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Research article

Modeling and analysis of release strategies of sterile mosquitoes incorporating stage and sex structure of wild ones

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Abstract: This paper proposes and studies a switched interactive model of wild and sterile mosquitoes with stage and sex structure. Sterile males are released periodically and impulsively and remain sexually active for time $\bar{T}$. We investigate the dynamical behavior of the system when the release period T is shorter than the sexual lifespan $\bar{T}$, corresponding to a relatively frequent release. We first determine two important thresholds, m_1^* and m_2^*, for the release amount m and prove the exponential asymptotic stability of the extinction equilibrium. Using fixed point theory, we establish the existence of positive periodic solutions for $0 < m < m_1^*$ and $m_1^* \leq m < m_2^*$. Furthermore, by applying the comparison theorem of monotone systems, we demonstrate that the extinction equilibrium is globally asymptotically stable when $m \geq m_2^*$. Finally, numerical examples are presented to confirm our theoretical results.

Keywords: switched system; extinction equilibrium; periodic solution; stability

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1. Introduction

Mosquito-borne infectious diseases are spread by mosquitoes and are prevalent worldwide, including malaria, dengue fever, filariasis, yellow fever, and other highly harmful infectious diseases. Research shows that there are more than 80 such diseases, many of which have caused significant losses to human lives and health in different countries and regions. In recent years, the risk of mosquito-borne infectious disease outbreaks has increased due to climate change, as well as the rise of commercial trade and personnel exchanges.

Due to the lack of an effective vaccine for mosquito-borne diseases, controlling the population of mosquitoes has become crucial in curbing the spread of such diseases. Traditional physicochemical control methods have significant disadvantages such as negative environmental effects and drug resistance. In recent years, new biological control approaches such as insect sterility technology (SIT) and its derivatives have been extensively studied and practiced. These approaches involve the release of sterile or *Wolbachia*-infected mosquitoes into the wild to disrupt or suppress the development of the wild mosquito population. Certain studies specifically explored the use of sterile insect release for targeted population control [1–3]. Other research considered how the endosymbiotic bacterium *Wolbachia* can induce resistance to the dengue virus in *Aedes* aegypti [4], and there has been research into the control of particular mosquito-borne diseases [5]. Furthermore, in [6], a combination of incompatible and sterile insect techniques has been implemented to eradicate wild mosquitoes. During the exploration of these technologies, detailed analyses have been conducted on specific population characteristics such as population diffusion, periodic changes, the Allee effect, and others [7–11]. To ensure successful suppression of mosquito populations, continuous release over a long period is often necessary. Researchers have studied release strategies for different situations through various methods. For instance, uninterrupted continuous release strategies, based on ODE or DDE models, have undergone extensive examination [12–15]. More recently, impulsive release strategies, which better align with practical operations, have received considerable attention. Switched systems have been widely employed in simulating these release strategies [16–20]. In addition, many researchers have shown a preference for impulsive differential dynamical systems [21–24]. Some have integrated the duration of sexual activity of sterile mosquitoes with impulsive release behavior to investigate release strategies that effectively suppress mosquito populations [25–27].

Typically, male sterile mosquitoes are released into the wild to mate with female wild mosquitoes, rendering them unable to lay or hatch eggs. In [18,26–28], the authors proposed a mathematical model that considers the sexual lifespan of sterile mosquitoes, including only those within their sexually active period. Due to the short sexual lifespan of sterile mosquitoes, natural deaths during the sexual activity period are often neglected. Instead of using a single equation to describe the change in the number of sterile mosquitoes, the number of sterile mosquitoes is used as a control function [15, 18, 26–29]. Based on these ideas, the authors provided a nearly complete characterization of the release strategy using the following model.

$$\frac{d\omega}{dt} = \frac{a\omega^2}{\omega + g} - \mu\omega - \xi(\omega + g)\omega \tag{1.1}$$

where $\omega(t)$ and $g(t)$ represent the number of wild mosquitoes and the number of sterile male mosquitoes in their sexually active period. The birth rate per wild mosquito, denoted by a, follows logistic growth. Additionally, ξ and μ represent the density-dependent and independent death rates, respectively.

We know that mosquito growth includes three aquatic stages, namely egg, larva, and pupa, as well as an adult stage. Intraspecific competition significantly affects the aquatic stages, while the adult stage is rarely affected. Building on model (1.1), the authors in [30] proposed and studied a stage-structured model for suppressing mosquito populations as follows:

$$\begin{cases} \frac{dJ}{dt} = \frac{\beta A^2}{A+g} - \alpha J - (\mu_0 + \xi J)J, \\ \frac{dA}{dt} = \alpha J - \mu_1 A, \end{cases} \tag{1.2}$$

where J and A represent the number of wild mosquitoes in the aquatic and adult stages, respectively, while g represents the number of sterile males. The birth rate of adults is denoted by β. μ_i and ξ represent the natural mortality and intraspecific competition coefficient, respectively, where $i = 0, 1$. The parameter α represents the emergence rate from larvae to adults.

