Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(7): 3963–3979.
DOI: 10.3934/era.2023201
Received: 14 January 2023
Revised: 13 April 2023
Accepted: 21 April 2023
Published: 16 May 2023

Research article

The uniform asymptotic behavior of solutions for 2D g-Navier-Stokes equations with nonlinear dampness and its dimensions

Xiaoxia Wang*and Jinping Jiang

College of Mathematics and Computer Science,Yan'an University, Yan'an 716000, China.

* **Correspondence:** Email: yd-wxx@163.com.

Abstract: In this paper, the uniform asymptotic behavior of solutions for 2D g-Navier-Stokes equations with nonlinear dampness is studied in unbounded domain. The uniform asymptotic properties of the process family is proved with the energy equation method and the uniform attractor is obtained. Finally, the dimension of the uniform attractor is estimated in the quasi-periodical case.

Keywords: g-Navier-Stokes equations; energy equation method; nonlinear dampness; global well-posedness; dimensional estimation

1. Introduction

It is well-known that the Navier-Stokes equations are the typical evolution equations and widely used in the field of science and engineering. The attractors of Navier-Stokes equations are studied by many scholars in the fields of dynamical systems for a long time (see [1–15] and reference therein). Especially in recent years, there are many research achievements on g-Navier-Stokes equation. In [16–18], Roh deduced the 2D g-Navier-Stokes equations from 3D Navier-Stokes equations on thin region. It can be viewed as a perturbation of the usual Navier-Stokes equations. Bae et al. studied the well-posedness of weak solution for the 2D g-Navier-Stokes equations. Kwak et al. researched the global attractor and its fractal dimension of 2D g-Navier-Stokes equations in [19]. In [20–24], Jiang et al. studied global and the pullback attractor for g-Navier-Stokes equation. Moreover, the long-time behavior for 2D non-autonomous g-Navier-Stokes equations and the stability of solutions to stochastic 2D g-Navier-Stokes equations were studied by Anh in [25,26], The stationary solutions and its pullback attractor are researched in [27]. On the basis of the above research, we have studied the long time properties for g-Navier-Stokes equation with weakly dampness and time delay in [28] recently.

In this manuscript, the uniform attractor of the g-Navier-Stokes equations with nonlinear dampness is researched. Its usual form is as follows:

$$\begin{array}{ll} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + c|u|^{\beta-1}u + \nabla p = f(x,t) & \text{in } [\tau,+\infty)\times\Omega \\ \nabla\cdot(gu) = 0 & \text{in } [\tau,+\infty)\times\Omega \\ u(x,t) = 0 & \text{in } [\tau,+\infty)\times\partial\Omega \\ u(\tau,x) = u_\tau(x) & x\in\Omega \end{array} \tag{1.1}$$

In (1.1), we can see that $u(t,x) \in R^2$ and $p(t,x) \in R$ denote the velocity and pressure respectively. $\nu > 0$ is the viscosity coefficient, $c|u|^{\beta-1}u$ denotes nonlinear dampness. $c > 0$ and $\beta \geq 1$ are positive constant. $f = f(x,t)$ is the external force term, $0 < m_0 \leq g = g(x_1,x_2) \leq M_0$ and $g = g(x_1,x_2)$ is a suitable smooth function, Let $c = 0$ and $g = 1$, the Eq (1.1) will become the usual 2D Navier-Stokes equations.

This manuscript is organized as follows. In Section 2, we recall some basic results of 2D g-Navier-Stokes equations, then we give the concept about process families and uniform attractor. In Section 3, the global well-posedness of weak solutions for 2D g-Navier-Stokes equations with nonlinear dampness is studied. In Section 4, by the energy equation method, the existence of the uniform attractor of 2D g-Navier-Stokes equation with nonlinear dampness is proved on the unbounded domain. In Section 5, the dimension estimation of the uniform attractor in the quasi-periodic case is obtained.

2. Preliminaries

We assume Ω is a smooth unbounded domian of R^2, Let $L^2(g) = (L^2(\Omega))^2$ and we denote $(u,v) = \int_\Omega u\cdot v g dx$ and $|\cdot| = (\cdot,\cdot)^{1/2}$, $u, v \in L^2(g)$. Let $H_0^1(g) = (H_0^1(\Omega))^2$, Set

$$((u,v)) = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j g dx,$$

and $\|\cdot\| = ((\cdot,\cdot))^{1/2}, u = (u_1,u_2), v = (v_1,v_2) \in H_0^1(g)$. We denote $D(\Omega)$ be the space of C^∞ functions with compact support contained in Ω. So we have the following spaces

$$H = \{v \in (D(\Omega))^2 : \nabla\cdot gv = 0 \;\; in \;\; \Omega\};$$

$$H_g = closure\ of\ H\ in\ L^2(g);$$

$$V_g = closure\ of\ H\ in\ H_0^1(g).$$

where H_g and V_g endowed with the inner product and norm of $L^2(g)$ and $H_0^1(g)$ respectively.

We assume that there exists $\lambda_1 > 0$, such that

$$|u|^2 \leq \frac{1}{\lambda_1}\|u\|^2, \forall u \in V_g. \tag{2.1}$$

This Poincaré-type inequality imposes some restrictions on the geometry of the domain Ω.

The g-Laplacian operator is defined as follows:

$$-\Delta_g u = -\frac{1}{g}(\nabla\cdot g\nabla)u = -\Delta u - \frac{1}{g}\nabla g\cdot\nabla u.$$

The first equation of (1.1) can be rewritten as follows:

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{\nabla g}{g}\cdot\nabla u + c|u|^{\beta-1}u + (u,\nabla)u + \nabla p = f. \tag{2.2}$$

In [16], g-orthogonal projection and g-Stokes operator are defined respectively by $P_g : L^2(g) \to H_g$ and $A_g u = -P_g(\frac{1}{g}(\nabla\cdot(g\nabla u)))$. Applying the projection P_g on the Eq (2.2), we have the following weak formulation of (1.1).

$$\frac{d}{dt}(u,v) + \nu((u,v)) + c(|u|^{\beta-1}u,v) + b_g(u,u,v) + \nu(Ru,v) = \langle f,v\rangle \quad \forall v\in V_g, \forall t>0, \tag{2.3}$$

$$u(0) = u_0, \tag{2.4}$$

where $b_g : V_g\times V_g\times V_g \to \mathrm{R}$ and

$$b_g(u,v,w) = \sum_{i,j=1}^{2}\int u_i\frac{\partial v_j}{\partial x}w_j g dx,$$

we have

$$Ru = P_g[\frac{1}{g}(\nabla g\cdot\nabla)u], \forall u\in V_g.$$

Then the formula (2.3) and (2.4) are equivalent to the following functional equations

$$\frac{du}{dt} + \nu A_g u + c|u|^{\beta-1}u + Bu + \nu Ru = f \tag{2.5}$$

$$u(0) = u_0 \tag{2.6}$$

We denote

$$\langle A_g u, v\rangle = ((u,v)), \forall u,v\in V_g. \tag{2.7}$$

From [16,17,19], we have

$$\|B(u)\|_{V_g'} \le c|u|\|u\|, \quad \|Ru\|_{V_g'} \le \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|, \quad \forall u\in V_g.$$

where $B(u) = B(u,u) = P_g(u\cdot\nabla)u$ is defined by

$$\langle B(u,v),w\rangle = b_g(u,v,w), \forall u,v,w\in V_g.$$

A family of two parametric maps $\{U_f(t,\tau)\} = \{U_f(t,\tau)|t\ge\tau, \tau\in\mathrm{R}\}$ is defined in H_g as follows:

$$U_f(t,\tau) : E\to E, \quad t\ge\tau, \quad \tau\in\mathrm{R}.$$

The following concepts and conclusions are given from [7]. $\forall f\in L^\infty(\mathrm{R}^+;V_g')$, the translation operator is defined in $L^\infty(\mathrm{R}^+;V_g')$ as follows.

