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Research article

Global stability of multi-group SEIQR epidemic models with stochastic perturbation in computer network

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Abstract: In this paper, a class of multi-group SEIQR models with random perturbation in computer network is investigated. The existence and uniqueness of global positive solution with any positive initial value are obtained. The sufficient conditions on the asymptotic behavior of solutions around the disease-free equilibrium and endemic equilibrium of the corresponding deterministic model are established. Furthermore, the existence and uniqueness of stationary distribution are also obtained. Lastly, the analytical results are illustrated by the numerical simulations.

Keywords: computer network; multi-group stochastical SEIQR epidemic model; asymptotic behavior; lyapunov function; numerical simulation

1. Introduction

As is well known, the internet world has brought great changes in the society. In reality, we know that cyber world is being threatened by the attack of malicious objects. Malicious object is a code that infects computer systems. There are different kinds of malicious objects such as: Worm, Virus, Trojan horse, etc., which differ according to the way they attack computer systems and the malicious actions they perform (see [1–3]). With the development of the computer network, malicious objects be widely spread through a network, through an online service, through shared computer software or through a mobile storage tool, and so on. Because of the similarity between the transmission of human infectious diseases and transmission of malicious objects in the computer network, some authors employ the epidemic models to describe the transmission of malicious objects in the cyber world (see [2–17]).

Considering different contact patterns, different anti-virus software, or distinct number of contacts etc., it is more appropriate to divide individual hosts into groups in modeling epidemic disease. Therefore, it is reasonable to propose multi-group models to describe the transmission dynamics of malicious objects in heterogeneous host populations on computer network. At present, many scholars have focused their study on various forms of multi-group epidemic models (see [18–23]). They have also proved the global stability of the unique endemic equilibrium through Lyapunov function, which is one of the main mathematical challenges in analyzing multi-group models. Particularly, Wang et al. [23] proposed the following multi-group SEIQR epidemic model for describing the transmission of malicious objects in computer network

$$\begin{cases} dS_k(t) = & [\Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k(t) I_j(t) - d_k^S S_k]dt, \\ dE_k(t) = & [\sum_{j=1}^{n} \beta_{kj} S_k(t) I_j(t) - (d_k^E + \epsilon_k) E_k]dt, \\ dI_k(t) = & [\epsilon_k E_k - (d_k^I + \alpha_k + \delta_k + \gamma_k) I_k]dt, \\ dQ_k(t) = & [\delta_k I_k - (d_k^Q + \alpha_k + \mu_k) Q_k]dt, \\ dR_k(t) = & [\gamma_k I_k + \mu_k Q_k - d_k^R R_k]dt, \quad 1 \le k \le n, \end{cases} \tag{1.1}$$

where the total network nodes are divided into n groups of nodes, $n \ge 2$ is an integer. $S_k(t)$, $E_k(t)$, $I_k(t)$, $Q_k(t)$ and $R_k(t)$ express the numbers of susceptible nodes, exposed (infected but not yet infectious) nodes, infectious nodes, quarantined nodes and recovered nodes at time t in the k-th group ($1 \le k \le n$), respectively. The definitions of all parameters in model (1.1) are listed in Table 1. We assume that the parameters d_k^S, d_k^E, d_k^I, d_k^Q, d_k^R and Λ_k are positive and the rest of parameters in model (1.1) ia nonnegative for all k. In particular, $\beta_{kj} = 0$ if there is no transmission of the disease between compartments S_k and I_j. In model (1.1) , the basic reproduction number $R_0 = \rho(M_0)$, the spectral radius of matrix $M_0 = (\frac{\beta_{kj}\epsilon_k \frac{\Lambda_k}{d_k}}{(d_k^E+\epsilon_k)(d_k^I+\alpha_k+\delta_k+\gamma_k)})_{n\times n}$, is a threshold which completely determines the persistence or extinction of the disease. It is shown that, if $R_0 \le 1$, the disease-free equilibrium E_0 is globally stable in the feasible region and the disease always dies out, and if $R_0 > 1$, a unique endemic equilibrium E^* exists and is globally stable in the interior of the feasible region, and once the disease appears, it eventually persists at the unique endemic equilibrium level.

Table 1. Description of parameters in model (1.1) .

Symbol	Description
Λ_k	influx of individuals into the kth group
β_{kj}	transmission coefficient between compartments S_k and I_j
$d_k^S, d_k^E, d_k^I, d_k^Q, d_k^R$	natural death rates of S_k, E_k, I_k, Q_k, R_k compartments in the kth group
ϵ_k	the rate constant for nodes leaving the exposed class E_k for infective compartment in the kth group
δ_k	the rate constant for nodes leaving the infective compartment I_k for quarantine compartment in the kth group
α_k	the disease related death rate(crashing of nodes due to the attack of malicious objects)constant in the compartments
γ_k and μ_k	the rates at which nodes recover temporarily after the runof anti-malicious software and return to recovered class R from compartments I_k and Q_k in the kth group

On the other hand, there exist uncertainties and random phenomena everywhere in nature [23–27]. Environmental noises are usually considered to be harmful, which will lead to the disorder of the dynamics [20, 21]. Nevertheless, the noises also play a positive role in the dynamics of complex nonlinear systems, especially in interdisciplinary physical models and biomathematics models, such as noise induced resonances, noise enhanced stability (NES) and so on [22–24, 28–30]. According to the noise source, the noises can be divided into the additive noise and the multiplicative noise. The former is not controlled by the system and can be directly introduced to the system, while the latter is related to system parameters and variables. The multiplicative noises can always ensure the nonnegativity of the solution. The two main peculiarities of the presence of the multiplicative noise are the presence of the absorbing barrier in zero population density and the phenomenon of the anomalous fluctuations [25, 31]. The noise existing in biological systems is caused by environmental fluctuations, which is usually considered as the multiplicative white noise. For example, Caruso et al. [26] described the dynamic behavior of an ecosystem of two competing species by a stochastic Lotka-Volterra model with the multiplicative white noise. The multiplicative noise models the interaction between the environment and the species.

For human disease related epidemics, the nature of epidemic growth and spread is random due to the unpredictability in person to person contacts. Because of environmental noises, the deterministic approach has some limitations in the mathematical modeling transmission of an infectious disease, several authors began to consider the effect of white noise on the computer network systems (see [23–27]).

There are different approaches used in the literature to introduce random perturbations into population models, both from a mathematical and biological perspective (see [23–29, 31]). One is to perturb the positive equilibria in order for making robust the equilibria of deterministic models. In this situation, the essence of the investigation using the approach is to check if the asymptotic stability of the positive equilibria of deterministic models can be preserved. For example, Wang et al. [23] investigated a multi-group SEIQR model with random perturbation around the positive equilibrium of corresponding deterministic model, which revealed that the stochastic stability of endemic equilibrium depends on the magnitude of the intensity of noise as well as the parameters involved within the model. The other important approach is with parameters perturbation. We find that there are many literatures on this approach, see [25–27] and the references cited therein. In epidemic models, the natural death rate and the disease transmission rate are two of the key parameters to disease transmission. And in the real situation, the natural death rate and the disease transmission rate always fluctuate around some average value due to continuous fluctuation in the environment. For example, El Ansari et al. [25] considered a stochastic version of model (1.1) with noises introduced in the rate at which nodes are crashed due to reasons other than the attacks of viruses and the transmission rate, and they proved the various conditions that control the extinction and stability of a nonlinear mathematical spread model with stochastic perturbations.

We now turn to a continuous time SEIQRS model which takes random effects into account. In SEIQRS model (1.1), the natural death rate $d_k^{X_i}$, where $1 \leq k \leq n$ and $(X_1, X_2, X_3, X_4, X_5) = (S, E, I, Q, R)$, is one of the key parameters to disease transmission. May [30] pointed out that all the parameters involved in the population model exhibit random fluctuation as the factors controlling them are not constant. And in the real situation, the natural death rate d always fluctuate around some average value due to continuous fluctuation in the environment.

In this sense, $d_k^{X_i}$ can seem as a random variable $\tilde{d}_k^{X_i}$. More precisely, in $[t, t+dt)$,

$$-\tilde{d}_k^{X_i} dt = -d_k^{X_i} dt + \sigma_{ik} dB_{ik}(t), \ 1 \le k \le n, \ i = 1, 2, 3, 4, 5,$$

where $B_{ik}(t)$ $(1 \le k \le n, i = 1, 2, 3, 4, 5)$ are the independent standard Brownian motion defined on the complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions, and σ_{ik}^2 is the intensity of $B_{ik}(t)$. The reason of adopting σ_{ik}^2 $(1 \le k \le n, i = 1, 2, 3, 4, 5)$ as the intensity of the noise for the group S_k, E_k, I_k, Q_k and R_k, respectively, is considering the difference between the group mobility response to infection risks. And then, in $[t, t+dt)$, $-\tilde{d}_k^{X_i} dt$ is normally distributed with mean $\mathbb{E}(-\tilde{d}_k^{X_i} dt) = -d_k^{X_i} dt$ and variance $\mathbf{Var}(-\tilde{d}_k^{X_i} dt) = \sigma_i^2 dt$. Due to $\mathbf{Var}(-\tilde{d}_k^{X_i} dt) = \sigma_i^2 dt \to 0$ as $dt \to 0$, this is a biologically reasonable assumption. Indeed this is a well-established way of introducing stochastic environmental noise into biologically realistic population dynamic models.

Therefore, replace $-d_k^{X_i} dt$ in model (1.1) with $-\tilde{d}_k^{X_i} dt = -d_k^{X_i} dt + \sigma_{ik} dB_{ik}(t)$ $(1 \le k \le n, i = 1, 2, 3, 4, 5)$, and for simplicity, we replace $-\tilde{d}_k^{X_i}$ with $d_k^{X_i}$ again, then we can obtain the same SDE epidemic model as the following model (1.2) that is analog to its deterministic version model (1.1) by introducing stochastic perturbation terms to the growth equations of susceptible, infectious, recovered individuals to incorporate the effect of randomly fluctuating environments:

$$\begin{cases} \mathrm{d}S_k = & [\Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k(t) I_j(t) - d_k^S S_k]\mathrm{d}t + \sigma_{1k} S_k dB_{1k}, \\ \mathrm{d}E_k = & [\sum_{j=1}^{n} \beta_{kj} S_k(t) I_j(t) - (d_k^E + \epsilon_k) E_k]\mathrm{d}t + \sigma_{2k} E_k dB_{2k}, \\ \mathrm{d}I_k = & [\epsilon_k E_k - (d_k^I + \alpha_k + \delta_k + \gamma_k) I_k]\mathrm{d}t + \sigma_{3k} I_k dB_{3k}, \\ \mathrm{d}Q_k = & [\delta_k I_k - (d_k^Q + \alpha_k + \mu_k) Q_k]\mathrm{d}t + \sigma_{4k} Q_k dB_{4k}, \\ \mathrm{d}R_k = & [\gamma_k I_k + \mu_k Q_k - d_k^R R_k]\mathrm{d}t + \sigma_{5k} R_k dB_{5k}, \quad 1 \le k \le n. \end{cases} \tag{1.2}$$

Throughout this paper, we always assume that model (1.2) is defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contain all P-null sets). Furthermore, we also always assume that the infection rate matrix $B = (\beta_{kj})_{n\times n}$ in model (1.2) is irreducible.

In this paper, we will study the asymptotic behavior of positive solutions of model (1.2) around the disease-free and endemic equilibria of corresponding deterministic model (1.1) in probability meaning by using the theory of graphs, Lyapunov functions method, Itô's formula and the theory of stochastic analysis. Then by using the theory of stationary distributions of stochastic process we will study the existence of stationary distribution of model (1.2).

The paper is organized as follows. In Section 2, the criterion on the asymptotic behavior of positive solutions of model (1.2) around the disease-free equilibrium of the corresponding deterministic model is stated and proved. In Section 3, the sufficient condition the asymptotic behavior of positive solutions of model (1.2) around the endemic equilibrium of corresponding deterministic model and the existence of stationary distribution are stated and proved. In Section 4, we make some numerical simulations to illustrate our analytical results. Finally, in Section 5, we give a brief conclusion.

2. Asymptotic behavior around disease-free equilibrium of model (1.1)

We first give a lemma to show that for any positive initial value model (1.2) has a unique positive solution defined on $[0, \infty)$.

Lemma 1. *For any initial value in R_+^{5n} model (1.2) has a unique positive solution defined for all $t \geq 0$ and the solution remain in R_+^{5n} with probability one.*

This lemma can be easily proved by using the standard arguments as in [14,18] and with the help of Lyapunov function

$$\begin{aligned} & V(S_k, E_k, I_k, Q_k, R_k, 1 \leq k \leq n) \\ = & \sum_{k=1}^{n}[(S_k - a - a\log\frac{S_k}{ac_k}) + (E_k - 1 - \log E_k) \\ & +(I_k - 1 - \log I_k) + (Q_k - 1 - \log Q_k) + (R_k - 1 - \log R_k)], \end{aligned}$$

where positive constant a satisfies $a \leq \min\{\frac{d_k^I + \alpha_k}{\sum_{j=1}^n \beta_{jk}}, k = 1, 2, \cdots, n\}$.

