Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(7): 4266–4277.
DOI: 10.3934/era.2023217
Received: 28 February 2023
Revised: 08 May 2023
Accepted: 22 May 2023
Published: 01 June 2023

Theory article

Derivations of finite-dimensional modular Lie superalgebras $\overline{K}(n,m)$

Dan Mao[1,2] **and Keli Zheng**[1,*]

[1] Department of Mathematics, School of Science, Northeast Forestry University, Harbin 150040, China

[2] School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

* **Correspondence:** Email: zhengkl@nefu.edu.cn; Tel: +8615645011029.

Abstract: This paper is aimed at determining the derivation superalgebra of modular Lie superalgebra $\overline{K}(n,m)$. To that end, we first describe the $\mathbb{Z}$-homogeneous derivations of $\overline{K}(n,m)$. Then we obtain the derivation superalgebra $Der(\overline{K})$. Finally, we partly determine the derivation superalgebra $Der(K)$ by virtue of the invariance of $K(n,m)$ under $Der(\overline{K})$.

Keywords: Lie superalgebra; modular Lie superalgebra; derivation superalgebra; associative superalgebra; contact type

1. Introduction

Lie superalgebras, which originated from the research of quantum physics (see [1]), can be considered as the natural generalization of Lie algebras. Lie superalgebras are closely connected with mathematical physics as well as numerous branches of mathematics (see [2,3]). Based on the study of Lie algebras, the theory of Lie superalgebras developed rapidly, including the completed classification of finite dimensional simple Lie superalgebras in 1977 (see [4]). However, the classification of finite dimensional simple modular Lie superalgebras has not been accomplished up to now. Since the main difference between modular Lie superalgebras and Lie superalgebras in characteristic zero is the algebras of cartan type, we pay more attention to the related researches on modular Lie superalgebras of Cartan type. In [5,6], authors investigated the associative forms of modular Lie superalgebras of Cartan type. The natural filtrations (see [7–10]) and automorphisms (see [10,11]) of some Cartan type modular Lie superalgebras are studied. In addition, the cohomologies (see [6, 12, 13]) of some modular Lie superalgebras have also been determined.

It is known to all that the determination of derivation superalgebras is crucial to Lie superalgebras. The related research results in Cartan type modular Lie superalgebras are also quite rich. The derivation superalgebras of some finite dimensional simple modular Lie superalgebras of Cartan type such as $K(m,n,\underline{t})$, $W(m,n,\underline{t})$, $S(m,n,\underline{t})$, $HO(n,n;\underline{t})$, $KO(n,n+1,\underline{t})$, $SHO(m,m,\underline{t})^{(2)}$ (see [14–18]) are determined, respectively. Moreover, the derivation superalgebras of some nonsimple ones are also described, where we are most interested in the correlative results of $\overline{W}(n,m)$, $H(n,m)$ and $S(n,m)$ (see [19–21]). They all possess the derivation of Θ-type. In [22], we have constructed a class of finite dimensional modular Lie superalgebra of Contact type which is denoted by $\overline{K}(n,m)$. This paper is aimed at determining the derivation superalgebras of $\overline{K}(n,m)$.

The present paper is arranged as follows. In Section 2, certain essential notations and concepts are recalled. In Section 3, the $\mathbb{Z}$-homogeneous components of $Der(\overline{K})$, the derivation superalgebras of $\overline{K}(n,m)$, are described, respectively. Therefore, we determine $Der(\overline{K})$. In order to give a description of $Der(\overline{K})$, we prove that $K(n,m)$ is invariant under $Der(\overline{K})$.

2. Preliminaries

Hereafter $\mathbb{F}$ denotes a field of characteristic $p \geq 3$; $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ is the ring of integers modulo 2. Apart from the standard notation $\mathbb{Z}$, let $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and nonnegative integers, respectively. A simple description of construction of the modular Lie superalgebra $\overline{K}(n,m)$ in [22] will be given.

Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{F}$ in n variables $x_1, x_2, \ldots, x_n$. Suppose that $\mathbb{B}_k = \{\langle i_1, i_2, \ldots, i_k\rangle \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ and $\mathbb{B}(n) = \bigcup_{k=0}^{n} \mathbb{B}_k$, where $\mathbb{B}_0 = \emptyset$. For $u = \langle i_1, i_2, \ldots, i_k\rangle \in \mathbb{B}_k$, set $|u| = k$, $\{u\} = \{i_1, i_2, \ldots, i_k\}$ and $x^u = x_{i_1} x_{i_2} \cdots x_{i_k}$ ($|\emptyset| = 0, x^{\emptyset} = 1$). Then $\{x^u | u \in \mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of $\Lambda(n)$.

Let $\mathfrak{U} = \Lambda(n) \otimes \mathbb{T}(m)$ be the tensor product, where $\mathbb{T}(m)$ is the truncated polynomial algebra satisfying $y_i^p = 1$ for all $i = 1, 2, \ldots, m$ (see [20]). Then $\mathfrak{U}$ is an associative superalgebra with $\mathbb{Z}_2$-gradation, which is induced by the trivial $\mathbb{Z}_2$-gradation of $\mathbb{T}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Namely, $\mathfrak{U} = \mathfrak{U}_{\overline{0}} \oplus \mathfrak{U}_{\overline{1}}$, where $\mathfrak{U}_{\overline{0}} = \Lambda(n)_{\overline{0}} \otimes \mathbb{T}(m)$ and $\mathfrak{U}_{\overline{1}} = \Lambda(n)_{\overline{1}} \otimes \mathbb{T}(m)$.

