Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(7): 4088–4101.
DOI: 10.3934/era.2023208
Received: 03 March 2023
Revised: 06 May 2023
Accepted: 08 May 2023
Published: 26 May 2023

Research article

Finite-/fixed-time synchronization of leakage and discrete delayed Hopfield neural networks with diffusion effects

Minglei Fang, Jinzhi Liu and Wei Wang*

School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan 232001, China

* **Correspondence:** E-mail: wwangaust@126.com.

Abstract: In this paper, the problem on finite-/fixed-time synchronization (FFTS) is investigated for a class of diffusive Hopfield neural networks with leakage and discrete delays. Some new and useful criteria independent on time delays but dependent on the diffusion coefficients are established to guarantee the FFTS for the addressed network model under a unified framework. In sharp contrast to the existed results which can only finite-timely or fixed-timely synchronize the systems with both diffusion effects and leakage delays, the theoretical results of this paper are more general and practical. Finally, a numerical example is presented to show the effectiveness of the proposed control methods.

Keywords: Hopfield neural network; finite-/fixed-time synchronization; leakage and discrete delay; diffusion effect

1. Introduction

In the past four decades, Hopfield neural networks (HNNs), were firstly proposed by Hopfield [1,2], have drawn much attention and in-depth research since their potential applications in many fields such as object recognition [3], signal and image processing [4], mathematical programming [5], etc. It is widely recognized that these engineering applications rely heavily on the dynamic behaviors of neural networks. Therefore, the study of their dynamics became a hot research topic and significant progress has been made in recent years, see [6–8] and the references therein.

In the hardware implementation of neural networks, however, time delays are inevitable since the finite switching speed of amplifiers, and a neural network has much more complicated dynamics due to the incorporation of time delays. For example, it has been shown that oscillation and bifurcation phenomena will appear when a discrete delay was introduced into HNNs [9, 10], which are of great significance to practical applications of neural networks. Particularly, in a real nervous system, there perpetually encounters a representative time delay, which is essentially different from the conventional delays, named leakage delay, and it broadly exists in the negative feedback terms of the system which are identified as leakage terms, the leakage delay is usually incorporated in the study of network modeling, such a type of time delay often has a tendency to destabilize the neural networks and is difficult to handle [11]. Nevertheless, just as the traditional time delays, the leakage delay also has great influence on the dynamics of many different kinds of NNs, see, e.g., [12–16]. On the other hand, diffusion phenomenon inevitably appears in NNs and electric circuits once electrons transmit in a inhomogeneous electromagnetic field, which means that the whole structure and dynamic behavior of NNs are not only dependent on the time evolution but also on the spatial location [17, 18]. Therefore, it is of great importance and significance to investigate the dynamics of leakage and discrete delayed HNNs with diffusion effects.

Synchronization, as an important class of neurodynamics, has been extensively studied since its potential applications in such as secure communication [19], image processing [20] and so on. Among many control strategies in practical applications, finite-time and fixed-time control is an important and effective tool to realize synchronization of network systems, and it is shown that such a control technique has better robustness and disturbance rejection properties [22]. Because of these merits, the finite-/fixed-time synchronization problem of kinds of neural networks has been deeply studied in recent years (see for example, [23–30] and the references therein). It is notable, however, that few works have considered simultaneously the leakage delay, discrete delay and diffusion effects in FFTS dynamics of neural networks under consideration.

Motivated by the above discussions, we study the FFTS problem of leakage and discrete delayed HNNs with diffusion effects. Firstly, based on the finite-/fixed-time convergence theorem, a novel negative exponential state feedback controller is designed. Then, by applying the inequality technique, and Lyapunov-Krasovskii functional method, some new sufficient conditions are established to realize the FFTS for the considered network model. Finally, a numerical example is given to verify the theoretical results established in this paper.

The plan of the paper is organized as follows. In Section 2, the model description and some necessary preliminaries are presented. Section 3 is devoted to designing a negative exponential controller and the FFTS criteria are established. In Section 4, a numerical example is given to illustrate the effectiveness of the obtained results. Finally, a brief conclusion is drawn in Section 5.

2. Model description and preliminaries

In this paper, we are interested in the following diffusive Hopfield neural networks with leakage and discrete delays:

$$\begin{aligned}\frac{\partial u_i(t,x)}{\partial t} =& \sum_{k=1}^{m} D_{ik}\frac{\partial^2 u_i(t,x)}{\partial x_k^2} - a_i u_i(t-\sigma,x) + \sum_{j=1}^{n} b_{ij} f_j(u_j(t,x)) + \sum_{j=1}^{n} c_{ij} g_j(u_j(t-\tau,x)) \\ &+ I_i, \quad i,j = 1,2,...,n, \end{aligned} \tag{2.1}$$

where n is the number of neurons in the networks, $x = (x_1, x_2, ..., x_m)^T \in \Omega$ is the space vector, in which $\Omega \subseteq R^m$ represents a bounded set with smooth boundary $\partial\Omega$, and mes $\Omega > 0$ means the measure of Ω; $u_i(t,x)$ denotes the state of ith neuron at time t and in space x, $D_{ik} \geq 0$ represents the transmission diffusion coefficient along the ith neuron; σ and τ, respectively, stand for the leakage delay and transmission delay; $a_i > 0$ indicates the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; b_{ij} is the strength of the jth neuron on the ith neuron at time t and in space x; c_{ij} is the strength of the jth neuron on the ith neuron at time $t-\tau$ and in space x; $f_j(\cdot)$ and $g_j(\cdot)$, respectively, denotes the activation functions of the jth neuron without and with the time delay, and I_i denotes a constant external input.