For deterministic model (1.1), in [23] the authors have obtained that there is a disease-free equilibrium $E_0 = (S_1^0, 0, 0, 0, 0, S_2^0, 0, 0, 0, 0, \cdots, S_n^0, 0, 0, 0, 0)$, where $S_k^0 = \frac{\Lambda_k}{d_k^S}$, and if $R_0 \leq 1$, then E_0 is globally asymptotically stable, which means the disease will die out. Therefore, it is interesting to study the stability of disease-free equilibrium for controlling the spread of infectious disease. However, for stochastic model (1.2) there is not any disease-free equilibrium. Therefore, it is natural to ask how we can consider the disease will be extinct. In this section we mainly through estimating the asymptotic oscillation around equilibrium E_0 of any positive solutions of stochastic model (1.2) to reflect whether the disease in stochastic model (1.2) will die out. We have the following result.

Theorem 1. *Assume that $R_0 \leq 1$ and the following conditions hold*

$$\begin{aligned} & d_k^S > \sigma_{1k}^2,\ d_k^I + \alpha_k + \delta_k + \gamma_k > \frac{1}{2}\sigma_{3k}^2,\ d_k^E + \epsilon_k > \frac{1}{2}\sigma_{2k}^2, \\ & d_k^Q + \alpha_k + \mu_k > \frac{1}{2}\sigma_{4k}^2,\ d_k^R > \frac{1}{2}\sigma_{5k}^2,\ 1 \leq k \leq n. \end{aligned} \tag{2.1}$$

Then for any positive solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), 1 \leq k \leq n)$ of model (1.2) one has

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t}E\int_0^t \sum_{k=1}^{n}\{A_k(S_k(r) - \frac{\Lambda_k}{d_k^S})^2 + B_kE_k^2(r) + C_kI_k^2(r) \\ +D_kQ_k^2(r) + F_kR_k^2(r)\}dr \leq \sum_{k=1}^{n}(ba_k + 1)(\frac{\sigma_{1k}\Lambda_k}{d_k^S})^2, \end{aligned}$$

where $A_k = (d_k^S - \sigma_{1k}^2)$, $B_k = \frac{1}{4}(d_k^E + \epsilon_k - \frac{1}{2}\sigma_{2k}^2)$ and

$$\begin{aligned} C_k = & \ c_k[2(d_k^I + \alpha_k + \delta_k + \gamma_k) - \sigma_{3k}^2 - \frac{4c_k\epsilon_k^2}{[(d_k^E + \epsilon_k) - \frac{1}{2}\sigma_{2k}^2]}] \\ & -\frac{d_k\delta_k^2}{d_k^Q + \alpha_k + \mu_k - \frac{1}{2}\sigma_{4k}^2} - \frac{e_k\gamma_k^2}{d_k^R - \frac{1}{2}\sigma_{5k}^2}, \\ D_k = & \ d_k(d_k^Q + \alpha_k + \mu_k - \frac{1}{2}\sigma_{4k}^2),\ F_k = e_k(d_k^R - \frac{1}{2}\sigma_{5k}^2 - \frac{e_k\mu_k^2}{d_k(d_k^Q + \alpha_k + \mu_k - \frac{1}{2}\sigma_{4k}^2)}), \end{aligned}$$

and positive constants a_k, d_k, c_k, e_k $(1 \le k \le n)$ and b will be confirmed in the proof of the theorem.

Proof. Let $u_k = S_k - \frac{\Lambda_k}{d_k^S}, v_k = E_k, w_k = I_k, y_k = Q_k, z_k = R_k$ $(1 \le k \le n)$, then model (1.2) becomes into

$$\begin{cases} \mathrm{d}u_k = & [-\sum_{j=1}^n \beta_{kj} u_k(t) w_j(t) - \sum_{j=1}^n \beta_{kj} w_j(t) \frac{\Lambda_k}{d_k^S} - d_k^S u_k]\mathrm{d}t + \sigma_{1k}(u_k + \frac{\Lambda_k}{d_k^S}) dB_{1k}, \\ \mathrm{d}v_k = & [\sum_{j=1}^n \beta_{kj} u_k(t) w_j(t) + \sum_{j=1}^n \beta_{kj} w_j(t) \frac{\Lambda_k}{d_k^S} - (d_k^E + \epsilon_k) v_k]\mathrm{d}t + \sigma_{2k} v_k dB_{2k}, \\ \mathrm{d}w_k = & [\epsilon_k v_k - (d_k^I + \alpha_k + \delta_k + \gamma_k) w_k]\mathrm{d}t + \sigma_{3k} w_k dB_{3k}, \\ \mathrm{d}y_k = & [\delta_k w_k - (d_k^Q + \alpha_k + \mu_k) y_k]\mathrm{d}t + \sigma_{4k} y_k dB_{4k}, \\ \mathrm{d}z_k = & [\gamma_k w_k + \mu_k y_k - d_k^R z_k]\mathrm{d}t + \sigma_{5k} z_k dB_{5k}. \end{cases}$$

Since $B = (\beta_{kj})_{n\times n}$ is irreducible, then M_0 is also nonnegative and irreducible. Hence, by Lemma A.1 in [3], M_0 has a positive left eigenvector $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ such that

$$(\eta_1, \eta_2, \cdots, \eta_n)\rho(M_0) = (\eta_1, \eta_2, \cdots, \eta_n) M_0. \tag{2.2}$$

Define a Lyapunov function as follows.

$$V = V_1 + b(V_2 + V_3) + V_4 + V_5 + V_6$$

with $V_1 = \frac{1}{2}\sum_{k=1}^n (u_k + v_k)^2$, $V_2 = \frac{1}{2}\sum_{k=1}^n a_k u_k^2$, $V_3 = \sum_{k=1}^n \frac{\epsilon_k \eta_k}{(d_k^E+\epsilon_k)(d_k^I+\alpha_k+\delta_k+\gamma_k)}(v_k + \frac{d_k^E+\epsilon_k}{\epsilon_k} w_k)$, $V_4 = \sum_{k=1}^n c_k w_k^2$, $V_5 = \sum_{k=1}^n d_k y_k^2$ and $V_6 = \sum_{k=1}^n e_k z_k^2$, where positive constants a_k, c_k, d_k, e_k $(1 \le k \le n)$ and b will be determined later. By Itô's formula, we get

$$\begin{aligned} dV = & LVdt + \sum_{k=1}^n \sigma_{1k}(u_k + \frac{\Lambda_k}{d_k^S})[(1 + ba_k)u_k + v_k]dB_{1k} + \sum_{k=1}^n \sigma_{2k} v_k [u_k + v_k \\ & + \frac{b\omega_k \epsilon_k}{(d_k^E + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)}]dB_{2k} + \sum_{k=1}^n \sigma_{3k} w_k [\frac{b\omega_k}{d_k^I + \alpha_k + \delta_k + \gamma_k} \\ & + c_k w_k]dB_{3k} + \sum_{k=1}^n d_k \sigma_{4k} y_k^2 dB_{4k} + \sum_{k=1}^n e_k \sigma_{5k} z_k^2 dB_{5k} \end{aligned} \tag{2.3}$$

with $LV = LV_1 + b(LV_2 + LV_3) + LV_4 + LV_5 + LV_6$, where

$$\begin{aligned} LV_1 = & \sum_{k=1}^n (u_k + v_k)[-\sum_{j=1}^n \beta_{kj} u_k(t) w_j(t) - \sum_{j=1}^n \beta_{kj} w_j(t) \frac{\Lambda_k}{d_k^S} - d_k^S u_k + \sum_{j=1}^n \beta_{kj} u_k(t) w_j(t) \\ & + \sum_{j=1}^n \beta_{kj} w_j(t) \frac{\Lambda_k}{d_k^S} - (d_k^E + \epsilon_k) v_k] + \sum_{k=1}^n [\sigma_{1k}^2 (u_k + \frac{\Lambda_k}{d_k^S})^2 + \sigma_{2k}^2 v_k^2] \\ \le & -\sum_{k=1}^n \{(d_k^S - \sigma_{1k}^2) u_k^2 + [d_k^E + \epsilon_k - \frac{1}{2}\sigma_{2k}^2] v_k^2 + (d_k^S + d_k^E + \epsilon_k) u_k v_k - (\frac{\Lambda_k}{d_k^S})^2 \sigma_{1k}^2\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} LV_2 = & \sum_{k=1}^{n} a_k u_k [-\sum_{j=1}^{n} \beta_{kj} u_k(t) w_j(t) - \sum_{j=1}^{n} \beta_{kj} w_j(t) \frac{\Lambda_k}{d_k^S} - d_k^S u_k] \\ & + \sum_{k=1}^{n} a_k \sigma_{1k}^2 u_k^2 + \sum_{k=1}^{n} a_k \sigma_{1k}^2 (\frac{\Lambda_k}{d_k^S})^2 \\ \leq & -\sum_{k=1}^{n} a_k [(d_k^S - \sigma_{1k}^2) u_k^2 + \sum_{j=1}^{n} \beta_{kj} \frac{\Lambda_k}{d_k^S} u_k(t) w_j(t) - (\frac{\sigma_{1k}\Lambda_k}{d_k^S})^2] \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} LV_3 = & \sum_{k=1}^{n} \frac{\omega_k \epsilon_k}{(d_k^E + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)} [\sum_{j=1}^{n} \beta_{kj} u_k(t) w_j(t) + \sum_{j=1}^{n} \beta_{kj} w_j(t) \frac{\Lambda_k}{d_k^S} \\ & -(d_k^E + \epsilon_k) v_k + \epsilon_k v_k - \frac{d_k^E + \epsilon_k}{\epsilon_k} (d_k^I + \alpha_k + \delta_k + \gamma_k) w_k] \\ \leq & \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\beta_{kj} \omega_k \epsilon_k}{(d_k^E + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)} u_k(t) w_j(t) - \sum_{k=1}^{n} \omega_k w_k \\ & + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\beta_{kj} \omega_k \epsilon_k}{(d_k^E + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)} \frac{\Lambda_k}{d_k^S} w_j(t). \end{aligned}$$

Note from (2.2) that

$$-\eta_k w_k + \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\beta_{kj} \omega_k \epsilon_k}{(d_k^E + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)} \frac{\Lambda_k}{d_k^S} w_j(t) = (R_0 - 1) \eta w,$$

where $w = (w_1, w_2, \cdots, w_n)^T$. If $R_0 \leq 1$, then

$$LV_3 \leq \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\beta_{kj} \eta_k \epsilon_k}{(d_k^E + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)} u_k(t) w_j(t). \tag{2.6}$$

Furthermore, we also have

$$\begin{aligned} LV_4 = & -\sum_{k=1}^{n} c_k [2(d_k^I + \alpha_k + \delta_k + \gamma_k) - \sigma_{3k}^2] w_k^2 + 2 \sum_{k=1}^{n} c_k \epsilon_k w_k v_k, \\ LV_5 = & -\sum_{k=1}^{n} d_k [2(d_k^Q + \alpha_k + \mu_k) - \sigma_{4k}^2] y_k^2 + 2 \sum_{k=1}^{n} d_k \delta_k w_k y_k, \\ LV_6 = & -\sum_{k=1}^{n} e_k [2 d_k^R - \sigma_{5k}^2] z_k^2 + 2 \sum_{k=1}^{n} e_k \gamma_k w_k z_k + 2 \sum_{k=1}^{n} e_k \mu_k y_k z_k. \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} & 2 c_k \epsilon_k w_k v_k \leq \frac{1}{4} [(d_k^E + \epsilon_k) - \frac{1}{2} \sigma_{2k}^2] v_k^2 + \frac{4 c_k^2 \epsilon_k^2}{[(d_k^E + \epsilon_k) - \frac{1}{2} \sigma_{2k}^2]} w_k^2, \\ & 2 d_k \delta_k w_k y_k \leq d_k [(d_k^Q + \alpha_k + \mu_k) - \frac{1}{2} \sigma_{4k}^2] y_k^2 + \frac{d_k \delta_k^2}{d_k^Q + \alpha_k + \mu_k - \frac{1}{2} \sigma_{4k}^2} w_k^2, \\ & 2 e_k \gamma_k w_k z_k \leq e_k (d_k^R - \frac{1}{2} \sigma_{5k}^2) z_k^2 + \frac{e_k \gamma_k^2}{d_k^R - \frac{1}{2} \sigma_{5k}^2} w_k^2, \\ & 2 e_k \mu_k y_k z_k \leq d_k (d_k^Q + \alpha_k + \mu_k - \frac{1}{2} \sigma_{4k}^2) y_k^2 + \frac{e_k^2 \mu_k^2}{d_k (d_k^Q + \alpha_k + \mu_k - \frac{1}{2} \sigma_{4k}^2)} z_k^2. \end{aligned} \tag{2.8}$$

Choosing $a_k = \frac{d_k^S \eta_k \epsilon_k}{(d_k^E+\epsilon_k)(d_k^I+\alpha_k+\delta_k+\gamma_k)\Lambda_k}$ $(1 \le k \le n)$ and $b = \max_{1\le k\le n}\{\frac{(d_k^S+d_k^E+\epsilon_k)^2}{2a_k(d_k^E+\epsilon_k-\frac{1}{2}\sigma_{2k}^2)}\}$, then from (2.4)–(2.8) we finally obtain

$$LV \le -\sum_{k=1}^{n}\{A_k u_k^2 + B_k v_k^2 + C_k w_k^2 + D_k y_k^2 + F_k z_k^2\} + \sum_{k=1}^{n}(ba_k+1)(\frac{\sigma_{1k}\Lambda_k}{d_k^S})^2, \tag{2.9}$$

where A_k, B_k, C_k, D_k and F_k are given in the above.

