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http://www.aimspress.com/journal/era

ERA, 31(7): 4077–4087.
DOI: 10.3934/era.2023207
Received: 26 April 2023
Revised: 19 May 2023
Accepted: 23 May 2023
Published: 26 May 2023

*Research article*

# Uniqueness criteria for initial value problem of conformable fractional differential equation

**Yumei Zou and Yujun Cui***

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

* **Correspondence:** Email: cyj720201@163.com.

**Abstract:** This paper presents four uniqueness criteria for the initial value problem of a differential equation which depends on conformable fractional derivative. Among them is the generalization of Nagumo-type uniqueness theory and Lipschitz conditional theory, and advances its development in proving fractional differential equations. Finally, we verify the main conclusions of this paper by providing four concrete examples.

**Keywords:** conformable fractional differential equations; uniqueness of solution; initial value problem

## 1. Introduction

Fractional calculus is an effective assistant for explaining the mathematical analysis process in various research fields of finance, control systems and mechanics and so forth [1,2]. Latest results related to fractional differential equations, we recommend reference [3–5].

Uniqueness results play an integral role in the foundation of many of the results in applied science. Therefore, different types of initial value problem (IVP) and boundary value problems (BVP) for differential equation and differential system have been studied, we refer the reader to [2,6–13]. For example, [2] studied the existence and uniqueness results for a class of fractional differential equations (FDE), and the results obtained by Diethelm are very similar to the classical theorem in the first-order differential equation.

Mathematical models involving initial value problems established in scientific and engineering applications are described by ODE or FDE. Several examples are more convincing:

$$\begin{cases} \varphi'' = f(y, \varphi, \varphi'), & \varphi \in [0, T], \\ \varphi(0) = \alpha, & \varphi'(0) = \beta. \end{cases}$$

This nonlinear IVP is widely used in many places. It is used in [14] to give the description of the spatial variation for physical system. The initial value problem has also been significantly studied in chemical process design, [15] introduced the comparison of two interval methods to discuss the initial value problem of ODE.

Long ago, Nagumo [7] discussed the initial value problem consisting of the equation

$$\varphi' = f(y, \varphi(y)), \quad 0 \le y \le a,$$

and the initial condition

$$\varphi(0) = 0,$$

where $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies

$$f(y, 0) = 0; \quad |f(y, \xi) - f(y, \eta)| \le \frac{|\xi - \eta|}{y}, \quad y > 0, \quad |\xi|, |\eta| \le M.$$

Then, it is concluded that $\varphi(t) \equiv 0$ is the only solution of the above equation and is further generalized in [8]. It should be noted that for the $n$-th order equation in [8] when the coefficient $\frac{1}{y}$ in the above inequality is replaced by $\frac{w^{(n)}(y)}{w(y)}$, where $w$ is an absolutely continuous function with $w(0) = 0$, $w^{(n)}(y) > 0$ on $[0, a]$, and if

$$\frac{|f(y, \xi) - f(y, \eta)|}{w^{(n)}(y)} \to 0, \quad \text{when } y \to 0^+ \text{ and } \xi, \eta \to 0,$$

uniqueness is also established at this time.

In [11, 12], K. Diethelm introduced a uniqueness theorem for Caputo-type fractional differential equation together with initial value problems using a mean value theorem for Caputo-type fractional derivative when $0 < \alpha \le 1$.

Motivated by the above works, the task of this paper is to analyze the following nonlinear conformal fractional differential equations

$$\begin{cases} D_\alpha \varphi(y) = f(y, \varphi(y)), \quad y \in [0, b], \\ \varphi(0) = 0, \end{cases} \tag{1.1}$$

where $b$ is a nonnegative constant, $\alpha \in (0, 1]$ and $D_\alpha \varphi(y)$ is the standard conformable fractional derivative. The conformable fractional derivative, regarded as a new simple fractional derivative, is introduced by the authors [16]. In 2015 Abdeljawad [17] improved the definitions of conformable fractional derivative by introducing a slight modification. In 2019, AbreuBlaya et al. [18] introduced a generalized conformable fractional derivative. Also in 2018, Nazli et al. [19] introduced multi-variable conformable derivative for a vector valued function with several variables. Now, conformable fractional calculus have drawn significant interest due to its wide range of applications in different fields of sciences and engineering [20–25], and the nature of these definitions combines all the requirements of the standard derivative such as chain rule, fractional integration by parts formulas and fractional power series expansion, mean value theorem. For recent results and applications on conformable fractional calculus we refer the reader to [26–30] and references therein.