As for system (2.1), the initial-boundary value conditions are supplemented with

$$u_i(s,x) = \varphi_i(s,x),\ (s,x) \in [-\max\{\sigma,\tau\},0] \times \Omega,$$

and

$$u_i(t,x) = 0,\ (t,x) \in [-\max\{\sigma,\tau\},\infty) \times \partial\Omega, \tag{2.2}$$

in which $\varphi_i \in C([-\max\{\sigma,\tau\},0]\times\Omega,\mathbb{R})$ representing the Banach space of continuous functions defined on the region $[-\max\{\sigma,\tau\},0] \times \Omega,\ i = 1,2,...,n.$

To achieve the FFTS goal, we refer to system (2.1) as the drive system, the response system is given as follows:

$$\begin{aligned}\frac{\partial v_i(t,x)}{\partial t} =& \sum_{k=1}^{m} D_{ik}\frac{\partial^2 v_i(t,x)}{\partial x_k^2} - a_i v_i(t-\sigma,x) + \sum_{j=1}^{n} b_{ij} f_j(v_j(t,x)) + \sum_{j=1}^{n} c_{ij} g_j(v_j(t-\tau,x)) \\ &+ I_i + U_i(t,x),\end{aligned} \tag{2.3}$$

where $v_i(t,x)$ represents the response state, $U_i(t,x)$ means the control input. Associated with (2.3), the initial-boundary value conditions are shown as follows:

$$v_i(s,x) = \phi_i(s,x),\ (s,x) \in [-\max\{\sigma,\tau\},0] \times \Omega,$$

and

$$v_i(t,x) = 0,\ (t,x) \in [-\max\{\sigma,\tau\},\infty) \times \partial\Omega, \tag{2.4}$$

in which $\phi_i \in C([-\max\{\sigma,\tau\},0] \times \Omega,\mathbb{R}),\ i = 1,2,...,n.$

Denote $e_i(t,x) = v_i(t,x) - u_i(t,x)$, and then subtract (2.1) from (2.3), we deduce that the error system can be written as follows:

$$\begin{aligned}\frac{\partial e_i(t,x)}{\partial t} =& \sum_{k=1}^{m} D_{ik}\frac{\partial^2 e_i(t,x)}{\partial x_k^2} - a_i e_i(t-\sigma,x) + \sum_{j=1}^{n} b_{ij}(f_j(v_j(t,x)) - f_j(u_j(t,x))) \\ &+ \sum_{j=1}^{n} c_{ij}(g_j(v_j(t-\tau,x)) - g_j(u_j(t-\tau,x))) + U_i(t,x).\end{aligned} \tag{2.5}$$

In view of (2.2) and (2.4), the initial-boundary value conditions of system (2.5) are equipped as follows:

$$e_i(s,x) = \phi_i(s,x) - \varphi_i(s,x),\ (s,x) \in [-\max\{\sigma,\tau\},0] \times \Omega.$$

and

$$e_i(t,x) = 0,\ (t,x) \in [-\max\{\sigma,\tau\},\infty) \times \partial\Omega, \tag{2.6}$$

We are now in a position to give some preliminary preparations.

Definition 2.1. *If for a suitable designed controller and any initial state $e_i(s,x) = \phi_i(s,x) - \varphi_i(s,x)$, $s \in [-\max\{\sigma,\tau\}, 0]$, there is a time $\mathbb{T}$ that depends (or does not depend) on the initial values such that*

$$\lim_{t\to\mathbb{T}} |e_i(t,x)| = 0,$$

and

$$|e_i(t,x)| \equiv 0, \text{ for } t \geq \mathbb{T}, \quad i = 1,2,...,n.$$

Then the drive system (2.1) *and response system* (2.3) *are said to achieve finite-time (or fixed-time) synchronization.*

Lemma 1 (see [31]). *Suppose that $V(x) : \mathbb{R}^n \to \mathbb{R}$ is a C-regular, positive definite and radially unbounded function and $x(\cdot) : [0,\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact subinterval of $[0,\infty)$. Let $\mathscr{K}(V) = k_1 V^\alpha + k_2 V^\beta$ satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) \leq -\mathscr{K}(V)$$

and

$$\mathbb{T} = \int_0^{V(0)} \frac{1}{\mathscr{K}(\sigma)} d\sigma < \infty.$$

Then,

1) if $0 \leq \alpha, \beta < 1$, $V(t)$ will reach zero in a finite time and the settling time is bounded by $\mathbb{T} \leq \min\left\{\frac{V^{1-\alpha}(0)}{k_1(1-\alpha)}, \frac{V^{1-\beta}(0)}{k_2(1-\beta)}\right\}$;
2) if $\alpha > 1, 0 \leq \beta < 1$, $V(t)$ will reach zero in a fixed time and the settling time is bounded by $\mathbb{T} \leq \frac{1}{k_1}\frac{1}{\alpha-1} + \frac{1}{k_2}\frac{1}{1-\beta}$.

Lemma 2 (Poincaré's inequality, see [32]). *Suppose that $u(x) \in H_0^1(\Omega)$, then we have the following inequality*

$$\int_\Omega u^2(x)dx \leq \eta \int_\Omega \sum_{k=1}^{m} (\frac{\partial u}{\partial x_k})^2 dx,$$

where η is a constant independent of u, and $H_0^1(\Omega)$ is a Sobolev space expressed by

$$H_0^1(\Omega) = \Big\{u | u \in L^2, \frac{\partial u}{\partial x_k} \in L^2, u|_{\partial\Omega} = 0, 1 \leq k \leq m\Big\}.$$

Remark 2.1. *As pointed out by Chen et al. in [33], the constant η can be calculated as follows: $\eta = d^2$, where*

$$d = \min_{1\leq k\leq m} \Big\{d_k = \sup\{|x_k - y_k|\}, \text{ for any } (x_1,...,x_m), (y_1,...,y_m) \in \Omega\Big\}.$$

Lemma 3 (see [34]). *If $a_1, a_2, ..., a_n$ are positive numbers and $0 < r < p$, then*

$$\Big(\sum_{i=1}^n a_i^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{i=1}^n a_i^r\Big)^{\frac{1}{r}} \leq n^{\frac{1}{r}-\frac{1}{p}} \Big(\sum_{i=1}^n a_i^p\Big)^{\frac{1}{p}}.$$

To realize the FFTS, the following assumption is given.