If (2.1) holds, then $A_k > 0$, $B_k > 0$ and $D_k > 0$. Further, we can choose c_k, d_k and e_k such that

$$\begin{aligned}
0 < c_k &< \frac{[(d_k^E+\epsilon_k)-\frac{1}{2}\sigma_{2k}^2]}{4\epsilon_k^2}[2(d_k^I+\alpha_k+\delta_k+\gamma_k)-\sigma_{3k}^2],\\
0 < d_k &< \frac{c_k}{\eta_k}[2(d_k^I+\alpha_k+\delta_k+\gamma_k)-\sigma_{3k}^2-\frac{4c_k\epsilon_k^2}{[(d_k^E+\epsilon_k)-\frac{1}{2}\sigma_{2k}^2]}],\\
0 < e_k &< \frac{d_k(d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2)(d_k^R-\frac{1}{2}\sigma_{5k}^2)}{\mu_k^2}.
\end{aligned}$$

Particularly, we can take

$$\begin{aligned}
c_k &= \frac{[(d_k^E+\epsilon_k)-\frac{1}{2}\sigma_{2k}^2]}{8\epsilon_k^2}[2(d_k^I+\alpha_k+\delta_k+\gamma_k)-\sigma_{3k}^2],\\
d_k &= \frac{c_k}{2\eta_k}[2(d_k^I+\alpha_k+\delta_k+\gamma_k)-\sigma_{3k}^2-\frac{4c_k\epsilon_k^2}{[(d_k^E+\epsilon_k)-\frac{1}{2}\sigma_{2k}^2]}],\\
e_k &= \frac{d_k(d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2)(d_k^R-\frac{1}{2}\sigma_{5k}^2)}{2\mu_k^2},
\end{aligned}$$

where $\eta_k = \frac{\delta_k^2}{d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2} + \frac{\gamma_k^2(d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2)}{\mu_k^2} > 0$. Thus, we have

$$\begin{aligned}
C_k = {} & c_k[2(d_k^I+\alpha_k+\delta_k+\gamma_k)-\sigma_{3k}^2-\frac{4c_k\epsilon_k^2}{[(d_k^E+\epsilon_k)-\frac{1}{2}\sigma_{2k}^2]}]\\
& -\frac{d_k\delta_k^2}{d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2}-\frac{e_k\gamma_k^2}{d_k^R-\frac{1}{2}\sigma_{5k}^2}\\
> {} & c_k[2(d_k^I+\alpha_k+\delta_k+\gamma_k)-\sigma_{3k}^2-\frac{4c_k\epsilon_k^2}{[(d_k^E+\epsilon_k)-\frac{1}{2}\sigma_{2k}^2]}]\\
& -d_k[\frac{\delta_k^2}{d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2}+\frac{\gamma_k^2(d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2)}{\mu_k^2}] > 0
\end{aligned}$$

and $F_k = e_k(d_k^R - \frac{1}{2}\sigma_{5k}^2 - \frac{e_k\mu_k^2}{d_k(d_k^Q+\alpha_k+\mu_k-\frac{1}{2}\sigma_{4k}^2)}) > 0$. By integration and taking expectation of both sides of (2.3), from (2.9) we obtain

$$\begin{aligned}
& E(V(t)) - E(V(0)) = E[\int_0^t LV(r)dr]\\
\le {} & -E\int_0^t \sum_{k=1}^{n}\{A_k u_k^2(r) + B_k v_k^2(r) + C_k w_k^2(r) + D_k y_k^2(r) + F_k z_k^2(r)\}dr + \sum_{k=1}^{n}(ba_k+1)(\frac{\sigma_{1k}\Lambda_k}{d_k^S})^2.
\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\sum_{k=1}^n\{A_ku_k^2(r)+B_kv_k^2(r)+C_kw_k^2(r)+D_ky_k^2(r)+F_kz_k^2(r)\}dr\le\sum_{k=1}^n(ba_k+1)(\frac{\sigma_{1k}\Lambda_k}{d_k^S})^2.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\sum_{k=1}^n\{A_k(S_k(r)-\frac{\Lambda_k}{d_k^S})^2+B_kE_k^2(r)+C_kI_k^2(r)$$
$$+D_kQ_k^2(r)+F_kR_k^2(r)\}dr\le\sum_{k=1}^n(ba_k+1)(\frac{\sigma_{1k}\Lambda_k}{d_k^S})^2.$$

This completes the proof. □

Remark 1. *From Theorem 1, we see that under some conditions the solution of model (1.2) will oscillates around the disease-free equilibrium of deterministic model (1.1), and the intensity of fluctuation is only relation to the intensity of the white noise $B_{1k}(t)$, but do not relation to the intensities of the other white noises. In a biological interpretation, as the intensity of stochastic perturbations is small, the solution of model (1.2) will be close to the disease-free equilibrium of model (1.1) most of the time.*

As a special case of model (1.2), when $\sigma_{1k}=0$, then model (1.2) becomes into

$$\begin{cases} dS_k = & [\Lambda_k-\sum_{j=1}^n\beta_{kj}S_k(t)I_j(t)-d_k^SS_k]dt,\\ dE_k = & [\sum_{j=1}^n\beta_{kj}S_k(t)I_j(t)-(d_k^E+\epsilon_k)E_k]dt+\sigma_{2k}E_kdB_{2k},\\ dI_k = & [\epsilon_kE_k-(d_k^I+\alpha_k+\delta_k+\gamma_k)I_k]dt+\sigma_{3k}I_kdB_{3k},\\ dQ_k = & [\delta_kI_k-(d_k^Q+\alpha_k+\mu_k)Q_k]dt+\sigma_{4k}Q_kdB_{4k},\\ dR_k = & [\gamma_kI_k+\mu_kQ_k-d_k^RR_k]dt+\sigma_{5k}R_kdB_{5k}.\end{cases}\tag{2.10}$$

Obviously, E_0 is also the disease-free equilibrium of model (2.10). From the proof of Theorem 2, we get

$$LV\le-\sum_{k=1}^n\Big\{2a_kd_k^S(S_k(r)-\frac{\Lambda_k}{d_k^S})^2+B_kE_k^2(r)+C_kI_k^2(r)+D_kQ_k^2(r)+F_kR_k^2(r)\Big\},$$

which is negative definite if for each $1\le k\le n$

$$d_k^I+\alpha_k+\delta_k+\gamma_k>\frac{1}{2}\sigma_{3k}^2,\ d_k^R>\frac{1}{2}\sigma_{5k}^2,\ d_k^E+\epsilon_k>\frac{1}{2}\sigma_{2k}^2,\ d_k^Q+\alpha_k+\mu_k>\frac{1}{2}\sigma_{4k}^2.\tag{2.11}$$

Therefore, as a consequence of Theorem 1 we have the following result.

Corollary 1. *Assume that $R_0\le 1$ and condition (2.11) holds. Then disease-free equilibrium E_0 of model (2.9) is globally stochastically asymptotically stable.*

3. Asymptotic behavior around endemic equilibrium of model (1.1)

Firstly, we introduce some concepts and conclusions of graph theory (see [10]). A directed graph $g = (V, E)$ contains a set $V = \{1, 2, \cdots, n\}$ of vertices and a set E of arcs (k, j) leading from initial vertex k to terminal vertex j. A subgraph H of g is said to be spanning if H and g have the same vertex set. A directed digraph g is weighted if each arc (k, j) is assigned a positive weight a_{kj}. Given a weighted digraph g with n vertices, define the weight matrix $A = (a_{kj})_{n\times n}$ whose entry a_{kj} equals the weight of arc (k, j) if it exists, and 0 otherwise. A weighted digraph is denoted by (g, A). A digraph g is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other and it is well known that a weighted digraph (g, A) is stronly connected if and only if the weight matrix A is irreducible (see [32]).

The Laplacian matrix of graph (g, A) is defined by

$$L_A = \begin{pmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k\neq n} a_{nk} \end{pmatrix}.$$

Let $c_k\,(1 \leq k \leq n)$ denote the cofactor of the k-th diagonal element of L_A. The following lemmas are the classical results of graph theory (see [21, 33]) which will be used in this paper.

Lemma 2. *Assume that A is a irreducible matrix and* $n \geq 2$. *Then* $c_k > 0$ *for all* $1 \leq k \leq n$.

Lemma 3. *Assume that A is a irreducible matrix and* $n \geq 2$. *Then the following equality holds*

$$\sum_{k=1}^{n}\sum_{j=1}^{n} c_k a_{kj} G_k(x_k) = \sum_{k=1}^{n}\sum_{j=1}^{n} c_k a_{kj} G_j(x_j),$$

where $G_k(x_k)\,(1 \leq k \leq n)$ *are arbitrary functions.*

For model (1.2), we see that there is not any endemic equilibrium. Therefore, in order to study the persistence of disease in model (1.2), we need to study the asymptotic behavior of the endemic equilibrium of model (1.2) which is surrounding the deterministic model (1.1), we obtain the following result.

Theorem 2. *Assume that* $R_0 > 1$ *and the following conditions hold*

$$\sigma_{1k}^2 < d_k^S,\ \sigma_{2k}^2 < \frac{1}{2}d_k^E,\ \sigma_{3k}^2 < \frac{1}{2}(d_k^I+\alpha_k+\delta_k+\gamma_k), \sigma_{4k}^2 < \frac{1}{2}(d_k^Q+\alpha_k+\mu_k),\ \sigma_{5k}^2 < \frac{1}{2}d_k^R,\ 1 \leq k \leq n. \quad (3.1)$$

Then for any positive solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), 1 \le k \le n)$ of model (1.2) one has

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t\{&\sum_{k=1}^n\{c_k r\frac{d_k^S-\sigma_{1k}^2}{S_k^*}-2a_kD_k\}(S_k(s)-S_k^*)^2+2\sum_{k=1}^n(d_k^E-2\sigma_{2k}^2)(E_k(s)-E_k^*)^2\\&+\sum_{k=1}^n\{a_k(d_k^I+\alpha_k+\delta_k+\gamma_k-2\sigma_{3k}^2)-b_k\frac{\delta_k^2}{d_k^Q+\alpha_k+\mu_k}-d_k\frac{\gamma_k^2}{d_k^R}\}(I_k(s)-I_k^*)^2\\&+\sum_{k=1}^n\{b_k(d_k^Q+\alpha_k+\mu_k-2\sigma_{4k}^2)-\mu_k^2\}(Q_k(s)-Q_k^*)^2\\&+\sum_{k=1}^n d_k\{(d_k^R-2\sigma_{5k}^2)-d_k\}(R_k(s)-R_k^*)^2\}ds\le\sum_{k=1}^n\rho_k,\end{aligned}$$

where $E^* = (S_k^*, E_k^*, I_k^*, Q_k^*, R_k^*, 1 \le k \le n)$ be the endemic equilibrium of model (1.1), and

$$\begin{aligned}\rho_k=\;&2\sum_{k=1}^n a_k\{\sigma_{1k}^2(S_k^*)^2+\sigma_{2k}^2(E_k^*)^2+(1+\frac{d_k^E+d_k^I+\alpha_k+\delta_k+\gamma_k}{\epsilon_k})\sigma_{3k}^2(I_k^*)^2\}\\&+2\sum_{k=1}^n b_k\sigma_{4k}^2(Q_k^*)^2+2\sum_{k=1}^n d_k\sigma_{5k}^2(R_k^*)^2+\frac{1}{2}\sum_{k=1}^n c_k[(K+2)\sigma_{1k}^2S_k^*\\&+(K+1)\sigma_{2k}^2E_k^*+(K+1)\frac{d_k^E+\epsilon_k}{\epsilon_k}\sigma_{3k}^2I_k^*],\end{aligned}$$

and positive constants r, a_k, b_k, c_k and D_k $(1 \le k \le n)$ will be confirmed in the proof of the theorem.

Proof. When $R_0 > 1$, from [23] there exits an endemic equilibrium E^* of model (1.1), then

$$\begin{gathered}\Lambda_k=\sum_{j=1}^n\beta_{kj}S_k^*I_j^*+d_k^SS_k^*,\ \sum_{j=1}^n\beta_{kj}S_k^*I_j^*=(d_k^E+\epsilon_k)E_k^*,\\ \epsilon_kE_k^*=(d_k^I+\alpha_k+\delta_k+\gamma_k)I_k^*,\ \delta_kI_k^*=(d_k^Q+\alpha_k+\mu_k)Q_k^*,\\ \gamma_kI_k^*+\mu_kQ_k^*=d_k^RR_k^*,\ 1\le k\le n.\end{gathered}$$

Let matrix $A = (a_{kj})_{n\times n}$ with $a_{kj} = \beta_{kj}S_k^*I_j^*,\ k, j = 1, 2, \cdots, n$. Since $B = (\beta_{kj})_{n\times n}$ is irreducible, then A also is irreducible.

