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Research article

Correlation properties of interleaved Legendre sequences and Ding-Helleseth-Lam sequences

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Abstract: Sequences with optimal autocorrelation properties play an important role in wireless communication, radar and cryptography. Interleaving is a very important method in constructing the optimal autocorrelation sequence. Tang and Gong gave three different constructions of interleaved sequences (generalized GMW sequences, twin prime sequences and Legendre sequences). Su et al. constructed a series of sequences with optimal autocorrelation magnitude via interleaving Ding-Helleseth-Lam sequences. In this paper we further study the correlation properties of interleaved Legendre sequences and Ding-Helleseth-Lam sequences.

Keywords: fourth-order; correlation; interleaving; Legendre sequence; Ding-Helleseth-Lam sequence

1. Introduction and results

1.1. Optimal autocorrelation sequence

Given a binary sequence $v = (v(0), v(1), \cdots, v(N-1)) \in \{0, 1\}^N$ of period N, $\mathbb{Z}_N$ is the integer ring of modulo N, the set $C_v = \{t \in \mathbb{Z}_N : \ v(t) = 1\}$ is called the support of v. Let $\tau \in \{1, \cdots, N-1\}$, the autocorrelation function $R_v(\tau)$ of v is defined by

$$R_v(\tau) = \sum_{i=0}^{N-1} (-1)^{v(i)+v(i+\tau)},$$

where $i + \tau$ is operating on modulo N. Easy to verity

$$R_v(\tau) = N - 4\left|(C_v + \tau) \cap C_v\right|.$$

Thus $R_v(\tau) \equiv N \pmod 4$. In particular, $R_v(\tau) \equiv 0 \pmod 4$ for $N \equiv 0 \pmod 4$.

If $R_v(\tau) \in \{0, -4\}$ or $R_v(\tau) \in \{0, +4\}$ where $\tau \in \{1, \cdots, N-1\}$, then v is referred to as a sequence with optimal autocorrelation value. If $R_v(\tau) \in \{0, \pm 4\}$ where $\tau \in \{1, \cdots, N-1\}$, then v is referred to as a sequence with optimal autocorrelation magnitude. Sequences with optimal autocorrelation properties play an important role in wireless communication, radar and cryptography [1], so they have always been the focus of scholars' research (refer to [2–11]).

1.2. Interleaved method

Gong [12] introduced powerful interleaved method to design sequences. The key idea of this method is to obtain long sequences with good correlation from shorter ones. Let

$$v_k = (v_k(0), v_k(1), \cdots, v_k(N-1))$$

be a sequence of period N, where $0 \le k \le M-1$. Define the $N \times M$ matrix $U = (U_{ij})$ as follows

$$U = \begin{bmatrix} v_0(0) & v_1(0) & \cdots & v_{M-1}(0) \\ v_0(1) & v_1(1) & \cdots & v_{M-1}(1) \\ \vdots & \vdots & & \vdots \\ v_0(N-1) & v_1(N-1) & \cdots & v_{M-1}(N-1) \end{bmatrix}.$$

An interleaved sequence $u = (u(t))$ of period MN is defined by

$$u_{iM+j} = U_{ij} = v_j(i), \qquad 0 \le i < N,\ 0 \le j < M.$$

For convenience we write

$$u = I(v_0, v_1, \cdots, v_{M-1}).$$

Interleaving is a very important method in constructing optimal autocorrelation sequences. The autocorrelation of the sequence u is determined by the autocorrelation and cross-correlation of the sequences $v_0, v_1, \cdots, v_{M-1}$.

1.3. Interleaved Legendre sequences

Tang and Gong [10] gave three different constructions of interleaved sequences with the following structure:

$$u = I\left(a_0 + b(0),\ L^{d+\eta}(a_1) + b(1),\ L^{2d}(a_2) + b(2),\ L^{3d+\eta}(a_3) + b(3)\right), \tag{1.1}$$

where η is an integer with $0 \le \eta < N$, d is an integer with $4d \equiv 1 \pmod N$, $b = (b(0), b(1), b(2), b(3))$ is a binary perfect sequence, a_i, $i = 0, 1, 2, 3$, are sequences of period N taken from three different pairs of sequences (generalized GMW sequences, twin prime sequences and Legendre sequences).