Assumption 1. *For any $u, v \in \mathbb{R}$, there exist positive constants p_j and q_j, such that*

$$|f_j(v) - f_j(u)| \leq p_j |v - u|, \quad |g_j(v) - g_j(u)| \leq q_j |v - u|, \ j = 1,2,...,n.$$

3. Main results

In this section, the FFTS problem of drive-response systems (2.1)–(2.3) is discussed under a unified framework and the corresponding sufficient criteria are established.

Firstly, we design the controller by

$$U_i(t,x)=\begin{cases}-\lambda_{1i}e_i(t,x)-\dfrac{e_i(t,x)}{\|e(t,\cdot)\|_2^2}\displaystyle\sum_{i=1}^{n}2[E_i^{\alpha}(t)+E_i^{\frac{1}{2}}(t)]-\lambda_{2i}e_i(t,x)[\|e(t,\cdot)\|_2^{2\alpha-2}\\ \qquad +\|e(t,\cdot)\|_2^{-1}],\ \text{if } \|e(t,\cdot)\|\neq 0,\\ 0,\quad \text{if } \|e(t,\cdot)\|=0,\end{cases}\tag{3.1}$$

where

$$E_i(t)=\int_\Omega \max_{t-\max\{\sigma,\tau\}\le s\le t} e_i^2(s,x)\mathrm{d}x,\quad \|e(t,\cdot)\|_2=\Big(\int_\Omega\sum_{i=1}^{n}|e_i(t,x)|^2\mathrm{d}x\Big)^{\frac{1}{2}},$$

and λ_{1i}, $\lambda_{2i}>0$ are feedback control parameters to be determined, $i=1,2,...,n$.

Our main result can be stated as follows.

Theorem 3.1. *Suppose that Assumption 1 holds. If the control parameters in* (3.1) *satisfy*

$$\lambda_{1i}\ge -\frac{\underline{D}_i}{\eta}+a_i+\frac{1}{2}\sum_{j=1}^{n}(|b_{ij}|p_j+|b_{ji}|p_i+|c_{ij}|q_j+|c_{ji}|q_i),\tag{3.2}$$

where

$$\underline{D}_i=\min_{1\le k\le m}\{D_{ik}\}.$$

Then, (I) *the drive-response systems* (2.1) *and* (2.3) *can achieve finite-time synchronization when* $0<\alpha<1$*;* (II) *the drive-response systems* (2.1) *and* (2.3) *can achieve fixed-time synchronization when* $\alpha>1$.

Proof. Define a Lyapunov-Krasovskii functional as follows:

$$V(t)=V_1(t)+V_2(t)+V_3(t)\tag{3.3}$$

with

$$V_1(t)=\sum_{i=1}^{n}\int_\Omega e_i^2(t,x)\mathrm{d}x,\ V_2(t)=\sum_{i=1}^{n}\int_\Omega\int_{t-\sigma}^{t}a_ie_i^2(s,x)\mathrm{d}s\mathrm{d}x,$$

and

$$V_3(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega\int_{t-\tau}^{t}|c_{ji}|q_ie_i^2(s,x)\mathrm{d}s\mathrm{d}x.$$

We differentiate $V(t)$ along solutions of system (2.5), and obtain from Assumption 1 that

$$\frac{\mathrm{d}}{\mathrm{dt}}V(t)=\int_\Omega\Big[2\sum_{i=1}^{n}e_i(t,x)\frac{\partial e_i(t,x)}{\partial t}+\sum_{i=1}^{n}a_i(e_i^2(t,x)-e_i^2(t-\sigma,x))$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}|q_i(e_i^2(t,x)-e_i^2(t-\tau,x))\Big]\mathrm{d}x\\
=&\int_\Omega\Big\{2\sum_{i=1}^{n}e_i(t,x)\Big[\sum_{k=1}^{m}D_{ik}\frac{\partial^2 e_i(t,x)}{\partial x_k^2}-a_ie_i(t-\sigma,x)+\sum_{j=1}^{n}b_{ij}\big(f_j(v_j(t,x))-f_j(u_j(t,x))\big)\\
&+\sum_{j=1}^{n}c_{ij}\big(g_j(v_j(t-\tau,x))-g_j(u_j(t-\tau,x))\big)+U_i\Big]+\sum_{i=1}^{n}a_i(e_i^2(t,x)-e_i^2(t-\sigma,x))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}|q_i(e_i^2(t,x)-e_i^2(t-\tau,x))\Big\}\mathrm{d}x\\
\le&\int_\Omega 2\sum_{i=1}^{n}e_i(t,x)\sum_{k=1}^{m}D_{ik}\frac{\partial^2 e_i(t,x)}{\partial x_k^2}\mathrm{d}x+\int_\Omega\Big[2\sum_{i=1}^{n}a_i|e_i(t,x)||e_i(t-\sigma,x)|\\
&+2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|p_j|e_i(t,x)||e_j(t,x)|+2\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|q_j|e_i(t,x)||e_j(t-\tau,x)|\\
&-2\sum_{i=1}^{n}\lambda_{1i}e_i^2(t,x)+\sum_{i=1}^{n}a_ie_i^2(t,x)+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}|q_ie_i^2(t,x)-\sum_{i=1}^{n}a_ie_i^2(t-\sigma,x)\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}|q_ie_i^2(t-\tau,x)\Big]\mathrm{d}x-4\sum_{i=1}^{n}[E_i^{\alpha}(t)+E_i^{\frac{1}{2}}(t)]\\
&-2\min_{1\le i\le n}\{\lambda_{2i}\}(\|e(t,\cdot)\|_2^{2\alpha}+\|e(t,\cdot)\|_2),
\end{aligned}\tag{3.4}
$$

Recalling from Lemma 2, we have

$$
\begin{aligned}
&\int_\Omega 2\sum_{i=1}^{n}e_i(t,x)\sum_{k=1}^{m}D_{ik}\frac{\partial^2 e_i(t,x)}{\partial x_k^2}\mathrm{d}x\\
=&-\int_\Omega 2\sum_{i=1}^{n}\sum_{k=1}^{m}D_{ik}\Big(\frac{\partial e_i(t,x)}{\partial x_k}\Big)^2\mathrm{d}x\\
\le&-2\sum_{i=1}^{n}\frac{D_i}{\eta}\int_\Omega e_i^2(t,x)\mathrm{d}x.
\end{aligned}\tag{3.5}
$$