$$T(h)f(s) = f(s+h), \quad \forall\ h\ge 0,\ s\in\mathrm{R}.$$

Obviously

$$\|T(h)f\|_{L^\infty(\mathrm{R}^+;V_g')} \le \|f\|_{L^\infty(\mathrm{R}^+;V_g')},\ \forall\ h\ge 0,\ f\in L^\infty(\mathrm{R}^+;V_g').$$

We set $\Sigma = \{T(h)f(x,s) = f(x, s+h),\ \forall\ h \in \mathrm{R}\}$, where $T(\cdot)$ is the positive invariant semigroups which act on Σ and satify $T(h)\Sigma \subset \Sigma,\ \forall\ h \geq 0$ and

$$U_{T(h)f}(t,\tau) = U_f(t+h, \tau+h),\ \forall\ h \geq 0,\ t \geq \tau \geq 0.$$

Let $\rho_{\mathcal{F}} > 0$ be constant, $\Sigma \subset \{f \in L^\infty(\mathrm{R}^+; V_g') : \|f\|_{L^\infty(\mathrm{R}^+;V_g')} \leq \rho_{\mathcal{F}}\}$. For $\{U_f(t,\tau)\}$ with $f \in \Sigma$, we call the parameter f as the symbols of the process family $\{U_f(t,\tau)\}$, and Σ as the symbol space.

Definition 2.1 [7] A family of two-parametric maps $\{U(t,\tau)\}$ is called a process in H_g, if

(1)$U_f(t,s)U_f(s,\tau) = U_f(t,\tau), \quad \forall\ t \geq s \geq \tau, \quad \tau \in \mathrm{R}$,

(2)$U_f(\tau,\tau) = Id, \quad \tau \in \mathrm{R}$.

Let E be the Banach space, $\mathcal{B}(E)$ is denoted the set of all bounded sets on E, then

Definition 2.2 [7] A set $B_0 \subset E$ is said to be uniformly absorbing for the family of processes $\{U_f(t,\tau)\}, f \in \Sigma\}$, if for any $\tau \in \mathrm{R}$ and each $B \in \mathcal{B}(E)$, there exists $t_0 = t_0(\tau, B) \geq \tau$, such that for all $t \geq t_0$,

$$\bigcup_{f\in\Sigma} U_f(t,\tau)B \subseteq B_0.$$

Definition 2.3 [7] A set $P \subset E$ is said uniformly atttracting set of $\{U_f(t,\tau)\}, f \in \Sigma\}$, if for any $\tau \in \mathrm{R}$, there is

$$\lim_{t\to+\infty} (\sup_{f\in\Sigma} dist_E(U_f(t,\tau)B, P)) = 0.$$

Definition 2.4 [7] A closed set $\mathcal{A}_\Sigma \subset E$ is said to be the uniform attractor of the family of processes $\{U_f(t,\tau)\},\ f \in \Sigma\}$, if

(1) $\mathcal{A}_\Sigma \subset E$ is uniformly attractive;

(2) $\mathcal{A}_\Sigma \subset E$ is included in any uniformly attracting set $\mathcal{A}'$ of $\{U_f(t,\tau)\}, f \in \Sigma\}$, that is $\mathcal{A}_\Sigma \subset \mathcal{A}'$.

3. The well-posedness of the solution for 2D g-Navier-Stokes equations with nonlinear dampness in unbounded domain

In the section we will prove the well-posedness of the solution for 2D g-Navier-Stokes equations with nonlinear dampness by the Faedo-Galerkin method.

Definition 3.1 Let $u_0 \in H_g, f \in L^2_{Loc}(\mathrm{R}; V_g')$, For any $\tau \in \mathrm{R}$, $u \in L^\infty(\tau, T; V_g) \cap L^2(\tau, T; V_g) \cap L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega)), \forall T > \tau$ is called a weak solution of problem (1.1) if it fulfils

$$\frac{d}{dt}u(t) + \nu A_g u(t) + B(u(t)) + c|u|^{\beta-1}u + \nu R(u(t)) = f(x,t) \quad \text{on } \mathcal{D}'(\tau, +\infty; V_g'),$$
$$u(\tau) = u_0.$$

Theorem 3.1 Let $\beta \geq 1$, $f \in L^2_{Loc}(\mathrm{R}; V_g')$, Then for every $u_\tau \in V_g$, the equations (1.1) have a unique weak solution $u(t) = u(t;\tau,u_\tau) \in L^\infty(\tau,T;V_g) \cap L^2(\tau,T;V_g) \cap L^{\beta+1}(\tau,T;L^{\beta+1}(\Omega))$, and $u(t)$ is continuously depending on the initial value in V_g.

Proof. Let $\{w_j\}_{j\geq1}$ be the eigenfunctions of $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions, Its corresponding eigenvalues are $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, Obviously, $\{w_j\}_{j\geq1} \subset V_g$ forms a Hilbert basis in H_g, given $u_\tau \in V_g$ and $f \in L^2_{Loc}(\mathrm{R}; V_g')$.

For any positive integer $n \geq 1$, we structure the Galerkin approximate solutions as $u_n(t) = u_n(t; T, u_\tau)$, It has the following form

$$u_n(t,T;u_\tau) = \sum_{j=1}^{n} \gamma_{n,j}(t) w_j.$$

where $\gamma_{n,j}(t)$ is determined from the initial values of the following system of nonlinear ordinary differential equations.

$$(u_n^{'}(t), w_j) + \nu((u_n(t), w_j)) + c(|u_n(t)|^{\beta-1}u_n(t), w_j) + b(u_n(t), u_n(t), w_j) + b(\frac{\nabla g}{g}, u_n(t), w_j)$$
$$= \langle f(x,t), w_j\rangle, t > \tau, j = 1, 2, \ldots n \tag{3.1}$$
$$((u_n(t), w_j)) = ((u_\tau, w_j)).$$

where $\langle\cdot\rangle$ is dual product of V_g and $V_g^{'}$.

According to the results of the initial value problems of ordinary differential equations, we have that there exists a unique local solution of (3.1). In the following, we prove that the time interval of the solution can be extended to $[\tau, \infty)$.