Let $N = p$ be an odd prime. The Legendre sequence $l = (l(0), l(1), \cdots, l(p-1))$ of period p is defined by

$$l(i) = \begin{cases} 0 \text{ or } 1, & \text{if } i = 0, \\ 1, & \text{if } i \text{ is quadratic residue modulo } p, \\ 0, & \text{if } i \text{ is quadratic non-residue modulo } p. \end{cases}$$

In particular, l is called the first type Legendre sequence if $l(0) = 1$ otherwise the second type Legendre sequence. Let l and l' be the first type and the second type of Legendre sequence of period p, respectively.

Tang and Gong [10] showed that the constructions of interleaved Legendre sequence have optimal autocorrelation magnitude. Li and Tang [6] studied the linear complexity of binary sequences given in [10].

1.4. Interleaved Ding-Helleseth-Lam sequences

Let $p = 4f + 1$ be an odd prime, where f is a positive integer, α is a generator of $\mathbb{F}_p$. Define sets

$$D_i = \left\{\alpha^{i+4j} : \ 0 \le j \le f-1\right\}, \qquad i = 0, 1, 2, 3.$$

Then D_i, $i = 0, 1, 2, 3$ are called the cyclotomic classes of order 4 with respect to $\mathbb{F}_p$. It is very easy to verity that D_i, $i = 0, 1, 2, 3$ constitute a partition of $\mathbb{F}_p^*$.

Ding et al. [13] constructed some binary sequences based on the cyclotomic classes of order 4 of $\mathbb{F}_p$, and proved that these sequences are three-level autocorrelation.

Proposition 1.1. *Let $p = 4f + 1 = x^2 + 4y^2$ be an odd prime, where f, x, y are integers. Let s_1, s_2, s_3, s_4 be binary sequences of period p with supports $D_0 \cup D_1$, $D_0 \cup D_3$, $D_1 \cup D_2$ and $D_2 \cup D_3$, respectively. Then $R_{s_i}(\tau) \in \{1, -3\}$ for all $1 \le \tau < p$ if and only if f is odd and $y = \pm 1$.*

Let a_0, a_1, a_2, a_3 be four binary sequences of period p, $b = (b(0), b(1), b(2), b(3))$ be a binary sequence of length 4. Construct a binary sequence $u = u(t)$ of length $4p$ as follows:

$$u = I\left(a_0 + b(0),\ L^d(a_1) + b(1),\ L^{2d}(a_2) + b(2),\ L^{3d}(a_3) + b(3)\right), \tag{1.2}$$

where d is a integer with $4d \equiv 1 \pmod p$, L is a cyclic left shift operator, when $c = (c(0), c(1), \cdots, c(N-1))$, $L(c) = (c(1), c(2), \cdots, c(N-1), c(0))$.

Su et al. [8] constructed a series sequences with optimal autocorrelation magnitude of period $4p$ via interleaving Ding-Helleseth-Lam sequences.

Proposition 1.2. *Let $p = 4f + 1 = x^2 + 4y^2$ be an odd prime, where x is an integer, $y = \pm 1$, f is odd. Let $b(0) = b(2)$, $b(1) = b(3)$, and (a_0, a_1, a_2, a_3) be chosen from*

$$\begin{aligned} &(s_3, s_2, s_1, s_1), \quad (s_2, s_3, s_1, s_1), \quad (s_1, s_4, s_2, s_2), \quad (s_4, s_1, s_2, s_2),\\ &(s_1, s_4, s_3, s_3), \quad (s_4, s_1, s_3, s_3), \quad (s_3, s_2, s_4, s_4), \quad (s_2, s_3, s_4, s_4). \end{aligned} \tag{1.3}$$

Then the binary sequence u by (1.2) has $R_u(\tau) \in \{0, \pm 4\}$ for all $1 \le \tau < 4p$.

Later Fan [14] showed that the above sequences have large linear complexity. Sun et al. [15] proved that the sequences have good 2-adic complexity.

1.5. Results of this paper

This paper will further study the fourth-order correlation properties of interleaved Legendre sequences and Ding-Helleseth-Lam sequences. Our results are as follows.