Firstly, define the C^2-function $V_1 : R_+^{3n} \to R_+$ by

$$\begin{aligned}&V_1(S_k, E_k, I_k, 1\le k\le n)\\=\;&\sum_{k=1}^n c_k[(S_k-S_k^*-S_k^*\log\frac{S_k}{S_k^*})+(E_k-E_k^*-E_k^*\log\frac{E_k}{E_k^*})+\frac{d_k^E+\epsilon_k}{\epsilon_k}(I_k-I_k^*-I_k^*\log\frac{I_k}{I_k^*})],\end{aligned}$$

where c_k $(1 \le k \le n)$ are the cofactor of the k-th diagonal element of L_A. V_1 is positive definite. From

Itô's formula, by calculating we can get

$$\begin{aligned}
LV_1 = & \sum_{k=1}^{n} c_k[3\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^* + 2d_k^S S_k^* - d_k^S S_k - \frac{(S_k^*)^2 d_k^S}{S_k} - \sum_{j=1}^{n}\frac{\beta_{kj}(S_k^*)^2 I_j^*}{S_k} \\
& + \sum_{j=1}^{n}\beta_{kj}S_k^*I_j - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{S_k I_j E_k^*}{S_k^* E_k I_j^*} - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{I_k^* E_k}{E_k^* I_k} \\
& - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*}] + \frac{1}{2}\sum_{k=1}^{n}c_k(\sigma_{1k}^2 S_k^* + \sigma_{2k}^2 E_k^* + \frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*) \\
= & \sum_{k=1}^{n} c_k d_k^S S_k^*(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}) + \sum_{k=1}^{n} c_k[3\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^* - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{S_k^*}{S_k} \\
& - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{S_k I_j E_k^*}{S_k^* E_k I_j^*} - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{I_k^* E_k}{E_k^* I_k}] + \sum_{k=1}^{n} c_k[\sum_{j=1}^{n}\beta_{kj}S_k^*I_j \\
& - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*}] + \frac{1}{2}\sum_{k=1}^{n}c_k(\sigma_{1k}^2 S_k^* + \sigma_{2k}^2 E_k^* + \frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*).
\end{aligned} \tag{3.2}$$

By Lemma 2, we obtain

$$\begin{aligned}
& \sum_{k=1}^{n} c_k[\sum_{j=1}^{n}\beta_{kj}S_k^*I_j - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*}] \\
= & \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_j}{I_j^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*} \\
= & \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*} = 0.
\end{aligned} \tag{3.3}$$

Similarly, we also get

$$\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_k^* E_k}{E_k^* I_k} = \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_j^* E_j}{E_j^* I_j}.$$

Hence

$$\begin{aligned}
& \sum_{k=1}^{n} c_k[3\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^* - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{S_k^*}{S_k} - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{S_k I_j E_k^*}{S_k^* E_k I_j^*} - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{I_k^* E_k}{E_k^* I_k}] \\
= & \sum_{k=1}^{n} c_k\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*[3 - \frac{S_k^*}{S_k} - \frac{S_k I_j E_k^*}{S_k^* E_k I_j^*} - \frac{I_j^* E_j}{E_j^* I_j}] \\
\le & \sum_{k=1}^{n} c_k\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*[3 - 3 - \ln\frac{E_k^*}{E_k} - \ln\frac{E_j}{E_j^*}] \\
= & \sum_{k=1}^{n} c_k\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\ln\frac{E_k}{E_k^*} - \sum_{k=1}^{n} c_k\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\ln\frac{E_j}{E_j^*} = 0,
\end{aligned} \tag{3.4}$$

where the last equality is derived from Lemma 3. Substituting (3.3) and (3.4) into (3.2), we have

$$LV_1 \leq \sum_{k=1}^n c_k d_k^S S_k^*(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}) + \frac{1}{2}\sum_{k=1}^n c_k(\sigma_{1k}^2 S_k^* + \sigma_{2k}^2 E_k^* + \frac{d_k^E+\epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*). \quad (3.5)$$

Secondly, define the C^2-function $V_2 : R_+^{2n} \to R_+$ as follows.

$$V_2(E_k, I_k, 1 \leq k \leq n) = \sum_{k=1}^n c_k[(E_k - E_k^* - E_k^* \log \frac{E_k}{E_k^*}) + \frac{d_k^E + \epsilon_k}{\epsilon_k}(I_k - I_k^* - I_k^* \log \frac{I_k}{I_k^*})],$$

where c_k $(1 \leq k \leq n)$ are given as in V_1. V_2 is positive definite. It follows from Itô's formula that

$$\begin{aligned} LV_2 = & \sum_{k=1}^n c_k[\sum_{j=1}^n \beta_{kj} S_k I_j - \frac{(d_k^E + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)}{\epsilon_k} I_k \\ & - \sum_{j=1}^n \beta_{kj} S_k I_j \frac{E_k^*}{E_k} + (d_k^E + \epsilon_k)E_k^* - \frac{(d_k^E + \epsilon_k)E_k I_k^*}{I_k} \\ & + \frac{(d_k^E + \alpha_k + \delta_k + \epsilon_k)(d_k^I + \alpha_k + \delta_k + \gamma_k)}{\epsilon_k} I_k^*] + \frac{1}{2}\sum_{k=1}^n c_k(\sigma_{2k}^2 E_k^* + \frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*)] \\ = & \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^*[1 + \frac{S_k}{S_k^*} \\ & - \frac{S_k E_k^* I_j}{S_k^* E_k I_j^*} - \frac{E_k I_k^*}{E_k^* I_k}] + \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^* \frac{I_j}{I_j^*} \\ & - \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^* \frac{I_k}{I_k^*} + \frac{1}{2}\sum_{k=1}^n c_k(\sigma_{2k}^2 E_k^* + \frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*)]. \end{aligned} \quad (3.6)$$

We have

$$\begin{aligned} & \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^*[1 + \frac{S_k}{S_k^*} - \frac{S_k E_k^* I_j}{S_k^* E_k I_j^*} - \frac{E_k I_k^*}{E_k^* I_k}] \\ \leq & \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^*[\frac{S_k}{S_k^*} - 1 - \log \frac{S_k E_k^* I_j}{S_k^* E_k I_j^*} - \log \frac{E_k I_k^*}{E_k^* I_k}] \\ = & \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^*[\frac{S_k}{S_k^*} - 1 - \log \frac{S_k}{S_k^*} - \log \frac{I_j}{I_j^*} - \log \frac{I_k^*}{I_k}] \\ \leq & \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^*[\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2] - \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^*[\log \frac{I_j}{I_j^*} + \log \frac{I_k^*}{I_k}] \\ = & \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^*[\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2], \end{aligned} \quad (3.7)$$

where the last equality is derived from Lemma 3 such that

$$\sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_k^* \log \frac{I_j}{I_j^*} - \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_k^* \log \frac{I_k}{I_k^*} = 0.$$

We further get

$$\sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^* \frac{I_j}{I_j^*} - \sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} S_k^* I_j^* \frac{I_k}{I_k^*} = 0. \quad (3.8)$$

Substituting (3.7) and (3.8) into (3.6), we have

$$\begin{aligned} LV_2 \leq & \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*[\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2] \\ & + \frac{1}{2}\sum_{k=1}^{n} c_k(\sigma_{2k}^2 E_k^* + \frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*)]. \end{aligned} \tag{3.9}$$

Thirdly, define the C^2-function $V_3 : R_+^n \to R_+$ by

$$V_3(S_k, 1 \leq k \leq n) = \sum_{k=1}^{n} c_k \frac{(S_k - S_k^*)^2}{2S_k^*},$$

where c_k $(1 \leq k \leq n)$ are given as in V_1. We obtain

$$\begin{aligned} LV_3 = & -\sum_{k=1}^{n} c_k \frac{d_k^S (S_k - S_k^*)^2}{S_k^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}\frac{(S_k - S_k^*)^2 I_j}{S_k^*} \\ & + \frac{1}{2}\sum_{k=1}^{n} c_k S_k^2 \sigma_{1k}^2 - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}(S_k - S_k^*)(I_j - I_j^*) \\ \leq & -\sum_{k=1}^{n} c_k \frac{(d_k^S - \sigma_{1k}^2)(S_k - S_k^*)^2}{S_k^*} \\ & -\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^{n} c_k S_k^* \sigma_{1k}^2. \end{aligned} \tag{3.10}$$

Choose $K = \sum_{j=1}^{n} \beta_{kj}\frac{I_k^*}{d_k^S}$, then (3.5) together with (3.9) and (3.10) implies

$$\begin{aligned} & L(KV_1 + V_2 + V_3) \\ \leq & \sum_{k=1}^{n} Kc_k d_k^S S_k^*(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}) + \frac{1}{2}\sum_{k=1}^{n} Kc_k(\sigma_{1k}^2 S_k^* + \sigma_{2k}^2 E_k^* + \frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*) \\ & + \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*[\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2] \\ & + \frac{1}{2}\sum_{k=1}^{n} c_k(\sigma_{2k}^2 E_k^* + \frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*)] - \sum_{k=1}^{n} c_k \frac{(d_k^S - \sigma_{1k}^2)(S_k - S_k^*)^2}{S_k^*} \\ & - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^{n} c_k S_k^* \sigma_{1k}^2 \\ \leq & -\sum_{k=1}^{n} c_k \frac{(d_k^S - \sigma_{1k}^2)(S_k - S_k^*)^2}{S_k^*} + A_k, \end{aligned} \tag{3.11}$$

where $A_k = \frac{1}{2}\sum_{k=1}^{n} c_k[(K+2)\sigma_{1k}^2 S_k^* + (K+1)\sigma_{2k}^2 E_k^* + (K+1)\frac{d_k^E + \epsilon_k}{\epsilon_k}\sigma_{3k}^2 I_k^*]$.

Next, define the C^2-function $V_4 : R_+^{3n} \to R_+$ by

$$V_4(S_k, E_k, I_k, 1 \leq k \leq n) = \sum_{k=1}^{n} a_k(S_k - S_k^* + E_k - E_k^* + I_k - I_k^*)^2,$$

where a_k $(1 \leq k \leq n)$ are positive constants to be determined later. By calculating, we can get

$$\begin{aligned} LV_4 = & -2\sum_{k=1}^{n} a_k[d_k^S(S_k - S_k^*)^2 + d_k^E(E_k - E_k^*)^2 + (d_k^I + \alpha_k + \delta_k + \gamma_k)(I_k - I_k^*)^2] \\ & -2\sum_{k=1}^{n}\{a_k(d_k^S + d_k^E)(S_k - S_k^*)(E_k - E_k^*) + (d_k^S + d_k^I + \alpha_k + \delta_k + \gamma_k) \\ & \times(S_k - S_k^*)(I_k - I_k^*) + (d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k)(E_k - E_k^*)(I_k - I_k^*)\} \\ & + \sum_{k=1}^{n} a_k(\sigma_{1k}^2 S_k^2 + \sigma_{2k}^2 E_k + \sigma_{3k}^2 I_k). \end{aligned}$$

Since $2(d_k^S + d_k^E)(S_k - S_k^*)(E_k - E_k^*) \leq \frac{(d_k^S + d_k^E)^2}{d_k^E}(S_k - S_k^*)^2 + d_k^E(E_k - E_k^*)^2$ and

$$\begin{aligned} & 2(d_k^S + d_k^I + \alpha_k + \delta_k + \gamma_k)(S_k - S_k^*)(I_k - I_k^*) \\ \leq & \frac{(d_k^S + d_k^I + \alpha_k + \delta_k + \gamma_k)^2}{(d_k^I + \alpha_k + \delta_k + \gamma_k)}(S_k - S_k^*)^2 + (d_k^I + \alpha_k + \delta_k + \gamma_k)(I_k - I_k^*)^2, \end{aligned}$$

we further obtain

$$\begin{aligned} LV_4 \leq \; & 2\sum_{k=1}^{n} a_k[D_k(S_k - S_k^*)^2 - (d_k^E - 2\sigma_{2k}^2)(E_k - E_k^*)^2 \\ & -(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2)(I_k - I_k^*)^2] \\ & -2\sum_{k=1}^{n} a_k(d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k)(E_k - E_k^*)(I_k - I_k^*) \\ & +2\sum_{k=1}^{n} a_k(\sigma_{1k}^2(S_k^*)^2 + \sigma_{2k}^2(E_k^*)^2 + \sigma_{3k}^2(I_k^*)^2), \end{aligned} \tag{3.12}$$

where $D_k = d_k^S + d_k^E + \frac{(d_k^S)^2}{d_k^E} + \frac{(d_k^S + d_k^I + \alpha_k + \delta_k + \gamma_k)^2}{d_k^I + \alpha_k + \delta_k + \gamma_k} + \sigma_{1k}^2$.