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + \nu\|u_n(t)\|^2 + c|u_n(t)|_{\beta+1}^{\beta+1} + b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t)) = \langle f(x,t), u_n(t)\rangle \tag{3.2}$$

Using Cauchy's inequality and Young's inequality, we have

$$\langle f(x,t), u_n(t)\rangle \le \|f(x,t)\|_* \cdot \|u_n(t)\|$$
$$\le \frac{\nu}{2}\|u_n\|^2 + \frac{1}{2\nu}\|f(x,t)\|_*^2 \tag{3.3}$$

where $\|\cdot\|_*$ is norm of $V_g^{'}$. We take (3.3) into (3.2) to obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + \nu\|u_n(t)\|^2 + c|u_n(t)|_{\beta+1}^{\beta+1} + b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t))$$
$$\le \frac{\nu}{2}\|u_n\|^2 + \frac{1}{2\nu}\|f(x,t)\|_*^2$$
$$\frac{d}{dt}|u_n(t)|_2^2 + 2\nu\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} + 2b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t))$$
$$\le \nu\|u_n\|^2 + \frac{1}{\nu}\|f(x,t)\|_*^2$$
$$\frac{d}{dt}|u_n(t)|_2^2 + \nu\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} + 2b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t)) \le \frac{1}{\nu}\|f(x,t)\|_*^2 \tag{3.4}$$

That is

$$\frac{d}{dt}|u_n(t)|_2^2 + \nu\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} \le \frac{1}{\nu}\|f(x,t)\|_*^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u_n(t)\|^2$$
$$\frac{d}{dt}|u_n(t)|_2^2 + \nu(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} \le \frac{1}{\nu}\|f(x,t)\|_*^2 \tag{3.5}$$

By integrating (3.5) from τ to t, we have

$$|u_n(t)|^2 + \nu(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\int_\tau^t \|u_n(s)\|^2 ds + 2c\int_\tau^t |u_n(s)|_{\beta+1}^{\beta+1} ds$$

$$\leq |u_n(\tau)|^2 + \frac{1}{\nu}\int_\tau^t \|f(x,s)\|_*^2 ds.$$

For any $T > 0$ and $\beta \geq 1$, we obtain

$$\sup_{\tau\leq t\leq T}(|u_n(t)|^2) + \nu(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\int_\tau^t \|u_n(s)\|^2 ds + 2c\int_\tau^t |u_n(s)|_{\beta+1}^{\beta+1} ds$$

$$\leq \|u_n(\tau)\|^2 + \frac{1}{\nu}\int_\tau^t \|f(x,s)\|_*^2 ds \leq C.$$

So we can obtain that $\{u_n(t)\}$ is bounded in $L^\infty(\tau, T; V_g)$, (3.6)

$\{u_n(t)\}$ is bounded in $L^2(\tau, T; V_g)$, (3.7)

and $\{u_n(t)\}$ is bounded in $L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega))$. (3.8)

So $u_n(t) \in L^\infty(\tau, T; V_g)$. Therefore $B(u_n(t)) \in L^\infty(\tau, T; V_g^{'})$, $|u_n(t)|^{\beta-1}u_n(t) \in L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega))$. As a result,

$$\frac{d}{dt}\langle u_n(t), v\rangle = \langle f(x,t) - c|u_n(t)|^{\beta-1}u_n(t) - \nu Au_n(t) - B(u_n(t)) - \nu R(u_n(t)), v\rangle, \quad \forall v \in V_g.$$

so $\{u_n^{'}(t)\}$ is bounded in $L^2(\tau, T; V_g)$.

Then we deduce that there is a subsequence in $\{u_n(t)\}$, which is still denoted by $\{u_n(t)\}$. We obtain $u_n(t) \in L^2(\tau, T; V_g)$ and $u_n^{'}(t) \in L^2(\tau, T; V_g)$ such that

(i)$u_n(t) \to u(t)$ is weakly * convergent in $L^\infty(\tau, T; V_g)$;

(ii)$u_n(t) \to u(t)$ is weakly convergent in $L^2(\tau, T; V_g)$;

(iii)$|u_n(t)|^{\beta-1}u_n(t) \to \xi$ is weakly convergent in $L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega))$;

(iv)$u_n^{'}(t) \to u^{'}(t)$ is weakly convergent in $L^2(\tau, T; V_g)$;

(v)$u_n(t) \to u(t)$ is strongly convergent in $L^2(\tau, T; H_g)$;

(vi)$u_n(t) \to u(t)$, $a\ e\ (x,t) \in \Omega \times [\tau, T]$.

From Lemma 1.3 of [29], we can see $\xi = |u|^{\beta-1}u$. Since $\bigcup_{n\in N^+} span\{w_1, w_2, \cdots, w_n\}$ is denseness in V_g, Taking the limit $n \to \infty$ on both sides of Eq (3.1), we can obtain that u is a weak solution of (1.1).

In the following, the solution is proved to be unique and continuously dependent on initial values. Let u_1, u_2 be two weak solutions of (1.1) corresponding to the initial values $u_{1\tau}, u_{2\tau} \in V_g$, We take $u = u_1 - u_2$, From (2.3) we have

$$\frac{1}{2}\frac{d}{dt}(|u|^2) + \nu\|u\|^2 + c(|u_1|^{\beta-1}u_1 - |u_2|^{\beta-1}u_2, u) + \nu(Ru, u) = \langle B(u_2) - B(u_1), u\rangle. \tag{3.9}$$

Using Hölder inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}(|u_1|^{\beta-1}u_1 - |u_2|^{\beta-1}u_2, u) &= \int_\Omega (|u_1|^{\beta-1}u_1 - |u_2|^{\beta-1}u_2)(u_1 - u_2)dx \\ &\geq \int_\Omega (|u_1|^{\beta+1} - |u_1|^\beta|u_2| - |u_2|^\beta u_1 + |u_2|^{\beta+1})dx \\ &= \int_\Omega (|u_1|^\beta - |u_2|^\beta)(|u_1| - |u_2|)dx \geq 0. \end{aligned} \tag{3.10}$$

we have

$$
\begin{aligned}
|\langle B(u_2)-B(u_1),u\rangle &= |\langle B(u_2,u_2-u_1)-B(u_1-u_2,u_1),u\rangle| \\
&\le C_1\|u_2\|\|u_2-u_1\|\|u\|+C_1\|u_1-u_2\|\|u_1\|\|u\| \\
&= C_1\|u\|^2(\|u_1+\|u_2\|) \\
&\le C_1\|u\|^2 \qquad (3.11)
\end{aligned}
$$

where $C_1>0$ is any constant.

$$
\begin{aligned}
\nu(Ru,u) &\le \nu\frac{\|\nabla g\|_\infty}{m_0\lambda_1^{1/2}}\|u\||u| \\
&\le \frac{\nu\|\nabla g\|_\infty}{2m_0\lambda_1^{1/2}}(\|u\|^2+|u|^2) \\
&= \alpha(\|u\|^2+|u|^2). \qquad (3.12)
\end{aligned}
$$

where $\alpha=\frac{\nu\|\nabla g\|_\infty}{2m_0\lambda_1^{1/2}}$. so

$$\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2\le C_1\|u\|^2+\alpha(\|u\|^2+|u|^2).$$

$$\frac{d}{dt}|u|^2+2(\nu-C_1-\alpha)\lambda_1|u|^2\le\alpha|u|^2.$$

Thus

$$\frac{d}{dt}|u|^2\le[\alpha-2\lambda_1(\nu-C_1-\alpha)]|u|^2.$$

Let C be a constant and $C=\alpha-2\lambda_1(\nu-C_1-\alpha)>0$, then

$$\frac{d}{dt}|u|^2\le C|u|^2.$$

Therefore

$$|u|_2^2\le e^{C(t-\tau)}|u_\tau|_2^2.$$

So we prove the continuous dependence on the initial value. When $u_{1\tau}=u_{2\tau}$, that is $u_\tau=0$, then the uniqueness of the solution holds.