On the other hand, one can easily deduce that

$$
2\sum_{i=1}^{n}a_i|e_i(t,x)||e_i(t-\sigma,x)|\le\sum_{i=1}^{n}a_i(e_i^2(t,x)+e_i^2(t-\sigma,x)),\tag{3.6}
$$

$$
\begin{aligned}
2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|p_j|e_i(t,x)||e_j(t,x)|&\le\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|p_j(e_i^2(t,x)+e_j^2(t,x))\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}(|b_{ij}|p_j+|b_{ji}|p_i)e_i^2(t,x)
\end{aligned}\tag{3.7}
$$

and

$$2\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|q_j|e_i(t,x)||e_j(t-\tau,x)| \le \sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|q_j(e_i^2(t,x)+e_j^2(t-\tau,x))$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|q_je_i^2(t,x)+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}|q_ie_i^2(t-\tau,x). \tag{3.8}$$

Therefore, substituting (3.5)–(3.8) into (3.4), we deduce from (3.2) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}V(t) \le& \int_\Omega\sum_{i=1}^{n}\Big(-\frac{2\underline{D}_i}{\eta}+2a_i+\sum_{j=1}^{n}(|b_{ij}|p_j+|b_{ji}|p_i+|c_{ij}|q_j+|c_{ji}|q_i)-2\lambda_{1i}\Big)e_i^2(t,x)\mathrm{d}x\\ &-4\sum_{i=1}^{n}[E_i^\alpha(t)+E_i^{\frac{1}{2}}(t)]-2\min_{1\le i\le n}\{\lambda_{2i}\}(\|e(t,\cdot)\|_2^{2\alpha}+\|e(t,\cdot)\|_2)\\ \le&-4\sum_{i=1}^{n}[E_i^\alpha(t)+E_i^{\frac{1}{2}}(t)]-2\min_{1\le i\le n}\{\lambda_{2i}\}(V_1^\alpha(t)+V_1^{\frac{1}{2}}(t)).\end{aligned} \tag{3.9}$$

Case I: When $0<\alpha<1$, with the definition of $E_i(t)$ and Lemma 3, we can see that

$$\begin{aligned}&-4\sum_{i=1}^{n}[E_i^\alpha(t)+E_i^{\frac{1}{2}}(t)]\\ \le&-2\sum_{i=1}^{n}\Big[\Big(\frac{1}{\sigma}\int_\Omega\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^\alpha+\Big(\frac{1}{\tau}\int_\Omega\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^\alpha\\ &+\Big(\frac{1}{\sigma}\int_\Omega\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}+\Big(\frac{1}{\tau}\int_\Omega\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}\Big]\\ \le&-\frac{2}{\sigma^\alpha}\Big(\sum_{i=1}^{n}\int_\Omega\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^\alpha-\frac{2}{\sqrt{\sigma}}\Big(\sum_{i=1}^{n}\int_\Omega\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}\\ &-\frac{2}{\tau^\alpha}\Big(\sum_{i=1}^{n}\int_\Omega\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^\alpha-\frac{2}{\sqrt{\tau}}\Big(\sum_{i=1}^{n}\int_\Omega\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}\\ \le&-\frac{2}{(\gamma_1\sigma)^\alpha}V_2^\alpha(t)-\frac{2}{\sqrt{\gamma_1\sigma}}V_2^{\frac{1}{2}}(t)-\frac{2}{(\gamma_2\tau)^\alpha}V_3^\alpha(t)-\frac{2}{\sqrt{\gamma_2\tau}}V_3^{\frac{1}{2}}(t),\end{aligned} \tag{3.10}$$

where

$$\gamma_1=\frac{1}{\max\limits_{1\le i\le n}\{a_i\}},\quad \gamma_2=\frac{1}{\max\limits_{1\le i\le n}\{\sum\limits_{1\le j\le n}|c_{ji}|q_i\}}.$$

Putting (3.10) into (3.9), it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}V(t)\le&-\frac{2}{(\gamma_1\sigma)^\alpha}V_2^\alpha(t)-\frac{2}{\sqrt{\gamma_1\sigma}}V_2^{\frac{1}{2}}(t)-\frac{2}{(\gamma_2\tau)^\alpha}V_3^\alpha(t)-\frac{2}{\sqrt{\gamma_2\tau}}V_3^{\frac{1}{2}}(t)\\ &-2\min_{1\le i\le n}\{\lambda_{2i}\}(V_1^\alpha(t)+V_1^{\frac{1}{2}}(t))\end{aligned}$$

$$\leq -2\min\{\min_{1\leq i\leq n}\{\lambda_{2i}\},(\gamma_1\sigma)^{-\alpha},(\gamma_1\sigma)^{-\frac{1}{2}},(\gamma_2\tau)^{-\alpha},(\gamma_2\tau)^{-\frac{1}{2}}\}(V^{\alpha}(t)+V^{\frac{1}{2}}(t)). \tag{3.11}$$

Based on Case I in Lemma 1, we arrive at the conclusion that systems (2.1)–(2.3) can achieve the finite-time synchronization and the settling time is exactly estimated by

$$\begin{aligned}\mathbb{T}_1 &= \int_0^{V(0)} \frac{1}{2\min\{\min_{1\leq i\leq n}\{\lambda_{2i}\},(\gamma_1\sigma)^{-\alpha},(\gamma_1\sigma)^{-\frac{1}{2}},(\gamma_2\tau)^{-\alpha},(\gamma_2\tau)^{-\frac{1}{2}}\}(V^{\alpha}+V^{\frac{1}{2}})}dV \\ &\leq \min\left\{V^{\frac{1}{2}}(0),\frac{V^{1-\alpha}(0)}{2-2\alpha}\right\}\frac{1}{\min\{\min_{1\leq i\leq n}\{\lambda_{2i}\},(\gamma_1\sigma)^{-\alpha},(\gamma_1\sigma)^{-\frac{1}{2}},(\gamma_2\tau)^{-\alpha},(\gamma_2\tau)^{-\frac{1}{2}}\}}.\end{aligned} \tag{3.12}$$