As mentioned earlier, in order to suppress the wild mosquito population, male sterile mosquitoes are released in most scenarios. In addition, the role of female and male mosquitoes in interacting with sterile mosquitoes is different. Motivated by the previous research works, this paper aims to further investigate and develop a wild mosquito suppression model with both stage and sex structure.

The paper is organized as follows: In Section 2, we develop a mosquito population suppression model with both stage and sex structure. The model is a time-switched system with the number of sexually active sterile males serving as a control function. In Section 3, we first define two important thresholds for the release amount and then determine the conditions for the stability of the wild mosquito extinction equilibrium and the existence of positive periodic solutions. In Section 4, we provide numerical results to validate our theoretical findings. Finally, we conclude with a brief summary in Section 5.

2. Model formulation

L. Almeida et al. in [31] describes the dynamics of the mosquito population as follows

$$\begin{cases} \frac{dE}{dt} = \beta_E F(1-\frac{E}{K}) - (\nu_E + \delta_E)E, \\ \frac{dL}{dt} = \tau_E E - L(cL + \tau_E + \delta_L), \\ \frac{dP}{dt} = \tau_L L - (\tau_P + \delta_P)P, \\ \frac{dF}{dt} = \nu\tau_P P - \delta_F F, \\ \frac{dM}{dt} = (1-\nu)\tau_P P - \delta_M M, \end{cases} \tag{2.1}$$

where $E(t)$, $L(t)$, $P(t)$, $F(t)$ and $M(t)$ stands for the numbers of eggs, larvae, pupa, adult females and adult males at time t, respectively. The parameters β_E, K, and τ_E represent the oviposition rate, environmental capacities and hatching rate for eggs, respectively. Pupas will develop into males or females, and $\nu \in (0,1)$ reflects the corresponding proportion. τ_L, τ_P are transition rates, c is the intraspecific competition of larvae, and δ_E, δ_L, δ_P, δ_F and δ_M are the corresponding death rates.

To simplify the mosquito population model, we make the following assumptions, similar to those in [31]: (i) the dynamics of larvae and pupae are fast, and intraspecific competition at the larvae stage is negligible (*i.e.*, $c \ll 1$); (ii) the probability of a pupa developing into a female is the same as that of a male ($\nu = \frac{1}{2}$); (iii) male and female mosquitoes have the same death rate ($\delta_F = \delta_M$).

With these assumptions and the third and fourth functions of system (2.1), we obtain the equilibrium equations for larvae and pupae, which are:

$$P = \frac{\tau_L \tau_E E}{(\tau_P + \delta_P)(\tau_L + \delta_L)}, \quad L = \frac{\tau_E E}{\tau_L + \delta_L}. \tag{2.2}$$

After substituting Eq (2.2) into system (2.1) and introducing a release function $u(\cdot)$ for sterile male mosquitoes, we can derive the following interactive dynamical system describing the population dynamics of two kinds of mosquitoes

$$\begin{cases} \frac{dE}{dt} = \beta_E F(1-\frac{E}{K})\frac{F}{F+\gamma M_s} - (\nu_E + \delta_E)E, \\ \frac{dF}{dt} = \nu\beta_F E - \delta_F F, \\ \frac{dM_s}{dt} = u(\cdot) - \delta_M M, \end{cases} \tag{2.3}$$

where $M_s(t)$ is the number of sterile males in the field, γ measures the mating competitiveness of sterile males, and $\beta_F = \frac{\tau_P \tau_L \tau_E}{(\tau_P+\delta_P)(\tau_L+\delta_L)}$.

In this paper, we also only consider sexually active sterile mosquitoes in the dynamics. Let $M_s(t)$ be the number of sexually active sterile males at time t and ignore their natural death as in previous works [15, 18, 26–29]. By taking $M_s(t)$ as a control function, we convert the system (2.1) into

$$\begin{cases} \frac{dE}{dt} = \beta_E F(1-\frac{E}{K})\frac{F}{F+\gamma M_s} - (\nu_E+\delta_E)E, \\ \frac{dF}{dt} = \nu\beta_F E - \delta_F F. \end{cases} \tag{2.4}$$

Let T be the waiting time between two consecutive release, and sterile mosquitoes are released with an amount of m at time $t = kT$, $k = 0, 1, 2, \cdots$. We use $\bar{T}$ to represent the sexual lifespan of sterile mosquitoes, ignoring their natural death as in [15, 18, 26–29]. There are three possible cases for T and $\bar{T}$: $T = \bar{T}$, $T > \bar{T}$, and $T < \bar{T}$. Previous works [30] studied the effect of releasing sterile mosquitoes on wild mosquito populations in the first two cases.