For $f \in \Lambda(n)$ and $\alpha \in \mathbb{T}(m)$, we abbreviate $f \otimes \alpha$ as $f\alpha$. Then the elements $x^u y^\lambda$ with $u \in \mathbb{B}(n)$ and $\lambda \in G$ form an $\mathbb{F}$-basis of $\mathfrak{U}$. Obviously, $\mathfrak{U} = \bigoplus_{i=0}^{n} \mathfrak{U}_i$ is a $\mathbb{Z}$-graded superalgebra, where $\mathfrak{U}_i = \mathrm{span}_{\mathbb{F}}\{x^u y^\lambda \mid u \in \mathbb{B}(n), |u| = i, \lambda \in G\}$. In particular, $\mathfrak{U}_0 = \mathbb{T}(m)$ and $\mathfrak{U}_n = \mathrm{span}_{\mathbb{F}}\{x^\pi y^\lambda \mid \lambda \in G\}$, where $\pi := \langle 1, 2, \ldots, n\rangle \in \mathbb{B}(n)$.

In this paper, let $\mathrm{hg}(A) = A_{\overline{0}} \cup A_{\overline{1}}$, where $A = A_{\overline{0}} \oplus A_{\overline{1}}$ is a superalgebra. If x is a $\mathbb{Z}_2$-homogeneous element of A , then $\mathrm{deg}x$ denotes the $\mathbb{Z}_2$-degree of x.

Set $\mathrm{Y} = \{1, 2, \ldots, n\}$. Given $i \in \mathrm{Y}$, let $\partial/\partial x_i$ be the partial derivative on $\Lambda(n)$ with respect to x_i. For $i \in Y$, let D_i be the linear transformation on $\mathfrak{U}$ such that $D_i(x^u y^\lambda) = (\partial x^u/\partial x_i) y^\lambda$ for all $u \in \mathbb{B}(n)$ and $\lambda \in G$. Let $\mathrm{Der}\mathfrak{U}$ denote the derivation superalgebra of $\mathfrak{U}$(see [12]). Then $D_i \in \mathrm{Der}_{\overline{1}}\mathfrak{U}$ for all $i \in \mathrm{Y}$ since $\partial/\partial x_i \in \mathrm{Der}_{\overline{1}}(\Lambda(n))$ (see [23]).

Suppose that $u \in \mathbb{B}_k \subseteq \mathbb{B}(n)$ and $i \in \mathrm{Y}$. When $i \in \{u\}$, $u - \langle i\rangle$ denotes the uniquely determined element of $\mathbb{B}_{k-1}$ satisfying $\{u - \langle i\rangle\} = \{u\} \setminus \{i\}$ .Then the number of integers less than i in $\{u\}$ is denoted by $\tau(u, i)$. When $i \notin \{u\}$, we set $\tau(u, i) = 0$ and $x^{u-\langle i\rangle} = 0$. Therefore, $D_i(x^u) = (-1)^{\tau(u,i)} x^{u-\langle i\rangle}$ for all $i \in \mathrm{Y}$ and $u \in \mathbb{B}(n)$.

We define $(fD)(g) = fD(g)$ for $f, g \in \mathrm{hg}(\mathfrak{U})$ and $D \in \mathrm{hg}(\mathrm{Der}\mathfrak{U})$. Since the multiplication of $\mathfrak{U}$ is supercommutative, fD is a derivation of $\mathfrak{U}$. Let

$$W(n,m) = \mathrm{span}_{\mathbb{F}}\{x^u y^\lambda D_i \mid u \in \mathbb{B}(n), \lambda \in G, i \in \mathrm{Y}\}.$$

Then $W(n,m)$ is a finite dimensional Lie superalgebra contained in Der$\mathfrak{U}$. A direct computation shows that

$$[fD_i, gD_j] = fD_i(g)D_j - (-1)^{\deg(fD_i)\deg(gD_j)} gD_j(f)D_i.$$

where $f, g \in \mathrm{hg}(\mathfrak{U})$ and $i, j \in \mathrm{Y}$.

Set $J = \{1, ..., n-1\}$. Let $\widetilde{D}_k : \mathfrak{U} \longrightarrow W(n,m)$ be the linear map such that

$$\widetilde{D}_k(f) = \sum_{i \in J} f_i D_i + f_n x_n D_n,$$

where $f \in \mathrm{hg}(\mathfrak{U})$, $f_i = (-1)^{degf} (x_i x_n D_n(f) + D_i(f))$, $i \in J$ and $f_n = 2f - \sum_{i\in J} x_i D_i(f)$.

Let $\overline{K}(n,m)$ =span$_{\mathbb{F}} \left\{\widetilde{D}_k(f) | f \in \mathfrak{U}\right\}$. Then $\overline{K}(n,m)$ is a subspace of $W(n,m)$.

Let

$$G_i = D_i + x_i x_n D_n, \forall i \in J, G_n = 2x_n D_n.$$

By direct calculation, we have

$$\left[G_i, G_j\right] = \delta_{ij} G_n, \left[G_n, G_j\right] = 0,$$

where $i, j \in J$ and δ_{ij} is Kronecker delta.

It is easy to prove that $\widetilde{D}_k(f) = \sum_{i\in J} (-1)^{degf} G_i(f) G_i + fG_n$

For $f \in \mathfrak{U}_\theta$ and $g \in \mathfrak{U}_\mu$, where $\theta, \mu \in \mathbb{Z}_2$, set $\langle f, g\rangle = \widetilde{D}_k(f)(g) - G_n(f)(g)$. In [22], we have proved that $\left[\widetilde{D}_k(f), \widetilde{D}_k(g)\right] = \widetilde{D}_k(\langle f, g\rangle)$. Namely, $\overline{K}(n,m)$ is a subalgebra of $W(n,m)$.

If we define an operator $[,]$ in $\mathfrak{U}$ such that $[f, g] = \widetilde{D}_k(f)(g) - G_n(f)(g)$ for any $f, g \in \mathfrak{U}$. Then $\overline{K}(n,m) \cong (\mathfrak{U}, [,])$. Moreover, for any $f, g \in \overline{K}(n,m)$, we have

$$\begin{aligned}[f, g] =& \left(2f - \sum_{i\in J} x_i D_i(f)\right) x_n D_n(g) \\ & - (-1)^{deg(f)deg(g)} \left(2g - \sum_{i\in J} x_i D_i(g)\right) x_n D_n(f) \\ & + \sum_{i\in J} (-1)^{degf} D_i(f) D_i(g).\end{aligned}$$

Let $K(n,m)$ be the derived algebra of $\overline{K}(n,m)$, then $K(n,m) = \mathrm{span}_{\mathbb{F}}\{x^u y^\lambda | x^u y^\lambda \in \mathfrak{U}, x^u y^\lambda \neq x^{\hat{u}} y^\lambda\}$, where $\hat{u} = \langle 1, ..., n-1\rangle$. By [22], we know that modular Lie superalgebra $K(n,m)$ is not simple.