Case II: When $\alpha > 1$, a similar procedure as applied to derive (3.10) gives

$$\begin{aligned}&-4\sum_{i=1}^{n}[E_i^{\alpha}(t)+E_i^{\frac{1}{2}}(t)] \\ \leq& -2\sum_{i=1}^{n}\Big[\Big(\frac{1}{\sigma}\int_{\Omega}\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\alpha}+\Big(\frac{1}{\tau}\int_{\Omega}\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\alpha} \\ &+\Big(\frac{1}{\sigma}\int_{\Omega}\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}+\Big(\frac{1}{\tau}\int_{\Omega}\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}\Big] \\ \leq& -\frac{2n^{1-\alpha}}{\sigma^{\alpha}}\Big(\sum_{i=1}^{n}\int_{\Omega}\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\alpha}-\frac{2}{\sqrt{\sigma}}\Big(\sum_{i=1}^{n}\int_{\Omega}\int_{t-\sigma}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}} \\ &-\frac{2n^{1-\alpha}}{\tau^{\alpha}}\Big(\sum_{i=1}^{n}\int_{\Omega}\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\alpha}-\frac{2}{\sqrt{\tau}}\Big(\sum_{i=1}^{n}\int_{\Omega}\int_{t-\tau}^{t}e_i^2(s,x)\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}} \\ \leq& -\frac{2n^{1-\alpha}}{(\gamma_1\sigma)^{\alpha}}V_2^{\alpha}(t)-\frac{2}{\sqrt{\gamma_1\sigma}}V_2^{\frac{1}{2}}(t)-\frac{2n^{1-\alpha}}{(\gamma_2\tau)^{\alpha}}V_3^{\alpha}(t)-\frac{2}{\sqrt{\gamma_2\tau}}V_3^{\frac{1}{2}}(t).\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.9), we find from Lemma 3 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}V(t) \leq& -4\sum_{i=1}^{n}[E_i^{\alpha}(t)+E_i^{\frac{1}{2}}(t)]-2\min_{1\leq i\leq n}\{\lambda_{2i}\}(V_1^{\alpha}(t)+V_1^{\frac{1}{2}}(t)) \\ \leq& -2\min\left\{3^{1-\alpha}\min_{1\leq i\leq n}\{\lambda_{2i}\},\frac{3n}{(3n\gamma_1\sigma)^{\alpha}},(\gamma_1\sigma)^{-\frac{1}{2}},\frac{3n}{(3n\gamma_2\tau)^{\alpha}},(\gamma_2\tau)^{-\frac{1}{2}}\right\}(V^{\alpha}(t)+V^{\frac{1}{2}}(t)).\end{aligned}$$

According to Case II in Lemma 1, it follows that the response system (2.3) can fixed-timely synchronize to the drive system (2.1). Moreover, the settling time can be precisely bounded by

$$\begin{aligned}\mathbb{T}_2 &= \int_0^{V(0)}\frac{1}{2\min\left\{3^{1-\alpha}\min_{1\leq i\leq n}\{\lambda_{2i}\},\frac{3n}{(3n\gamma_1\sigma)^{\alpha}},(\gamma_1\sigma)^{-\frac{1}{2}},\frac{3n}{(3n\gamma_2\tau)^{\alpha}},(\gamma_2\tau)^{-\frac{1}{2}}\right\}(V^{\alpha}+V^{\frac{1}{2}})}dV \\ &\leq \frac{2\alpha-1}{2(\alpha-1)\min\left\{3^{1-\alpha}\min_{1\leq i\leq n}\{\lambda_{2i}\},\frac{3n}{(3n\gamma_1\sigma)^{\alpha}},(\gamma_1\sigma)^{-\frac{1}{2}},\frac{3n}{(3n\gamma_2\tau)^{\alpha}},(\gamma_2\tau)^{-\frac{1}{2}}\right\}}.\end{aligned} \tag{3.14}$$

This ends the proof of Theorem 3.1.

Remark 3.1. It is readily seen from (3.12) and (3.14) that the settling time is closely related not only to the controller parameters but also to the leakage and discrete delays, the finite or fixed time synchronization goal can be realized by adjusting the related parameters flexibly. One the other side, the synchronization criteria established in this paper are independent on the leakage and discrete delays, which are easier to check than the delay dependent ones. Therefore, the obtained theoretical results show that the control parameters and time delays including both leakage and discrete delys affect the synchronization mechanism from different aspects.

Remark 3.2. In [23, 25, 27, 28], the authors considered other types of neural networks with diffusion effects, some novel criteria have been established to realize the FFTS synchronization, but the leakage delay is not considered. In [27], by designing new negative exponential controllers, the finite-/fixed-time anti-synchronization problem of discontinuous neural networks with leakage delays is well studied. Unfortunately, the diffusion effects are still not considered. As we all know that network system with a diffusion term can be described by a partial differential equations which is a infinite dimensional dynamical system, which undoubtedly increases the complexity of the model. On the other hand, the leakage delay in negative feedback term also enhances the nonlinearity of the model itself, these undoubtedly bring many theoretical difficulties. To the best of our knowledge, there is no research on FFTS of neural networks considering both diffusion and delay effects until now. Thus, the obtained theoretical results are completely new and can be seen as a continuation work of the aforementioned references.

4. A numerical example

In this section, we provide an example and its numerical simulations to support the theoretical result.

Example 4.1. For $n = 2,\ m = 1$, the parameters of the drive-response systems (2.1)–(2.3) are chosen as follows: $A = diag(a_1, a_2) = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}, B = (b_{ij})_{2\times 2} = \begin{pmatrix} 3.12 & -2.34 \\ -4.2 & -1.68 \end{pmatrix}, C = (c_{ij})_{2\times 2} = \begin{pmatrix} -2.1 & 1.4 \\ -3.2 & 2.6 \end{pmatrix}, D_1 = 0.15,\quad D_2 = 0.35$, and $I_1 = 1,\ I_2 = -1,\ \sigma = 0.5,\ \tau = 0.2$.