4. The uniform attractor of 2D g-Navier-Stokes equations with nonlinear dampness in unbounded domain

In the following we have that the family of processes $\{U_f(t,\tau)\}$, $f\in\Sigma$ is uniformly bounded ($w.r.t. f\in\Sigma$) and it has uniform absorbing sets.

Firstly, the existence of uniformly absorbing sets is proved. Taking the inner product of (2.5) with u, we have

$$\frac{d}{dt}|u|^2+2\nu\|u\|^2+2c|u|^{\beta+1}=2\langle f,u\rangle-2\nu((\frac{\nabla g}{g}\cdot\nabla)u,u),$$

Then

$$\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 + 2c|u|^{\beta+1} \le \frac{\|f\|^2_{L^\infty(\mathrm{R}^+;V_g')}}{\nu} + \nu\|u\|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2,$$

For $\beta \ge 1$, we obtain

$$\frac{d}{dt}|u|^2 + \nu\lambda_1\gamma|u|^2 \le \frac{d}{dt}|u|^2 + \nu\gamma\|u\|^2 \le \frac{\|f\|^2_{L^\infty(\mathrm{R}^+;V_g')}}{\nu},$$

where $\gamma = 1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$ for sufficiently small $|\nabla g|_\infty$. Using the Gronwall inequality, we have

$$|u(t)|^2 \le |u_0|^2 e^{-\nu\lambda_1\gamma t} + \frac{\|f\|^2_{L^\infty(\mathrm{R}^+;V_g')}}{\nu^2\lambda_1\gamma}, \quad \forall\, t > 0.$$

and from

$$\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 + 2c|u|^{\beta+1} \le \frac{\|f\|^2_{L^\infty(\mathrm{R}^+;V_g')}}{\nu} + \nu\|u\|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2,$$

we have

$$\frac{d}{dt}|u|^2 + \nu\|u\|^2 + 2c|u|^{\beta+1} \le \frac{\|f\|^2_{L^\infty(\mathrm{R}^+;V_g')}}{\nu} + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2.$$

So

$$\frac{d}{dt}|u|^2 + \nu(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\|u\|^2 \le \frac{\|f\|^2_{L^\infty(\mathrm{R}^+;V_g')}}{\nu}. \tag{4.1}$$

Integrating (4.1) in s from 0 to t, we have

$$\frac{1}{t}\int_0^t \|u(s)\|^2 ds \le \frac{|u_0|^2}{t\nu\gamma} + \frac{\|f\|^2_{L^\infty(\mathrm{R}^+;V_g')}}{\nu^2\gamma}, \quad \forall\, t > 0.$$

then we know that the family of processes corresponding to u is uniformly bounded, and

$$B_0 = \{u \in H_g : |u| \le \rho_0 = \frac{1}{\nu}\sqrt{\frac{2}{\lambda_1\gamma}}\|f\|_{L^\infty(\mathrm{R}^+;V_g')}\}$$

is uniformly absorbing set in H_g. Then the following lemma holds.

Lemma 4.1 Let Σ be symbolic space, The process family corresponding to Eq (1.1) is uniformly bounded in $L^\infty(\mathrm{R}^+; H_g) \cap L^2(\tau, T; V_g)$ and there is a uniform absorbing set in H_g.

Lemma 4.2 Let $\tau \ge 0$, u_{τ_n} be the sequence in H_g that weakly converges to $u_\tau \in H_g$, $f_n \in \Sigma$ is the sequence in $L^\infty(\mathrm{R}^+; V_g')$ that weakly converges to f, then

(1) For $\forall\, t > \tau$, $U_{f_n}(t,\tau)u_{\tau_n}$ is weakly converges to $U_f(t,\tau)u_\tau$ in H_g;

(2) For $\forall\, T > \tau$, $U_{f_n}(\cdot,\tau)u_{\tau_n}$ is weakly converges to $U_f(\cdot,\tau)u_\tau$ in $L^2(\tau, T; V_g)$.

The proof is similar to Lemma 3.2 of [7], so it is omitted.

As we know, when u_{τ_n} is bounded in H_g, $f_n \subset \Sigma$, $t_n \to +\infty$. If $\{U_{f_n}(t_n,\tau)u_{\tau_n}\}$ is precompact in H_g, then the family of processes $\{U_f(t,\tau)\}$, $f \in \Sigma$ is asymptotically compact. So we construct an energy functional $[\cdot,\cdot] : V_g \times V_g \to \mathrm{R}$ as follows:

$$[u,v] = \nu((u,v)) + \frac{\nu}{2}((\frac{\nabla g}{g},\nabla)u,v) + \frac{\nu}{2}((\frac{\nabla g}{g},\nabla)v,u) - \frac{\nu\lambda_1}{4}(u,v) + c(|u|^{\beta-1}u,v), \quad \forall\, u,v \in V_g.$$

Obviously $[\cdot,\cdot]$ is bilinear and symmetric, and

$$\begin{aligned}[u]^2 = [u,u] &= \nu\|u\|^2 + \nu((\frac{\nabla g}{g}\cdot\nabla)u,u) - \frac{\nu\lambda_1}{4}|u|^2 + c|u|^{\beta+1} \\ &\geq \nu\|u\|^2 - \nu(\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + \frac{1}{4})\|u\|^2 \\ &\geq \frac{\nu}{2}\|u\|^2. \qquad (4.2)\end{aligned}$$

Let $|\nabla g|_\infty$ be sufficiently small in (4.2), such that $\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} < \frac{1}{4}$. Hence

$$\frac{\nu}{2}\|u\|^2 \leq [u]^2 \leq \frac{3}{2}\nu\|u\|^2, \qquad \forall\, u \in V_g.$$

Since

$$\frac{d}{dt}|u|^2 + \frac{\nu\lambda_1}{2}|u|^2 + 2[u]^2 = 2(f,u),$$

Given $u = u(t) = U_f(t,\tau)u_\tau,\ \ u_\tau \in H_g,\ \ t \geq \tau \geq 0$, Then we have

$$|U_f(t,\tau)u_\tau|^2 = |u_\tau|^2 e^{-\nu\lambda_1(t-\tau)/2} + 2\int_\tau^t e^{-\nu\lambda_1(t-s)/2}((f,U_f(s,\tau)u_\tau) - [U_f(s,\tau)u_\tau]^2)ds.$$

That is $\forall\, u_\tau \in H_g,\ t \geq \tau \geq 0$, we obtain

$$\begin{aligned}|U_f(t,\tau)u_\tau|^2 &= |u_\tau|^2 e^{-\nu\lambda_1(t-\tau)/2} \\ &+2\int_0^{t-\tau} e^{-\nu\lambda_1(t-\tau-s)/2}((T(\tau)f(s),U_{T(\tau)f}(s,0)u_\tau) - [U_{T(\tau)f}(s,0)u_\tau]^2)ds.\end{aligned}$$

Lemma 4.3 Let $\{U_f(t,\tau)\}_{f\in\Sigma}$ is the family of processes of Eq (1.1), then $\{U_f(t,\tau)\}_{f\in\Sigma}$ is uniformly asymptotically compact.