A function $\varphi(y)$ is called a solution of Eq (1.1) if $\varphi \in C[0, b]$, $D_\alpha \varphi(y)$ exists and $\varphi(y)$ satisfies Eq (1.1). We derived the uniqueness result of Eq (1.1) employing the mean value theorem of the conformable fractional calculus are proved in [17]. We then introduce the uniqueness of the initial value problem for conformable fractional differential operators. This uniqueness theorem extends the classical Nagumo theorem of first-order differential equations (see [10]); later we further extend the Athanassov-like term and the classical Lipschitz condition as the uniqueness theorem. In addition, These effective methods are also derived from the ideas in [6] and [13].

## 2. Preliminaries

In this section, it is not doubtful that we introduce some necessary definitions, lemmas, and some related properties.

**Definition 2.1.** [16] *Let $\varphi : [0, +\infty) \to \mathbb{R}$, $y > 0$ and $\alpha \in (0, 1]$. Then The $\alpha$-conformable fractional derivative of a function $\varphi(y)$ is defined by*

$$D_\alpha \varphi(y) = \lim_{\xi \to 0} \frac{\varphi(y + \xi y^{1-\alpha}) - \varphi(y)}{\xi}$$

*for $y > 0$ and the conformable fractional derivative at 0 is defined as $D_\alpha \varphi(0) = \lim_{y \to 0^+} (D_\alpha \varphi)(y)$. If $\varphi$ is differentiable then $D_\alpha \varphi(y) = y^{1-\alpha} \varphi'(y)$.*

**Definition 2.2.** [16] *The conformable fractional integral of a function $\varphi(y)$ of order $\alpha$ is given as*

$$I_\alpha \varphi(y) = \int_0^y s^{\alpha-1} \varphi(s) ds. \tag{2.1}$$

**Lemma 2.3.** [17] *Let $\varphi : (0, +\infty) \to \mathbb{R}$ be differentiable and $\alpha \in (0, 1]$. Then,*

$$I_\alpha D_\alpha \varphi(y) = \varphi(y) - \varphi(0), \quad y > 0.$$

**Lemma 2.4** [17] *Let $\alpha \in (0, 1]$, $\gamma, k, k_1, k_2 \in \mathbb{R}$, and the function $u$, $v$ be $\alpha$-differentiable on $[0, +\infty)$, then:*

*(i) $D_\alpha u(y) = 0$ for all constant functions $u(y) = k$;*

*(ii) $D_\alpha (k_1 u + k_2 v) = k_1 D_\alpha u(x) + k_2 D_\alpha v(x)$;*

*(iii) $D_\alpha y^\gamma = \gamma y^{\gamma - \alpha}$;*

*(iv) $D_\alpha (uv) = u(y) D_\alpha v(y) + v(y) D_\alpha u(y)$;*

*(v) $D_\alpha \left( \frac{u}{v} \right) = \frac{v(y) D_\alpha u(y) - u(y) D_\alpha v(y)}{v^2}$.*

**Lemma 2.5.** [17] (Mean value theorem) *Let $b > a > 0$, and $u : [a, b] \to \mathbb{R}$ be a given function that satisfies*

*(i) $u$ is continuous on $[a, b]$,*

*(ii) $u$ is $\alpha$-differentiable for some $\alpha \in (0, 1)$.*

*Then there exists a $\zeta \in (a, b)$, such that $D_\alpha u(\zeta) = \frac{u(b) - u(a)}{\frac{1}{\alpha} b^\alpha - \frac{1}{\alpha} a^\alpha}$.*

**Lemma 2.6.** [31] *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $\alpha$-differentiable for some $\alpha \in (0, 1)$. Then we have the following:*

*1). $u$ is increasing on $[a, b]$ if $D_\alpha u(x) > 0$ for any $x \in (a, b)$.*

*2). $u$ is decreasing on $[a, b]$ if $D_\alpha u(x) < 0$ for any $x \in (a, b)$.*

## 3. Main results

This section is the most exciting part of this article, some uniqueness results for the IVP involving conformable fractional differential equation are stated as follows.