Theorem 1.1. *Let $p > 2$ be a prime, $b = (b(0), b(1), b(2), b(3))$ be a binary perfect sequence, η be an integer with $0 < \eta < p$ and let the binary sequence u be generated by (1.1). For $n \in \{0, 1, \cdots, 4p-1\}$ we have*

$$\begin{aligned}&(-1)^{u_n + u_{n+p} + u_{n+2p} + u_{n+3p}}\\ &= \begin{cases} (-1)^{a_0(p-\eta) + a_1(0) + a_2(p-\eta) + a_3(0) + 1}, & n \equiv -4\eta \pmod p,\\ (-1)^{a_0(0) + a_1(\eta) + a_2(0) + a_3(\eta) + 1}, & n \equiv 0 \pmod p \ \textit{and}\ n \not\equiv -4\eta \pmod p,\\ -1, & n \not\equiv 0 \pmod p \ \textit{and}\ n \not\equiv -4\eta \pmod p. \end{cases}\end{aligned}$$

Theorem 1.2. *Let $p = 4f + 1$ be an odd prime, $b(0) = b(2)$, $b(1) = b(3)$ and let (a_0, a_1, a_2, a_3) be chosen from* (1.3)*. Let the binary sequence u be defined as in* (1.2)*. For $n \in \{0, 1, \cdots, 4p - 1\}$ we have*

$$(-1)^{u_n+u_{n+p}+u_{n+2p}+u_{n+3p}} = \begin{cases} +1, & n \equiv 0 \pmod{p}, \\ -1, & n \not\equiv 0 \pmod{p}. \end{cases}$$

Our results show that the fourth-order correlation for interleaved Legendre sequences and Ding-Helleseth-Lam sequences u we have $(-1)^{u_n+u_{n+p}+u_{n+2p}+u_{n+3p}} = -1$ which the number of n is at least $4p - 5$ for all $n \in \{0, 1, \cdots, 4p-1\}$. This means that there are at least four subsequences that do not satisfy lower mutual interference. Therefore, researchers should pay careful attention to these four subsequences when consider their application in communication systems.

2. Proof of Theorem 1.1

Write

$$\begin{aligned} u &= I(v_0,\ v_1,\ v_2,\ v_3) \\ &= I\left(a_0 + b(0),\ L^{d+\eta}(a_1) + b(1),\ L^{2d}(a_2) + b(2),\ L^{3d+\eta}(a_3) + b(3)\right). \end{aligned}$$

Then

$$(-1)^{v_0(n)} = (-1)^{b(0)}(-1)^{a_0(n)}, \tag{2.1}$$
$$(-1)^{v_1(n)} = (-1)^{b(1)}(-1)^{a_1(n+d+\eta)}, \tag{2.2}$$
$$(-1)^{v_2(n)} = (-1)^{b(2)}(-1)^{a_2(n+2d)}, \tag{2.3}$$
$$(-1)^{v_3(n)} = (-1)^{b(3)}(-1)^{a_3(n+3d+\eta)}. \tag{2.4}$$

Case I: $p \equiv 1 \pmod{4}$. We have

$$p = 4 \cdot \frac{p-1}{4} + 1, \quad 2p = 4 \cdot \frac{2(p-1)}{4} + 2, \quad 3p = 4 \cdot \frac{3(p-1)}{4} + 3, \quad d = \frac{3p+1}{4}.$$

Write $n = 4i + j$, where $0 \le i \le p - 1$ and $0 \le j \le 3$. Hence,

$$\begin{aligned} &u_n + u_{n+p} + u_{n+2p} + u_{n+3p} \\ &= u_{4i+j} + u_{4i+j+4\cdot\frac{p-1}{4}+1} + u_{4i+j+4\cdot\frac{2(p-1)}{4}+2} + u_{4i+j+4\cdot\frac{3(p-1)}{4}+3}. \end{aligned} \tag{2.5}$$