To realize the FFTS between the drive-response systems (2.1)–(2.3), we design the following feedback controllers:

$$\begin{cases} U_1(t,x) = & -\lambda_{11}e_1(t,x) - \frac{e_1(t,x)}{\|e(t,\cdot)\|_2^2}\sum\limits_{i=1}^{2} 2[E_i^{\alpha}(t) + E_i^{\frac{1}{2}}(t)] \\ & -\lambda_{21}e_1(t,x)[\|e(t,\cdot)\|_2^{2\alpha-2} + \|e(t,\cdot)\|_2^{-1}], \\ U_2(t,x) = & -\lambda_{12}e_2(t,x) - \frac{e_2(t,x)}{\|e(t,\cdot)\|_2^2}\sum\limits_{i=1}^{2} 2[E_i^{\alpha}(t) + E_i^{\frac{1}{2}}(t)] \\ & -\lambda_{22}e_2(t,x)[\|e(t,\cdot)\|_2^{2\alpha-2} + \|e(t,\cdot)\|_2^{-1}]. \end{cases} \tag{4.1}$$

Case I (Finite-time synchronization):

Let $\Omega = [-8, 8]$, and choose the following activation functions

$$f_j(z) = 0.3\tanh(z) + 0.7\sin(z),\ g_j(z) = 1.6z.$$

Clearly, $f_j(\cdot)$ and $g_j(\cdot)$ satisfy Assumption 1 with $p_j = 1,\ q_j = 1.6,\quad j = 1, 2$.

Since $\Omega = [-8, 8]$, we obtain $\eta = 4$. Choosing $\lambda_{11} = 15$, $\lambda_{12} = 15$, $\lambda_{21} = 1$, $\lambda_{22} = 1, \alpha = 0.5$ in (4.1), we have

$$\lambda_{11} = 15 \geq -\frac{D_1}{\eta} + a_1 + \frac{1}{2}\sum_{j=1}^{2}(|b_{1j}|p_j + |b_{j1}|p_1 + |c_{1j}|q_j + |c_{j1}|q_1) = 13.5925,$$

$$\lambda_{12} = 15 \geq -\frac{D_2}{\eta} + a_2 + \frac{1}{2}\sum_{j=1}^{2}(|b_{2j}|p_j + |b_{j2}|p_2 + |c_{2j}|q_j + |c_{j2}|q_2) = 13.1025.$$

From Theorem 3.1, we can conclude that error system (2.5) is finite-time stable, and thus, the drive system (2.1) and response system (2.3) realize the finite-time synchronization under the designed controller (4.1) with $\alpha = 0.5$.

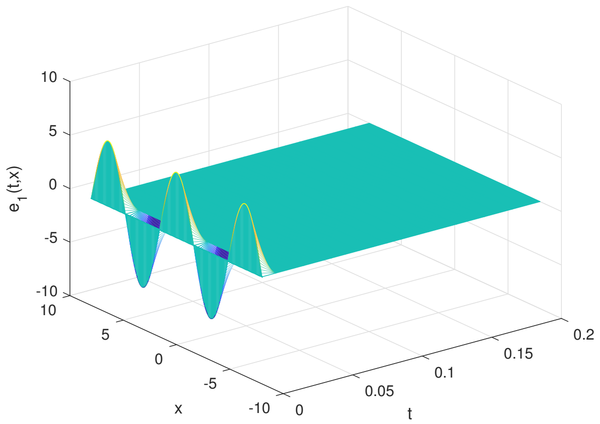

Figure 1. Spatiotemporal evolution trajectories of $e_i(t, x)$ of Case I in Example 4.1 under the initial conditions: $u_1(t, x) = -2.4\cos(\frac{5\pi x}{16})$, $u_2(t, x) = 0.6\sin(\frac{\pi x}{4}), v_1(t, x) = 3.7\cos(\frac{5\pi x}{16})$, $v_2(t, x) = -1.9\sin(\frac{\pi x}{4})$, $(t, x) \in [-0.5, 0] \times [-8, 8], i = 1, 2$.

The spatiotemporal evolution trajectories of the error system are shown in Figure 1, which strongly support the desired finite-time synchronization results.

Case II (Fixed-time synchronization):

Take activation functions as follows:

$$f_j(z) = |z + 1| - |z - 1|, \; g_j(z) = 2.5\arctan(z).$$

It is easy to verify that Assumption 1 is satisfied with $p_j = 2$, $q_j = 2.5$, $j = 1, 2$.

Let $\Omega = [-\frac{9}{2}, \frac{9}{2}]$ and design the controller (4.1) with $\lambda_{11} = 25, \lambda_{12} = 24, \lambda_{21} = 1$, $\lambda_{22} = 1, \alpha = 1.5$. By simple calculation, we obtain

$$\lambda_{11} = 25 \geq -\frac{D_1}{\eta} + a_1 + \frac{1}{2}\sum_{j=1}^{2}(|b_{1j}|p_j + |b_{j1}|p_1 + |c_{1j}|q_j + |c_{j1}|q_1) = 23.93,$$

$$\lambda_{12} = 24 \geq -\frac{D_2}{\eta} + a_2 + \frac{1}{2}\sum_{j=1}^{2}(|b_{2j}|p_j + |b_{j2}|p_2 + |c_{2j}|q_j + |c_{j2}|q_2) \approx 22.43.$$

It has been verified that all of the conditions in Theorem 3.1 are valid. Then the fixed-time synchronization can be realized for systems (2.1) and (2.3) under the feedback controller (4.1). Simulation results of the error system are shown in Figure 2, which demonstrate that the synchronization errors $e_1(t, x)$ and $e_2(t, x)$ converge to zero in a fixed time.