Proof. Let $B \subset H_g$ is bounded, $u_{\tau_n} \in B,\ f_n \in \Sigma$ and $t_n \in \mathcal{R}^+$ is satisfied $t_n \to +\infty (n \to +\infty)$. From Lemma 4.1, we have a constant $M(B,\tau) > \tau$ and

$$U_f(t,\tau)B \subset B_0, \quad \forall\, t \geq M(B,\tau),\ \ f \in \Sigma.$$

There exists sufficiently large $t_n \geq M(B,\tau)$, such that $U_{f_n}(t_n,\tau)B \subset B_0$. then $\{U_{f_n}(t_n,\tau)u_{\tau_n}\}$ is weakly precompact in H_g. For $w \in B_0 \subset H_g$, we can deduce that $U_{f_{n^s}}(t_{n^s},\tau)u_{\tau_{n^s}}$ is weakly convergent to w in H_g. Similarly $\forall\, T > 0$ and $t_{n^s} \geq T + M(B,\tau)$, we obtain $U_{f_{n^s}}(t_{n^s} - T,\tau)u_{\tau_{n^s}} \in B_0$. The same to

$w_T \in B_0$, we can take n^s, $\forall T > 0$, so we have $u_{t_{n^s}} = U_{f_{n^s}}(t_{n^s} - T, \tau)u_{\tau_{n^s}}$ is weakly convergent to w_T in H_g. According to the definition of process and translation operator, we have

$$U_{f_{n^s}}(t_{n^s}, \tau) = U_{T(t_{n^s}-T)f_{n^s}}(T, 0) \circ U_{f_{n^s}}(t_{n^s} - T, \tau).$$

Let $g_{T,n^s} = T(t_{n^s} - T)f_{n^s}$, we denote $\lim_{n^s H_w}$ as weak limit in H_g, then

$$w = \lim_{n^s H_w} U_{f_{n^s}}(t_{n^s}, \tau)u_{\tau_{n^s}} = \lim_{n^s H_w} U_{g_{T,n^s}}(T, 0)u_{t_{n^s}} = U_{g_T}(T, 0)w_T,$$

thus

$$|w| \le \liminf_{n^s} |U_{f_{n^s}}(t_{n^s}, \tau)u_{\tau_{n^s}}| = \liminf_{n^s} |U_{g_{T,n^s}}(T, 0)u_{t_{n^s}}|.$$

Now we will prove

$$\limsup_{n^s} |U_{f_{n^s}}(t_{n^s}, \tau)u_{\tau_{n^s}}| \le |w|.$$

$\forall T > 0$, we have $w^k = U_{g_T^k}(T, 0)w_T$. When $t_{n^s} \ge T + M(B, \tau)$, we obtain

$$\begin{aligned} &U_{g^k_{T,n^s}}(T, 0)u_{t_{n^s}} \\ &= 2\int_0^t e^{-\nu\lambda_1(T-s)/2}((g^k_{T,n^s}(s), U_{g^k_{T,n^s}}(s, 0)u_{t_n^s}) - [U_{g^k_{T,n^s}}(s, 0)u_{t_{n^s}}]^2)ds + |u_{t_{n^s}}|^2 e^{-\nu\lambda_1 T/2} \end{aligned}$$

Obviously

$$\limsup_{n^s} (e^{-\nu\lambda_1 T/2}|u_{t_{n^s}}|^2) \le \rho_0^2 e^{-\nu\lambda_1 T/2}.$$

From Lemma 4.2, we obtain

$$\int_0^T e^{-\nu\lambda_1 T/2}[U_{g_T^k}(s, 0)w_T]^2 ds \le \liminf_{n^s} \int_0^T e^{-\nu\lambda_1 T/2}[U_{g^k_{T,n^s}}(s, 0)u_{t_{n^s}}]^2 ds.$$

So

$$\begin{aligned} &\limsup_{n^s} -2\int_0^T e^{-\nu\lambda_1 T/2}[U_{g^k_{T,n^s}}(s, 0)u_{t_{n^s}}]^2 ds \\ &= -2\liminf_{n^s} \int_0^T e^{-\nu\lambda_1 T/2}[U_{g^k_{T,n^s}}(s, 0)u_{t_{n^s}}]^2 ds \\ &\le -2\int_0^T e^{-\nu\lambda_1 T/2}[U_{g_T^k}(s, 0)w_T]^2 ds. \end{aligned}$$

For

$$\lim_{n^s\to\infty} \int_0^T e^{-\nu\lambda_1(T-s)/2}(g^k_{T,n^s}(s), U_{g^k_{T,n^s}}(s, 0)u_{t_{n^s}})ds = \int_0^T e^{-\nu\lambda_1(T-s)/2}(g_T^k(s), U_{g_T^k}(s, 0)w_T)ds$$

thus

$$\begin{aligned} \limsup_{n^s} |U_{g^k_{T,n^s}}(T, 0)u_{t_{n^s}}|^2 \le 2\int_0^T e^{-\nu\lambda_1(T-s)/2}((g_T^k(s), U_{g_T^s}(s, 0)w_T) - [U_{g_T^s}(s, 0)w_T]^2)ds \\ +\rho_0^2 e^{-\nu\lambda_1 T/2}. \end{aligned}$$

From $w^k = U_{g_T^s}(s,0)w_T$, we have

$$|w^k|^2 = |U_{g_T^s}(s,0)w_T|^2$$

$$= e^{-\nu\lambda_1 T/2}|w_T|^2 + 2\int_0^T e^{-\nu\lambda_1(T-s)/2}((g_t^k(s), U_{g_T^k}(s,0)w_T) - [U_{g_T^k}(s,0)w_T]^2)ds.$$

$\forall T > 0$, we have

$$\limsup_{n^s} |U_{g^k_{T,n^s}}(T,0)u_{t_{n^s}}|^2 \le |w^k|^2 + (\rho_0^2 - |w_T|^2)e^{-\nu\lambda_1 T/2} \le |w^k|^2 + \rho_0^2 e^{-\nu\lambda_1 T/2}.$$

From $w = U_{g_T}(T,0)w_T$, by the Lemma 3.3 of [7] and Lemma 4.2, we can obtain $w^k \to w$ in H_g. So there exists any sufficiently small $\varepsilon > 0$, such that $|w^k|^2 \le |w|^2 + \varepsilon$. Since

$$\sup_{n^s\to+\infty} |U_{f_{n^s}}(t_{n^s},\tau)u_{\tau_{n^s}}|^2 = \sup_{n^s\to+\infty} |U_{g_{T,n^s}}(T,0)u_{t_{n^s}}|^2 \le |w|^2 + \varepsilon + \rho_0^2 e^{-\nu\lambda_1 T/2}.$$