Further, define the C^2-function $V_5 : R_+^n \to R_+$ by

$$V_5(I_k, 1 \leq k \leq n) = \sum_{k=1}^{n} a_k \frac{(d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k)}{\epsilon_k}(I_k - I_k^*)^2.$$

We obtain

$$\begin{aligned} LV_5 = & -2\sum_{k=1}^{n} a_k[\frac{(d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k)}{\epsilon_k}(d_k^I + \alpha_k + \delta_k + \gamma_k \\ & -\sigma_{3k}^2)(I_k - I_k^*)^2 - (d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k)(E_k - E_k^*)(I_k - I_k^*)] \\ & +2\sum_{k=1}^{n} a_k \frac{(d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k)}{\epsilon_k}\sigma_{3k}^2(I_k^*)^2. \end{aligned} \tag{3.13}$$

Finally, define the C^2 functions V_6 and $V_7 : R_+^n \to R_+$ as follows.

$$V_6(Q_k, 1 \leq k \leq n) = \sum_{k=1}^{n} b_k(Q_k - Q_k^*)^2, \;\; V_7(R_k, 1 \leq k \leq n) = \sum_{k=1}^{n} d_k(R_k - R_k^*)^2,$$

where b_k, d_k $(1 \le k \le n)$ are positive constants to be determined later. We get

$$
\begin{aligned}
LV_6 = &\ -2\sum_{k=1}^{n} b_k(d_k^Q + \alpha_k + \mu_k - \sigma_{4k}^2)(Q_k - Q_k^*)^2 \\
&+2\sum_{k=1}^{n} b_k\delta_k(Q_k - Q_k^*)(I_k - I_k^*) + 2\sum_{k=1}^{n} b_k\sigma_{4k}^2(Q_k^*)^2 \\
\le &\ -\sum_{k=1}^{n} b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2)(Q_k - Q_k^*)^2 \\
&+\sum_{k=1}^{n} b_k\frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k}(I_k - I_k^*)^2 + 2\sum_{k=1}^{n} b_k\sigma_{4k}^2(Q_k^*)^2
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
LV_7 = &\ -2\sum_{k=1}^{n} d_k(d_k^R - \sigma_{5k}^2)(R_k - R_k^*)^2 + 2\sum_{k=1}^{n} d_k\gamma_k(R_k - R_k^*)(I_k - I_k^*) \\
&+2\sum_{k=1}^{n} d_k\mu_k(Q_k - Q_k^*)(R_k - R_k^*) + 2\sum_{k=1}^{n} d_k\sigma_{5k}^2(R_k^*)^2 \\
\le &\ -\sum_{k=1}^{n} d_k(d_k^R - 2\sigma_{5k}^2)(R_k - R_k^*)^2 + \sum_{k=1}^{n} d_k\frac{\gamma_k^2}{d_k^R}(I_k - I_k^*)^2 \\
&+\sum_{k=1}^{n} \mu_k^2(Q_k - Q_k^*)^2 + \sum_{k=1}^{n} d_k^2(R_k - R_k^*)^2 + 2\sum_{k=1}^{n} d_k\sigma_{5k}^2(R_k^*)^2,
\end{aligned}
\tag{3.15}
$$

where the last equality is derived by the inequality $2ab \le a^2 + b^2$.

From (3.12)–(3.15) we obtain

$$
\begin{aligned}
&L(V_4 + V_5 + V_6 + V_7) \\
\le &\ 2\sum_{k=1}^{n} a_k D_k(S_k - S_k^*)^2 - 2\sum_{k=1}^{n}(d_k^E - 2\sigma_{2k}^2)(E_k - E_k^*)^2 \\
&-\sum_{k=1}^{n}\{a_k(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2) - b_k\frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k} - d_k\frac{\gamma_k^2}{d_k^R}\}(I_k - I_k^*)^2 \\
&-\sum_{k=1}^{n}\{b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2) - \mu_k^2\}(Q_k - Q_k^*)^2 \\
&-\sum_{k=1}^{n} d_k\{(d_k^R - 2\sigma_{5k}^2) - d_k\}(R_k - R_k^*)^2 + \sum_{k=1}^{n} C_k,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
C_k = &\ 2\sum_{k=1}^{n} a_k\{\sigma_{1k}^2(S_k^*)^2 + \sigma_{2k}^2(E_k^*)^2 + (1 + \frac{d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k}{\epsilon_k})\sigma_{3k}^2(I_k^*)^2)\} \\
&+2\sum_{k=1}^{n} b_k\sigma_{4k}^2(Q_k^*)^2 + 2\sum_{k=1}^{n} d_k\sigma_{5k}^2(R_k^*)^2.
\end{aligned}
$$

From condition (3.1), we can choose positive numbers r, a_k, b_k and d_k for $k = 1, 2, \cdots, n$ satisfying

$d_k < d_k^R - 2\sigma_{5k}^2$ and

$$r > \frac{2S_k^* D_k a_k}{(d_k^S - \sigma_{1k}^2)c_k},\ a_k > \frac{[b_k \frac{\delta_k^2}{d_k^Q+\alpha_k+\mu_k} + d_k \frac{\gamma_k^2}{d_k^R}]}{(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2)},\ b_k > \frac{\mu_k^2}{(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2)}$$

such that for each $1 \leq k \leq n$

$$a_k(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2) - [b_k \frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k} + d_k \frac{\gamma_k^2}{d_k^R}] > 0,\ d_k^R - 2\sigma_{5k}^2 - d_k > 0,$$
$$b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2) - \mu_k^2 > 0,\ c_k r - \frac{S_k^*}{d_k^S - \sigma_{1k}^2} 2a_k D_k > 0.$$

Lastly, define a Lyapunov function as follows

$$V = r(KV_1 + V_2 + V_3) + V_4 + V_5 + V_6 + V_7.$$

By It$\hat{o}$'s formula, we obtain

$$\begin{aligned} dV = \ & LVdt + \textstyle\sum_{k=1}^n \sigma_{1k}[c_k r(K + \frac{S_k}{S_k^*})(S_k - S_k^*) + 2a_k(S_k - S_k^* + E_k - E_k^* + I_k - I_k^*)S_k]dB_{1k} \\ & +2\sum_{k=1}^n \sigma_{2k}[c_k r K(E_k - E_k^*) + a_k(S_k - S_k^* + E_k - E_k^* + I_k - I_k^*)E_k]dB_{2k} \\ & + \sum_{k=1}^n \sigma_{3k}\{[r(K+1)c_k \frac{d_k^E + \epsilon_k}{\epsilon_k} + a_k \frac{d_k^E + d_k^I + \alpha_k + \delta_k + \gamma_k}{\epsilon_k} I_k](I_k - I_k^*) + 2a_k \\ & \times(S_k - S_k^* + I_k - I_k^* + E_k - E_k^*)I_k\}dB_{3k} + \sum_{k=1}^n \sigma_{4k} b_k(Q_k - Q_k^*)Q_k dB_{4k} \\ & + \sum_{k=1}^n \sigma_{5k} d_k(R_k - R_k^*)R_k dB_{5k}, \end{aligned} \tag{3.17}$$

where (3.11) together with (3.16) implies

$$\begin{aligned} LV \leq \ & -\sum_{k=1}^n [\{c_k r \frac{(d_k^S - \sigma_{1k}^2)}{S_k^*} - 2a_k D_k\}(S_k - S_k^*)^2 + 2(d_k^E - 2\sigma_{2k}^2)(E_k - E_k^*)^2 \\ & +\{a_k(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2) - b_k \frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k} - d_k \frac{\gamma_k^2}{d_k^R}\}(I_k - I_k^*)^2 \\ & +\{b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2) - \mu_k^2\}(Q_k - Q_k^*)^2 \\ & +\{(d_k^R - 2\sigma_{5k}^2) - d_k\}(R_k - R_k^*)^2] + \sum_{k=1}^n \rho_k. \end{aligned} \tag{3.18}$$

By integration and taking expectation of both sides of (3.17), we obtain

$$
\begin{aligned}
&E(V(t)) - E(V(0)) = E[\int_0^t LV(r)dr] \\
\leq\ & -E\int_0^t \sum_{k=1}^{n}[\{c_k r\frac{(d_k^S - \sigma_{1k}^2)}{S_k^*} - 2a_k D_k\}(S_k - S_k^*)^2 + 2(d_k^E - 2\sigma_{2k}^2)(E_k - E_k^*)^2 \\
&+\{a_k(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2) - b_k\frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k} - d_k\frac{\gamma_k^2}{d_k^R}\}(I_k - I_k^*)^2 \\
&+\{b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2) - \mu_k^2\}(Q_k - Q_k^*)^2 \\
&+\{(d_k^R - 2\sigma_{5k}^2) - d_k\}(R_k - R_k^*)^2]dr + t\sum_{k=1}^{n}\rho_k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \sum_{k=1}^{n}&[\{c_k r\frac{(d_k^S - \sigma_{1k}^2)}{S_k^*} - 2a_k D_k\}(S_k - S_k^*)^2 + 2(d_k^E - 2\sigma_{2k}^2)(E_k - E_k^*)^2 \\
&+\{a_k(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2) - b_k\frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k} - d_k\frac{\gamma_k^2}{d_k^R}\}(I_k - I_k^*)^2 \\
&+\{b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2) - \mu_k^2\}(Q_k - Q_k^*)^2 \\
&+\{(d_k^R - 2\sigma_{5k}^2) - d_k\}(R_k - R_k^*)^2]dr \leq \sum_{k=1}^{n}\rho_k.
\end{aligned}
$$

This completes the proof. □

As a consequence of Theorem 2, we have the following result on the existence and uniqueness of stationary distribution for model (1.2).

Theorem 3. *Assume that all conditions in Theorem 2 hold. Then model (1.2) has a unique stationary distribution $\mu(\cdot)$ in R_+^{5n}.*

Proof. Choose region Ω in ([34], Lemma 2.5) by $\Omega = R_+^{5n}$. Consider the following inequality

$$
\begin{aligned}
&\sum_{k=1}^{n}\{c_k\frac{(d_k^S - \sigma_{1k}^2)}{S_k^*} - 2a_k B_k\}(S_k - S_k^*)^2 + 2\sum_{k=1}^{n}(d_k^E - 2\sigma_{2k}^2)(E_k - E_k^*)^2 \\
&+\sum_{k=1}^{n}\{a_k(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2) - b_k\frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k} - d_k\frac{\gamma_k^2}{d_k^R}\}(I_k - I_k^*)^2 \\
&+\sum_{k=1}^{n}\{b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2) - \mu_k^2\}(Q_k - Q_k^*)^2 \\
&+\sum_{k=1}^{n}d_k\{(d_k^R - 2\sigma_{5k}^2) - d_k\}(R_k - R_k^*)^2 \leq H.
\end{aligned}
$$

Let region U_1 denote all points $(S_k, E_k, I_k, Q_k, R_k, 1 \leq k \leq n)$ which satisfy the above inequality with $H = 2\sum_{k=1}^{n}\rho_k$ and region U_2 denote all points $(S_k, E_k, I_k, Q_k, R_k, 1 \leq k \leq n)$ which satisfy the above inequality with $H = 3\sum_{k=1}^{n}\rho_k$. Obviously, U_2 is a neighborhood of U_1 and the closure $\bar{U}_2 \subset \Omega$. Then from (3.18), for any $x \in \Omega \setminus U_1$,

$$\begin{aligned} LV \leq\ & -\sum_{k=1}^{n}[\{c_k r\frac{(d_k^S - \sigma_{1k}^2)}{S_k^*} - 2a_k D_k\}(S_k - S_k^*)^2 + 2(d_k^E - 2\sigma_{2k}^2)(E_k - E_k^*)^2 \\ & +\{a_k(d_k^I + \alpha_k + \delta_k + \gamma_k - 2\sigma_{3k}^2) - b_k\frac{\delta_k^2}{d_k^Q + \alpha_k + \mu_k} - d_k\frac{\gamma_k^2}{d_k^R}\}(I_k - I_k^*)^2 \\ & +\{b_k(d_k^Q + \alpha_k + \mu_k - 2\sigma_{4k}^2) - \mu_k^2\}(Q_k - Q_k^*)^2 \\ & +\{(d_k^R - 2\sigma_{5k}^2) - d_k\}(R_k - R_k^*)^2] + \sum_{k=1}^{n}\rho_k \leq -\sum_{k=1}^{n}\rho_k, \end{aligned}$$

which implies condition (ii) in ([35], Lemma 2.5) is satisfied.