**Theorem 3.1.** *Let* $0 < \alpha \leq 1$. *Assume that* $\lim\limits_{(y,u)\to(0,0)} f(y,u) = f(0,0)$*, and for all* $y \in [0,b]$ *and* $u, v \in \mathbb{R}$*, the function* $f$ *satisfies the inequality*

$$y^{\alpha}|f(y,u) - f(y,v)| \leq k|u - v|, \quad k \leq \alpha. \tag{3.1}$$

*Then Eq* (1.1) *has at most one solution.*

**Proof.** Suppose that $\varphi_1$, $\varphi_2$ are two different continuous solution of Eq (1.1) on $[0,b]$, it is clear that $\varphi_1(0) = \varphi_2(0) = 0$. We need to prove that $\varphi_1(y) = \varphi_2(y)$ for $y \in (0,b]$. Let $\varphi(y) = \varphi_1(y) - \varphi_2(y)$, now we define a function $\psi(y)$ by

$$\psi(y) = \begin{cases} y^{-\alpha}|\varphi(y)|, & y \in (0,b], \\ 0, & y = 0. \end{cases}$$

Since both $\varphi_1$ and $\varphi_2$ are two solutions of Eq (1.1) and on account of Lemma 2.5, we conclude

$$\begin{aligned} \psi(y) &= y^{-\alpha}|\varphi_1(y) - \varphi_2(y)| \\ &= y^{-\alpha}|[\varphi_1(y) - \varphi_1(0)] - [\varphi_2(y) - \varphi_2(0)]| \\ &= y^{-\alpha}|\alpha^{-1}y^{\alpha}D_{\alpha}\varphi_1(\eta) - \alpha^{-1}y^{\alpha}D_{\alpha}\varphi_2(\eta)| \\ &= \alpha^{-1}|D_{\alpha}\varphi_1(\eta) - D_{\alpha}\varphi_2(\eta)| \\ &= \alpha^{-1}|f(\eta,\varphi_1(\eta)) - f(\eta,\varphi_2(\eta))|, \ \eta \in (0,t). \end{aligned}$$

Subsequently, we can get $\eta \to 0$, and $\varphi_1(\eta), \varphi_2(\eta) \to 0$ when $y \to 0$. From the continuity of $f$, we conclude that

$$\psi(y) = \alpha^{-1}|f(\eta,\varphi_1(\eta)) - f(\eta,\varphi_2(\eta))| \to \alpha^{-1}|f(0,0) - f(0,0)| = \psi(0).$$

That is to say that $\psi(y)$ is nonnegative continuous on $[0,b]$. If $\psi(y) \neq 0$ on $(0,b]$, then it is easy to conclude that there is a $y_0 \in (0,b]$ such that $\max\limits_{y\in[0,b]} \psi(y) = \psi(y_0) > 0$ and that $\psi(y_1) < \psi(y_0)$ for $y_1 \in (0,y_0)$. By help of Lemma 2.5 and Eq (3.1), we derive that

$$\begin{aligned} \psi(y_0) &= y_0^{-\alpha}|\varphi_1(y_0) - \varphi_2(y_0)| \\ &= y_0^{-\alpha}|[\varphi_1(y_0) - \varphi_1(0)] - [\varphi_2(y_0) - \varphi_2(0)]| \\ &= \alpha^{-1}|D_{\alpha}\varphi_1(y_1) - D_{\alpha}\varphi_2(y_1)| \\ &= \alpha^{-1}|f(y_1,\varphi_1(y_1)) - f(y_1,\varphi_2(y_1))| \\ &\leq \alpha^{-1}y_1^{-\alpha}k|\varphi_1(y_1) - \varphi_2(y_1)| \\ &\leq \alpha^{-1}y_1^{-\alpha}\alpha|\varphi_1(y_1) - \varphi_2(y_1)| \\ &= \psi(y_1), \quad y_1 \in (0,y_0). \end{aligned}$$

The contradiction show that $\psi(y) \equiv 0$, $y \in [0,b]$, thus, $\varphi(y) \equiv 0$, $y \in [0,b]$, in other words, we have $\varphi_1(y) = \varphi_2(y)$ for $y \in [0,b]$. The proof of Theorem 3.1 is completed.