First we consider the case of $j = 1$. By (2.1)–(2.5) we get

$$\begin{aligned} &u_n + u_{n+p} + u_{n+2p} + u_{n+3p} \\ &= u_{4i+1} + u_{4(i+\frac{p-1}{4})+2} + u_{4(i+\frac{2(p-1)}{4})+3} + u_{4(i+\frac{3(p-1)}{4})+4} \\ &= v_1(i) + v_2\left(i + \frac{p-1}{4}\right) + v_3\left(i + \frac{2(p-1)}{4}\right) + v_0\left(i + \frac{3p+1}{4}\right) \\ &= a_1(i + d + \eta) + b(1) + a_2\left(i + \frac{p-1}{4} + 2d\right) + b(2) \\ &\quad + a_3\left(i + \frac{2(p-1)}{4} + 3d + \eta\right) + b(3) + a_0\left(i + \frac{3p+1}{4}\right) + b(0) \end{aligned}$$

$$= b(0) + b(1) + b(2) + b(3) + a_0(i + d) + a_1(i + d + \eta) \\ +a_2(i + d) + a_3(i + d + \eta). \tag{2.6}$$

For all $j \in \{0, 1, 2, 3\}$ we can also have

$$u_n + u_{n+p} + u_{n+2p} + u_{n+3p} = b(0) + b(1) + b(2) + b(3) \\ +a_0(i + jd) + a_1(i + jd + \eta) + a_2(i + jd) + a_3(i + jd + \eta). \tag{2.7}$$

Noting that $4d \equiv 1 \pmod p$ and $n = 4i + j$, thus we get

$$i + jd \equiv 0 \pmod p \Longleftrightarrow 4i + j \equiv 0 \pmod p \Longleftrightarrow n \equiv 0 \pmod p,$$
$$i + jd + \eta \equiv 0 \pmod p \Longleftrightarrow 4i + j + 4\eta \equiv 0 \pmod p \Longleftrightarrow n + 4\eta \equiv 0 \pmod p.$$

Therefore

$$\begin{aligned}
(-1)^{u_n+u_{n+p}+u_{n+2p}+u_{n+3p}} &= (-1)^{a_0(i+jd)+a_1(i+jd+\eta)+a_2(i+jd)+a_3(i+jd+\eta)} \\
&= \begin{cases} (-1)^{a_0(0)+a_1(0)+a_2(0)+a_3(0)+1}, & n \equiv 0 \pmod p \ \text{ and } n + 4\eta \equiv 0 \pmod p, \\ (-1)^{a_0(p-\eta)+a_1(0)+a_2(p-\eta)+a_3(0)+1}, & n \not\equiv 0 \pmod p \ \text{ and } n + 4\eta \equiv 0 \pmod p, \\ (-1)^{a_0(0)+a_1(\eta)+a_2(0)+a_3(\eta)+1}, & n \equiv 0 \pmod p \ \text{ and } n + 4\eta \not\equiv 0 \pmod p, \\ -1, & n \not\equiv 0 \pmod p \ \text{ and } n + 4\eta \not\equiv 0 \pmod p, \end{cases} \\
&= \begin{cases} (-1)^{a_0(p-\eta)+a_1(0)+a_2(p-\eta)+a_3(0)+1}, & n \equiv -4\eta \pmod p, \\ (-1)^{a_0(0)+a_1(\eta)+a_2(0)+a_3(\eta)+1}, & n \equiv 0 \pmod p \ \text{ and } n \not\equiv -4\eta \pmod p, \\ -1, & n \not\equiv 0 \pmod p \ \text{ and } n \not\equiv -4\eta \pmod p. \end{cases}
\end{aligned}$$

Case II: $p \equiv 3 \pmod 4$. We have

$$p = 4 \cdot \frac{p-3}{4} + 3, \quad 2p = 4 \cdot \frac{2(p-1)}{4} + 2, \quad 3p = 4 \cdot \frac{3p-1}{4} + 1, \quad d = \frac{p+1}{4}.$$

Similarly we can get

$$\begin{aligned}
&(-1)^{u_n+u_{n+p}+u_{n+2p}+u_{n+3p}} \\
&= \begin{cases} (-1)^{a_0(p-\eta)+a_1(0)+a_2(p-\eta)+a_3(0)+1}, & n \equiv -4\eta \pmod p, \\ (-1)^{a_0(0)+a_1(\eta)+a_2(0)+a_3(\eta)+1}, & n \equiv 0 \pmod p \ \text{ and } n \not\equiv -4\eta \pmod p, \\ -1, & n \not\equiv 0 \pmod p \ \text{ and } n \not\equiv -4\eta \pmod p. \end{cases}
\end{aligned}$$

This proves Theorem 1.1.

