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Research article

Stochastic travelling wave solution of the N-species cooperative systems with multiplicative noise

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Abstract: The current paper is devoted to the stochastic N-species cooperative system with a moderately strong noise. By the theory of monotone random systems and the technique of suitable marker of wavefront, the existence of the travelling wave solution is established. By applying the Feynman-Kac formula and sup-sub solution technique, the upper and lower bounded of the asymptotic wave speed are also obtained. Finally, we give an example for stochastic 3-species cooperative systems.

Keywords: N-species stochastic cooperative system; Feynman-Kac formula; stochastic travelling waves; asymptotic wave speed

1. Introduction

There are many papers investigating the stochastic travelling waves of population dynamical system with multiplicative noise, most of them focus on the scaler Fisher-KPP equation. For instance, Tribe [1] studied the KPP equation with nonlinear multiplicative noise $\sqrt{u}dW_t$, and Müeller et al. [2–4] studied the KPP equation with $\sqrt{u(1-u)}dW_t$. Both of their work take the Heaviside function as the initial data, and they also gave the estimates of the wave speed with an upper bound and a lower bound. Zhao et al. [5–7] showed that only if the strength of noise is moderately, for example the multiplicative noise $k(t)dW_t$, the effects of noise would present or the solution would tend to be zero or converge to the deterministic travelling wave solution. Shen [8] developed a theoretical random variational framework to show the existence of random travelling waves, and then Shen and his collaborators [9, 10] also studied the random travelling waves in reaction-diffusion equations with Fisher-KPP nonlinearity, Nagumo nonlinearity and ignition nonlinearity, in random media. Furthermore, Huang et al. [11–14] investigated the bifurcations of asymptotic behaviors of solution induced by strength of the dual noises for stochastic Fisher-KPP equation. Recently, Wang and Zhou [15] discovered that the same results still hold even if the decrease restrictions on the growth function are removed. Moreover, they showed that with increasing the noise intensity, the original equation with Fisher-KPP nonlinearity evolves into first the one with degenerated Fisher-KPP nonlinearity and then the one with Nagumo nonlinearity, and we refer it to [15] for details.

It is worthy to point out that the above mentioned papers mainly focus on the scalar stochastic reaction-diffusion equation. Recently, Wen et al. [16] applied the theory of random monotone dynamical systems developed by Cheushov [17] and Kolmogorov tightness criterion to obtain the existence of stochastic travelling wave solution for stochastic two-species cooperative system

$$\begin{cases} du = [u_{xx} + u(1 - a_1 u + b_1 v)]dt + \epsilon u dW_t, \\ dv = [v_{xx} + v(1 - a_2 v + b_2 u)]dt + \epsilon v dW_t, \\ u(0) = u_0, v(0) = v_0, \end{cases} \tag{1.1}$$

where $W(t)$ is a white noise as in [11], u_0, v_0 are both Heaviside functions, and a_i, b_i are positive constants satisfying $\min\{a_i\} > \max\{b_i\}$. The element "1" of $1 - a_1 u + b_1 v$ and $1 - a_2 v + b_2 u$ in Eq (1.1) is the formal environment carrying capacity, and then by constructing upper and lower solution and applying Feynman-Kac formula they obtained the estimation of upper bound and lower bound for wave speed, respectively. Moreover, Wen et al. [18] established the existence of stochastic travelling wave solution for stochastic two-species competitive system, and they obtained the upper bound and lower bound of the asymptotic wave. To the best of our knowledge, there are few papers concerning the stochastic travelling waves for cooperative N-species systems ($N \geq 3$), which leads to the motivation of the current work.

There are some papers that study the stability and stochastic persistence for the stochastic N-species system without space diffusion. For example, Cui and Chen [19] proved that there exists a unique globally asymptotically stable positive ω-periodic solution for the N-species time dependent Lotka-Volterra periodic mutualistic system

$$\dot{x}_i = x_i(r_i(t) + \sum_{j=1}^{n} a_{ij}(t)x_j), \quad i = 1, 2, \cdots, n, \tag{1.2}$$

provided with $(-1)^k det(\max_{0 \leq t < +\infty} a_{ij}(t))_{1 \leq i,j \leq k} > 0$. Subsequently, Ji et al. [20] studied the N-species Lotka-Volterra mutualism system with stochastic perturbation

$$dx_i(t) = x_i(t)\Big[\Big(b_i - \sum_{j=1}^{n} a_{ij}x_j(t)\Big)dt + \sigma_i dB_i(t)\Big], \quad i = 1, 2, \cdots, n,$$

and proved the sufficient criteria for persistence in mean and stationary distribution of the system. Moreover, they also showed the large white noise make the system nonpersistent, we refer the readers to [20, 21] for details.

In this paper, we consider the travelling wave solution of the following stochastic N-species cooperative systems,

$$\begin{cases} du^{(i)} = [u^{(i)}_{xx} + u^{(i)}(a_i - b_{ii}u^{(i)} + \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} b_{ij}u^{(j)})]dt + \epsilon u^{(i)} dW_t, i = 1, 2, \cdots, n, \\ u^{(i)}(0, x) = u^{(i)}_0 = p_i \chi_{(-\infty,0]}, i = 1, 2, \cdots, n, \end{cases} \tag{1.3}$$

where $W(t)$ is a Brownian motion, $u_0^{(i)}(i = 1, 2, \cdots, n)$ are Heaviside functions, a_i represents the environment carrying capacity, and b_{ij} are positive constants satisfying $\min\{b_{ii}\} > 2n\max_{j\neq k}\{b_{jk}\}$, $rank\{(b_{ij})_{n\times n}\} = n$.

To study the existence of stochastic travelling wave solution for stochastic N-species cooperative systems (1.3), it needs to introduce a suitable wavefront marker for system (1.3). The comparison method is applied to prove the boundedness of the solutions based on the random monotonicity and the Feynman-Kac formula. The existence of the travelling wave solution is focused on verifying the trajectory property, connecting the two states poses the support compactness propagation (SCP) property, defined by Shiga in [22].

Denote

$$\begin{aligned} Y &= (u^{(1)}, u^{(2)}, \cdots, u^{(n)})^T, \\ Y_0 &= (u_0^{(1)}, u_0^{(2)}, \cdots, u_0^{(n)})^T, \end{aligned}$$

and

$$F_i(Y) = u^{(i)}(a_i - b_{ii}u^{(i)} + \sum_{\substack{j=1 \\ j\neq i}}^{n} b_{ij}u^{(j)}),\ F(Y) = (F_1(Y), \cdots, F_n(Y))^T,$$

then the stochastic cooperative system (1.3) can be rewritten as the following vector equation

$$\begin{cases} dY = [Y_{xx} + F(Y)]dt + \epsilon Y dW_t, \\ Y(0, x) = Y_0. \end{cases} \tag{1.4}$$

For any matrix $M = (m_{ij})_{n\times m}$, define the norm $|\cdot|$ as $|M| = \sum_{i=1}^{n}\sum_{j=1}^{n} |m_{ij}|$, and the vector norm is defined as $|A|_\infty = \max_i(A_i)$ for vector $A = (a_i)_{n\times 1}$. Let Ω be the space of temper distributions, $\mathcal{F}$ be the σ-algebra on Ω, and $(\Omega, \mathcal{F}, \mathbb{P})$ be the white noise probability space.

In order to apply the Feynman-Kac formula in [7], we can define

$$\beta_t(k) := e^{\int_0^t k(s)dW_s - \frac{1}{2}\int_0^t k^2(s)ds}, \qquad 0 \leq t < \infty.$$

Denote by

$$\begin{aligned} &\phi_\lambda(x) = e^{-\lambda|x|}, \qquad |f|_\lambda = \sup_{x\in R}(|f(x)\phi_\lambda(x)|), \\ &C^+ = \{f | f : R \to [0, \infty)\ and\ f\ is\ continuous\}, \\ &C_\lambda^+ = \{f \in C^+ | f\ is\ continuous, and\ |f(x)\phi_\lambda(x)| \to 0\ as\ x \to \pm\infty\}, \\ &C_{tem}^+ = \bigcap_{\lambda>0} C_\lambda^+. \end{aligned}$$

$C_{C[0,1]}^+ = \{f | f : R \to [0, 1]\}$ is space of nonnegative functions with compact support, $\Phi = \{f : \|f\|_\lambda < \infty\ for\ some\ \lambda < 0\}$ is the space of functions with exponential decay, and C_{tem}^+ is the space of vector valued functions whose each component belongs to C_{tem}^+.

