;
$$

$$
\left|\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(abc)\right| \le p^2 \quad \text{and} \quad \left|\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(dec)\right| \le p^2.
$$

In fact, if $abc \equiv de \bmod p$ and $\lambda^i(a)\lambda^j(b)\lambda^k(c)\lambda^h(d)\lambda^s(e) \neq 1$, $0 \leq i,\ j,\ k,\ h,\ s \leq 2$, then we can also prove the estimate

$$\left| \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda^i(a)\lambda^j(b)\lambda^k(c)\lambda^h(d)\lambda^s(e) \right| \leq p^2.$$

**Proof.** From the properties of the orthogonality of the characters we have

$$\begin{aligned} & \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left( \sum_{a=1}^{p-1} \lambda(a)\chi(a)e\left(\frac{ma}{p}\right)\right)^2 \left( \sum_{a=1}^{p-1} \chi(a)e\left(\frac{ma}{p}\right)\right) \\ & \times \left( \sum_{b=1}^{p-1} \overline{\chi}(b)e\left(\frac{-mb}{p}\right)\right)^2 e\left(\frac{-m}{p}\right) \\ = \quad & p(p-1) \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \lambda(ab). \end{aligned} \tag{2.17}$$

On the other hand, from the properties of the classical Gauss sums we also have

$$\begin{aligned} & \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left( \sum_{a=1}^{p-1} \lambda(a)\chi(a)e\left(\frac{ma}{p}\right)\right)^2 \left( \sum_{a=1}^{p-1} \chi(a)e\left(\frac{ma}{p}\right)\right) \\ & \times \left( \sum_{b=1}^{p-1} \overline{\chi}(b)e\left(\frac{-mb}{p}\right)\right)^2 e\left(\frac{-m}{p}\right) \\ = \quad & \sum_{\chi \bmod p} \tau^2(\lambda\chi)\, \tau(\chi) \overline{\tau(\chi)}^2 \sum_{m=1}^{p-1} \overline{\lambda}(m)\overline{\chi}(m) e\left(\frac{-m}{p}\right) \\ = \quad & \sum_{\chi \bmod p} |\tau(\lambda\chi)|^2\, |\tau(\chi)|^2\, \tau(\lambda\chi)\, \overline{\tau(\chi)} \end{aligned}$$

or

$$\begin{aligned} & \left| \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left( \sum_{a=1}^{p-1} \lambda(a)\chi(a)e\left(\frac{ma}{p}\right)\right)^2 \left( \sum_{a=1}^{p-1} \chi(a)e\left(\frac{ma}{p}\right)\right) \right| \\ & \times \left| \left( \sum_{b=1}^{p-1} \overline{\chi}(b)e\left(\frac{-mb}{p}\right)\right)^2 e\left(\frac{-m}{p}\right) \right| \\ = \quad & \left| \sum_{\chi \bmod p} |\tau(\lambda\chi)|^2\, |\tau(\chi)|^2\, \tau(\lambda\chi)\, \overline{\tau(\chi)} \right| \leq p^3(p-1). \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18) we may immediately deduce the estimate

$$\left|\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\lambda(ab)\right| \le p^2.$$

Similarly, we can also deduce the other estimations in Lemma 4.

**Lemma 5.** Let $p$ be an odd prime with $p \equiv 1 \bmod 3$. Then we have the identity

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+e^3+1 \bmod p \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = 7p^2 - 29p + 44.$$

**Proof.** From the properties of the congruence equation modulo $p$ we have

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+e^3+1 \bmod p \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ (a+b+c)^3-a^3-b^3-c^3\equiv(d+e+1)^3-d^3-e^3-1 \bmod p \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1$$

$$= \sum_{\substack{a=1 \\ (a+b+c)\left(a^2+b^2+c^2-d^2-e^2-1\right)\equiv 0 \bmod p \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a+b+c\equiv 0 \bmod p \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1$$

$$+ \sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 - \sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p \\ a+b+c\equiv d+e+1\equiv 0 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1. \qquad (2.19)$$

Now we calculate the three sums in (2.19) separately. First we have

$$\sum_{\substack{a=1 \\ a+b+c\equiv 0 \bmod p \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ da+eb+c\equiv 0 \bmod p \\ d+e+1\equiv 0 \bmod p \\ abc\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1 \\ da+eb+\overline{ab}\equiv 0 \bmod p \\ d+e+1\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1$$

$$= \sum_{\substack{a=1 \\ da-(d+1)b+\overline{ab}\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} 1 - \sum_{\substack{a=1 \\ -a+\overline{ab}\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1$$

$$= \sum_{\substack{a=1 \\ d(a-b)\equiv b-\overline{ab} \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} 1 - (p-1)$$