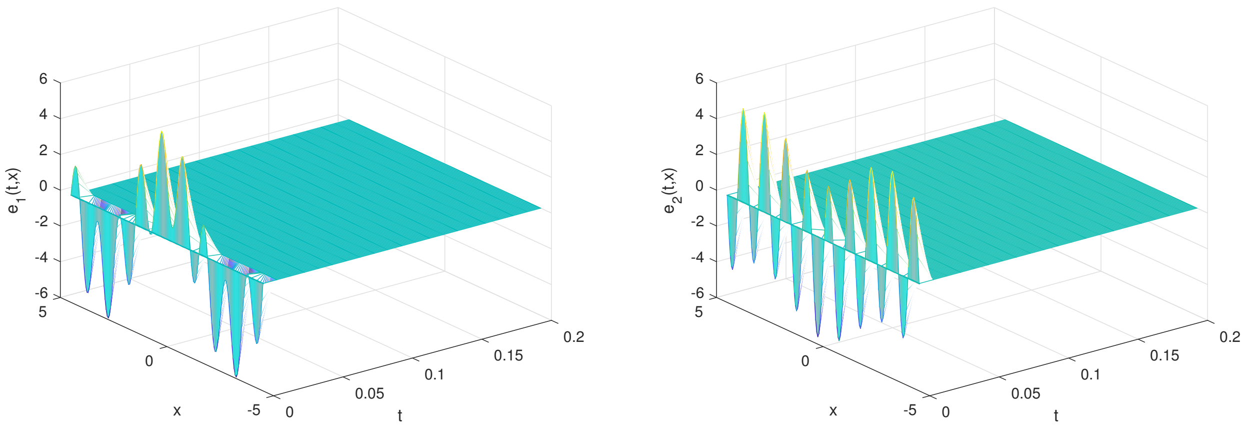

Figure 2. Spatiotemporal evolution trajectories of $e_i(t, x)$ of Case II in Example 4.1 under the initial conditions: $u_1(t, x) = -2.57\sin(2\pi x), u_2(t, x) = 1.28\cos(\frac{\pi x}{3}), v_1(t, x) = 3.46\cos(\frac{\pi x}{3}), v_2(t, x) = -4.35\sin(2\pi x), (t, x) \in [-0.5, 0] \times [-\frac{9}{2}, \frac{9}{2}], i = 1, 2.$

5. Conclusions

In this paper, we studied the FFTS problem for leakage and discrete delayed HNNs with diffusions. By designing a novel negative exponential state feedback controller and constructing a Lyapunov-Krasovskii functional, and then by the aid of finite-/fixed-time convergence theorem, we obtained some novel and useful FFTS criteria under a unified framework, which can ensure the FFTS of the considered systems. Moreover, a numerical example is given to support the effectiveness and feasibility of the proposed approach. Compared with the asymptotical or exponential synchronization results on the leakage delayed NNs with diffusion effects, the established results greatly shorten the convergence time and hence have a better applicability. In the future work, we intend to study the dynamic behaviors of other kinds of network models such discontinuous HNNs or stochastic HNNs with leakage and diffusive effects.

Data Availability

The data used to support the findings of this study is available from the corresponding author upon request.

Acknowledgment

This work was jointly supported by the Anhui Provincial Natural Science Foundation (2208085MA04), Key Program of University Natural Science Research Fund of Anhui Province

(KJ2021A0451), Key Program of Anhui Province University Outstanding Young Talents Support Plan (gxyqZD2022035), Open Research Fund of Anhui Province Engineering Laboratory for Big Data Analysis and Early Warning Technology of Coal Mine Safety (CSBD2022-ZD02).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, *Proc. Natl. Acad. Sci. USA*, **79** (1982), 2554–2558. https://doi.org/10.1073/pnas.79.8.2554
2. J. J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, *Proc. Natl. Acad. Sci. USA*, **81** (1984), 3088–3092. https://doi.org/10.1073/pnas.81.10.3088
3. N. M. Nasrabadi, W. Li, Object recognition by a Hopfield neural network, *IEEE Trans. Syst. Man Cybern.*, **21** (1991), 1523–1535. https://doi.org/10.1109/21.135694
4. G. Pajares, A Hopfield neural network for image change detection, *IEEE Trans. Neural Networks*, **17** (2006), 1250–1264. https://doi.org/10.1109/TNN.2006.875978
5. U. P. Wen, K. M. Lan, H. S. Shih, A review of Hopfield neural networks for solving mathematical programming problems, *Eur. J. Oper. Res.*, **198** (2009), 675–687. https://doi.org/10.1016/j.ejor.2008.11.002
6. H. Zhang, Z. Wang, D. Liu, A comprehensive review of stability analysis of continuous-time recurrent neural networks, *IEEE Trans. Neural Networks Learn. Syst.*, **25** (2014), 1229–1262. https://doi.org/10.1109/TNNLS.2014.2317880
7. H. Lin, C. Wang, F. Yu, J. Sun, S. Du, Z. Deng, et al., A review of chaotic systems based on memristive Hopfield neural networks, *Mathematics*, **11** (2023), 1369. https://doi.org/10.3390/math11061369
8. H. Lin, C. Wang, Y. Sun, T. Wang, Generating n-scroll chaotic attractors from a memristor-based magnetized Hopfield neural network, *IEEE Trans. Circuits Syst. II Express Briefs*, **70** (2022), 311–315. https://doi.org/10.1109/TCSII.2022.3212394
9. J. Bélair, S. A. Campbell, P. van den Driessche, Frustration, stability, and delay-induced oscillations in a neural network model, *SIAM J. Appl. Math.*, **56** (1996), 245–255. https://doi.org/10.1137/S0036139994274526
10. H. Zhao, L. Wang, C. Ma, Hopf bifurcation and stability analysis on discrete-time Hopfield neural network with delay, *Nonlinear Anal. Real World Appl.*, **9** (2008), 103–113. https://doi.org/10.1016/j.nonrwa.2006.09.005
11. K. Gopalsamy, Leakage delays in BAM, *J. Math. Anal. Appl.*, **325** (2007), 1117–1132. https://doi.org/10.1016/j.jmaa.2006.02.039
12. X. Li, J. Cao, Delay-dependent stability of neural networks of neutral type with time delay in the leakage term, *Nonlinearity*, **23** (2010), 1709. https://doi.org/10.1088/0951-7715/23/7/010