The rest of the paper is organized as follows. In Section 2, the existence of stochastic travelling wave solution is established. In Section 3, the upper and lower bound of asymptotic wave speed are obtained. An example of 3-species stochastic cooperative system is also presented in Section 4.

2. Existence of the stochastic travelling wave solutions

In this section, we establish the existence of stochastic travelling wave solution. We first provide with the definition of stochastic travelling wave solution, which is from [1]. To the end, it needs to define some state space follows as

$$\begin{aligned}\mathcal{D}_{[0,\infty)} &= \{\phi : R \to [0,\infty), \phi \text{ is right continuous and decreasing,}\\ &\qquad \phi_{-\infty} = \lim_{x\to-\infty} \phi \text{ exists}\}.\\ \mathcal{D}_{[0,1]} &= \{\phi : R \to [0,1], \phi \text{ is right continuous and decreasing}\}.\\ \mathcal{D} &= \{\phi \in \mathcal{D}_{[0,1]} : \ \ \phi(-\infty) = 1, \phi(\infty) = 0\}.\end{aligned}$$

We endow $\mathcal{D}_{[0,\infty)}$ with the topology induced from $L^1_{loc}(R)$ metric. Then $\mathcal{D}_{[0,1]}$ and $\mathcal{D}$ are the measurable subset of $\mathcal{D}_{[0,\infty)}$. It follows from [13] that $\mathcal{D}_{[0,\infty)}$,$\mathcal{D}_{[0,1]}$ and $\mathcal{D}$ are Polish spaces and compact.

Consider the following stochastic reaction diffusion equation with Heaviside data

$$\begin{cases} du = [Du_{xx} + f(u)]dt + \sigma(u)dW_t,\\ u(0) = \chi_{x\le 0}.\end{cases} \tag{2.1}$$

Definition 2.1 (Stochastic travelling wave solution)**.** A stochastic travelling wave is a solution $u = (u(t) : t \ge 0)$ to (2.1) with values in $\mathcal{D}$ and for which the centered process $(\tilde{u}(t) = u(t, \cdot + R_0(t)) : t \ge 0)$ is a stationary process with respect to time, where $R_0(t)$ is a wave front marker. The law of a stochastic travelling wave is the law of $\tilde{u}(0)$ on $\mathcal{D}$.

Then, we prove the following Lemmas 2.2 and 2.3 by the idea of Tribe [1].

Lemma 2.2. *For any Heaviside functions Y_0, and a.e. $\omega \in \Omega$, there exists a unique solution to* (1.4) *in law with the form*

$$\begin{aligned} Y(t,x) &= \int_R G(t,x,y)Y_0 dy\\ &\quad + \int_0^t \int_R G(t-s,x,y)F(Y)dsdy + \epsilon \int_0^t \int_R G(t-s,x,y)YdW_s dy,\end{aligned} \tag{2.2}$$

where $G(t,x,y)$ is Green function, and $Y(t,x) \in C^+_{tem}$.

Lemma 2.3. *All solutions to* (1.4) *started at Y_0 have the same law which we denote by $Q^{Y_0,a_i,b_{ij}}$, and the map $(Y_0, a_i, b_{ij}) \to Q^{Y_0,a_i,b_{ij}}$ is continuous. The law $Q^{Y_0,a_i,b_{ij}}$ for Y_0 as a Heaviside function forms a strong Markov family.*

Next, we estimate the term $Y(t,x)$, which is key tools to prove the existence of stochastic travelling wave solutions.

Theorem 2.4. *For any Heaviside functions $u_0^{(i)}$, and $t > 0$ fixed, a.e. $\omega \in \Omega$, it permits that*

$$\mathbb{E}[\sum_{i=1}^n u^{(i)}(t,x)] \le C(\epsilon,t)(\sum_{i=1}^n u_0^{(i)} + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}),\ \forall x \in R, \tag{2.3}$$

where $C(\epsilon,t)$ is a constant, $k = \frac{\min_i\{b_{ii}\}-(n-1)\max_{i\neq j}\{b_{ij}\}}{n}$, $\alpha = \max_i\{a_i\}$.

Proof. Denote by $\phi(t,x) = \sum\limits_{i=1}^{n} u^{(i)}(t,x)$, we have

$$\begin{cases} d\phi = [\phi_{xx} + \sum\limits_{i=1}^{n} u^{(i)}(t,x)(a_i - b_{ii}u^{(i)} + \sum\limits_{\substack{j=1\\ j\neq i}}^{n} b_{ij}u^{(j)})]dt + \epsilon\phi dW_t, \\ \phi(0,x) = \phi_0 = \sum\limits_{i=1}^{n} u_0^{(i)}, \end{cases} \tag{2.4}$$

Since $\min\{b_{ii}\} > 2n \max\limits_{j\neq k}\{b_{jk}\}$, then

$$\begin{aligned} &\sum_{i=1}^{n} u^{(i)}(a_i - b_{ii}u^{(i)} + \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}u^{(j)}) \\ &\leq \alpha \sum_{i=1}^{n} u^{(i)} - \min_i\{b_{ii}\} \sum_{i=1}^{n} (u^{(i)})^2 + 2\max_{i\neq j}\{b_{ij}\} \sum_{\substack{i,j=1\\ i<j}}^{n} u^{(i)}u^{(j)} \\ &\leq \alpha \sum_{i=1}^{n} u^{(i)} - k \sum_{i=1}^{n} (u^{(i)})^2 \leq \sum_{i=1}^{n} u^{(i)}(\alpha - k\sum_{i=1}^{n} u^{(i)}). \end{aligned}$$

Let ψ be the solution of the following equation

$$\begin{cases} d\psi = [\psi_{xx} + \psi(\alpha - k\psi)]dt + \epsilon\psi dW_t, \\ \psi_0 = \sum\limits_{i=1}^{n} u_0^{(i)}, \end{cases} \tag{2.5}$$

then, $u^{(i)}(t,x) \leq \psi(t,x)$ a.s., $i = 1, 2, \cdots, n$.

Let ζ be a solution to the following equation

$$\begin{cases} \zeta_t = \zeta_{xx} + \zeta(\alpha - k\zeta) - \frac{\epsilon^2}{2}\zeta, \\ \zeta_0 = \psi_0. \end{cases} \tag{2.6}$$

We claim that for every $(t,x) \in [0,\infty) \times R$, it follows

$$e^{\inf\limits_{0\leq r\leq t} \int_r^t \epsilon dW_s} \zeta(t,x) \leq \psi(t,x) \leq e^{\sup\limits_{0\leq r\leq t} \int_r^t \epsilon dW_s} \zeta(t,x) \qquad a.s. \tag{2.7}$$

In fact, we prove this claim by contradiction. We suppose that there is $(t_0, x_0) \in [0,\infty) \times R$ such that

$$\psi(t_0, x_0) > e^{\sup\limits_{0\leq r\leq t_0} \int_r^{t_0} \epsilon dW_s} \zeta(t_0, x_0), \tag{2.8}$$

which implies that

$$\psi(t_0, x_0) > \zeta(t_0, x_0).$$

To construct a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, and denote $\hat{W} = (\hat{W}(t) : t \geq 0)$ be a Brownian motion over the new probability space. Let $X_s^{t_0,x_0} = (t_0 - s, x_0 + \sqrt{2}\hat{W}(s))$, $s > 0$, and define a stopping time

$$\tau = \inf\{s > 0 : \zeta(X_s^{t_0,x_0}) = \psi(X_s^{t_0,x_0})\},$$

for each $\omega \in \hat{\Omega}$. Using the stochastic Feynman-Kac formula and by the strong Markov property, we have almost surely

$$\begin{aligned}\psi(t_0, x_0) =&\hat{\mathbb{E}}[\psi(X_\tau^{t_0,x_0}) \exp(\int_0^\tau (\alpha - k\psi(X_\tau^{t_0,x_0}))] \times \exp(\int_{t_0-\tau}^{t_0} \epsilon dW_s - \frac{1}{2}\int_{t_0-\tau}^{t_0} \epsilon^2 ds)\\ \leq&\hat{\mathbb{E}}[\zeta(X_\tau^{t_0,x_0})e^{\int_0^\tau (\alpha - k\zeta(X_\tau^{t_0,x_0}))\,ds}] \times e^{\int_{t_0-\tau}^{t_0} \epsilon dW_s - \frac{1}{2}\int_{t_0-\tau}^{t_0} \epsilon^2 ds}\\ =&e^{\sup\limits_{0\leq r\leq t_0} \int_{t_0-r}^{t_0} \epsilon dW_s} \zeta(t_0, x_0),\end{aligned}$$

which contradicts (2.8) and the upper bound is proved.