When $\varepsilon \to 0$, $T \to \infty$, we have

$$\limsup_{n^s} |U_{f_{n^s}}(t_{n^s},\tau)u_{\tau_{n^s}}|^2 \le |w|^2.$$

Let $B \subset H_g$ be any bounded set, we have

$$\omega_{\tau,\Sigma}(B) = \bigcap_{t\ge\tau}\overline{\bigcup_{f\in\Sigma}\bigcup_{s\ge t} U_f(s,\tau)B}.$$

and $v \in \omega_{\tau,\Sigma}(B)$ iff there exists a sequence $v_n \in B$, $f_n \in \Sigma$, $t_n \in [\tau,+\infty)$. When $n \to \infty$, we have $t_n \to +\infty$ and $U_{f_n}(t_n,\tau)v_n \to v$ in H_g. When $\{U_f(t,\tau)\}$, $f \in \Sigma\}$ is uniformly asymptotically compact, $t \to +\infty$, we have

$$\sup_{f\in\Sigma} dist_{H_g}(U_f(t,\tau)B, \omega_{\tau,\Sigma}(B)) \to 0.$$

We will obtain the minimization of the uniform attractor in the following.

Lemma 4.4 Let $\{U_f(t,\tau)\}$, $f \in \Sigma$ is any the family of processes, B_0 is uniformly absorbing set, $\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0)$. then $\mathcal{A}_\Sigma$ is contained in any uniform absorbing set of $\{U_f(t,\tau)\}$, $f \in \Sigma$.

Proof. $\forall\ \tau > 0$, $\forall\ B \subset H_g$, Suppose there is another bounded closed set $P \subset H_g$ which satisfies

$$\lim_{t\to\infty}\sup_{f\in\Sigma} dist_{H_g}(U_f(t,\tau)B, P) = 0,$$

where $\mathcal{A}_\Sigma$ is not contained in the P. We deduce there is at least one $v \in \mathcal{A}_\Sigma$ and $v \notin P$. Since $v \in \mathcal{A}_\Sigma = \omega_{0,\mathcal{F}}(B_0)$, From the definition of the uniform ω limit set, there is a sequence $v_n \in B$, $f_n \in \Sigma$, $t_n \in [\tau,+\infty)$, as $n \to \infty$, we have $t_n \to +\infty$, then $U_{f_n}(t_n,0)v_n \to v$ is obtained in H_g. Given $\tilde{v}_n = U_{f_n}(t_n,0)v_n$, when $n \to +\infty$, There must be $U_{f_n}(t_n,\tau)\tilde{v}_n \to v$. Let $\tilde{v}_n \in B$, then we obtain $v \in P$, It is contradiction, so $\mathcal{A}_\Sigma \subset P$.

Theorem 4.1 Let $\{U_f(t,\tau)$, $f \in \Sigma\}$ is a family of processes of Eq (1.1), Then the process family has a unique compact uniform attractor $\mathcal{A}_\Sigma \subset H_g$. where $\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0)$, B_0 is any uniform absorbing set corresponding to a family of processes.

5. The dimension estimation of the uniform attractor in the quasi-periodical case

When $f(x,t) = f(x, w_1(t), w_2(t), \ldots, w_k(t))$ is a quasi-periodic function, That is, there exists a set of rational independent real numbers $\alpha_1, \ldots, \alpha_k$ which satisfies $f(x, \alpha_1 t, \ldots, \alpha_i t + 2\pi, \ldots, \alpha_k t) = f(x, \alpha_1 t, \ldots, \alpha_i t, \ldots, \alpha_k t)$ $(1 \le i \le k)$. Here $w_1(t+\alpha_1) = w_1(t), w_2(t+\alpha_2) = w_2(t), \ldots, w_k(t+\alpha_k) = w_k(t)$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are rational independent.

Let $\alpha t = (\alpha_1 t, \ldots, \alpha_k t)$, $\alpha = (\alpha_1, \ldots, \alpha_k)$, $w(t) = (w_1(t), \ldots, w_k(t)) = [\alpha t + w_0] = (\alpha t + w_0) mod (2\pi)^k$, $w_0 = (w_{01}, \ldots, w_{0k}) \in T^k = [0, 2\pi]^k$, $F(x, w(t)) = f(x,t)$. we can obtain the following conclusion.

Theorem 5.1 Let $\mathcal{A}$ is the uniform attractor of (1.1), then its Hausdorff and Fractal dimensions are estimated as follows:

$$d_H(\mathcal{A}) \le \frac{4}{\nu m_1 \lambda_1}\left(\frac{C_q^2 |f|^2}{2\nu^3 \lambda_1 m_1 m_2} + \frac{G^2 k}{\nu m_1 \lambda_1}\right) + k + 1.$$

$$d_F(\mathcal{A}) \le \frac{16}{\nu m_1 \lambda_1}\left(\frac{C_q^2 |f|^2}{2\nu^3 \lambda_1 m_1 m_2} + \frac{G^2 k}{\nu m_1 \lambda_1}\right) + 2k + 2.$$

where

$$G = \left(\sum_{i=1}^{k} \left|\frac{\partial F}{\partial w_i}\right|^2_{BC(T^k, H_g)}\right)^{\frac{1}{2}},$$

$$m_1 = 1 + \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}, \quad m_2 = 1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}.$$

Proof. We transform the Eq (1.1) into the following forms of autonomous systems by semigroup $S(t)(u_0, w_0) = (U_{w_0}(t,0)u_0, T_1(t)w_0)$,

$$\frac{\partial u}{\partial t} - \nu \Delta_g u + B(u,u) + \nu\left(\frac{\nabla g}{g} \cdot \nabla\right)u + c|u|^{\beta-1}u + \nabla p = F(x, w(t)) \tag{5.1}$$

$$\frac{d}{dt} w(t) = \alpha \tag{5.2}$$

$$u|_{t=0} = u_0, \quad u(t)|_{t=0} = w_0, \quad u_0 \in H_g, \quad w_0 \in T^k. \tag{5.3}$$

Let $y(t) = (u(x,t), w(t))^T$, $M(y(t)) = (\nu \Delta_g u - B(u,u) - \nu(\frac{\nabla g}{g} \cdot \nabla)u - c|u|^{\beta-1}u - \nabla p + F(x, w(t)), \alpha)^T$.
Then we can write the Eqs (5.1) and (5.2) as follows,

$$\frac{\partial y(t)}{\partial t} = M(y(t)) \tag{5.4}$$

$$y(t)|_{t=0} = y_0 = (u_0, w_0) \tag{5.5}$$

$\forall y_0 \in \mathcal{A}$, where $y(t) = (u(t), w(t))^T$ is the solution of Eqs (5.1) and (5.2) and y_0 as initial value. The linearized equation of (5.1) in $y(t)$ is

$$\frac{\partial z(t)}{\partial t} = M'(y(t))z \tag{5.6}$$

$$z(t)|_{t=0} = z_0 \tag{5.7}$$

In the equation of (5.6), $z(t) = (v(t), w(t))^T$, $\mu(t) = (\mu_1(t), \cdots, \mu_k(t))$, $z_0 = (v_0, \mu_0)^T \in H_g \times T^k$,

$$M'(y(t))z = (\nu \Delta_g v - B(v,u) - B(u,v) - \nu\left(\frac{\nabla g}{g} \cdot \nabla\right)v - c|u|^{\beta-1}v - \nabla \tilde{p} + F'_w u, 0)^T.$$

while

$$(M^{'}(y(t))z,z) = -\nu\|v\|^2 - b(v,u,v) - \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|v\|^2 - c|u|^{\beta-1}|v|^2 - \int_\Omega F_w^{'}u\cdot vdx$$

$$\leq -\nu\|v\|^2 + \int_\Omega |\nabla u|\cdot|v|^2dx + \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|v\|^2 + \frac{c|u|^{\beta-1}}{\lambda_1}\|v\|^2 + \int_\Omega |F_w^{'}u|\cdot|v|dx$$