3. Determination of derivation superalgebras

In this section, we will abbreviate $\overline{K}(n,m)$, $K(n,m)$ as $\overline{K}$ and K, respectively.

In [22], we proved that $K(n,m)$ does not possess a $\mathbb{Z}$-graded structure as $W(n,m)$ (see [19]). In fact, $\overline{K}(n,m)$ does not possess $\mathbb{Z}$-gradation in the ordinary sense as well. If $\overline{K} = \oplus_{i=-r}^{s} \overline{K}_i$, then it does not satisfy that

$$\left[\overline{K}_i, \overline{K}_j\right] \subseteq \overline{K}_{i+j}, \forall i, j \in \{-r, -r+1, ..., s\}.$$

Now we give a "formal" $\mathbb{Z}$-gradation of $\overline{K}(n,m)$:

$$\overline{K}(n,m) = \oplus_{i=-2}^{n-2} \overline{K}(n,m)_i,$$

where $\overline{K}(n,m)_i = \text{span}_{\mathbb{F}}\{x^u y^\lambda | u \in \mathbb{B}(n), |u| = i+2, \lambda \in G\}$. Let

$$Der_t(\overline{K}) = \left\{\varphi \in Der(\overline{K}) | \varphi(\overline{K}_i) \subseteq \overline{K}_{t+i}, \forall i \in \mathbb{Z}\right\}.$$

It is easy to prove that $Der(\overline{K}) = \oplus_{t\in\mathbb{Z}} Der_t(\overline{K})$ is a $\mathbb{Z}$-graded Lie superalgebra (see [24]).

Lemma 3.1. *Let $\varphi \in Der(\overline{K})$, $f \in \overline{K}$ and $[f, x_i] = b_i$, $\forall i \in J$. If $\varphi(x_i) = \varphi(b_i) = 0$, $\forall i \in J$, then $\varphi(f) \in \overline{K}_{-2}$.*

Proof. By applying φ to $[f, x_i] = b_i$, we obtain $[\varphi(f), x_i] + (-1)^{deg\varphi deg f}[f, \varphi(x_i)] = \varphi(b_i)$. Since $\varphi(x_i) = \varphi(b_i) = 0$, we have $[\varphi(f), x_i] = 0$, $\forall i \in J$. Note that

$$\begin{aligned}[\varphi(f), 1] &= [\varphi(f), -[x_1, x_1]] \\ &= -[\varphi(f), [x_1, x_1]] \\ &= -([[\varphi(f), x_1], x_1] + [x_1, [\varphi(f), x_1]]) \\ &= 0.\end{aligned}$$

Therefore, $-2x_n D_n(\varphi(f)) = [\varphi(f), 1] = 0$. Then $D_n(\varphi(f)) = 0$. For all $i \in J$,

$$\begin{aligned}[\varphi(f), x_i] = &\left(2\varphi(f) - \sum_{t\in J} x_t D_t(\varphi(f))\right) x_n D_n(x_i) \\ &- (-1)^{deg(\varphi(f))deg(x_i)} \left(2x_i - \textstyle\sum_{t\in J} x_t D_t(x_i)\right) x_n D_n(\varphi(f)) \\ &+ \textstyle\sum_{t\in J} (-1)^{deg\varphi(f)} D_t(\varphi(f)) D_t(x_i) \\ = &(-1)^{deg\varphi(f)} D_i(\varphi(f)).\end{aligned}$$

Since $[\varphi(f), x_i] = 0$, we obtain $D_i(\varphi(f)) = 0, \forall i \in J$. Therefore, $\varphi(f) \in \overline{K}_{-2}$. □

Lemma 3.2. *Let $\varphi \in Der_{-t}(\overline{K})$, $t \geq 2$. If $\varphi\left(\overline{K}_{t-2}\right) = 0$, then $\varphi = 0$.*

Proof. If $s < t-2$, then $\varphi(\overline{K}_s) \subseteq \overline{K}_{s-t} = \{0\}$.

When $s \geqslant t-2$, we will use induction on s to prove that $\varphi(\overline{K}_s) = 0$. For $s = t-2$, we have $\varphi(\overline{K}_s) = 0$ with the hypothesis of the lemma. Suppose $s > t-2$. For any $y \in \overline{K}_s$, $i \in J$, set $[y, D_i] = y_i$. Then $y_i \in \overline{K}_{s'}$, where $s' < s$. According to the hypothesis of induction, we have $\varphi(y_i) = 0$. Noting that $\varphi(D_i) = 0$, we obtain $\varphi(y) \in \overline{K}_{-2}$. Therefore, $\varphi(y) \in \overline{K}_{-2} \cap \overline{K}_{s-t} = \{0\}$. Namely, $\varphi(y) = 0$. Then $\varphi(\overline{K}_s) = 0$. It follows that $\varphi = 0$. □

Proposition 3.3. $Der_{-t}(\overline{K}) = 0$, $t \geq 2$.