3. Proof of Theorem 1.2

Write

$$\begin{aligned} u &= I(v_0,\ v_1,\ v_2,\ v_3) \\ &= I\left(a_0 + b(0),\ L^d(a_1) + b(1),\ L^{2d}(a_2) + b(2),\ L^{3d}(a_3) + b(3)\right). \end{aligned}$$

Then

$$(-1)^{v_0(n)} = (-1)^{b(0)}(-1)^{a_0(n)}, \tag{3.1}$$

$$
\begin{aligned}
(-1)^{v_1(n)} &= (-1)^{b(1)}(-1)^{a_1(n+d)}, &(3.2)\\
(-1)^{v_2(n)} &= (-1)^{b(2)}(-1)^{a_2(n+2d)}, &(3.3)\\
(-1)^{v_3(n)} &= (-1)^{b(3)}(-1)^{a_3(n+3d)}. &(3.4)
\end{aligned}
$$

Let $b(0) = b(2)$ and $b(1) = b(3)$. Noting that $p \equiv 1 \pmod 4$, thus we have

$$p = 4 \cdot \frac{p-1}{4} + 1, \quad 2p = 4 \cdot \frac{2(p-1)}{4} + 2, \quad 3p = 4 \cdot \frac{3(p-1)}{4} + 3, \quad d = \frac{3p+1}{4}.$$

Write $n = 4i + j$, where $0 \le i \le p-1$ and $0 \le j \le 3$. Hence,

$$
\begin{aligned}
&u_n + u_{n+p} + u_{n+2p} + u_{n+3p}\\
&= u_{4i+j} + u_{4i+j+4\cdot\frac{p-1}{4}+1} + u_{4i+j+4\cdot\frac{2(p-1)}{4}+2} + u_{4i+j+4\cdot\frac{3(p-1)}{4}+3}. \qquad (3.5)
\end{aligned}
$$

For the case of $j = 1$, by (3.1)–(3.5) we get

$$
\begin{aligned}
&u_n + u_{n+p} + u_{n+2p} + u_{n+3p}\\
&= u_{4i+1} + u_{4(i+\frac{p-1}{4})+2} + u_{4(i+\frac{2(p-1)}{4})+3} + u_{4(i+\frac{3(p-1)}{4})+4}\\
&= v_1(i) + v_2\left(i + \frac{p-1}{4}\right) + v_3\left(i + \frac{2(p-1)}{4}\right) + v_0\left(i + \frac{3p+1}{4}\right)\\
&= a_1\,(i+d) + b(1) + a_2\left(i + \frac{p-1}{4} + 2d\right) + b(2)\\
&\quad + a_3\left(i + \frac{2(p-1)}{4} + 3d\right) + b(3) + a_0\left(i + \frac{3p+1}{4}\right) + b(0)\\
&= b(0) + b(1) + b(2) + b(3) + a_0(i+d) + a_1(i+d) + a_2(i+d) + a_3(i+d). \qquad (3.6)
\end{aligned}
$$

For all $j \in \{0, 1, 2, 3\}$ we can also have

$$
\begin{aligned}
&u_n + u_{n+p} + u_{n+2p} + u_{n+3p}\\
&= b(0) + b(1) + b(2) + b(3) + a_0(i+jd) + a_1(i+jd) + a_2(i+jd) + a_3(i+jd). \qquad (3.7)
\end{aligned}
$$

Clearly,

$$i + jd \equiv 0 \pmod p \Longleftrightarrow 4i + j \equiv 0 \pmod p \Longleftrightarrow n \equiv 0 \pmod p.$$

Let (a_0, a_1, a_2, a_3) be chosen from

$$
\begin{array}{llll}
(s_3, s_2, s_1, s_1), & (s_2, s_3, s_1, s_1), & (s_1, s_4, s_2, s_2), & (s_4, s_1, s_2, s_2),\\
(s_1, s_4, s_3, s_3), & (s_4, s_1, s_3, s_3), & (s_3, s_2, s_4, s_4), & (s_2, s_3, s_4, s_4).
\end{array}
$$