Similarly, we have almost surely

$$\psi(t_0, x_0) \geq \exp(\inf_{0\leq r\leq t_0} \int_r^{t_0} \epsilon dW_s)\zeta(t_0, x_0)\ a.s.$$

For arbitrary $t > 0$ fixed, for any $\sigma > 0$, multiplying $G(t - s + \sigma, x - y)$ in (2.6) and integrating over R, we obtain

$$\begin{aligned}&\frac{\partial}{\partial s}\int_R \zeta(s, y)G(t - s + \sigma, x - y)dy\\ &\quad\leq (\alpha - \frac{\epsilon^2}{2})\int_R \zeta(s, y)G(t - s + \sigma, x - y)dy - k(\int_R \zeta(s, y)G(t - s + \sigma, x - y)dy)^2.\end{aligned}$$

Let $\varphi(s) = \int_R \zeta(s, y)G(t - s + \sigma, x - y)dy$, thus we get

$$\begin{cases}\frac{d\varphi(s)}{ds} \leq (\alpha - \frac{\epsilon^2}{2})\varphi(s) - k\varphi^2(s),\\ \varphi_0 = \int_R \zeta_0 G(t + \sigma, x - y)dy.\end{cases} \tag{2.9}$$

In general, we have

$$\varphi(s) \leq \varphi_0 + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}, \tag{2.10}$$

which implies

$$\int_R \zeta(t, y)G(\sigma, x - y)dy \leq \int_R \zeta_0 G(t + \sigma, x - y)dy + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}. \tag{2.11}$$

Let $\sigma \to 0$, then

$$\zeta(t, x) \leq \int_R \zeta_0 G(t, x - y)dy + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}\ a.s. \tag{2.12}$$

Combining the above estimate with (2.7), we obtain

$$\sum_{i=1}^n u^{(i)}(t, x) \leq e^{\sup\limits_{0\leq r\leq t}\int_r^t \epsilon dW_s} \times (\int_R \psi_0 G(t, x - y)dy + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}) \quad a.s. \tag{2.13}$$

Fixing the initial data $u_0^{(i)} = p_i\chi_{(-\infty,0]}$, and taking the expectation, we get

$$\mathbb{E}[\sum_{i=1}^{n} u^{(i)}(t,x)] \leq C(\epsilon,t)(\sum_{i=1}^{n} u_0^{(i)} + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}), \tag{2.14}$$

where $C(\epsilon,t) = \mathbb{E}[e^{\sup_{0\leq r\leq t}\int_r^t \epsilon dW_s}]$. □

Lemma 2.5. *For any Heaviside functions $u_0^{(i)}$, a.e. $\omega \in \Omega$ and $t > 0$, one has*

$$\mathbb{E}[\sum_{i=1}^{n} |u^{(i)}(t)|^2] \leq \mathbb{E}[\sum_{i=1}^{n} |u_0^{(i)}|^2]e^{-t} + K(1-e^{-t}), \tag{2.15}$$

where $K = \frac{(\epsilon^6+2\alpha+1)^3 n}{54k^2}$.

Proof. Let $V(t) := \sum_{i=1}^{n} |u^{(i)}(t)|^2$, by Itô formula we have

$$\begin{aligned} dV(t) =& 2\sum_{i=1}^{n}\langle u^{(i)}, \triangle u^{(i)}\rangle dt + 2\sum_{i=1}^{n}\langle u^{(i)}, a_i u^{(i)} - b_{ii}(u^{(i)})^2 + \sum_{j=1}^{n} b_{ij}u^{(i)}u^{(j)}\rangle dt \\ &+ \epsilon^2\sum_{i=1}^{n}(u^{(i)})^2 dt + 2\epsilon\sum_{i=1}^{n}(u^{(i)})^2 dW_t. \end{aligned}$$

Integrate both sides on $[0,t]$ and take expectation, we have

$$\begin{aligned} \mathbb{E}[V(t)] =& \mathbb{E}\sum_{i=1}^{n}(u_0^{(i)})^2 + 2\mathbb{E}\sum_{i=1}^{n}\int_0^t \langle u^{(i)}, \triangle u^{(i)}\rangle ds + 2\mathbb{E}\sum_{i=1}^{n}\int_0^t \langle u^{(i)}, a_i u^{(i)} - b_{ii}(u^{(i)})^2 \\ &+ \sum_{j=1}^{n} b_{ij}u^{(i)}u^{(j)}\rangle ds + \epsilon^2\sum_{i=1}^{n}\mathbb{E}\int_0^t (u^{(i)})^2 ds \\ \leq& \mathbb{E}\sum_{i=1}^{n}(u_0^{(i)})^2 - 2\mathbb{E}\sum_{i=1}^{n}\int_0^t |\nabla u^{(i)}|^2 ds + 2\alpha\mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^2 ds \\ &- 2k\mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^3 ds + \epsilon^2\mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^2 ds \\ \leq& \mathbb{E}\sum_{i=1}^{n}(u_0^{(i)})^2 - 2k\mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^3 ds + 2\alpha\mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^2 ds \\ &+ \epsilon^2\mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^2 ds + \mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^2 ds - \mathbb{E}\sum_{i=1}^{n}\int_0^t (u^{(i)})^2 ds. \end{aligned}$$

By Young inequality we have

$$(2\alpha+1)\mathbb{E}\int_0^t \sum_{i=1}^{n}(u^{(i)})^2 ds \leq k\int_0^t \mathbb{E}\sum_{i=1}^{n}(u^{(i)})^3 ds + \frac{(2\alpha+1)^3 n}{54k^3}t, \tag{2.16}$$

and

$$\epsilon^2\mathbb{E}\int_0^t \sum_{i=1}^n (u^{(i)})^2 ds \leq k\int_0^t \mathbb{E}\sum_{i=1}^n (u^{(i)})^3 ds + \frac{\epsilon^6 n}{54k^2}t. \tag{2.17}$$

Combining (2.16) with (2.17) offers that

$$\mathbb{E}[\sum_{i=1}^n |u^{(i)}(t)|^2] \leq \mathbb{E}[\sum_{i=1}^n |u_0^{(i)}|^2] + \frac{(\epsilon^6+2\alpha+1)^3)n}{54k^2}t - \mathbb{E}\sum_{i=1}^n\int_0^t (u^{(i)})^2 ds.$$

Thus by Gronwall inequality we have

$$\mathbb{E}\sup_{0\leq t\leq T}[\sum_{i=1}^n |u^{(i)}(t)|^2] \leq \mathbb{E}[\sum_{i=1}^n |u_0^{(i)}|^2]e^{-t} + \frac{(\epsilon^6+2\alpha+1)^3)n}{54k^2}(1-e^{-t}).$$

□

Modifying the argument in Lemma 2.1 from [1], we can estimate how fast the compact support of $Y(t)$ can spread.

Lemma 2.6. *Let $Y(t,x)$ be a solution to (1.4) started at Y_0, suppose for some $R>0$ that Y_0 is supported outside $(-R-2,R+2)$, then for any $t\geq 1$,*

$$\mathbb{P}(\int_0^t\int_{-R}^R \|Y(s,x)\|_\infty dsdx > 0) \leq Ce^t\int \frac{\sqrt{t}}{|x|-(R+1)}exp(-\frac{(|x|-(R+1))^2}{2t})\|Y_0\|_\infty dx.$$

Proof. From Theorem 2.4, we know the solution $Y(t,x)$ is uniformly bounded, thus the sup-solution solves

$$\begin{cases} dv^{(i)} = [v_{xx}^{(i)} + v^{(i)}(k-bv^{(i)})]dt + \epsilon v^{(i)}dW_t, \\ v^{(i)}(0) = u_0^{(i)}, \quad i=1,2,\cdots,n, \end{cases} \tag{2.18}$$

where $k>0$ is a constant satisfying $F_i(Y)\leq u(k-bu)$. Refer to [1,23], the proof can be completed. □

Remark 1. When $R_0(t)$ is defined as a wavefront marker as in [1], the SCP property of $Y(t,x)$ can not hold. Additionally, we can not ensure the translational invariance of the solution $Y(t,x)$ with respect to $R_0(t)$. However thanks to Lemma 2.6, we can choose a suitable wavefront marker to ensure the SCP property of $Y(t)$ holds.