When $\alpha = 1$ in Theorem 3.1, the conformable fractional derivative of order $\alpha = 1$ coincides with the known usual derivatives. Thus Theorem 3.1 complement and extend the results in [10].

The following example shows that the restriction $k \leq \alpha$ imposed in Eq (3.1) is optimal.

**Example 3.1.** Consider IVP in the following form:

$$\begin{cases} D_\alpha \varphi(y) = f(y, \varphi(y)), \quad y \in [0, b], \\ \varphi(0) = 0, \end{cases} \tag{3.2}$$

where

$$f(y, u) = \begin{cases} 0, \quad y \in [0, b], u \le 0; \\ \dfrac{(\alpha + \xi)u}{y^\alpha}, \ y \in [0, b], \ 0 < u < y^\varepsilon, \ \varepsilon = \alpha + \xi > \alpha; \\ (\alpha + \xi) y^{\varepsilon - \alpha}, \ y \in [0, b], \ y^\varepsilon \le u. \end{cases}$$

For $0 < u < y^\varepsilon$, $f(y, u) = \frac{(\alpha+\xi)u}{y^\alpha}$. So we have

$$|f(y, u)| = \left| \frac{(\alpha + \xi)u}{y^\alpha} \right| < (\alpha + \xi) y^{\varepsilon - \alpha},$$

which easily implies that $f$ is continuous on $[0, b] \times \mathbb{R}$. Moreover, $f$ satisfies the Eq (3.1) on $[0, b] \times \mathbb{R}$ except that $k = \alpha + \xi > \alpha$. We have the following four cases to illustrate:

Case 1: When $0 < u, \overline{u} < y^\varepsilon$, we have

$$|f(y, u) - f(y, \overline{u})| = \left| \frac{(\alpha + \xi)u}{y^\alpha} - \frac{(\alpha + \xi)\overline{u}}{y^\alpha} \right| = \frac{(\alpha + \xi)}{y^\alpha} |u - \overline{u}|.$$

Case 2: When $0 < u < y^\varepsilon \le \overline{u}$, we have

$$\begin{aligned} |f(y, u) - f(y, \overline{u})| &= \left| \frac{(\alpha + \xi)u}{y^\alpha} - (\alpha + \xi) y^{\varepsilon - \alpha} \right| = \frac{(\alpha + \xi)}{y^\alpha} (y^\varepsilon - u) \\ &\le \frac{(\alpha + \xi)}{y^\alpha} |\overline{u} - u|. \end{aligned}$$

Case 3: When $u \le 0 < \overline{u} < y^\varepsilon$, we have

$$|f(y, u) - f(y, \overline{u})| = \left| 0 - \frac{(\alpha + \xi)\overline{u}}{y^\alpha} \right| \le \frac{(\alpha + \xi)}{y^\alpha} |u - \overline{u}|.$$

Case 4: When $u \le 0$, $y^\varepsilon \le \overline{u}$, we have

$$\begin{aligned} |f(y, u) - f(y, \overline{u})| &= |0 - (\alpha + \xi) y^{\varepsilon - \alpha}| = \frac{(\alpha + \xi)}{y^\alpha} y^\varepsilon \\ &\le \frac{(\alpha + \xi)}{y^\alpha} \overline{u} \le \frac{(\alpha + \xi)}{y^\alpha} |u - \overline{u}|. \end{aligned}$$

However, Eq (3.2) has multiple solutions $\varphi(y) = c y^{\alpha + \xi}$ with $c \in (0, 1)$.