Proof. Let $\varphi \in Der_{-t}(\overline{K}), t \geq 2$. We will prove $\varphi\left(\overline{K}_{t-2}\right) = 0$, where

$$\overline{K}_{t-2} = span_{\mathbb{F}}\{x^{u_1} x_n y^\lambda, x^{u_2} y^\eta | u_1, u_2 \in \mathbb{B}(n), |u_1| = t-1, |u_2| = t, \lambda, \eta \in G\}.$$

Note that $\varphi(\overline{K}_{t-2}) \subseteq \overline{K}_{-2}$. Without loss of generality, we put $\varphi(x^{u_1} x_n y^\lambda) = ay^\mu$, $\varphi(x^{u_2} y^\eta) = by^\mu$, where $a, b \in \mathbb{F}$, $\mu \in G$. Applying φ to $[x_n, x^{u_1} x_n y^\lambda] = 0$, we obtain

$$\left[\varphi(x_n), x^{u_1} x_n y^\lambda\right] + (-1)^{deg\varphi deg x_n}\left[x_n, \varphi\left(x^{u_1} x_n y^\lambda\right)\right] = 0. \tag{3.1}$$

Since $\varphi(x_n) \in \overline{K}_{-t-1} = \{0\}$, which combined with (3.1) yields $\left[x_n, \varphi\left(x^{u_1}x_ny^{\lambda}\right)\right] = 0$. Namely, $[x_n, ay^{\mu}] = 0$. In fact, $[x_n, ay^{\mu}] = -2ay^{\mu}x_n$. Therefore, $-2ay^{\mu}x_n = 0$. Then

$$\varphi(x^{u_1}x_ny^{\lambda}) = ay^{\mu} = 0. \tag{3.2}$$

For $i \in \{u_2\}$, we have

$$[x_ix_n, x^{u_2}y^{\eta}] = (-1)^{\tau(u_2,i)}x^{u_2-\langle i\rangle}x_ny^{\eta}. \tag{3.3}$$

Without loss of generality, let $x^{u_2-\langle i\rangle}x_ny^{\eta} = x^{u}x_ny^{\eta}$, where $|u| = t-1$. Then we may write the Eq (3.3) as

$$[x_ix_n, x^{u_2}y^{\eta}] = (-1)^{\tau(u_2,i)}x^{u}x_ny^{\eta}. \tag{3.4}$$

By virtue of the Eq (3.2), we have $\varphi(x^{u}x_ny^{\eta}) = 0$. Applying φ to the Eq (3.4), we obtain

$$[\varphi(x_ix_n), x^{u_2}y^{\eta}] + (-1)^{deg\varphi deg(x_ix_n)}[x_ix_n, \varphi(x^{u_2}y^{\eta})] = 0. \tag{3.5}$$

If $t = 2$, it follows from the Eq (3.2) that $\varphi(x_ix_n) = 0$. If $t > 2$, then $\varphi(x_ix_n) \in \overline{K}_{-t} = \{0\}$. Consequently, $\varphi(x_ix_n) = 0$. Therefore, by virtue of the Eq (3.5) , we have $[x_ix_n, \varphi(x^{u_2}y^{\eta})] = 0$. Namely, $[x_ix_n, by^{\mu}] = 0$. In fact, $[x_ix_n, by^{\mu}] = -2by^{\mu}x_ix_n$. Therefore, $-2by^{\mu}x_ix_n = 0$. Then

$$\varphi(x^{u_2}y^{\eta}) = by^{\mu} = 0. \tag{3.6}$$

It follows from the Eqs (3.2) and (3.6) that $\varphi\left(\overline{K}_{t-2}\right) = 0$. By virtue of Lemma 3.2, we have $\varphi = 0$. Therefore, $Der_{-t}(\overline{K}) = 0$, $t \geq 2$. □

Lemma 3.4. *Let $\varphi \in Der_t(\overline{K})$, $t \in \mathbb{Z}$. Suppose that $\varphi(\overline{K}_j) = 0$, $j = -2, -1, ..., l$. If $t + l \geq -2$, then $\varphi = 0$.*

Proof. By virtue of Lemma 3.1, the proof is completely analogous to [24, Lemma 2.8]. □

Proposition 3.5. $Der_{-1}(\overline{K}) = 0$.

Proof. Let $\varphi \in Der_{-1}(\overline{K})$. Then $\varphi(\overline{K}_{-2}) = 0$. In order to prove $\varphi = 0$, we need to obtain $\varphi(\overline{K}_{-1}) = 0$. Without loss of generality, we put $\varphi(x_ny^{\lambda}) = ay^{\mu}$, where $a \in \mathbb{F}, \mu \in G$. Applying φ to $[1, x_ny^{\lambda}] = 2x_ny^{\lambda}$ yields $(-1)^{deg\varphi deg1}[1, ay^{\mu}] = 2ay^{\mu}$. Therefore, $2ay^{\mu} = 0$. Then $\varphi(x_ny^{\lambda}) = ay^{\mu} = 0$. Similarly, we can prove that $\varphi(x_iy^{\lambda}) = 0$, $i \in J$. Therefore, $\varphi(\overline{K}_{-1}) = 0$. Then $\varphi = 0$. □

Let $\Theta = \mathbb{T}(m) \times ... \times \mathbb{T}(m)$. For every $\theta = (h_1(y), ..., h_m(y)) \in \Theta$, we define $\widetilde{\theta} : G \to \mathbb{T}(m)$. (see [20]) For every $\theta \in \Theta$,we define $D_{\theta} : \overline{K} \to \overline{K}$ such that $D_{\theta}(\widetilde{D}_k(x^uy^{\lambda})) = \widetilde{\theta}(\lambda)\widetilde{D}_k(x^uy^{\lambda})$, for $x^uy^{\lambda} \in \mathfrak{U}$. A direct computation shows that $D_{\theta} \in Der_{\overline{0}}(\overline{K})$, for all $\theta \in \Theta$. Put $\Omega = \{D_{\theta}|\theta \in \Theta\}$.

Proposition 3.6. $Der_0(\overline{K}) = ad\overline{K}_{-2} + \Omega$.