It is easy to verify that $Y(t,x)$ satisfy Kolmogorov tightness criterion, and $Y(t,x)\in K(C,\delta,\mu,\gamma)$, which helps constructing a probability measure sequence, which is convergent.

Lemma 2.7. *For any Heaviside functions $u_0^{(i)}$, $t>0$, fixed $p\geq 2$ and a.e. $\omega\in\Omega$, if $|x-x'|\leq 1$, there exists positive constant $C(t)$, such that*

$$Q^{Y_0}(|Y(t,x)-Y(t,x')|^p) \leq C(t)|x-x'|^{p/2-1}.$$

Proof. Referring to [1], it is not difficult to complete the proof. □

Define Q^{Y_0} as the law of the unique solution to Eq (1.4) with initial data $Y(0) = Y_0$. For a probability measure ν on C^+_{tem}, we define

$$Q^{\nu}(A) = \int_{C^+_{tem}} Q^{Y_0}(A)\nu(dY_0).$$

In order to construct the travelling wave solution to Eq (1.3), we must ensure that the translation of solution with respect to a wavefront marker is stationary and the solution poses the SCP property. However, $R_0(Y(t))$ does not satisfy this condition. So we have to choose a new suitable wavefront marker. As the solution to (1.4) with Heaviside initial condition is exponentially small almost surely as $x \to \infty$, with the stochastic Feynman-Kac formula we may turn to $R_1(t) : C^+_{tem} \to [-\infty, \infty]$ defined as

$$R_1(f) = ln \int_R e^x f dx, \quad R_1(u^{(i)}(t)) = ln \int_R e^x u^{(i)}(t)dx,$$

and

$$R_1(t) := R_1(Y(t)) = \max_i \{R_1(u^{(i)}(t))\}.$$

The marker $R_1(t)$ is an approximation to $R_0(Y(t)) = \max_i \{R_0(u^{(i)}(t))\}$.

Let

$$\begin{aligned} Z(t) &= Y(t, \cdot + R_1(t)) = (Z_1(t), Z_2(t), \cdots, Z_n(t))^T, \\ Z_0(t) &= Y(t, \cdot + R_0(Y(t))), \end{aligned}$$

and define

$$Z(t) = \begin{cases} (0, 0, \cdots, 0)^T, & R_1(t) = -\infty, \\ (u^{(1)}(t, \cdot + R_1(t)), u^{(2)}(t, \cdot + R_1(t)) \cdots, u^{(n)}(t, \cdot + R_1(t)))^T, & \\ & -\infty < R_1(t) < \infty \\ (p_1, p_2, \cdots, p_n)^T, & R_1(t) = \infty. \end{cases}$$

Hence $Z(t)$ is the wave shifted so that the wavefront marker $R_1(t)$ lies at the origin. Note that whenever $R_0(Y_0) < \infty$, the compact support property implies that $R_0(t) < \infty$, $\forall t > 0$, Q^{Y_0}-a.s.

Remark 2. Here we define $R_1(t)$ in the maximum form, not only since it simplifies the discussion about boundedness, but also the asymptotic wave speed is the minimum wave speed which keeps the travelling wave solution monotonic. As mentioned before, we approximate the asymptotic wave speed via $c = \lim_{t\to\infty} \frac{R_1(t)}{t}$. Therefore, the wavefront marker $R_1(t)$ defined in such form can ensure the travelling wave solutions of the two subsystems monotonic.

Define

$$\nu_T = \textit{ the law of } \frac{1}{T}\int_0^T Z(s)ds \textit{ under } Q^{Y_0}.$$

Now we summarise the method for constructing the travelling wave solution. With the initial data $(u_0^{(1)} = p_1\chi_{(-\infty,0]}, u_0^{(2)} = p_2\chi_{(-\infty,0]}, \cdots, u_0^{(n)} = p_n\chi_{(-\infty,0]}) \in C^+_{tem}$ as Heaviside function, we shall show that the sequence $\{\nu_T\}_{T\in\mathbb{N}}$ is tight (see Lemma 2.9) and any limit point is nontrivial (see Theorem 2.10). Hence for any limit point ν (the limit is not unique), Q^{ν} is the law of a travelling wave solution. Two parts constituting the proof of tightness are Kolmogorov tightness criterion for the unshifted waves (see Lemma 2.7) and the control on the movement of the wavefront marker $R_1(t)$ to ensure the shifting will not destroy the tightness (see Lemma 2.8).

Lemma 2.8. *For any Heaviside functions $u_0^{(i)}$, $t \geq 0$, $d > 0$, $T \geq 1$, and a.e. $\omega \in \Omega$ there exists a positive constant $C(t) < \infty$, such that*

$$Q^{\nu_T}(|R_1(t)| > d) \leq \frac{C(t)}{d}. \tag{2.19}$$

Proof. By the comparison principle proposed in [24, 25], we can construct a sup-solution satisfying, for $i = 1, 2, \cdots, n$,

$$\begin{cases} d\tilde{u}^{(i)} = [\tilde{u}^{(i)}_{xx} + k_0\tilde{u}^{(i)}]dt + \epsilon\tilde{u}dW_t, \\ \tilde{u}^{(i)}_0 = u^{(i)}_0, \end{cases} \tag{2.20}$$

where the constant $k_0 > 0$ can be obtained by Theorem 2.4 such that $F_i(Y) < k_0u^{(i)}$. Therefore, we know that $u^{(i)}(t) \leq \tilde{u}^{(i)}(t)$ hold on $[0, T]$ uniformly, and for a.e. $\omega \in \Omega$ the solution $\tilde{Y}(t, x)$ to Eq (2.20) is

$$\tilde{Y}(t, x) = \int_R e^{k_0t}G(t, x - y)Y_0(y)dy + \epsilon \int_R \int_0^t G(t - s, x - y)\tilde{Y}dW_s dy. \tag{2.21}$$

Applying the comparison method yields, for any $i = 1, 2, \cdots, n$ we have

$$Q^{u_0^{(i)}}(\int_R u^{(i)}(t, x)e^x dx) \leq \mathbb{E}[\int_R \tilde{u}^{(i)}(t, x)e^x dx] = e^{k_0t+t} \int_R u_0^{(i)}(x)e^x dx.$$

Without loss of generality, we assume that $R_1(t) = R_1(u^{(1)}(t))$, then

$$\int_R u^{(1)}(t, x + R_1(t))e^x dx = e^{-R_1(t)} \int_R u^{(1)}(t, x)e^x dx = 1.$$

Combing with the above arguments, we deduce that

$$Q^{\nu_T}(R_1(t) \geq d) = \frac{1}{T}\int_0^T Q^{u_0^{(1)}}(Q^{u^{(1)}(s)}(e^{-d}\int_R u^{(1)}(t, x)e^x dx \geq 1))ds \leq e^{-d}e^{k_0t+t}.$$

Then the Jensen's inequality gives

$$Q^{u_0^{(1)}}(R_1(t)) \leq \ln(e^{k_0t+t}\int_R u_0^{(1)}(x)e^x dx) \leq k_0t + t + R_1(u_0^{(1)}).$$

Direct calculation implies

$$\begin{aligned} &\frac{1}{T}Q^{u_0^{(1)}}(\int_t^{T+t} R_1(s)ds - \int_0^T R_1(s)ds) \\ \leq &\frac{1}{T}\int_0^T \int_0^\infty Q^{u_0^{(1)}}(R_1(t + s) - R_1(s) \geq y)dyds \end{aligned}$$

$$
\begin{aligned}
&- \frac{d}{T}\int_0^T Q^{u_0^{(1)}}(R_1(t+s)-R_1(s)\le -d)ds \\
&= \int_0^\infty Q^{\nu_T}(R_1(t)\ge y)dy - dQ^{\nu_T}(R_1(t)\le -d).
\end{aligned}
$$

Thus, by rearranging the above inequalities

$$
Q^{\nu_T}(R_1(t)\le -d) \le \frac{1}{d}\int_0^\infty Q^{\nu_T}(R_1(t)\ge y)dy + \frac{1}{dT}\int_0^T Q^{\nu_T}(R_1(s))ds \le \frac{C(t)}{d},
$$

which completes the proof of Lemma 2.8. □

We will prove the marker $R_1(t)$ is bounded, which helps to prove the sequence $\{\nu_T : T\in\mathbb{N}\}$ is tight and the wavefront marker $R_0(t)$ is bounded. Next, we will show the tightness of $\{\nu_T : T\in\mathbb{N}\}$ with $Y(t,x)\in K(C,\delta,\mu,\gamma)$.