For model (1.2), the diffusion matrix is

$$A(x) = \mathrm{diag}(\sigma_{1k}^2 S_k^2, \sigma_{2k}^2 E_k^2, \sigma_{3k}^2 I_k^2, \sigma_{4k}^2 Q_k^2, \sigma_{5k}^2 R_k^2, 1 \leq k \leq n).$$

Choose a positive constant $M \geq \inf_{\bar{U}_2}\{\sigma_{1i}^2 S_i^2, \sigma_{2i}^2 E_i^2, \sigma_{3i}^2 I_i^2, \sigma_{4i}^2 Q_i^2, \sigma_{5i}^2 R_i^2, 1 \leq i \leq n\}$. Then,

$$\begin{aligned} \sum_{i,j=1}^{5n} a_{ij}\xi_i\xi_j =\ & \sum_{i=1}^{n}\sigma_{1i}^2 S_i^2\xi_{5i-4}^2 + \sum_{i=1}^{n}\sigma_{2i}^2 E_i^2\xi_{5i-3}^2 + \sum_{i=1}^{n}\sigma_{3i}^2 I_i^2\xi_{5i-2}^2 \\ & + \sum_{i=1}^{n}\sigma_{4i}^2 Q_i^2\xi_{5i-1}^2 + \sum_{i=1}^{n}\sigma_{5i}^2 R_i^2\xi_{5i}^2 \geq M\|\xi\|^2, \end{aligned}$$

for all $(S_i, E_i, I_i, Q_i, R_i,\ 1 \leq i \leq n) \in \bar{U}_2$ and $\xi \in R^{5n}$. This implies condition (i) in ([34], Lemma 2.5) is also satisfied. Therefore, by ([34], Lemma 2.5), model (1.2) has a unique stationary distribution μ in R_+^{5n}. This completes the proof. □

4. Numerical simulation

In this section, we analyse the stochastic behaviour of model (1.2) by means of the numerical simulations in order to make readers understand our results more better. The numerical simulation method can be found in [36]. The corresponding discretization system of

$$\begin{cases} S_{k,i+1} = & S_{k,i} + [\Lambda_k - \beta_{k1}S_{k,i}I_{1,i} - \beta_{k2}S_{k,i}I_{2,i} - d_k^S S_{k,i}]\Delta t \\ & +\sigma_{1k}S_{k,i}\sqrt{\Delta t}\varepsilon_{1k,i} + \frac{\sigma_{1k}^2 S_{k,i}}{2}(\varepsilon_{1k,i}^2\Delta t - \Delta t), \\ E_{k,i+1} = & E_{k,i} + [\beta_{k1}S_{k,i}I_{1,i} + \beta_{k2}S_{k,i}I_{2,i} - (d_k^E + \epsilon_k)E_{k,i}]\Delta t \\ & +\sigma_{2k}E_{k,i}\sqrt{\Delta t}\varepsilon_{2k,i} + \frac{\sigma_{2k}^2 E_{k,i}}{2}(\varepsilon_{2k,i}^2\Delta t - \Delta t), \\ I_{k,i+1} = & I_{k,i} + [\epsilon_k E_{k,i} - (d_k^I + \alpha_k + \delta_k + \gamma_k)I_{k,i}]\Delta t \\ & +\sigma_{3k}I_{k,i}\sqrt{\Delta t}\varepsilon_{3k,i} + \frac{\sigma_{3k}^2 I_{k,i}}{2}(\varepsilon_{3k,i}^2\Delta t - \Delta t), \\ Q_{k,i+1} = & Q_{k,i} + [\delta_k I_{k,i} - (d_k^Q + \alpha_k + \mu_k)Q_{k,i}]\Delta t + \sigma_{4k}Q_{k,i}\sqrt{\Delta t}\varepsilon_{4k,i} \\ & +\frac{\sigma_{4k}^2 Q_{k,i}}{2}(\varepsilon_{4k,i}^2\Delta t - \Delta t), \\ R_{k,i+1} = & R_{k,i} + [\gamma_k I_{k,i} + \mu_k Q_{k,i} - d_k^R R_{k,i}]\Delta t + \sigma_{5k}R_{k,i}\sqrt{\Delta t}\varepsilon_{5k,i} \\ & +\frac{\sigma_{5k}^2 R_{k,i}}{2}(\varepsilon_{5k,i}^2\Delta t - \Delta t), \end{cases}$$

where time increment $\Delta t > 0$, and $\varepsilon_{1k,i}$, $\varepsilon_{2k,i}$, $\varepsilon_{3k,i}$, $\varepsilon_{4k,i}$, $\varepsilon_{5k,i}$ for $1 \leq k \leq n$ are $N(0,1)$-distributed independent random variables which be generated numerically by pseudo-random number generators.

Example 1. In model (1.2), we choose $n = 2$ and the parameters $\Lambda_1 = 3.2$, $\epsilon_1 = 0.1$, $\alpha_1 = 0.1$, $\beta_{11} = 0.409$, $d_1^S = 0.9$, $d_1^E = 0.7$, $d_1^I = 0.81$, $d_1^Q = 0.2$, $d_1^R = 0.65$, $\mu_1 = 0.3$, $\gamma_1 = 0.04$, $\beta_{12} = 0.02$, $\delta_1 = 0.1$, $\sigma_{11} = 0.15$, $\sigma_{21} = 0.1$, $\sigma_{31} = 0.41$, $\sigma_{41} = 0.2$, $\sigma_{51} = 0.3$, $\Lambda_2 = 7.5$, $\epsilon_2 = 2.4$, $\alpha_2 = 0.2$, $\beta_{21} = 0.05$, $d_2^S = 0.49$, $d_2^E = 0.25$, $d_2^I = 0.15$, $d_2^Q = 0.25$, $d_2^R = 0.39$, $\mu_2 = 0.5$, $\gamma_2 = 0.15$, $\beta_{22} = 0.0014$, $\delta_2 = 0.43$, $\sigma_{12} = 0.2$, $\sigma_{22} = 0.6$, $\sigma_{32} = 0.5$, $\sigma_{42} = 0.8$ and $\sigma_{52} = 0.8$.

By computing, we have $R_0 \doteq 0.8675 < 1$ and disease-free equilibrium $E_0 = (3.56, 0, 0, 0, 0, 15.31, 0, 0, 0, 0)$ for corresponding deterministic model (1.1), and the conditions in Theorem 1 are satisfied. Therefore, according to the conclusion in Theorem 1 by numerical calculation we can obtain that for the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ satisfying the initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$ one has

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \sum_{k=1}^{2} \{A_k(S_k(r) - S_k^0)^2 + B_k E_k^2(r) + C_k I_k^2(r) + D_k Q_k^2(r) + F_k R_k^2(r)\} dt \leq 10.49, \tag{4.1}$$

where $S_1^0 = 3.56$, $S_2^0 = 15.31$, $A_1 = 0.8775$, $A_2 = 0.48$, $B_1 = 0.1988$, $B_2 = 0.6175$, $C_1 = 18.21$, $C_2 = 0.04295$, $D_1 = 0.1482$, $D_2 = 0.0077$, $F_1 = 0.1507$ and $F_2 = 3.7551 \times 10^5$.

From the numerical simulations given in Figure 1 we easily see that the above formula (4.1) holds. That is, the solution of stochastic model (1.2) asymptotically oscillates in probability around disease-free equilibrium E_0.

In addition, from Figure 1 we also easily see that the mean of susceptible $S_k(t)$ $(k = 1, 2)$ tend to S_k^0 and all exposed E_k, infectious I_k, quarantined Q_k and recovered R_k for $k = 1, 2$ tend to zero in probability as $t \to \infty$.

Example 2. In model (1.2), we choose $n = 2$ and the parameters $\Lambda_1 = 0.8$, $\epsilon_1 = 0.1$, $\alpha_1 = 0.1$, $\beta_{11} = 0.109$, $d_1^S = 0.19$, $d_1^E = 1.107$, $d_1^I = 0.081$, $d_1^Q = 0.2$, $d_1^R = 0.65$, $\mu_1 = 0.3$, $\gamma_1 = 0.04$, $\beta_{12} = 0.02$, $\delta_1 = 0.01$, $\sigma_{11} = 1.15$, $\sigma_{21} = 1.1$, $\sigma_{31} = 1.41$, $\sigma_{41} = 01.2$, $\sigma_{51} = 1.3$, $\Lambda_2 = 1.5$, $\epsilon_2 = 2.4$, $\alpha_2 = 0.2$, $\beta_{21} = 0.05$, $d_2^S = 0.49$, $d_2^E = 0.25$, $d_2^I = 0.15$, $d_2^Q = 0.25$, $d_2^R = 0.39$, $\mu_2 = 0.5$, $\gamma_2 = 0.15$, $\beta_{22} = 0.0014$, $\delta_2 = 0.043$, $\sigma_{12} = 1.2$, $\sigma_{22} = 1.6$, $\sigma_{32} = 0.5$, $\sigma_{42} = 0.8$ and $\sigma_{52} = 0.8$.

By computing, we have $R_0 \doteq 0.5174 \leq 1$. Since $d_1^S - \sigma_{11}^2 = -1.13 < 0$, $d_2^S - \sigma_{12}^2 = -0.33 < 0$, $d_1^R - \frac{1}{2}\sigma_{51}^2 = -0.2 < 0$ and $d_2^R - \frac{1}{2}\sigma_{52}^2 = -0.46 < 0$, the condition (2.1) in Theorem 1 does not hold. However, from the numerical simulations given in Figure 2, we can see that the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model (1.2) with initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$ asymptotically oscillates in probability around the disease-free equilibrium $E_0 = (4.21, 0, 0, 0, 0, 3.06, 0, 0, 0, 0)$ of corresponding deterministic model (1.1). This example seems to indicate that the condition (2.1) in Theorem 1 can be weakened or taken out.

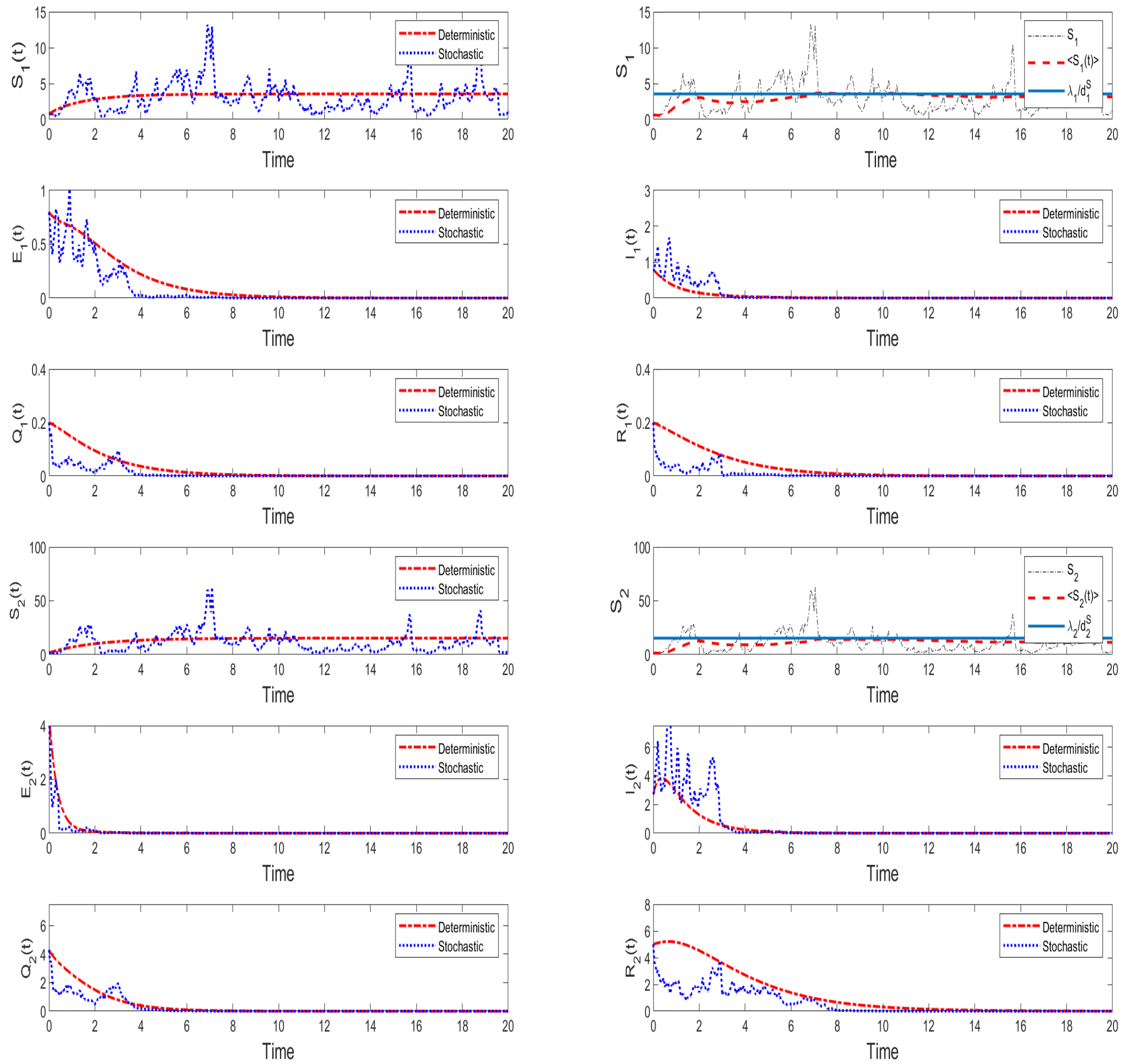

Figure 1. The numerical simulations of asymptotic oscillation in probability around disease-free equilibrium E_0 for the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model with initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$.