$$= \quad 3(p-1)+3(p-2)+(p-3)(p-4)-(p-1)=p^2-2p+4. \tag{2.20}$$

$$\begin{aligned}
&\sum_{\substack{a=1\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1\\ ab+ac+bc\equiv de+d+e \bmod p\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1\\ ab+(a+b)c\equiv abde+ad+eb \bmod p\\ a+b+c\equiv ad+be+1 \bmod p\\ c\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1\\
=\;& \sum_{\substack{a=1\\ (a-1)(b-1)(de-1)\equiv 0 \bmod p\\ a+b+de\equiv ad+be+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = 1 + \sum_{\substack{d=1\\ (d-1)(e-1)\equiv 0 \bmod p\\ (de-1,\ p)=1}}^{p-1}\sum_{e=1}^{p-1} 1 + 2\sum_{\substack{b=2\\ (d-1)(b-\overline{d})\equiv 0 \bmod p}}^{p-1}\sum_{d=1}^{p-1} 1\\
&+2\sum_{\substack{b=2\\ (e-1)(b-d)\equiv 0 \bmod p\\ (de-1,\ p)=1}}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + \sum_{\substack{a=2\\ a+b\equiv ad+be \bmod p\\ de\equiv 1 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1\\
=\;& 1+2(p-2)+4(p-2)+2(p-2)(2p-5)+\sum_{\substack{a=2\\ (a+b)e\equiv a+be^2 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\sum_{e=1}^{p-1} 1\\
=\;& 1+4(p-2)(p-1)+\sum_{\substack{a=2\\ (e-1)(eb-a)\equiv 0 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\sum_{e=1}^{p-1} 1\\
=\;& 1+4(p-2)(p-1)+2(p-2)^2-(p-2)=6p^2-21p+19.
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
&\sum_{\substack{a=1\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\ a+b+c\equiv d+e+1\equiv 0 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1\\ (a-1)(b-1)(de-1)\equiv 0 \bmod p\\ a+b+de\equiv ad+be+1\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{d=1\\ 2+de\equiv d+e+1\equiv 0 \bmod p}}^{p-1}\sum_{e=1}^{p-1} 1\\
&+2\sum_{\substack{b=2\\ 2+b\equiv d+b\overline{d}+1\equiv 0 \bmod p}}^{p-1}\sum_{d=1}^{p-1} 1 + 2\sum_{\substack{b=2\\ 1+b+de\equiv d+be+1\equiv 0 \bmod p\\ (de-1,\ p)=1}}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + \sum_{\substack{a=2\\ a+b+1\equiv ad+b\overline{d}+1\equiv 0 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\sum_{d=1}^{p-1} 1\\
=\;& 2+4+2\,(p-4+p-5)+p-4+p-5=6p-21.
\end{aligned} \tag{2.22}$$

Combining (2.19)–(2.22), we have the desired result

$$\sum_{\substack{a=1\\ a^3+b^3+c^3\equiv d^3+e^3+1 \bmod p\\ a+b+c\equiv d+e+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = 7p^2-29p+44.$$

**Lemma 6.** Let $p$ be an odd prime with $p \equiv 1 \bmod 3$. Then we have the asymptotic formula

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+e^3+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3 + O\left(p^2\right).$$

**Proof.** Let $\lambda$ be a third-order character modulo $p$. Note that if $abc \equiv de \bmod p$, then $\lambda(abc)\overline{\lambda}(de) = \overline{\lambda}(abc)\lambda(de) = 1$, so from the properties of the third-order characters modulo $p$ and Lemmas 1–4 we have the asymptotic formula

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+e^3+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \frac{1}{3}\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+e^3+1 \bmod p \\ a^3b^3c^3\equiv d^3e^3 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1$$

$$= \frac{1}{3}\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\left(1+\lambda(a)+\overline{\lambda}(a)\right)\left(1+\lambda(b)+\overline{\lambda}(b)\right)$$

$$\times\left(1+\lambda(c)+\overline{\lambda}(c)\right)\left(1+\lambda(d)+\overline{\lambda}(d)\right)\left(1+\lambda(e)+\overline{\lambda}(e)\right)$$

$$= \sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + O\left(p^2\right) = p^3 + O\left(p^2\right).$$

This proves Lemma 6.

## 3. Proofs of the theorems

Now we apply the lemmas in Section 2 to complete the proof of our theorem. Note that the trigonometrical identities

$$\sum_{a=0}^{p-1} e\left(\frac{na}{p}\right) = \begin{cases} p & \text{if } p \mid n; \\ 0 & \text{if } p \nmid n. \end{cases}$$

From the orthogonality of the characters sums, Lemma 5 and 6 we have

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right|^6$$

$$= p(p-1)\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+h^3+f^3 \bmod p \\ abc\equiv dhf \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{h=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{a+b+c-d-h-f}{p}\right)$$

$$= p(p-1)\sum_{\substack{a=1 \\ a^3+b^3+c^3\equiv d^3+h^3+1 \bmod p \\ abc\equiv dh \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{h=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{f(a+b+c-d-h-1)}{p}\right)$$

$$= p^2(p-1)\sum_{\substack{a=1\\ a^3+b^3+c^3\equiv d^3+h^3+1 \bmod p\\ a+b+c\equiv d+h+1 \bmod p\\ abc\equiv dh \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{h=1}^{p-1} 1 - p(p-1)\sum_{\substack{a=1\\ a^3+b^3+c^3\equiv d^3+h^3+1 \bmod p\\ abc\equiv dh \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{h=1}^{p-1} 1$$

$$= p^2(p-1)\left(7p^2-29p+44\right) - p(p-1)\left(p^3+O\left(p^2\right)\right) = 6p^5 + O\left(p^4\right).$$

This completes the proof of our theorem.

## 4. Conclusions

The main result of this paper is to give an asymptotic formula for the sixth power mean of one kind special two-term exponential sums. That is, for any prime $p$ with $p \equiv 1 \bmod 3$, we have the asymptotic formula

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right|^6 = 6p^5 + O\left(p^4\right).$$

Whether there exists an exact formula for the above power mean is still an open problem. We will keep working on it.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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