From (3.7) we get

$$
\begin{aligned}
(-1)^{u_n+u_{n+p}+u_{n+2p}+u_{n+3p}} &= (-1)^{a_0(i+jd)+a_1(i+jd)+a_2(i+jd)+a_3(i+jd)}\\
&= \begin{cases} +1, & n \equiv 0 \pmod p,\\ -1, & n \not\equiv 0 \pmod p. \end{cases}
\end{aligned}
$$

This completes the proof of Theorem 1.2.

Use of AI tools declaration

The authors have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. S. W. Golomb, G. Gong, *Signal Design for Good Correlation-for Wireless Communication, Cryptography and Radar*, Cambridge University Press, Cambridge, 2005. https://doi.org/10.1017/CBO9780511546907
2. K. T. Arasu, C. Ding, T. Helleseth, P. V. Kumar, H. Martinsen, Almost difference sets and their sequences with optimal autocorrelation, *IEEE Trans. Inf. Theory*, **47** (2001), 2934–2943. https://doi.org/10.1109/18.959271
3. Y. Cai, C. Ding, Binary sequences with optimal autocorrelation, *Theor. Comput. Sci.*, **410** (2009), 2316–2322. https://doi.org/10.1016/j.tcs.2009.02.021
4. D. Jungnickel, *Finite Fields: Structure and Arithmetics*, BI-Wissenschaftsverlag, Mannheim, 1993.
5. A. Lempel, M. Cohn, W. L. Eastman, A class of balanced binary sequences with optimal autocorrelation properties, *IEEE Trans. Inf. Theory*, **23** (1977), 38–42. https://doi.org/10.1109/TIT.1977.1055672
6. N. Li, X. Tang, On the linear complexity of binary sequences of period $4N$ with optimal autocorrelation value/magnitude, *IEEE Trans. Inf. Theory*, **57** (2011), 7597–7604. https://doi.org/10.1109/TIT.2011.2159575
7. V. M. Sidel'nikov, Some k-valued pseudo-random sequences and nearly equidistant codes, *Probl. Peredachi Inf.*, **5** (1969), 12–16.
8. W. Su, Y. Yang, C. Fan, New optimal binary sequences with period $4p$ via interleaving Ding-Helleseth-Lam sequences, *Des. Codes Cryptogr.*, **86** (2018), 1329–1338. https://doi.org/10.1007/s10623-017-0398-5
9. W. Su, Y. Yang, Z. Zhou, X. Tang, New quaternary sequences of even length with optimal autocorrelation, *Sci. China Inf. Sci.*, **61** (2018). https://doi.org/10.1007/s11432-016-9087-2
10. X. Tang, G. Gong, New constructions of binary sequences with optimal autocorrelation value/magnitude, *IEEE Trans. Inf. Theory*, **56** (2010), 1278–1286. https://doi.org/10.1109/TIT.2009.2039159

11. N. Yu, G. Gong, New binary sequences with optimal autocorrelation magnitude, *IEEE Trans. Inf. Theory*, **54** (2008), 4771–4779. https://doi.org/10.1109/TIT.2008.928999

12. G. gong, Theory and applications of q-ary interleaved sequences, *IEEE Trans. Inf. Theory*, **41** (1995), 400–411. https://doi.org/10.1109/18.370141

13. C. Ding, T. Helleseth, K. Y. Lam, Several classes of binary sequences with three-level autocorrelation, *IEEE Trans. Inf. Theory*, **45** (1999), 2606–2612. https://doi.org/10.1109/18.796414

14. C. Fan, The linear complexity of a class of binary sequences with optimal autocorrelation, *Des. Codes Cryptogr.*, **86** (2018), 2441–2450. https://doi.org/10.1007/s10623-018-0456-7

15. Y. Sun, T. Yan, Z. Chen, L. Wang, The 2-adic complexity of a class of binary sequences with optimal autocorrelation magnitude, *Cryptogr. Commun.*, **12** (2020), 675–683. https://doi.org/10.1007/s12095-019-00411-4