Proof. Assume that $y^\lambda \in \overline{K}_{-2}$. For $f \in \overline{K}_j$, $j \in \{-2, -1, ..., n-2\}$, we have

$$\begin{aligned}\left(ady^\lambda\right)(f) =& \left[y^\lambda, f\right]\\ =& \left(2y^\lambda - \sum_{i\in J} x_i D_i\left(y^\lambda\right)\right) x_n D_n(f)\\ &- (-1)^{deg\left(y^\lambda\right)deg(f)}\left(2f - \sum_{i\in J} x_i D_i(f)\right) x_n D_n\left(y^\lambda\right)\\ &+ \sum_{i\in J}(-1)^{degy^\lambda} D_i\left(y^\lambda\right) D_i(f)\\ =& 2y^\lambda x_n D_n(f) \in \overline{K}_j.\end{aligned}$$

Therefore, $ad\overline{K}_{-2} \subseteq Der_0(\overline{K})$. Obviously, $\Omega \subseteq Der_0(\overline{K})$. It follows that $ad\overline{K}_{-2} + \Omega \subseteq Der_0(\overline{K})$.

Conversely, let $\varphi \in Der_0(\overline{K})$. It is obvious that there exist $y^\mu \in \overline{K}_{-2}$ and $D_\theta \in \Omega$ such that $(\varphi - ady^\mu - D_\theta)\left(\overline{K}_{-2}\right) = 0$. Consequently, by lemma 3.4, we have $\varphi - ady^\mu - D_\theta = 0$. Then $\varphi = ady^\mu + D_\theta \in ad\overline{K}_{-2} + \Omega$. Thus, $Der_0(\overline{K}) \subseteq ad\overline{K}_{-2} + \Omega$.

□

Lemma 3.7. *Let $h_1, ..., h_n$ be the nonzero elements of $\mathfrak{U}$. If $G_i(h_j) = -G_j(h_i)$ for all distinct $i, j \in Y$, then there exists nonzero element $h \in \mathfrak{U}$ such that $G_i(h) = h_i$, for $i \in J$ and $G_n(h) = -h_n$.*

Proof. We use induction to prove there exist nonzero element $h' \in \mathfrak{U}$ such that $G_i(h') = h_i$ for all $i \in J$.

When $n - 1 = 1$, let $h' = x_1 h_1 \neq 0$. Then

$$G_1(h') = (D_1 + x_1 x_n D_n)(x_1 h_1) = h_1.$$

Suppose that there exists $0 \neq g \in \mathfrak{U}$ such that $G_i(g) = h_i$ for all $i \in \{1, ..., n-2\}$. For $i \in \{1, ..., n-2\}$,

$$G_i(h_{n-1}) = -G_{n-1}(h_i) = -G_{n-1}(G_i(g)) = G_i(G_{n-1}(g)).$$

Therefore, $G_i(h_{n-1} - G_{n-1}(g)) = 0$. Let $h' = g + x_{n-1}(h_{n-1} - G_{n-1}(g))$. For $i \in \{1, ..., n-2\}$,

$$G_i(h') = G_i(g) + G_i(x_{n-1}(h_{n-1} - G_{n-1}(g)) = G_i(g) = h_i.$$

On the other hand,

$$G_{n-1}(h') = G_{n-1}(g) + G_{n-1}(x_{n-1}(h_{n-1} - G_{n-1}(g)) = h_{n-1}.$$

Consequently, we have $G_i(h') = h_i$, for all $i \in \{1, ..., n-1\}$. For $i \in \{1, ..., n-1\}$,

$$G_i(h_n) = -G_n(h_i) = -G_n(G_i(h')) = -G_i(G_n(h')).$$

Then $G_i(h_n + G_n(h')) = 0$. Therefore, $h_n + G_n(h') = 0$. Namely, $G_n(h') = -h_n$. Consequently, the assertion follows from $h = h'$. □

Lemma 3.8. *Let $\varphi \in h(Der_t(\overline{K}))$, $t \in \mathbb{N}$. If $\varphi(G_j) = 0$, $\forall j \in Y$, then there exists $\theta \in \Theta$ such that $\varphi(\widetilde{D}_k(y^\lambda)) = D_\theta(\widetilde{D}_k(y^\lambda))$, for $\lambda \in G$.*

Proof. Assume that for every $i \in Y$, $\varphi(y^\lambda G_i) = \sum_{k=1}^n g_{ki\lambda} G_k$, where $g_{ki\lambda} \in \mathfrak{U}$. Applying φ to $[G_i, y^\lambda G_n] = 0, i \in Y$ yields $g_{kn\lambda} \in \mathbb{T}(m)$. Let $g(y)_{kn\lambda} = g_{kn\lambda}$. Then $\varphi(y^\lambda G_n) = \sum_{k=1}^n g(y)_{kn\lambda} G_k$. Note that $[G_j, y^\lambda G_i] = 0$ and $[G_i, y^\lambda G_i] = y^\lambda G_n$ for all $i \in J, i \neq j \in Y$. Applying φ to these equations, we know that $g_{ki\lambda}$ contains at most x_i in $\Lambda(n)$. Let $g_{ki\lambda} = g(x_i, y)_{ki\lambda}, i \in J$. For $j \in J$, let $\widetilde{D}_k(x_j) = \sum_{k=1}^n a_k G_k$. Since $[G_j, G_j] = G_n$ and $[G_j, \widetilde{D}_k(x_j)] = 0$, we obtain

$$[G_j, G_j + \widetilde{D}_k(x_j)] = G_n. \tag{3.7}$$

Applying φ to (3.7) yields $(-1)^{deg\varphi}[G_j, \varphi(G_j + \widetilde{D}_k(x_j))] = 0$. Then $[G_j, \varphi(\widetilde{D}_k(x_j))] = 0$. Namely, $[G_j, \sum_{k=1}^n a_k G_k] = 0$. Therefore, $G_j(a_k) = 0, k \in Y$. Applying φ to $[y^\lambda G_j, G_j + \widetilde{D}_k(x_j)] = y^\lambda G_n$ yields

$$[\varphi(y^\lambda G_j), G_j + \widetilde{D}_k(x_j)] + (-1)^{deg\varphi}[y^\lambda G_j, \varphi(\widetilde{D}_k(x_j))] = \varphi(y^\lambda G_n).$$

Therefore, $[\varphi(y^\lambda G_j), G_j + \widetilde{D}_k(x_j)] = \varphi(y^\lambda G_n)$. Namely,

$$[\sum_{k=1}^n g(x_j, y)_{kj\lambda} G_k, G_j + \widetilde{D}_k(x_j)] = \sum_{k=1}^n g(y)_{kn\lambda} G_k.$$