The fractional order integral equation is used in the proof process of the next uniqueness result.

**Theorem 3.2.** *Assume that there is a function $w \in C^\alpha[0, b] = \{\varphi : \varphi \in C[0, b], D_\alpha \varphi \in C[0, b]\}$ such that $w(y) > 0$ for all $y > 0$ and $w(0) = 0$. In addition, for $u, \overline{u} \in \mathbb{R}$, $w$ and $f$ satisfies*

$$|f(y, u) - f(y, \overline{u})| \le \frac{D_\alpha w(y)}{w(y)} |u - \overline{u}|, \quad y \in (0, b], \tag{3.3}$$

*and*

$$\lim_{y\to 0,u,\overline{u}\to 0}\frac{|f(y,u)-f(y,\overline{u})|}{D_\alpha w(y)}=0. \tag{3.4}$$

*Then Eq* (1.1) *has at most one solution.*

**Proof.** Assume that Eq (1.1) has two different continuous solutions $\varphi_1$ and $\varphi_2$. On account of Lemma 2.3, we can infer that

$$\varphi_1(y)=\int_0^y x^{\alpha-1}f(x,\varphi_1(x))dx,$$

and

$$\varphi_2(y)=\int_0^y x^{\alpha-1}f(x,\varphi_2(x))dx.$$

Then take $\psi(y)=\varphi_1(y)-\varphi_2(y)$, we get

$$\begin{aligned}|\psi(y)| &= |\varphi_1(y)-\varphi_2(y)|\\ &= \left|\int_0^y x^{\alpha-1}[f(x,\varphi_1(x))-f(x,\varphi_2(x))]dx\right|\\ &\le \int_0^y x^{\alpha-1}|f(x,\varphi_1(x))-f(x,\varphi_2(x))|dx.\end{aligned} \tag{3.5}$$

Now, for $y\in[0,b]$, define function $\phi(y)$ by

$$\phi(y)=\begin{cases}\dfrac{|\psi(y)|}{w(y)}, & y\in(0,b],\\ 0, & y=0.\end{cases}$$

In view of Eq (3.4) we obtain that for all $\varepsilon>0$ there exists a $\eta>0$ such that

$$\frac{|f(x,\varphi_1(x))-f(x,\varphi_2(x))|}{D_\alpha w(x)}<\varepsilon,\quad 0<x<\eta.$$

In other words,

$$|f(x,\varphi_1(x))-f(x,\varphi_2(x))|<\varepsilon D_\alpha w(x).$$

By Eqs (3.3) and (3.5), we obtain that

$$\begin{aligned}|\psi(y)| &\le \int_0^y x^{\alpha-1}|f(x,\varphi_1(x))-f(x,\varphi_2(x))|dx\\ &< \varepsilon\int_0^y x^{\alpha-1}D_\alpha w(x)dx\\ &= \varepsilon I_\alpha D_\alpha w(y)\\ &= \varepsilon w(y),\ 0<y<\eta.\end{aligned}$$

In other words,

$$\lim_{y\to 0}\frac{|\psi(y)|}{w(y)}=0.$$

Then, we get $\phi(y)$ is a nonnegative continuous on $[0, b]$. From the above assumption, it is clear that there exists a $y_0 \in (0, b]$ be such that $\max_{y\in[0,b]} \phi(y) = \phi(y_0) > 0$. Applying Eq (3.3), we can infer that

$$\begin{aligned} |\psi(y)| &\leq \int_0^y x^{\alpha-1} |f(x, \varphi_1(x)) - f(x, \varphi_2(x))| dx \\ &\leq \int_0^y x^{\alpha-1} \frac{D_\alpha w(x)}{w(x)} |\varphi_1(x) - \varphi_2(x)| dx \\ &= \int_0^y x^{\alpha-1} D_\alpha w(x) \phi(x) dx \\ &< \phi(y_0) \int_0^y x^{\alpha-1} D_\alpha w(x) dx. \end{aligned}$$

Thus,

$$\phi(y_0) = \frac{|\psi(y_0)|}{w(y_0)} < \frac{\phi(y_0)}{w(y_0)} \int_0^{y_0} x^{\alpha-1} D_\alpha w(x) dx = \frac{\phi(y_0)}{w(y_0)} w(y_0) = \phi(y_0).$$

From the above contradiction, we conclude that $\phi(y) \equiv 0$ on $[0, b]$. In other words, $\varphi_1(y) \equiv \varphi_2(y)$ for $y \in [0, b]$. So the proof is finished.