Lemma 2.9. *For any Heaviside functions $u_0^{(i)}$, and a.e. $\omega\in\Omega$, the sequence $\{\nu_T : T\in\mathbb{N}\}$ is tight.*

Proof. Following the idea to prove Lemma 2.8, we focus on the term $u^{(i)}(t,x)$. Since $Y(t,x)\in K(C,\delta,\mu,\gamma)$ gives $u^{(i)}(t,x)\in K(C,\delta,\mu,\gamma)$, then it is easy to prove that

$$
\begin{aligned}
\nu_T(K(C,\delta,\gamma,\mu)) =& \frac{1}{T}\int_0^T Q^{u_0^{(i)}}(u^{(i)}(t,\cdot+R_1(t))\in K(C,\delta,\gamma,\mu))ds \\
\ge& \frac{1}{T}\int_0^T Q^{u_0^{(i)}}((u^{(i)}(t,\cdot+R_1(t-1))\in K(Ce^{-\mu d},\delta,\gamma,\mu)) \\
&\times |R_1(t)-R_1(t-1)|\le d)ds \\
\ge& \frac{1}{T}\int_1^T Q^{u_0^{(i)}}(Q^{Z_1(t-1)}(u^{(i)}(1)\in K(Ce^{-\mu d},\delta,\gamma,\mu)))dt \\
&- \frac{1}{T}\int_1^T Q^{u_0^{(i)}}(|R_1(t)-R_1(t-1)|\ge d)dt \\
=:& I - II.
\end{aligned}
$$

With Lemma 2.8, $II\to 0$ as $d\to\infty$. Via the Kolmogorov tightness and Lemma 2.7, for given $d,\mu>0$, one can choose C,δ,γ to make I as close to $\frac{T-1}{T}$ as desired. In addition, we have

$$
\nu_T\{u_0^{(i)} : \int_R u_0^{(i)}(x)e^{-|x|}dx \le \int_R u_0^{(i)}(x)e^{x}dx = 1\} = 1.
$$

The definition of tightness implies that for given $\mu>0$, one can choose C,δ,γ such that $\nu_T(K(C,\delta,\mu,\gamma)\cap\{u_0^{(i)} : \int_R u_0^{(i)}(x)e^{-|x|}dx\})$ as close to 1 as desired for T and d sufficient large, which implies that the sequence $\{\nu_T : T\in\mathbb{N}\}$ is tight. □

Theorem 2.10. *For any Heaviside functions $u_0^{(i)}$, and for a.e. $\omega\in\Omega$, there is a travelling wave solution to Eq* (1.3)*, and Q^ν is the law of travelling wave solution.*

Proof. By the comparison method, we have

$$Z_i(t,x) \le e^{t-\frac{\epsilon^2}{2}t+\int_0^t \epsilon dW_s} \times \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{-\frac{x}{\sqrt{2}}} e^{-\frac{|y|^2}{2t}} dy \qquad a.s.,$$

under the law $Q^{u_0^{(i)}}$ for $t > 0$. Taking $u^{(1)}(t,x)$ together with Doob's inequality and (2.3), we have

$$\begin{aligned} u^{(1)}(1,x) \le & e^{t-\frac{\epsilon^2}{2}+\int_0^1 \epsilon dW_s} \times e^{-\frac{\epsilon^2}{2}(t-1)+\int_0^{t-1} \epsilon dW_s} \\ & \times \frac{1}{2\pi\sqrt{t-1}} \int_{-\infty}^{+\infty} \int_{-\infty}^{-\frac{x}{\sqrt{2}}-z} e^{-\frac{|y|^2}{2(t-1)}} dy e^{-\frac{|y|^2}{2}} dz \\ \le & e^{t-\frac{\epsilon^2}{2}+\int_0^1 \epsilon dW_s} \times e^{-\frac{\epsilon^2}{2}(t-1)+\int_0^{t-1} \epsilon dW_s - \frac{x^2}{4t}} \qquad a.s., \end{aligned} \tag{2.22}$$

for all $t > 1$. Integrate (2.22) in $[d, \infty)$ and take the expectation, we have

$$\lim_{d\to\infty} Q^{u_0}(Q^{Z_1(t-1)}(\int_d^\infty u^{(1)}(1,x)dx)) \le \lim_{d\to\infty} \sqrt{t} e^{t-\frac{d^2}{4t}} = 0. \tag{2.23}$$

Furthermore, it follows that

$$\lim_{d\to\infty} Q^{u_0^{(1)}}(Q^{Z_1(t-1)}(\int_d^\infty u^{(1)}(1,x)dx)) = 1, \tag{2.24}$$

and

$$\begin{aligned} & \nu_T(u_0^{(1)} : \lim_{d\to\infty} \int_{2d}^\infty u_0^{(1)}(x)dx = 0) \qquad (2.25) \\ = & \frac{1}{T}\int_0^T Q^{u_0^{(1)}}(\forall\delta > 0, \exists d_0, \int_{2d}^\infty Z_1(t,x)dx) < \delta \ \forall\ d > d_0)dt, \\ & |R_1(t) - R_1(t-1)| \le d, \forall d > d_0)dt \\ \ge & \frac{1}{T}\int_1^T Q^{u_0^{(1)}}(Q^{Z_1(t-1)}(\lim_{d\to\infty}\int_d^\infty u^{(1)}(1,x)dx = 0))dt - \lim_{d\to\infty} Q^{\nu_T}(|R_1(1)| \ge d). \end{aligned}$$

Thus by Lemma 2.8, combining (2.24) with (2.25) gives

$$\lim_{T\to\infty}\lim_{d\to\infty} \nu_T(u_0^{(1)} : \int_d^\infty u_0^{(1)}(x)dx = 0) = 1. \tag{2.26}$$

To prove the boundness of $R_0(t)$, it follows from $\nu_{T_n}(u_0^{(1)} : \int_R u_0^{(1)}(x)e^x dx = p_1) = 1$ that $\nu(u_0^{(1)} : \int_R u_0^{(1)}(x)e^x dx \le p_1) = 1$. Taking $e_1^d(x) = e^{d-|x-d|}$, we have

$$\begin{aligned} & \nu(u_0^{(1)} : (u_0^{(1)}, e^x) \ge p_1) \ge \nu(u_0^{(1)} : \int_R u_0^{(1)}(x)e_1^d(x)dx \ge p_1) \\ \ge & \limsup_{n\to\infty} \nu_{T_n}(u_0^{(1)} : \int_R u_0^{(1)}(x)e_1^d(x)dx = p_1) \\ = & \limsup_{n\to\infty} \nu_{T_n}(u_0^{(1)} : \int_R u_0^{(1)}(x)I_{(d,\infty)}dx) = 0) \to 1,\ as\ d \to \infty. \end{aligned}$$