A direct computation shows that $g(y)_{kn\lambda} = 0, k \in J$ and $g(x_j, y)_{jj\lambda} = g(y)_{nn\lambda}$. Then $\varphi(y^\lambda G_n) = g(y)_{nn\lambda} G_n$. We abbreviate $g(y)_{nn\lambda}$ as $g(y)_{n\lambda}$. Let $h_{n\lambda}(y) = g(y)_{n\lambda} y^{-\lambda}$. Then $\varphi(y^\lambda G_n) = h_{n\lambda}(y) y^\lambda G_n$. Let $\varphi(\widetilde{D}_k(x_j y^\eta)) = \sum_{k=1}^n b_k G_k$, where $b_k \in \mathfrak{U}$. Note that

$$[G_j, y^\eta G_j + \widetilde{D}_k(x_j y^\eta)] = y^\eta G_n \tag{3.8}$$

Applying φ to (3.8) yields

$$(-1)^{deg\varphi}[G_j, \varphi(y^\eta G_j) + \sum_{k=1}^n b_k G_k] = \varphi(y^\eta G_n). \tag{3.9}$$

Since $g(x_j, y)_{jj\lambda} = g(y)_{nn\lambda}$, we know $g(x_j, y)_{jj\eta} \in \mathbb{T}(m)$. We may abbreviate $g(x_j, y)_{jj\eta}$ as $g(y)_{j\eta}$. Let $h_{j\eta}(y) = g(y)_{j\eta} y^{-\eta}$. It follows from (3.9) that

$$(-1)^{deg\varphi} h_{j\eta}(y) y^\eta G_n + (-1)^{deg\varphi} \sum_{k=1}^n G_j(b_k) G_k = h_{n\eta}(y) y^\eta G_n.$$

Then $G_j(b_n) = (-h_{j\eta}(y) + (-1)^{deg\varphi} h_{n\eta}(y)) y^\eta$ and $G_j(b_i) = 0, i \in J$. Therefore, $b_n = (-h_{j\eta}(y) + (-1)^{deg\varphi} h_{n\eta}(y)) y^\eta x_j + q_n$ with $G_j(q_n) = 0$. Then

$$\varphi(\widetilde{D}_k(x_j y^\eta)) = \sum_{k=1}^n b_k G_k = (-h_{j\eta}(y) + (-1)^{deg\varphi} h_{n\eta}(y)) y^\eta x_j G_n + q_n G_n + \sum_{i \in J} b_i G_i.$$

Applying φ to $[y^\lambda G, y^\eta G_j + \widetilde{D}_k(x_j y^\eta)] = y^{\lambda+\eta} G_n$ yields

$$[\varphi(y^\lambda G), y^\eta G_j + \widetilde{D}_k(x_j y^\eta)] + (-1)^{deg\varphi}[y^\lambda G, \varphi(y^\eta G_j) + \varphi(\widetilde{D}_k(x_j y^\eta))] = \varphi(y^{\lambda+\eta} G_n).$$

A calculation shows that

$$h_{j\lambda}(y) + h_{n\eta}(y) = h_{n(\lambda+\eta)}(y). \tag{3.10}$$

Particularly, $h_{j\lambda}(y) + h_{n\lambda}(y) = h_{n(2\lambda)}(y) = h_{n\lambda}(y) + h_{n\lambda}(y)$. Therefore, $h_{j\lambda}(y) = h_{n\lambda}(y)$ for $j \in J$. Consequently, we may abbreviate $h_{i\lambda}(y)$ as $h_\lambda(y)$ for $i \in Y$. Then the Eq (3.10) can be equivalent to

$$h_\lambda(y) + h_\eta(y) = h_{\lambda+\eta}(y), \tag{3.11}$$

for $\lambda, \eta \in G$. Therefore, for $k \in Y, c \in \Pi$, we have

$$h_{cz_k}(y) = ch_{z_k}(y). \tag{3.12}$$

Let $\lambda = \sum_{j=1}^{m} \lambda_j z_j \in G$. It follows from (3.11) and (3.12) that

$$h_\lambda(y) = h_{\lambda_1 z_1+\ldots+\lambda_m z_m}(y) = \sum_{j=1}^{m} \lambda_j h_{z_j}(y). \tag{3.13}$$

We abbreviate $h_{z_j}(y)$ as $h_j(y)$ for $j = 1, \ldots, m$. Let $\theta = (h_1(y), \ldots, h_m(y)) \in \Theta$. It follows from (3.13) that $h_\lambda(y) = \sum_{j=1}^{m} \lambda_j h_{z_j}(y) = \widetilde{\theta}(\lambda)$. Therefore,

$$\varphi(\widetilde{D}_k(y^\lambda)) = \varphi(y^\lambda G_n) = h_\lambda(y) y^\lambda G_n = \widetilde{\theta}(\lambda) y^\lambda G_n = D_\theta(\widetilde{D}_k(y^\lambda)).$$

□

Lemma 3.9. *Let $\varphi \in h(Der_t(\overline{K}))$, $t \in \mathbb{N}$. Then there exist $B \in Nor_W(\overline{K}) = \{x \in W | [x, \overline{K}] \subseteq \overline{K}\}$ and $\theta \in \Theta$ such that $\varphi = adB + D_\theta$.*

Proof. Assume that for every $k \in J$, $\varphi(G_k) = \sum_{i=1}^{n} f_{ik} G_i$, where $f_{ik} \in \mathfrak{U}$ and $\varphi(G_n) = -(-1)^{deg\varphi} \sum_{i=1}^{n} f_{in} G_i$, where $f_{in} \in \mathfrak{U}$. For $k \neq l \in Y$, applying φ to $[G_k, G_l] = 0$ yields

$$G_l(f_{ik}) = -G_k(f_{il})$$

for all $i \in Y$. It follows from lemma 3.7 that there exist $g_i (i = 1, \ldots, n)$ such that

$$G_k(g_i) = f_{ik}, k \in J, G_n(g_i) = -f_{in}.$$

Let $B = -(-1)^{deg\varphi} \sum_{i=1}^{n} g_i G_i$. Then for any $k \in J$,

$$adB(G_k) = [B, G_k] = \sum_{i=1}^{n} f_{ik} G_i = \varphi(G_k),$$

$$adB(G_n) = [B, G_n] = -(-1)^{deg\varphi} \sum_{i=1}^{n} f_{in} G_i = \varphi(G_n).$$