Subsequently, an example is given to share the application of the Theorem 3.2.

**Example 3.2.** Consider initial value problem in the following form:

$$\begin{cases} D_\alpha \varphi(y) = y^{1-\alpha} \cos^2 \varphi(y) + e^y, \quad y \in [0, 1], \quad \alpha \in (0, 1], \\ \varphi(0) = 0. \end{cases} \tag{3.6}$$

Obviously, $f(y, u) = y^{1-\alpha} \cos^2 u + e^y$ and $b = 1$. New let $w(y) = y$, so we get

$$D_\alpha w(y) = y^{1-\alpha}.$$

It is easy to obtain that

$$\left|\cos^2 u - \cos^2 \overline{u}\right| = \frac{1}{2} |\cos 2u - \cos 2\overline{u}| \leq |u - \overline{u}|.$$

Subsequently, we observe that

$$|f(y, u) - f(y, \overline{u})| = y^{1-\alpha} |\cos^2 u - \cos^2 \overline{u}| \leq \frac{y^{1-\alpha}}{y} |u - \overline{u}| = \frac{D_\alpha w(y)}{w(y)} |u - \overline{u}|,$$

and

$$\begin{aligned} \lim_{y\to0,u,\overline{u}\to0} \frac{|f(y, u) - f(y, \overline{u})|}{D_\alpha w(y)} &= \lim_{y\to0,u,\overline{u}\to0} \frac{y^{1-\alpha} |\cos^2 u - \cos^2 \overline{u}|}{y^{1-\alpha}} \\ &= \lim_{y\to0,u,\overline{u}\to0} |\cos^2 u - \cos^2 \overline{u}| \\ &= 0. \end{aligned}$$

With the help of Theorem 3.2, we can conclude that Eq (3.6) has at most one solution.

**Theorem 3.3.** *Let $f \in C([0, b] \times \mathbb{R}, \mathbb{R})$ and $f(y, u)$ is nonincreasing in $u$ for every $y \in [0, b]$. Then Eq* (1.1) *has at most one solution in* $[0, b]$.

**Proof.** Assume that Eq (1.1) has two different solutions $\varphi_1$ , $\varphi_2$ on $[0, b]$. Then

$$D_\alpha\varphi_1(y) = f(y, \varphi_1(y)), \ \ y \in [0, b],$$

$$D_\alpha\varphi_2(y) = f(y, \varphi_2(y)), \ \ y \in [0, b],$$

and

$$\varphi_1(0) = \varphi_2(0) = 0.$$

Without loss of generality, we assume that there exist $y_1, y_2 \in [0, b]$ such that

$$\varphi_2(y) = \varphi_1(y), \ \ 0 \le y \le y_1,$$

$$\varphi_2(y) > \varphi_1(y), \ \ y_1 < y \le y_2. \tag{3.7}$$

Thus for $y \in [y_1, y_2]$, we have

$$D_\alpha(\varphi_2(y) - \varphi_1(y)) = f(y, \varphi_2(y)) - f(y, \varphi_1(y)) \le 0.$$

Applying Lemma 2.6 to the above inequality, we get $\varphi_2(y) - \varphi_1(y)$ is nonincreasing on $[y_1, y_2]$. Further, since $\varphi_2(y_1) = \varphi_1(y_1)$, we have $\varphi_2(y) \le \varphi_1(y)$ on $[y_1, y_2]$ which contradicts Eq (3.7). This contradiction shows that $\varphi_2(y) \equiv \varphi_1(y)$ for all $y \in [0, b]$. We complete the proof.

We shall illustrate Theorem 3.3 with an example.