Let

$$G = (\sum_{i=1}^k |\frac{\partial F}{\partial w_i}|^2_{BC(T^k,H)})^{1/2},$$

then

$$(M^{'}(y(t))z,z) \leq -\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \frac{c|u|^{\beta-1}}{\lambda_1})\|v\|^2 + \int_\Omega |\nabla u|\cdot|v|^2dx + \frac{bG}{2}|v|^2 + \frac{G}{2b}|u|^2.$$

b is any positive constant, Let $(M(y(t))z,z) = (M_1v,v) + (M_2\mu,\mu)$,

$$M_1v = -\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \frac{c|u|^{\beta-1}}{\lambda_1})\Delta_g v + (|\nabla u| + \frac{bG}{2})v, \quad M_2\mu = \frac{G}{2b}I_k\mu.$$

I_k is identity operator in R^k. Thus operator

$$\tilde{M}_1 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

is block operator, $(v_1,0),\cdots,(v_{n-k},0)$ are respectively solution of (2.7) with $(\xi_1,0),\cdots,(\xi_{n-k},0)$ as the initial value, where $\xi_1,\cdots,\xi_{n-k}$ is linearly independent basis in H_g, $\phi_1,\cdots,\phi_{n-k}$ is unit orthogonal basis of $span\{v_1,\cdots,v_{n-k}\}$, $\tilde{\mu}_{n-k+1},\cdots,\tilde{\mu}_n$ is unit orthogonal basis of R^k, then $(\phi_1,0),\cdots,(\phi_{n-k},0)$, $(0,\tilde{\mu}_{n-k+1}),\cdots,(0,\tilde{\mu}_n)$ is unit orthogonal basis of $H_g\times\mathrm{R}^k$. Let $\theta_i = (\phi_i,0)$ $(i = 1,\cdots,n-k), v_i = (0,\widetilde{\mu_i})$, $(n-k+1 \leq i \leq n)$.

$$q_n = \lim_{T\to\infty}\inf\sup_{y_0\in\mathcal{A}}(\frac{1}{T}\int_0^T\sum_1^n(M^{'}(y(s))\theta_i,\theta_i)ds).$$

$$\sum_1^n(M^{'}(y(s))\theta_i,\theta_i) \leq -\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \frac{c|u|^{\beta-1}}{\lambda_1})\sum_{i=1}^{n-k}\|\varphi_i\|^2 + \|u\|(\int_\Omega(\sum_{i=1}^{n-k}|\phi_i|^2)^2dx)^{1/2} + \frac{bG}{2}(n-k) + \frac{G}{2b}k.$$

We make $m_1 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \frac{c|u|^{\beta-1}}{\lambda_1}$, then

$$\sum_1^n(M^{'}(y(s))\theta_i,\theta_i) \leq -\nu m_1\sum_{i=1}^{n-k}\|\varphi_i\|^2 + \|u\|(\int_\Omega(\sum_{i=1}^{n-k}|\phi_i|^2)^2dx)^{1/2} + \frac{bG}{2}(n-k) + \frac{G}{2b}k$$

$$\leq -\frac{\nu m_1}{2}\sum_{i=1}^{n-k}\|\varphi_i\|^2 + \frac{C_q^2}{2\nu m_1}\|u\|^2 + \frac{bG}{2}(n-k) + \frac{G}{2b}k.$$

We take $b = \frac{\nu m_1\lambda_1}{2G}$, then

$$\sum_1^n(M^{'}(y(s))\theta_i,\theta_i) \leq -\frac{\nu m_1\lambda_1}{4}(n-k) + \frac{C_q^2}{2\nu m_1}\|u\|^2 + \frac{G^2k}{\nu m_1\lambda_1}.$$

Since

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 = (f, u) - \nu((\frac{\nabla g}{g} \cdot \nabla)u, u) - \nu(|u|^{\beta-1}u, u),$$

$$\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 \leq \frac{|f|^2}{\nu\lambda_1} + \nu\lambda_1|u|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2 + \frac{\nu|u|^{\beta-1}}{\lambda_1}\|u\|^2.$$

That is

$$\frac{d}{dt}|u|^2 + \nu(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \frac{|u|^{\beta-1}}{\lambda_1})\|u\|^2 \leq \frac{|f|^2}{\nu\lambda_1}.$$

We take $m_2 = 1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \frac{|u|^{\beta-1}}{\lambda_1}$, then

$$\frac{d}{dt}|u|^2 + \nu m_2\|u\|^2 \leq \frac{|f|^2}{\nu\lambda_1}.$$

So $\|u\|^2 \leq \frac{|f|^2}{\nu^2\lambda_1 m_2}$. therefore

$$q_n \leq -\frac{\nu m_1\lambda_1}{4}(n-k) + \frac{C_q^2|f|^2}{2\nu^3\lambda_1 m_1 m_2} + \frac{G^2 k}{\nu m_1\lambda_1}.$$

Let

$$n_0 = [\frac{4}{\nu m_1\lambda_1}(\frac{C_q^2|f|^2}{2\nu^3\lambda_1 m_1 m_2} + \frac{G^2 k}{\nu m_1\lambda_1})]_* + k + 1,$$

where $[\cdot]_*$ denotes trunc, then we have $q_{n_0} < 0$. so

$$d_H(\mathcal{A}) \leq \frac{4}{\nu m_1\lambda_1}(\frac{C_q^2|f|^2}{2\nu^3\lambda_1 m_1 m_2} + \frac{G^2 k}{\nu m_1\lambda_1}) + k + 1.$$

We take another

$$n_1 = [\frac{8}{\nu m_1\lambda_1}(\frac{C_q^2|f|^2}{2\nu^3\lambda_1 m_1 m_2} + \frac{G^2 k}{\nu m_1\lambda_1})]_* + k + 1,$$

then $q_{n_1} < 0$, and $\max_{1\leq j\leq n_1-1}\frac{(q_j)_+}{q_{n_1}} < 1$, thus

$$d_F(\mathcal{A}) \leq \frac{16}{\nu m_1\lambda_1}(\frac{C_q^2|f|^2}{2\nu^3\lambda_1 m_1 m_2} + \frac{G^2 k}{\nu m_1\lambda_1}) + 2k + 2.$$

6. Conclusions

In this paper, using a priori estimates of the solutions and the energy equation method, we show how to control the nonlinear dampness and obtain the uniform attractor of the g-Navier-Stokes equation on unbounded domain. Meanwhile, the dimension of the uniform attractor is estimated in the quasi-periodic case. The methods in this paper can bring some inspiration for the research of 3D Navier-Stokes equations in the future.