Therefore, $\varphi(G_j) = adB(G_j)$ for all $j \in Y$. Namely, $(\varphi - adB)(G_j) = 0$ for all $j \in Y$. According to lemma 3.8, there exists $\theta \in \Theta$ such that

$$(\varphi - adB)(\widetilde{D}_k(y^\lambda)) = D_\theta(\widetilde{D}_k(y^\lambda))$$

for all $\lambda \in G$. Let $\phi = \varphi - adB - D_\theta$. We use induction on j to prove $\phi(\overline{K}_j) = 0$, $j = -2, -1, 0, ..., n-2$. A direct computation shows that $\phi(\overline{K}_{-2}) = 0$. Assume that $j \geq -1$. For any $\xi \in \overline{K}_j$, let $[\xi, G_n] = \xi_1$. Applying ϕ to this equation, we obtain

$$[\phi(\xi), G_n] + (-1)^{deg\phi deg\xi}[\xi, \phi(G_n)] = \phi(\xi_1).$$

It follows from the induction hypothesis that $\phi(\xi_1) = 0$. Since $\phi(G_n) = 0$, we have $[\phi(\xi), G_n] = 0$. Then $\phi(\xi) \in \overline{K}_{-1} \bigcap \overline{K}_{j+t} = \{0\}$. Therefore, $\phi(\xi) = 0$. Since the choice of ξ was arbitrary, we have $\phi(\overline{K}_j) = 0$. Consequently, $\varphi = adB + D_\theta$. □

Lemma 3.10. *Let* $\widehat{K}(n, m) = \{\sum_{j \in Y} f_j G_j \in W | G_i(f_j) = -G_j(f_i) + (-1)^{degf}\delta_{ij}G_n(f), i, j \in J\}$. *Then* $\overline{K}(n, m) = \widehat{K}(n, m)$.

Proof. Assume that $\sum_{j \in Y} f_j G_j \in \widehat{K}(n, m)$. According to lemma 3.7, there exists $0 \neq f \in \mathfrak{U}$ such that $G_i(f) = f_i$ for all $i \in J$. Therefore, $\sum_{j \in Y} f_j G_j = \sum_{j \in J} G_j(f)G_j + f_n G_n \in \overline{K}(n, m)$. Then $\widehat{K}(n, m) \subseteq \overline{K}(n, m)$.

Conversely, let $\widetilde{D}_k(f) = \sum_{i=1}^{n} f_i G_i \in \overline{K}(n, m)$. For $i, j \in J$,

$$\begin{aligned} G_i(f_j) &= G_i((-1)^{degf}G_j(f)) \\ &= (-1)^{degf}(-G_j G_i(f) + \delta_{ij}G_n(f)) \\ &= -G_j((-1)^{degf}G_i(f)) + (-1)^{degf}\delta_{ij}G_n(f) \\ &= -G_j(f_i) + (-1)^{degf}\delta_{ij}G_n(f). \end{aligned}$$

Therefore, $\widetilde{D}_k(f) \in \widehat{K}(n, m)$. Then $\overline{K}(n, m) \subseteq \widehat{K}(n, m)$.

Consequently, $\overline{K}(n, m) = \widehat{K}(n, m)$. □

Proposition 3.11. *Let* $t \in \mathbb{N}$. *Then* $Der_t(\overline{K}) = ad(\overline{K}_t) + \Omega$.

Proof. It suffices to prove that $Der_t(\overline{K}) \subseteq ad(\overline{K}_t) + \Omega$. Let $\varphi \in Der_t(\overline{K})$. By virtue of Lemma 3.9, there exists $B \in Nor_W(\overline{K})$ such that $\varphi(G_k) = adB(G_k)$, for $k \in Y$. Assume that $B = \sum_{j=1}^{n} g_j G_j$ and plug it into the following equation, i.e.,

$$\begin{aligned} (-1)^{degBdegG_j}[G_j, [G_i, B]] + (-1)^{degG_j degG_i}[G_i, [B, G_j]], \\ +(-1)^{degG_i degB}[B, [G_j, G_i]] = 0 \end{aligned}$$

for $i, j \in J$. Accordingly, we have $G_i(f_j) + G_j(f_i) - (-1)^{degf}\delta_{ij}G_n(f) = 0, i, j \in J$. Therefore, $B \in \overline{K}(n, m)$. By virtue of Lemma 3.9, $\varphi = adB + D_\theta \in ad\overline{K} + \Omega$. Then $Der_t(\overline{K}) \subseteq ad(\overline{K}_t) + \Omega$. Consequently, $Der_t(\overline{K}) = ad(\overline{K}_t) + \Omega$. □

Lemma 3.12. *The following statements hold:*

(1) Ω *is a subspace of* $Der(\overline{K})$.

(2) $ad(\overline{K}) \bigcap \Omega = \{0\}$.

Proof. The proof is completely analogous to [20, Lemma 3.11]. □

Theorem 3.13. *$Der(\overline{K}) = ad(\overline{K}') \oplus \Omega$, where $\overline{K}' = \overline{K}_{-2} \oplus \oplus_{i=1}^{n-2} \overline{K}_i$.*

Proof. By virtue of Propositions 3.3, 3.5, 3.6, 3.11 and Lemma 3.12, this theorem can be easily proved. □

Proposition 3.14. *$K(n, m)$ is invariant under $Der(\overline{K})$.*

Proof. Let $\phi \in \Omega$. Obviously, $\phi(K) \subseteq K$.