13. R. Sakthivel, P. Vadivel, K. Mathiyalagan, A. Arunkumar, M. Sivachitra, Design of state estimator for bidirectional associative memory neural networks with leakage delays, *Inf. Sci.*, **296** (2015), 263–274. https://doi.org/10.1016/j.ins.2014.10.063

14. L. Duan, H. Wei, L. Huang, Finite-time synchronization of delayed fuzzy cellular neural networks with discontinuous activations, *Fuzzy Sets Syst.*, **361** (2019), 56–70. https://doi.org/10.1016/j.fss.2018.04.017

15. C. Xu, L. Chen, Effect of leakage delay on the almost periodic solutions of fuzzy cellular neural networks, *J. Exp. Theor. Artif. Intell.*, **30** (2018), 993–1011. https://doi.org/10.1080/0952813X.2018.1509895

16. C. Aouiti, F. Miaadi, Pullback attractor for neutral Hopfield neural networks with time delay in the leakage term and mixed time delays, *Neural Comput. Appl.*, **31** (2019), 4113–4122. https://doi.org/10.1007/s00521-017-3314-z

17. C. Hu, H. Jiang, Z. Teng, Impulsive control and synchronization for delayed neural networks with reaction-diffusion terms, *IEEE Trans. Neural Networks*, **21** (2010), 67–81. https://doi.org/10.1109/TNN.2009.2034318

18. X. Hu, L. Wang, C. Zhang, X. Wan, Y. He, Fixed-time stabilization of discontinuous spatiotemporal neural networks with time-varying coefficients via aperiodically switching control, *Sci. China Inf. Sci.*, **631** (2023), 241–255. https://doi.org/10.1007/s11432-022-3633-9

19. L. Shanmugam, P. Mani, R. Rajan, Y. H. Joo, Adaptive synchronization of reaction-diffusion neural networks and its application to secure communication, *IEEE Trans. Cybern.*, **50** (2018), 911–922. https://doi.org/10.1109/TCYB.2018.2877410

20. C. Zhou, C. Wang, W. Yao, H. Lin, Observer-based synchronization of memristive neural networks under DoS attacks and actuator saturation and its application to image encryption, *Appl. Math. Comput.*, **425** (2022), 127080. https://doi.org/10.1016/j.amc.2022.127080

21. L. Wang, H. He, Z. Zeng, Global synchronization of fuzzy memristive neural networks with discrete and distributed delays, *IEEE Trans. Fuzzy Syst.*, **28** (2020), 2022–2034. https://doi.org/10.1109/TFUZZ.2019.2930032

22. S. G. Nersesov, W. M. Perruquetti, Finite-time stabilization of nonlinear impulsive dynamical systems, *Nonlinear Anal. Hybrid Syst.*, **2** (2008), 832–845. https://doi.org/10.1016/j.nahs.2007.12.001

23. S. Wang, Z. Guo, S. Wen, T. Huang, S. Gong, Finite/fixed-time synchronization of delayed memristive reaction-diffusion neural networks, *Neurocomputing*, **375** (2020), 1–8. https://doi.org/10.1016/j.neucom.2019.06.092

24. X. Yang, X. Li, P. Duan, Finite-time lag synchronization for uncertain complex networks involving impulsive disturbances, *Neural Comput. Appl.*, **34** (2022), 5097–5106. https://doi.org/10.1007/s00521-021-05987-8

25. C. Zhou, C. Wang, Y. Sun, W. Yao, H. Lin, Cluster output synchronization for memristive neural networks, *Inf. Sci.*, **589** (2022), 459–477. https://doi.org/10.1016/j.ins.2021.12.084

26. L. Duan, J. Liu, C. Huang, Z. Wang, Finite-/fixed-time anti-synchronization of neural networks with leakage delays under discontinuous disturbances, *Chaos Solitons Fractals*, **155** (2022), 111639. https://doi.org/10.1016/j.chaos.2021.111639

27. L. Duan, M. Shi, C. Huang, X. Fang, Synchronization in finite-/fixed-time of delayed diffusive complex-valued neural networks with discontinuous activations, *Chaos Solitons Fractals*, **142** (2021), 110386. https://doi.org/10.1016/j.chaos.2020.110386

28. R. Sakthivel, N. Aravinth, C. Aouiti, K. Arumugam , Finite-time synchronization of hierarchical hybrid coupled neural networks with mismatched quantization, *Neural Comput. Appl.*, **33** (2021), 16881–16897. https://doi.org/10.1007/s00521-021-06049-9

29. Y. Fan, X. Huang, Y. Li, J. Xia, G. Chen, Aperiodically intermittent control for quasi-synchronization of delayed memristive neural networks: an interval matrix and matrix measure combined method, *IEEE Trans. Syst. Man Cybern. Syst.*, **49** (2018), 2254–2265. https://doi.org/10.1109/TSMC.2018.2850157

30. L. Wang, Z. Zeng, M. Ge, A disturbance rejection framework for finite-time and fixed-time stabilization of delayed memristive neural networks, *IEEE Trans. Syst. Man Cybern. Syst.*, **51** (2021), 905–915. https://doi.org/10.1109/TSMC.2018.2888867

31. X. Liu, D. W. C. Ho, Q. Song, J. Cao, Finite-/fixed-time robust stabilization of switched discontinuous systems with disturbances, *Nonlinear Dyn.*, **90** (2017), 2057–2068. https://doi.org/10.1007/s11071-017-3782-9

32. L. C. Evans, *Partial Differential Equations*, Providence, RI, USA: American Mathematical Society, 1998.

33. Z. Chen, X. Fu, D. Zhao, Anti-periodic mild attractor of delayed hopfield neural networks systems with reaction-diffusion terms, *Neurocomputing*, **99** (2013), 372–380. https://doi.org/10.1016/j.neucom.2012.07.022

34. G. Hardy, J. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1988.