As $\nu(u_0^{(1)}: \int_R u_0^{(1)}(x)e^x dx = p_1) = 1$, we obtain $\nu(u_0^{(1)}: R_0(u_0^{(1)}) > -\infty) = 1$. Now, we complete the half of the proof of the boundness of the wavefront marker $R_0(t)$. Take $\psi_d \in \Phi$ with $(\psi_d > 0) = (d, \infty)$, then

$$\begin{aligned}\nu(u_0^{(1)}: R_0(u_0^{(1)}) \le d) =&\nu(u_0^{(1)}: \int_R u_0^{(1)}(x)\psi_d(x)dx = 0)\\ \ge&\limsup_{n\to\infty}\nu_{T_n}(u_0^{(1)}: \int_R u_0^{(1)}(x)\psi_d(x)dx = 0)\\ =&\limsup_{n\to\infty}\nu_{T_n}(u_0^{(1)}: \int_R u_0^{(1)}(x)I_{(d,\infty)}dx = 0) \to 1,\ as\ d\to\infty,\end{aligned}$$

so we have $\nu(Y_0 : -\infty < R_0(Y_0) < \infty) = 1$ and complete the proof of the boundedness of the wavefront marker $R_0(t)$. To verify that the solution $Y(t)$ is nontrivial, let $R_1^d(t) = \ln \int \|Y(t)\|_\infty e_1^d(x)dx$, we have

$$\begin{aligned}Q^\nu(\exists s\le t,\ |Y(s)| = 0) \le&Q^\nu(R_1^d(t) < -d)\\ \le&\limsup_{n\to\infty}Q^{\nu_{T_n}}(R_1^d(t) < -d)\\ \le&\limsup_{n\to\infty}(Q^{\nu_{T_n}}(R_1(t) < -d) + Q^{\nu_{T_n}}(\int_R u^{(i)}(t,x)I_{(d,\infty)}dx > 0))\\ \le&\frac{C(t)}{d} \to 0,\ as\ d\to\infty.\end{aligned}$$

We now show that $Z(t)$ is a stationary process and Q^ν is the law of a travelling wave solution to (1.3). Let $F: C_{tem}^+ \to R$ be bounded and continuous, and take $u^{(i)}(t,x)$ for example, for any fixed $t > 0$

$$\begin{aligned}&|Q^{\nu_{T_n}}(F(Z_i(t))) - Q^\nu(F(Z_i(t)))|\\ \le&|Q^{\nu_{T_n}}(F(u^{(i)}(t,\cdot + R_1^d(t)))) - Q^\nu(F(u^{(i)}(t,\cdot + R_1^d(t))))|\\ &+ \|F(u_0^{(i)})\|_\infty(Q^{\nu_{T_n}}(R_1(t) \ne R_1^d(t)) + Q^\nu(R_1(t) \ne R_1^d(t))),\end{aligned}$$

since $\nu_{T_n}(u_0^{(i)}: \int_R u_0^{(i)}e^x dx = p_i) = 1$, we have

$$Q^{\nu_{T_n}}(R_1(t) \ne R_1^d(t)) \le Q^{\nu_{T_n}}(\int_R u^{(i)}(t,x)I_{(d,\infty)}dx > 0) \le C(k_0,t)/d, \tag{2.27}$$

and with $\nu(u_0^{(i)}: \int_R u_0^{(i)}e^x dx = p_i) = 1$, we have

$$Q^\nu(R_1(t) \ne R_1^d(t)) \le Q^\nu(\int_R u(t,x)I_{(d,\infty)}dx > 0) \le C(k_0,t)/d. \tag{2.28}$$

By the continuity of $u_0^{(i)} \to Q^{u_0^{(i)}}$, one have $Q^{\nu_{T_n}} \to Q^\nu$. Since F is bounded and continuous, we obtain that

$$|Q^{\nu_{T_n}}(F(u^{(i)}(t,\cdot + R_1^d(t)))) - Q^\nu(F(u^{(i)}(t,\cdot + R_1^d(t))))| \to 0,\ as\ n\to\infty.$$

Therefore, we have

$$Q^\nu(F(Z_i(t))) = \lim_{n\to\infty} Q^{\nu_{T_n}}(F(Z_i(t))) = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} Q^{u_0}(F(Z_i(s)))ds = \nu(F).$$

It is straightforward to check that $\{Z(t) : t \ge 0\}$ is Markov, hence $\{Z(t) : t \ge 0\}$ is stationary. Since the map $Y_0 \to Y_0(\cdot - R_0(Y_0))$ is measurable on C_{tem}^+, the process $\{Z_0(t) : t \ge 0\}$ is also stationary, which implies that Q^ν is the law of the travelling wave solution to Eq (1.3). □

3. Approximation of the asymptotic wave speed

In this section, we investigate the asymptotic property of the travelling wave solutions. By constructing the sup-solution and the sub-solution, we obtain the asymptotic wave speed for the two travelling wave solutions respectively. Then we have the estimation of the wave speed of travelling wave solutions to (1.3). Since the asymptotic wave speed c of the travelling wave solution defined as

$$c = \lim_{t\to\infty} \frac{R_0(t)}{t} \ a.s.,$$

we denote by $R_0(u^{(i)}(t)) = \sup\{x \in \mathbb{R} : u^{(i)}(t,x) > 0\}$ for the sub-systems of the cooperative system. Since the wavefront marker $R_0(t)$ of the cooperative system is $R_0(t) = \max_i\{R_0(u^{(i)}(t))\}$, and the asymptotic wave speed is the maximum value among $\lim_{t\to\infty} \frac{R_0(u^{(i)}(t))}{t}$, we can define the wave speed $c^\star$ as

$$c^\star = \lim_{t\to\infty} \frac{R_0(Y(t))}{t} \ a.s.$$

We now construct a sup-solution. Let $\bar{Y}(t,x) = (\bar{u}^{(1)}(t,x), \cdots, \bar{u}^{(n)}(t,x))^T$ satisfying

$$\begin{cases} d\bar{u}^{(i)} = [\bar{u}^{(i)}_{xx} + \bar{u}(p - b_{ii}\bar{u})]dt + \epsilon\bar{u}dW_t, \\ \bar{u}^{(i)}_0 = u^{(i)}_0, \quad i = 1,2,\cdots,n, \end{cases} \tag{3.1}$$

where $F_i(Y) \le u^{(i)}(p - b_{ii}u^{(i)})$, $p = \max_{i\neq j}\{b_{ij}\} \times \max_i\{\sqrt{\sum_{i=1}^n |u^{(i)}_0|^2 + K}, C(\epsilon,t)(\sum_{i=1}^n u^{(i)}_0 + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}), p_i\} + 1$. Then we construct a sub-solution, denote by $a = \min\{a_i\}$ and let $\underline{Y}(t,x) = (\underline{u}^{(1)}(t,x), \cdots, \underline{u}^{(n)}(t,x))^T$ satisfy

$$\begin{cases} d\underline{u}^{(i)} = [\underline{u}^{(i)}_{xx} + \underline{u}^{(i)}(a - b_{ii}\underline{u})]dt + \epsilon\underline{u}dW_t, \\ \underline{u}^{(i)}_0 = u^{(i)}_0, \quad i = 1,2,\cdots,n. \end{cases} \tag{3.2}$$

Obviously, $F_i(Y) \ge u^{(i)}(a - b_{ii}u^{(i)})$. With Eq (3.1) and (3.2), we have such following conclusion:

Theorem 3.1. *For any Heaviside functions $u^{(i)}_0$, let $c^\star$ be the asymptotic wave speed of Eq* (1.3)*, then*

$$\sqrt{4a - 2\epsilon^2} \le c^\star \le \sqrt{4p - 2\epsilon^2} \ a.s. \tag{3.3}$$

In order to prove Theorem 3.1, we need the following lemmas. We first introduce the comparison method for the asymptotic wave speed.