Let $\phi \in ad(\overline{K}')$. Without loss of generality, we may suppose that $\phi = adf$, where $f \in \overline{K}' \subseteq \overline{K}$. Since K is a derived algebra of $\overline{K}$, K is an ideal of $\overline{K}$. Therefore, $\phi(K) = adf(K) = [f, K] \subseteq K$.

Since $Der(\overline{K}) = ad(\overline{K}') \oplus \Omega$, we have $\phi(K) \subseteq K$ for any $\phi \in Der(\overline{K})$. Namely, $K(n, m)$ is invariant under $Der(\overline{K})$. □

An immediate corollary of this proposition is the following.

Corollary 3.15. *$Der(\overline{K})|_K \subseteq Der(K)$.*

Remark. *If $m = 0$, then derivation superalgebras of modular Lie superalgebras $K(n)$, which were mentioned in [22], will be obtained.*

Acknowledgments

The authors give special thanks to referees for many helpful suggestions. This work was supported by the Fundamental Research Funds for the Central Universities (No.2572021BC02) and the National Natural Science Foundation of China (Grant No.11626056)

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. B. DeWitt, *Supermanifolds*, 2nd edition, Cambridge University Press, Cambridge, 1992.
2. P. Fayet, S. Ferrara, Supersymmetry, *Phys. Rep.*, **32** (1977), 249–334. https://doi.org/10.1016/0370-1573(77)90066-7
3. V. S. Varadarajan, *Supersymmetry for Mathematicians: An Introduction*, American Mathematical Society, Rhode Island, 2004
4. V. G. Kac, Lie Superalgebras, *Adv. Math.*, **26** (1977), 8–96. https://doi.org/10.1016/0001-8708(77)90017-2
5. Y. Wang, Y. Zhang, The associative forms of the graded Lie superalgebras, *Adv. Math.*, **29** (2000), 65–70.
6. J. Yuan, W. Liu, W. Bai, Associative forms and second cohomologies of Lie superalgebras HO and KO, *J. Lie Theory*, **23** (2009), 203–215.

7. Q. Mu, Y. Zhang, Infinite-dimensional Hamiltonian Lie superalgebras, *Sci. China Math.*, **53** (2010), 1625–1634. https://doi.org/10.1007/s11425-010-3142-4

8. K. Zheng, Y. Zhang, The natural filtration of finite dimensional modular Lie superalgebras of special type, *Abstr. Appl. Anal.*, **2013** (2013), 891241. https://doi.org/10.1155/2013/891241

9. K. Zheng, Y. Zhang, W. Song, The natural filtrations of finite-dimensional modular Lie superalgebras of Witt and Hamiltonian type, *Pac. J. Math.*, **269** (2014), 199–218. https://doi.org/10.2140/pjm.2014.269.199

10. J. Yuan, W. Liu, Filtration, automorphisms, and classification of infinite-dimensional odd contact superalgebras, *Front. Math. China*, **8** (2013), 203–216. https://doi.org/10.1007/s11464-012-0185-6

11. W. Liu, Y. Zhang, Automorphism groups of restricted cartan-type Lie superalgebras, *Commun. Algebr.*, **34** (2006), 3767–37842. https://doi.org/10.1080/00927870600862615

12. W. Xie, Y. Zhang, Second cohomology of the modular Lie superalgebra of cartan type k, *Algebr. Colloq.*, **16** (2009), 309–324. https://doi.org/10.1142/S1005386709000303

13. S. Bouarroudj, P. Grozman, A. Lebedev, D. Leites, Divided power (co)homology. Presentations of simple finite dimensional modular Lie superalgebras with Cartan matrix, *Homol. Homotopy Appl.*, **12** (2010), 237–278. https://doi.org/10.4310/HHA.2010.v12.n1.a13

14. F. Ma, Q. Zhang, The derivation algebra of modular Lie superalgebra k-type, *J. Math.*, **20** (2000), 431–435. https://doi.org/10.13548/j.sxzz.2000.04.015

15. Q. Zhang, Y. Zhang, Derivation algebras of modular Lie superalgebras W and S of Cartan type, *Acta Math. Sci.*, **20** (2000), 137–144. https://doi.org/10.1016/s0252-9602(17)30743-9

16. W. Liu, Y. Zhang, X. Wang, The derivation algebra of the Cartan-type Lie superalgebra HO, *J. Algebr.*, **273** (2004), 176–205. https://doi.org/10.1016/j.jalgebra.2003.10.019

17. J. Fu, Q. Zhang, C. Jiang, The Cartan-type modular Lie superalgebra KO, *Commun. Algebr.*, **34** (2006), 107–128. https://doi.org/10.1080/00927870500346065

18. W. Bai, W. Liu, Superderivations for modular graded lie superalgebras of cartan-type, *Algebr. Represent. Theor.*, **17** (2014), 69–86. https://doi.org/10.1007/s10468-012-9387-6

19. S. Awuti, Y. Zhang, Modular Lie superalgebra $\overline{W}(n, m)$, *Northeast Norm. Univ. (Nat. Sci. Ed.)*, **40** (2008), 7–11.

20. L. Ren, Q. Mu, Y. Zhang, A class of finite-dimensional Lie superalgebras of hamiltonian type, *Algebr. Colloq.*, **18** (2011), 347–360. https://doi.org/10.1142/S1005386711000241

21. K. Zheng, Y. Zhang, J. Zhang, A class of finite dimensional modular Lie superalgebras of special type, *Bull. Malays. Math. Sci. Soc.*, **39** (2016), 381–390. https://doi.org/10.1007/s40840-015-0177-2

22. D. Mao, K. Zheng, Constructions and properties for a finite-dimensional modular Lie superalgebra $K(n, m)$, *J. Math.*, **2021** (2021), 7069556. https://doi.org/10.1155/2021/7069556

23. Y. Zhang, Finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic, *Chin. Sci. Bull.*, **42** (1997), 720–724. https://doi.org/10.1007/BF03186962

24. Y. Zhang, W. Liu, *Modular Lie Superalgebras*, Science Press, Beijing, 2004.