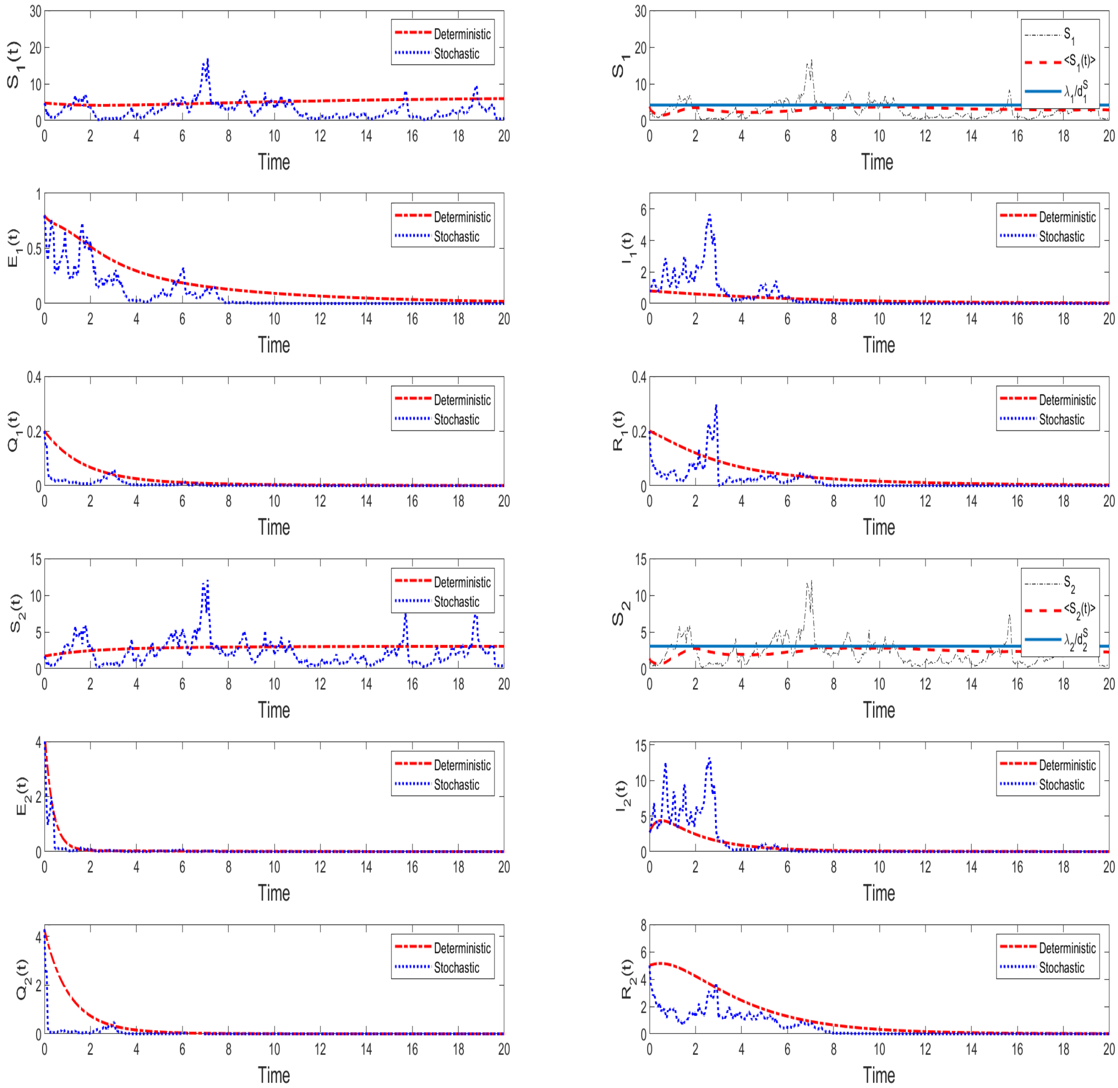

Figure 2. The numerical simulations of asymptotic oscillation in probability around disease-free equilibrium E_0 for the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model (1.2) with initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$.

Example 3. In model (1.2), we choose $n = 2$ and the parameters $\Lambda_1 = 4.5$, $\epsilon_1 = 1$, $\alpha_1 = 0.1$, $\beta_{11} = 1.55$, $d_1^S = 0.5$, $d_1^E = 0.15$, $d_1^I = 0.1$, $d_1^Q = 0.2$, $d_1^R = 0.65$, $\mu_1 = 0.3$, $\gamma_1 = 0.4$, $\beta_{12} = 1.35$, $\delta_1 = 0.6$, $\sigma_{11} = 0.3$, $\sigma_{21} = 0.5$, $\sigma_{31} = 0.4$, $\sigma_{41} = 0.2$, $\sigma_{51} = 0.4$, $\Lambda_2 = 7.5$, $\epsilon_2 = 2.4$, $\alpha_2 = 0.2$, $\beta_{21} = 1.5$, $d_2^S = 0.49$, $d_2^E = 0.25$, $d_2^I = 0.15$, $d_2^Q = 0.25$, $d_2^R = 0.39$, $\mu_2 = 0.5$, $\gamma_2 = 0.15$, $\beta_{22} = 1.24$, $\delta_2 = 0.43$, $\sigma_{12} = 0.2$, $\sigma_{22} = 0.6$, $\sigma_{32} = 0.5$, $\sigma_{42} = 0.8$ and $\sigma_{52} = 0.3$.

By computing, we have $R_0 \doteq 1.1032 > 1$ and the conditions in Theorem 2 are satisfied. The numerical simulations are given in Figures 3 and 4. Figure 3 shows that the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model (1.2) satisfying the initial values

$(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$ asymptotically oscillates in probability around the endemic equilibrium $E^* = (0.37, 3.35, 0.27, 0.38, 0.19, 0.79, 2.68, 6.93, 3.14, 6.68)$ of corresponding deterministic model (1.1). Figure 4 shows that the solution has a unique stationary distribution. Therefore, the conclusions of Theorem 3 are validated by the numerical example.

In addition, from Figure 3 we also easily see that the mean value of the solution for stochastic model (1.2) asymptotically oscillates in probability around the endemic equilibrium E^* of corresponding deterministic model (1.1). From Figure 5 we can find the relationship between variances of the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ and the intensities of noises $(\sigma_{ik}^2, \sigma_{2k}^2, \sigma_{3k}^2, \sigma_{4k}^2, \sigma_{5k}^2, k = 1, 2)$ as time t is enough large.

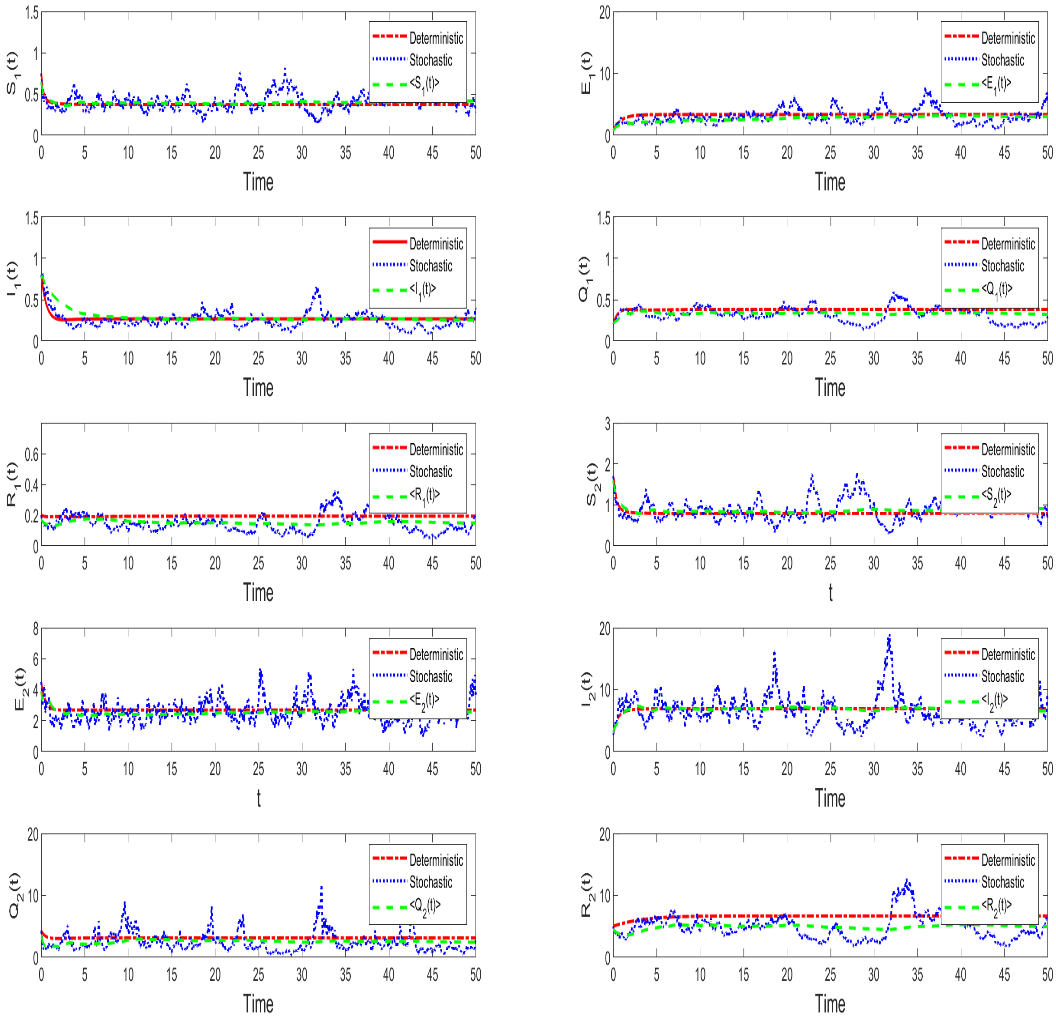

Figure 3. The numerical simulations of asymptotic oscillation in probability around endemic equilibrium E^* for the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model (1.2) with initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$.

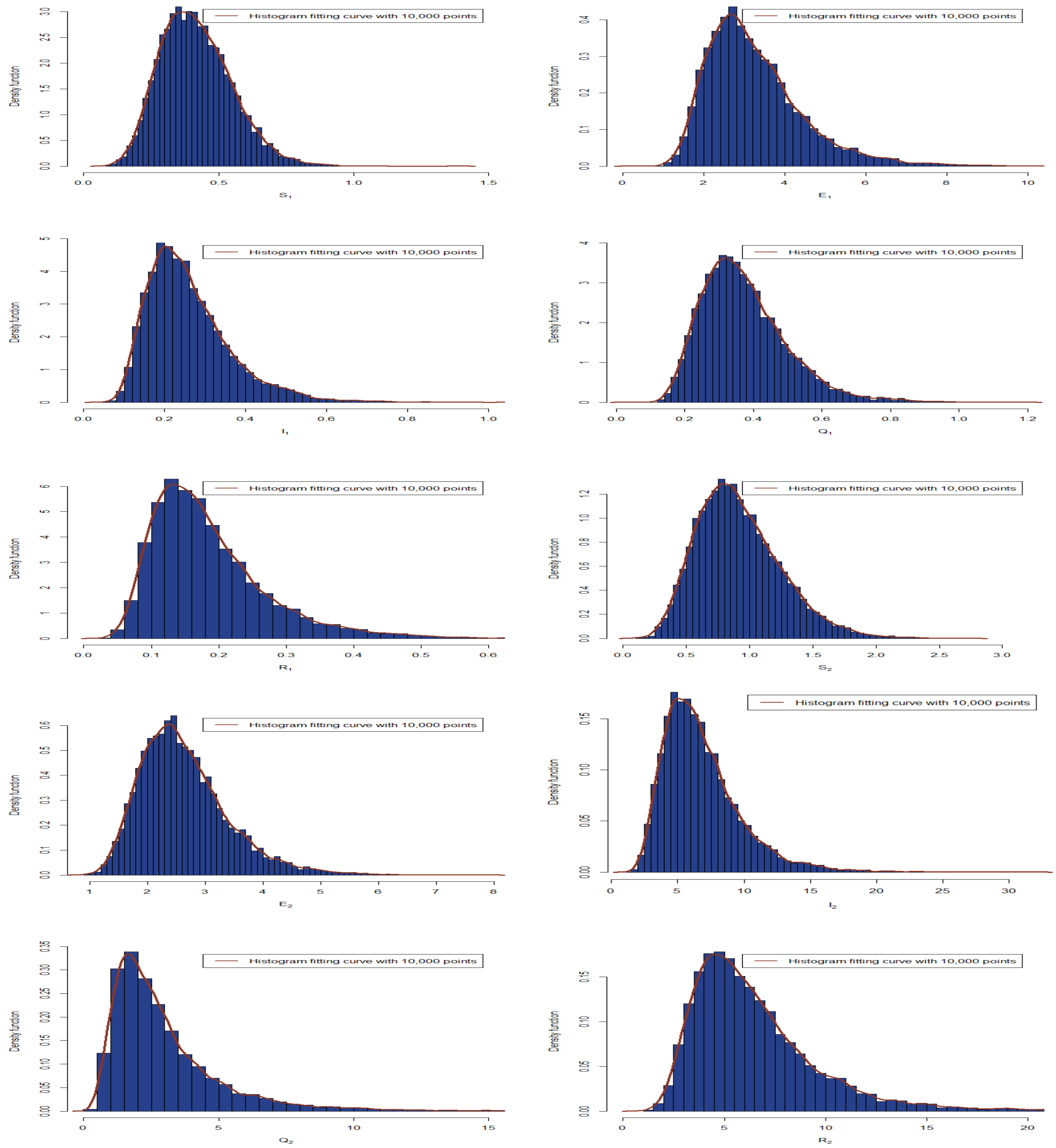

Figure 4. The stationary distribution of the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ for the stochastic model (1.2).