Lemma 3.2. *Let $\underline{Y}(t,x)$ and $\bar{Y}(t,x)$ be the solutions to* (3.2) *and* (3.1) *respectively, if $\underline{c}$ is the asymptotic wave speed of $\underline{Y}(t,\cdot + R_0(\underline{Y}(t)))$ and $\bar{c}$ is the asymptotic wave speed of $\bar{Y}(t,\cdot + R_0(\bar{Y}(t)))$, then*

$$\underline{c} \le c^\star \le \bar{c} \quad a.s.$$

Proof. The comparison method for the stochastic diffusion equation gives that $\underline{Y}(t,x) \le Y(t,x) \le \bar{Y}(t,x)$, which implies $\underline{u}^{(i)}(t,x) \le u^{(i)}(t,x) \le \bar{u}^{(i)}(t,x)$ a.s. and $\underline{v}^{(i)}(t,x) \le v^{(i)}(t,x) \le \bar{v}^{(i)}(t,x)$ a.s.. Denote the wavefront markers by $R_1(\underline{Y}(t))$, $R_1(Y(t))$ and $R_1(\bar{Y}(t))$, with the definition of asymptotic wave speed

$$c = \lim_{t\to\infty} \frac{R_1(t)}{t} \ a.s.,$$

and the definition of the wavefront marker

$$R_1(Y(t)) = \max_i\{\ln \int_R u^{(i)}(t,x)e^x dx\},$$

it gives

$$\lim_{t\to\infty} \frac{R_1(\underline{Y}(t))}{t} \leq \lim_{t\to\infty} \frac{R_1(Y(t))}{t} \leq \lim_{t\to\infty} \frac{R_1(\bar{Y}(t))}{t} \ a.s., \tag{3.4}$$

which implies $\underline{c} \leq c^\star \leq \bar{c}$ $a.s.$. Thus, the proof of Lemma 3.1 is complete. □

3.1. Asymptotic wave speed of sub-solution

Now we show the asymptotic property of the wavefront marker of the sub-solution. Consider Eq (3.2), for $i = 1, 2, \cdots, n$,

$$\begin{cases} d\underline{u}^{(i)} = [\underline{u}^{(i)}_{xx} + \underline{u}^{(i)}(a - b_{ii}\underline{u}^{(i)})]dt + \epsilon \underline{u}^{(i)} dW_t, \\ \underline{u}^{(i)}_0 = u_0. \end{cases}$$

Obviously $\underline{u}^{(i)}$ are independent from each other, thus we can divide (3.2) into n equations to study. For each equation one can have the asymptotic wave speed $c(\underline{u}^{(i)})$ respectively, so the asymptotic wave speed of (3.2) is $c(\underline{Y}) = \max_i\{c(\underline{u}^{(i)})\}$.

Theorem 3.3. *For any Heaviside functions $u^{(i)}_0$, $\underline{Y}(t,x)$ is solution to* (3.2)*, then the asymptotic wave speed $c(\underline{Y})$ satisfies*

$$c(\underline{Y}) = \sqrt{4a - 2\epsilon^2} \ a.s., \tag{3.5}$$

where $a = \min_i\{a_i\}$.

Proof. For any $h > 0$, take $\kappa \in (0, \frac{h^2}{4} + \sqrt{1 - \frac{\epsilon^2}{2}h})$ and define

$$\eta_t(\omega) = e^{\int_0^t \epsilon dW_s - \frac{1}{2}\int_0^t \epsilon^2 ds}, \ 0 \leq t \leq \infty,$$

construct new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, $\tilde{W} = (\tilde{W}(t) : t \geq 0)$ is a Brownian motion. Then there exists $T_1 > 0$, such that for $t \geq T_1$ and a.e. $\omega \in \Omega$

$$e^{-\frac{\epsilon^2}{2}t - \kappa t} \leq \eta_t(\omega) \leq e^{-\frac{\epsilon^2}{2}t + \kappa t}.$$

Thus the stochastic Feynman-Kac formula gives

$$\begin{aligned} \underline{u}^{(i)}(t,x) \leq & e^{at - \frac{1}{2}\epsilon^2 t + \kappa t}\tilde{\mathbb{P}}(\tilde{W}(t) \leq -\frac{x}{\sqrt{2}}) \\ \leq & e^{at - \frac{1}{2}\epsilon^2 t + \kappa t - \frac{x^2}{4t}} \ a.s., \end{aligned}$$

for $t \geq T_1$. For a constant k, set $x \geq (k+h)t$. Multiple e^x with both sides and integrate in $[(k+h)t, \infty)$, we have

$$\begin{aligned}\int_{(k+h)t}^{\infty} \underline{u}^{(i)}(t,x)e^x dx \leq & \int_{(k+h)t}^{\infty} \exp(at - \frac{1}{2}\epsilon^2 t + \kappa t - \frac{x^2}{4t} + x)dx \\ \leq & 2\sqrt{t}e^{at-\frac{1}{2}\epsilon^2 t+\kappa t+t} \int_{\frac{(k+h)t-2t}{\sqrt{4t}}}^{\infty} e^{-x^2} dx \\ \leq & \sqrt{t}e^{a+\kappa-\frac{k^2}{4}-\frac{kh}{2}-\frac{h^2}{4}-k-h-\frac{\epsilon^2}{2}t} \quad a.s.,\end{aligned}$$

for $t \geq T_1$. Let $k = \sqrt{4a - 2\epsilon^2 + 4} - 2$, then we obtain

$$\lim_{t\to\infty} \int_{(k+h)t}^{\infty} \underline{u}^{(i)}(t,x)e^x dx = 0 \qquad a.s. \tag{3.6}$$

Integrating $\underline{u}^{(i)}(t,x)e^x$ in $[(\sqrt{4a-2\epsilon^2}+h)t, (k-h)t)$ yields

$$\begin{aligned}&\int_{(\sqrt{4a-2\epsilon^2}+h)t}^{(k-h)t} \underline{u}^{(i)}(t,x)e^x dx \\ \leq & \int_{(\sqrt{4a-2\epsilon^2}+h)t}^{(k-h)t} \exp(at - \frac{1}{2}\epsilon^2 t + \kappa t - \frac{x^2}{4t} + x)dx \\ \leq & 2\sqrt{t}e^{at-\frac{1}{2}\epsilon^2 t+\kappa t+t} \int_{\frac{(\sqrt{4a-2\epsilon^2}+h)t-2t}{2\sqrt{t}}}^{\frac{(k-h)t-2t}{2\sqrt{2}}} e^{-x^2} dx \\ \leq & \sqrt{t}\exp(at - \frac{\epsilon^2}{2}t + \kappa t - \frac{4a-2\epsilon^2}{4}t - \frac{(\sqrt{4a-2\epsilon^2})h}{2}t - \frac{h^2}{4}t + \sqrt{4a-2\epsilon^2}t + ht) \\ & - \sqrt{t}\exp(at - \frac{\epsilon^2}{2}t + \kappa t - \frac{k^2}{4}t + \frac{kh}{2}t - \frac{h^2}{4}t + kt - ht) \\ \leq & \sqrt{t}e^{\kappa t+\sqrt{4a-2\epsilon^2}t-\frac{(\sqrt{4a-2\epsilon^2})h}{2}t-\frac{h^2}{4}t+ht} - \sqrt{t}e^{\kappa t-\frac{k^2}{4}t+\frac{kh}{2}t-\frac{h^2}{4}t-ht} \qquad a.s.,\end{aligned}$$

for $t \geq T_1$. Thus, we have

$$\begin{aligned}&\int_{(\sqrt{4a-2\epsilon^2}-h)t}^{(\sqrt{4a-2\epsilon^2}+h)t} \underline{u}^{(i)}(t,x)e^x dx \\ \leq & \sqrt{t}e^{\kappa t+\sqrt{4a-2\epsilon^2}t+\frac{\sqrt{4a-2\epsilon^2}h}{2}t-\frac{h^2}{4}t-ht} - \sqrt{t}e^{\kappa t+\sqrt{4a-2\epsilon^2}t-\frac{\sqrt{4a-2\epsilon^2}h}{2}t-\frac{h^2}{4}t+ht} \qquad a.s.,\end{aligned}$$

and

$$\int_{(k-h)t}^{(k+h)t} \underline{u}^{(i)}(t,x)e^x dx \leq \sqrt{t}e^{\kappa t+\frac{kh}{2}t-\frac{h^2}{4}t-ht} - \sqrt{t}e^{\kappa t-\frac{kh}{2}t-\frac{h^2}{4}t+ht} \qquad a.s.,$$

for $t \geq T_1$. Referring to [7], there exists $T_2 > 0$, such that for all $t \geq T_2$ and $x < (\sqrt{4a-2\epsilon^2} - h)t$, there exist $\rho_1, \rho_2 > 0$ satisfying

$$e^{-\rho_1\sqrt{2t\ln\ln t}} \leq \underline{u}^{(i)}(t,x) \leq e^{\rho_2\sqrt{2t\ln\ln t}} \qquad a.s., \tag{3.7}$$

which goes into

$$\int_{-\infty}^{(\sqrt{4a-2\epsilon^2}-h)t} \underline{u}^{(i)}(t,x)e^x dx \le e^{\rho_2\sqrt{2t\ln\ln t}+(\sqrt{4a-2\epsilon^2}-h)t} \qquad a.s. \tag{3.8}$$