**Example 3.3.** Consider IVP in the following form:

$$\begin{cases} D_\alpha\varphi(y) = -|\varphi(y)|^{\frac{1}{2}}\mathrm{sgn}\varphi(y), \ \ y \in [0, b], \\ \varphi(0) = 0, \end{cases} \tag{3.8}$$

Obviously, the function $f(y, u) = -|u|^{\frac{1}{2}}\mathrm{sgn}u$ is continuous on $[0, b] \times \mathbb{R}$ and it is nonincreasing in $u$ for $y \in [0, b]$. Thus, it follows from Theorem 3.3 that $\varphi(y) \equiv 0$ is the only solution of Eq (3.8).

However, the function $f$ in Eq (3.8) does not satisfy the Lipschitz condition. Then, the above description shows that the Lipschitz condition is a sufficient rather than a necessary condition to guarantee the uniqueness for Eq (1.1). The following example shows that the nonincreasing property in Theorem 3.3 cannot be replaced by nondecreasing property.

**Example 3.4.** Consider IVP in the following form:

$$\begin{cases} D_{\frac{1}{2}}\varphi(y) = |\varphi(y)|^{\frac{1}{2}}\mathrm{sgn}\varphi(y), \ \ y \in [0, b], \\ \varphi(0) = 0, \end{cases} \tag{3.9}$$

Clearly, $f(y, u) = |u|^{\frac{1}{2}}\mathrm{sgn}u$ is continuous on $[0, b] \times \mathbb{R}$ and it is nondecreasing in $u$ for $y \in [0, b]$. However, Eq (3.9) has two solutions $\varphi(y) \equiv 0$ and $\varphi(y) = y$.

**Theorem 3.4.** *Let $f \in C([0, b] \times \mathbb{R}, \mathbb{R})$. In addition, $f$ satisfies one-sided Lipschitz condition*

$$f(y, \overline{u}) - f(y, u) \le L(\overline{u} - u), \ y \in [0, b], \ u \le \overline{u}. \tag{3.10}$$

*Then Eq* (1.1) *has at most one solution in* $[0, b]$.

**Proof.** Let $\varphi(y)$ be a solution of (1.1). Take $\psi(y) = e^{-\frac{L}{\alpha}y^\alpha}\varphi(y)$. According to Lemma 2.4 it follows that

$$\begin{aligned} D_\alpha\psi(y) &= e^{-\frac{L}{\alpha}y^\alpha} D_\alpha\varphi(y) + \varphi(y) D_\alpha e^{-\frac{L}{\alpha}y^\alpha} \\ &= e^{-\frac{L}{\alpha}y^\alpha} f(y, \varphi(y)) - L e^{-\frac{L}{\alpha}y^\alpha}\varphi(y) \\ &= e^{-\frac{L}{\alpha}y^\alpha} f(y, e^{\frac{L}{\alpha}y^\alpha}\psi(x)) - L\psi(x). \end{aligned}$$

Therefore, $\varphi(y)$ solve of Eq (1.1) if and only if $\psi(y)$ satisfying the following conformable fractional differential equation

$$\begin{cases} D_\alpha u(y) = F(y, u(y)), & y \in [0, b], \\ u(0) = 0, \end{cases} \tag{3.11}$$

where $F(y, u) = e^{-\frac{L}{\alpha}y^\alpha} f(y, e^{\frac{L}{\alpha}y^\alpha} u) - Lu$. It follows from Eq (3.10) that the function $F(y, u)$ satisfying the conditions in Theorem 3.3. So, Eq (3.11) has at most one solution, equivalently, Eq (1.1) has at most one solution.

## 4. Conclusions

This paper is concerned with the study of uniqueness criteria for the initial value problem with the uses of a conformable fractional derivative. By using conformable fractional calculus, we present four uniqueness theorems which extends and complements the Nagumo-type uniqueness theory and Lipschitz conditional theory. Four concrete examples are given to better demonstrate our main results.

## Acknowledgments

The research is supported by the National Natural Science Foundation of China (11571207) and Shandong Natural Science Foundation (ZR2018MA011).

## Conflict of interest

The authors declare there is no conflicts of interest.

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