From a theoretical point of view, it is important to analysis the connection between Navier-Stokes equations and g-Navier-Stokes equations. So it is of great significance to study the dynamics for the g-Navier-Stokes equations. To obtain more research results for the study of g-Navier-Stokes equations in the next research, we may continue the research in this line, extending the case of Lebesgue space L^2 to the case of $L^{2,\lambda}$, for suitable $0 < \lambda < 2$. On the other hand, we may consider that the pullback asymptotic behavior of solutions for 2D g-Navier-Stokes equations with nonlinear dampness on the unbounded domain.

Acknowledgments

The author would like to thank the referees for the helpful suggestions. This work is supported by the National Natural Science Foundation of China (NSFC) (Grant No. 11971378).

Conflict of interest

The authors declare this work does not have any conflicts of interest.

References

1. G. R. Sell, Global attractor for the three dimensional Navier-Stokes equations, *J. Dynam. Differ. Equations*, **8** (1996), 1–33. https://doi.org/10.1007/BF02218613
2. R. Temam, *Infnite-Dimensional Dynamical Systems in Mechanics and Physics*, 2ed edition, Springer-Verlag, New York, 1997.
3. J. M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, *J. Nonlinear Sci.*, **7** (1997), 475–502. http://dx.doi.org/10.1007/s003329900050
4. F. Flandoli, B. Schmalfuβ, Weak solutions and attractors for three-dimensional Navier-Stokes equations with nonregular force, *J. Dyn. Differ. Equations*, **11** (1999), 355–398. https://doi.org/10.1023/A:1021937715194
5. J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction To Dissipative Parabolic Pdes And The Theory Of Global Attractors*, Cambridge University Press, 2001.
6. V. V. Chepyzhov, M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, 2002.
7. Y. R. Hou, K. T. Li, The uniform attractor for the 2D non-autonomous Navier-Stokes flow in some unbounded domain, *Nonlinear Anal.*, **58** (2004), 609–630. https://doi.org/10.1016/j.na.2004.02.031
8. A. Cheskidov, C. Foias, On global attractors of the 3D Navier-Stokes equations, *J. Differ. Equations*, **231** (2006), 714–754. https://doi.org/10.1016/j.jde.2006.08.021
9. A. V. Kapustyan, J. Valero, Weak and strong attractors for the 3D Navier-Stokes system, *J. Differ. Equations*, **240** (2007), 249–278. https://doi.org/10.1016/j.jde.2007.06.008
10. X. J. Cai, Q. S. Jiu, Weak and strong solutions for the incompressible Navier-Stokes equations with damping, *J. Math. Anal. Appl.*, **343** (2008), 799–809. https://doi.org/10.1016/j.jmaa.2008.01.041
11. A. Cheskidov, S. S. Lu, Uniform global attractors for the nonautonomous 3D Navier-Stokes equations, *Adv. Math.*, **267** (2014), 277–306. https://doi.org/10.1016/j.aim.2014.09.005

12. X. L. Song, Y. R. Hou, Uniform attractors for three-dimensional Navier-Stokes equations with nonlinear damping, *J. Math. Anal. Appl.*, **422** (2015), 337–351. https://doi.org/10.1016/j.jmaa.2014.08.044

13. A. M. Alghamdi, S. Gala, M. A. Ragusa, Global regularity for the 3D micropolar fluid flows, *Filomat*, **36** (2022), 1967–1970. https://doi.org/10.2298/FIL2206967A

14. E. M. Elsayed, R. S. Shah, K. Nonlaopon, The analysis of the fractional-order Navier-Stokes equations by a novel approach, *J. Function Spaces*, **2022** (2022). https://doi.org/10.1155/2022/8979447

15. H. Fang, Y. H. Fan, Y. P. Zhou, Energy equality for the compressible Navier-Stokes-Korteweg equations, *AIMS Math.*, **7** (2022), 5808–5820. https://doi.org/10.3934/math.2022321

16. J. Roh, *g-Navier-Stokes Equations*, Ph.D. thesis, Minnesota University, 2001.

17. H. O. Bae, J. Roh, Existence of solutions of the g-Navier-Stokes equations, Taiwan. *J. Math*, **8** (2004), 85–102. https://doi.org/10.11650/twjm/1500558459

18. J. Roh, Dynamics of the g-Navier-Stokes equations, *J. Differ. Equations*, **211** (2005), 452–484. https://doi.org/10.1016/j.jde.2004.08.016

19. M. Kwak, H. Kweana, J. Roh, The dimension of attractor of the 2D g-Navier-Stokes equations, *J. Math. Anal. Appl*, **315** (2006), 435–461. https://doi.org/10.1016/j.jmaa.2005.04.050

20. J. P. Jiang, Y. R. Hou, The global attractor of g-Navier-Stokes equations with linear dampness on R^2, *Appl. Math. Comput*, **215** (2009), 1068–1076. https://doi.org/10.1016/j.amc.2009.06.035

21. J. P. Jiang, X. X. Wang, Global attractor of 2D autonomous g-Navier-Stokes equations, *Appl. Math. Mech.*, **34** (2013), 385–394. https://doi.org/10.1007/s10483-013-1678-7

22. J. P. Jiang, Y. R. Hou, Pullback attractor of 2D non-autonomous g-Navier-Stokes equations on some bounded domains, *Appl. Math. Mech.*, **3** (2010), 697–708. https://doi.org/10.1007/s10483-010-1304-x

23. J. P. Jiang, Y. R. Hou, X. X. Wang, Pullback attractor of 2D nonautonomous g-Navier-Stokes equations with linear dampness, *Appl. Math. Mech.*, **32** (2011), 151–166. https://doi.org/10.1007/s10483-011-1402-x

24. J. P. Jiang, Y. R. Hou, X. X. Wang, The pullback asymptotic behavior of the solutions for 2D nonautonomous g-Navier-Stokes equations, *Adv. Appl. Math. Mech.*, **4** (2012), 223–237. https://doi.org/10.4208/aamm.10-m1071

25. C. T. Anh, D. T. Quyet, Long-time behavior for 2D non-autonomous g-Navier-Stokes equations, *Ann. Polonici Math*, **103** (2012), 277–302. https://doi.org/10.4064/ap103-3-5

26. C. T. Anh, N. V. Thanh, N. V. Tuan, On the stability of solutions to stochastic 2D g-Navier-Stokes equations with fnite delays, *Random Oper. Stoch. Equations*, **25** (2017), 1–14. https://doi.org/10.1515/rose-2017-0016

27. D. T. Quyet, Pullback attractors for strong solutions of 2D non-autonomous g-Navier-Stokes equations, *Acta Math. Vietnam*, **40** (2015), 637–651. https://doi.org/10.1007/s40306-014-0073-0

28. X. X. Wang, J. P. Jiang, The long time behavior of 2D non-autonomous g-Navier-Stokes equations with weakly dampness and time delay, *J. Function Space*, **2022** (2022). https://doi.org/10.1155/2022/2034264

29. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Nonlineaires*, Paris, 1969.