Since $\int_{(k+h)t}^{\infty} \underline{u}^{(i)}(t,x)e^x dx \le 1$, then we have

$$\int_R \underline{u}^{(i)}(t,x)e^x dx \le e^{\rho_2\sqrt{2t\ln\ln t}+(\sqrt{4a-2\epsilon^2}-h)t}(2+H(t)+G(t)) \qquad a.s., \tag{3.9}$$

where

$$H(t) = \sqrt{t}e^{\frac{1}{2}\epsilon^2-\frac{\epsilon^2}{2}t+\kappa t+\frac{kh}{2}t-\frac{h^2}{4}t-\rho_2\sqrt{2t\ln\ln t}-\sqrt{4a-2\epsilon^2}t} \le 1,$$

and

$$G(t) = \sqrt{t}e^{\frac{1}{2}\epsilon^2-\frac{\epsilon^2}{2}t+\kappa t-\frac{\sqrt{4a-2\epsilon^2}h}{2}t-\rho_2\sqrt{2t\ln\ln t}-\frac{h^2}{4}t+2ht}.$$

Since h and κ are arbitrary, we derive that $H(t) \le 1$ a.s. for large t. Direct calculation implies that almost surely

$$\frac{1}{t}\ln G(t) = \frac{1}{2t}\ln 4t - \frac{1}{t}(\ln 2 - \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}t) + \kappa - \frac{4a-2\epsilon^2}{4}h - \frac{h^2}{4} + 2h - \frac{1}{t}\rho_2\sqrt{2t\ln\ln t},$$

and

$$\lim_{t\to\infty}\frac{1}{t}\ln G(t) = 0. \tag{3.10}$$

Hence, we obtain the upper bound of the asymptotic wave speed of the travelling wave solution to (3.2)

$$\frac{R_1(t)}{t} \le \frac{1}{t}\rho_2\sqrt{2t\ln\ln t} + \sqrt{4a-2\epsilon^2} - h + \frac{1}{t}\ln 2 + \frac{1}{t}\ln G(t)\ a.s. \tag{3.11}$$

Moreover, it follows that

$$\limsup_{t\to\infty}\frac{R_1(t)}{t} \le \sqrt{4a-2\epsilon^2} \qquad a.s. \tag{3.12}$$

and

$$\frac{R_1(t)}{t} \ge -\frac{1}{t}\rho_1\sqrt{2\ln\ln t} + \sqrt{4a-2\epsilon^2} - h \qquad a.s. \tag{3.13}$$

Thus, we deduce that the lower bound followed as

$$\liminf_{t\to\infty}\frac{R_1(t)}{t} \ge \sqrt{4a-2\epsilon^2} \qquad a.s. \tag{3.14}$$

Combining (3.12) and (3.14), we can get

$$\lim_{t\to\infty}\frac{R_1(t)}{t} = \sqrt{4a-2\epsilon^2} \qquad a.s. \tag{3.15}$$

The proof of Theorem 3.3 is complete. □

3.2. *Asymptotic wave speed of sup-solution*

By the method used in Theorem 3.3, we consider the sup-solution $\bar{Y}(t,x)$ satisfying the following equation, for $i = 1, 2, \cdots, n$

$$\begin{cases} d\bar{u}^{(i)} = [\bar{u}^{(i)}_{xx} + \bar{u}^{(i)}(p - a_1\bar{u}^{(i)})]dt + \epsilon\bar{u}dW_t, \\ \bar{u}^{(i)}_0 = u^{(i)}_0. \end{cases}$$

Similar to the proof of Theorem 3.3, we obtain the following result:

Theorem 3.4. *For any Heaviside functions $u^{(i)}_0$, $\bar{Y}(t,x)$ is a solution to* (3.1)*, then the asymptotic wave speed $c(\bar{Y})$ satisfies*

$$c(\bar{Y}) = \sqrt{4p - 2\epsilon^2} \qquad a.s. \tag{3.16}$$

Based on discussion above, and combaining Theorem 3.3 and Theorem 3.4, with Lemma 3.2 we can achieve the conclusion:

$$\sqrt{4a - 2\epsilon^2} \leq c^\star \leq \sqrt{4p - 2\epsilon^2} \qquad a.s. \tag{3.17}$$

which ends of the proof of Theorem 3.1.

4. Example: 3-species stochastic cooperative system

Recently, Zhao and Shao [26] studied the asymptotic stability and stability of stochastic 3-species cooperative system without diffusion. Shao et al. [27] studied the stochastic permanence, stability and optimal harvesting policy of a 3-three species cooperative system with delays and Lévy jumps. In this section, we apply the above conclusions to the following 3-species stochastic cooperative system and give some results about stochastic travelling waves

$$\begin{cases} du = [u_{xx} + u(a_1 - b_1u + c_1v)]dt + \epsilon udW_t, \\ dv = [v_{xx} + v(a_2 - b_2v + c_2u + d_1w)]dt + \epsilon vdW_t, \\ dw = [w_{xx} + w(a_3 - b_3w + c_3v)]dt + \epsilon wdW_t, \\ u(0,x) = u_0, v(0,x) = v_0, w(0,x) = w_0. \end{cases} \tag{4.1}$$

If $\min\{b_i\} > \max\{c_i, d_1\}$ and $b_2 \geq b_1 + b_3$, it is easy to know that $(0,0,0)$ is unstable, and $(\frac{a_1}{b_1} + \frac{c_1}{b_1} \times \frac{a_2b_1b_3+a_1b_3c_2+a_3b_1d_1}{b_1b_2b_3-b_1c_3d_1-b_3c_1c_2}, \frac{a_2b_1b_3+a_1b_3c_2+a_3b_1d_1}{b_1b_2b_3-b_1c_3d_1-b_3c_1c_2}, \frac{a_3}{b_3} + \frac{c_3}{b_3} \times \frac{a_2b_1b_3+a_1b_3c_2+a_3b_1d_1}{b_1b_2b_3-b_1c_3d_1-b_3c_1c_2}) := (p_1, p_2, p_3)$ is the only stable point, which implies that 3-species coexist. Repeating the above argument on the stochastic cooperative systems (4.1), we have the following results:

Theorem 4.1. *For any Heaviside functions u_0, v_0, w_0, and a_i, b_i, c_i, d_1 are positive constants satisfying $\min\{b_i\} > \max\{c_i, d_1\}$, $b_2 \geq b_1 + b_3$, then for a.e. $\omega \in \Omega$, there exists a travelling wave solution to Eq* (4.1)*. Moreover, the asymptotic wave speed can be obtained*

$$\sqrt{4a - 2\epsilon^2} \leq c \leq \sqrt{4p - 2\epsilon^2} \quad a.s., \tag{4.2}$$

where

$$p = 2\max\{c_i, d_1\} \times \max\{\mathbb{E}[e^{\sup\limits_{0\le r\le t}\int_r^t \epsilon dW_s}](u_0 + v_0 + w_0 + \frac{\alpha}{k} - \frac{\epsilon^2}{2k}),$$
$$\sqrt{|u_0|^2 + |v_0|^2 + |w_0|^2 + \frac{\epsilon^6 + (2\alpha+1)^3}{18k^2}}, p_1, p_2, p_3\} + \alpha,$$

and $\alpha = \max\{a_i\}$, $a = \min\{a_i\}$, $k = \frac{\min\{b_i\} - \max\{c_i, d_1\}}{3}$.

5. Conclusions

This paper introduces the travelling wave solution of stochastic N-species cooperative systems with noise, and we obtain the existence of travelling wave solution in law and estimate its corresponding wave speed. The upper bound of asymptotic wave speed depends on all the coefficients and the strength and noise, while the lower bound only relies on the environment capacity and strength of the noise. In fact, the minimal propagation speed of travelling wave depends on the supporting capacity of the natural environment, and the maximum propagation speed relies on the interspecific interaction intensity and intrinsic growth rate.

Use of AI tools declaration

The authors have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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