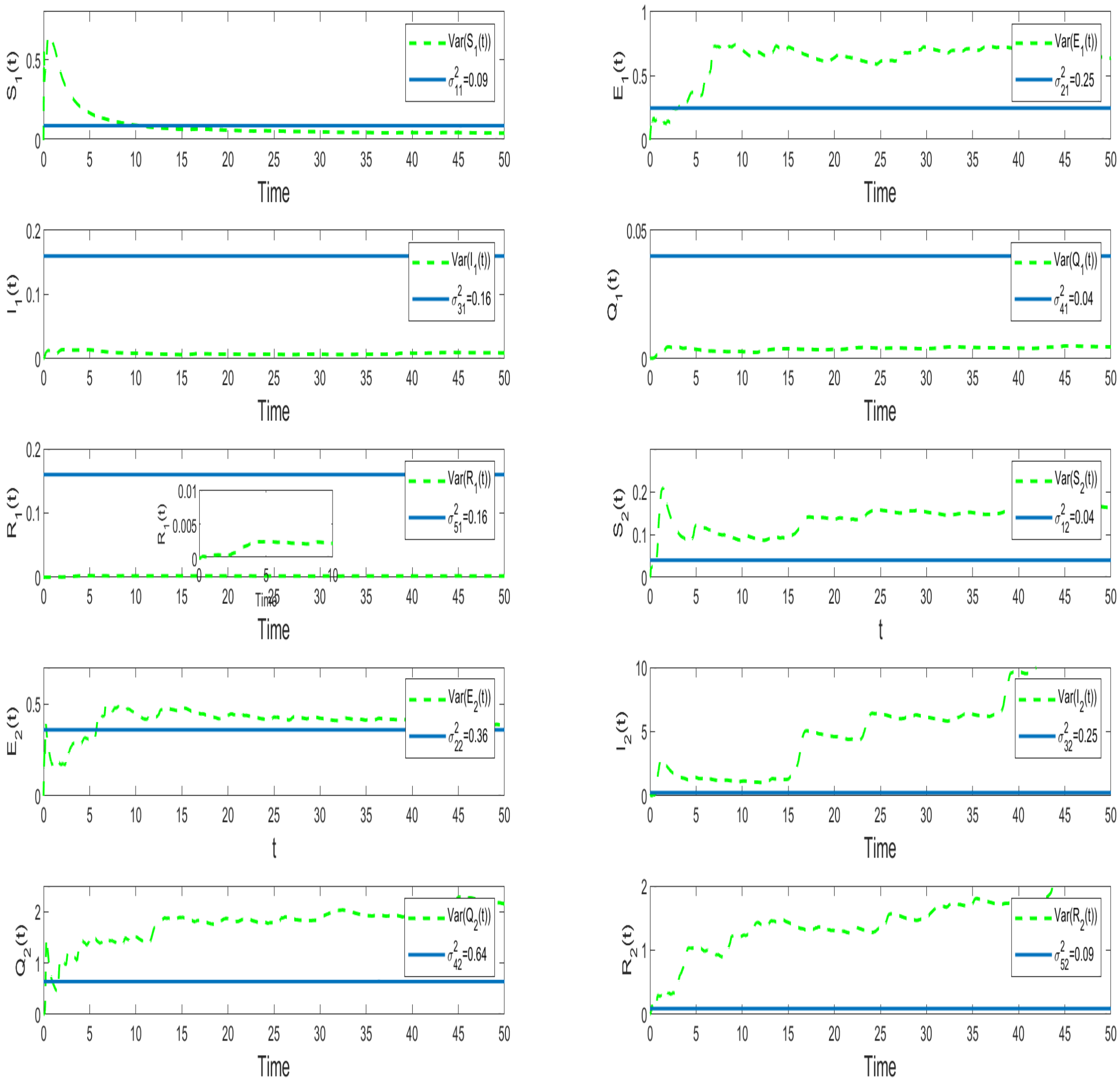

Figure 5. The numerical simulations of variances for the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model (1.2) with initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$.

Example 4. In model (1.2), we choose $n = 2$ and the parameters $\Lambda_1 = 4.5$, $\epsilon_1 = 0.1$, $\alpha_1 = 0.1$, $\beta_{11} = 1.55$, $d_1^S = 2.05$, $d_1^E = 1.015$, $d_1^I = 0.51$, $d_1^Q = 0.02$, $d_1^R = 0.65$, $\mu_1 = 0.3$, $\gamma_1 = 0.04$, $\beta_{12} = 1.35$, $\delta_1 = 0.6$, $\sigma_{11} = 2.3$, $\sigma_{21} = 1.5$, $\sigma_{31} = 0.5$, $\sigma_{41} = 0.4$, $\sigma_{51} = 0.4$, $\Lambda_2 = 7.5$, $\epsilon_2 = 2.4$, $\alpha_2 = 0.2$, $\beta_{21} = 1.5$, $d_2^S = 0.49$, $d_2^E = 0.25$, $d_2^I = 0.15$, $d_2^Q = 0.25$, $d_2^R = 0.39$, $\mu_2 = 0.5$, $\gamma_2 = 0.15$, $\beta_{22} = 0.24$, $\delta_2 = 0.43$, $\sigma_{12} = 1.2$, $\sigma_{22} = 0.6$, $\sigma_{32} = 0.5$, $\sigma_{42} = 0.8$ and $\sigma_{52} = 0.3$.

By computing, we have $R_0 \doteq 1.09013 > 1$. Since $d_1^S - \sigma_{11}^2 = -10.12 < 0$, $d_2^S - \sigma_{21}^2 = -0.43 < 0$ and $d_1^E - \frac{1}{2}\sigma_{21}^2 = -0.11 < 0$, the condition (3.1) in Theorem 2 does not hold. However, from the numerical simulations are given in Figures 6 we can see that the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model (1.2) with initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$ asymptotically oscillates in probability around the endemic equilibrium $E^* = (0.43, 3.24, 0.26, 0.37, 0.19, 3.34, 2.22, 5.7, 2.59, 5.51)$ of corresponding deterministic model (1.1). This example seems to indicate that the condition (3.1) in Theorem 2 can be weakened or taken out.

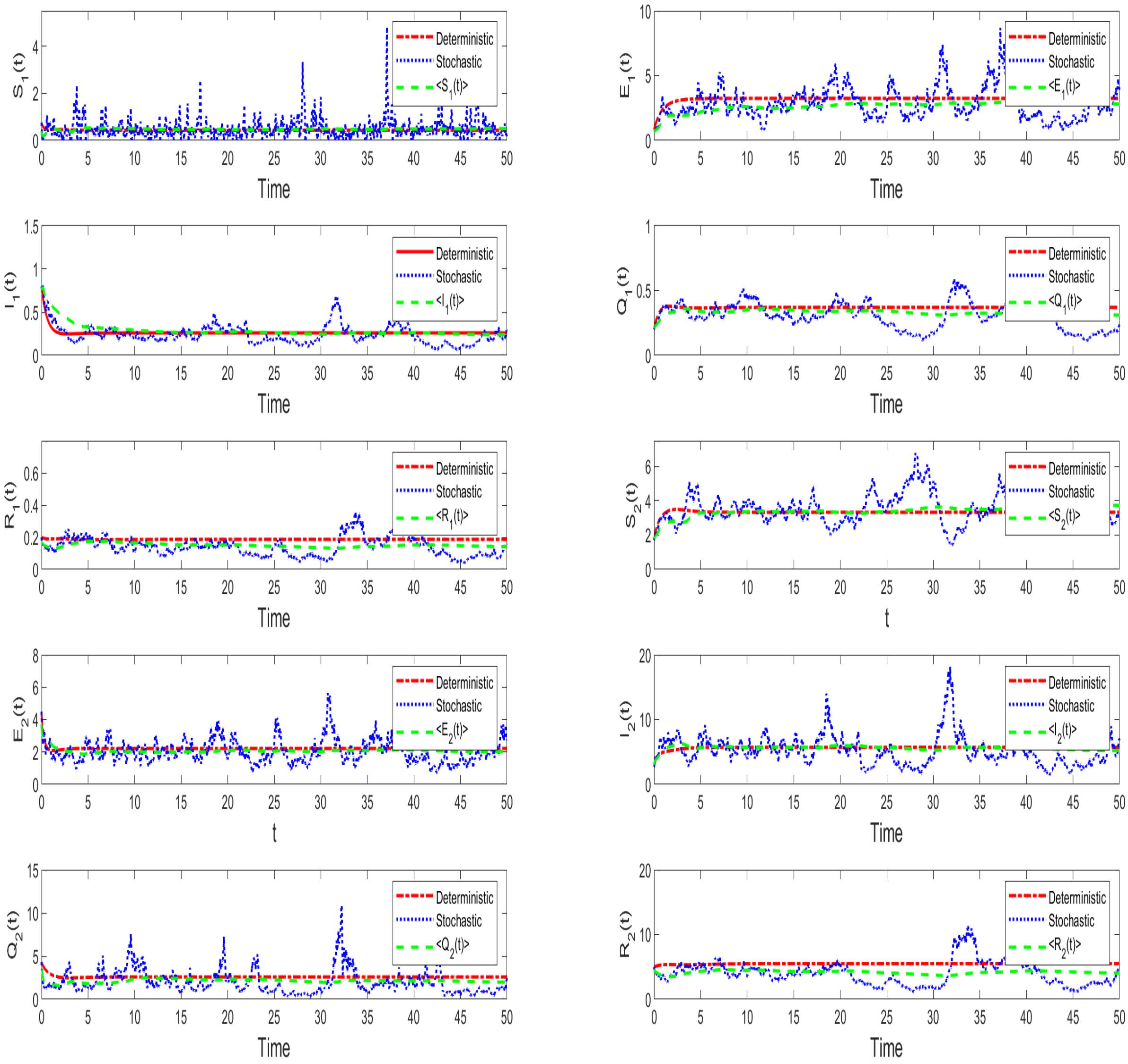

Figure 6. The numerical simulations of asymptotic oscillation in probability around endemic equilibrium E^* for the solution $(S_k(t), E_k(t), I_k(t), Q_k(t), R_k(t), k = 1, 2)$ of stochastic model (1.2) with initial values $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0)) = (0.75, 0.8, 0.8, 0.2, 0.2)$ and $(S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0)) = (1.7, 4.5, 2.7, 4.3, 5)$.

5. Conclusions

In this research we consider a class of stochastic multi-group SEIQR (susceptible, exposed, infectious, quarantined and recovered) models in computer network. For the deterministic system, if the reproduction number $R_0 > 1$, the system has unique endemic equilibrium which is globally stable, this means that the disease will persist at the endemic equilibrium level if it is initially present. It is clear that when the disease is endemic, the recovery nodes increases with the increasing quarantine nodes, and finally both reach the steady state values. Thus, it will be of great importance for one to run anti-malicious software to quarantine infected nodes. In order to study the asymptotic behavior of model (1.2), we first introduce the global existence of a positive solution. Then by using the theory of graphs, stochastic Lyapunov functions method, It$\hat{o}$'s formula and the theory of stochastic analysis, we carry out a detailed analysis on the asymptotic behavior of model (1.2). If $R_0 \leq 1$, the solution of model (1.2) oscillates around the disease-free equilibrium, while if $R_0 > 1$, the solution of model (1.2) fluctuates around the endemic equilibrium. The investigation of this stochastic model revealed that the stochastic stability of E^* depends on the magnitude of the intensity of noise as well as the parameters involved within the model system. finally, numerical methods are employed to illustrate the dynamic behavior of the model. The effect of quarantine on recovered nodes is also analyzed in the stochastic model.

Some interesting topics deserve further consideration. On the one hand, we can solve the corresponding probability density function of various stochastic epidemic models. On the other hand, we need to establish a more complete and systematic theory to obtain more accurate conditions and density function. The reader is referred to [37–45]. These problems are expected to be studied and solved as planned future work.

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Conflict of interest

The authors declare there is no conflicts of interest.

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