In this work, we mainly study the last case: $T < \bar{T}$. In fact, experimental studies have indicated that the lifespan of sterile mosquitoes can be as long as 51 days [32] such as the *Wolbachia*-infected mosquitoes, and their sexual lifespan can reach 14 days [33]. Nonetheless, the release of sterile mosquitoes is relatively frequent, such as in the experimental field of Shazai Island in Guangzhou, China, where they were released three times a week [32]. Hence, it is meaningful to consider the situation where the sexual lifespan of sterile mosquitoes exceeds the release period, that is $T < \bar{T}$.

Similar to the discussion in [17], when $T < \bar{T}$, there must exist a unique positive integer p and a non-negative number q that satisfy

$$\bar{T} = pT + q, \tag{2.5}$$

where $q \in [0, T)$ and $p = [\bar{T}/T]$, which is the integer closest to $\bar{T}/T$ and not greater than $\bar{T}/T$. Since $pT \le \bar{T}$, we can obtain the first p releases

$$M_s(t) = (k+1)m, \quad t \in (kT, (k+1)T]$$

where $k = 0, 1, 2, \cdots, p-1$. From the $(p+1)$th release, it is easy to get

$$M_s(t) = \begin{cases} (p+1)m & kT < t \le kT + q, \\ pm & kT + q < t \le (k+1)T, \end{cases} \qquad \text{if } q \ne 0,$$

where $k = p, p+1, \cdots$. If $q = 0$, then $M_s(t) = pm$, and the characteristics of the system are completely consistent with the simple system of constant value continuous release. In the present work, we assume that $0 < q < T$. As we are studying the asymptotic behavior of the system, for the sake of analysis convenience and without loss of generality, we can further assume that for $k = 0, 1, 2, \cdots$,

$$M_s(t) = \begin{cases} (p+1)m & kT < t \le kT + q, \\ pm & kT + q < t \le (k+1)T. \end{cases} \tag{2.6}$$

Based on the above assumption, the model (2.4) is transformed as follows:

$$\begin{cases} \frac{dE}{dt} = \beta_E F(1-\frac{E}{K})\frac{F}{F+\gamma(p+1)m} - (\tau_E+\delta_E)E, \\ \frac{dF}{dt} = \nu\beta_F E - \delta_F F, \end{cases} \qquad t \in (kT, kT+q], \tag{2.7}$$

$$\begin{cases} \frac{dE}{dt} = \beta_E F(1-\frac{E}{K})\frac{F}{F+\gamma pm} - (\tau_E+\delta_E)E, \\ \frac{dF}{dt} = \nu\beta_F E - \delta_F F, \end{cases} \qquad t \in (kT+q,(k+1)T], \tag{2.8}$$

where $k = 0, 1, 2, \cdots$.

Population dynamics of wild mosquitoes with sterile mosquito release then depend on the continuous switching between systems (2.7) and (2.8). In this paper, based on the switched systems (2.7) and (2.8), we mainly investigate the extinction and periodic change of wild mosquito population.

3. Dynamics of the switched system

3.1. preliminaries

First, we consider the dynamic behavior of the wild mosquito population without the release of sterile mosquitoes. Let $M_s(t) \equiv 0$, then the system (2.4) or (2.7) and (2.8) can be rewritten as follows:

$$\begin{aligned} \frac{dE}{dt} &= \beta_E F\left(1-\frac{E}{K}\right) - (\tau_E+\delta_E)E, \\ \frac{dF}{dt} &= \nu\beta_F E - \delta_F F. \end{aligned} \tag{3.1}$$

Let $\Omega = \{(E,F)|0 \le E \le K, 0 \le F\}$. Obviously, Ω is a positive invariant set of systems (3.1), (2.7) and (2.8), and every solution of systems (3.1), (2.7) and (2.8) with nonnegative initial values is positive and bounded. Therefore, in this work, we will investigate the dynamics of these systems in Ω. Additionally, it is easy to verify that system (3.1) is a monotonic system on Ω, so there are no closed orbits. By defining the basic offspring number of the wild mosquito as

$$R_0 := \frac{\nu\beta_E\beta_F}{\delta_F(\tau_E+\delta_E)},$$

we list the following conclusions about the system (3.1):

Lemma 1. (*i*) If $R_0 \le 1$, then the extinction equilibrium $A_0(0,0)$ is the unique equilibrium of system (3.1) that is globally asymptotically stable in Ω.

(*ii*) If $R_0 > 1$, then $A_0(0,0)$ is unstable, and the unique positive equilibrium $A^*(E^*,F^*)$ of system (3.1) is globally asymptotically stable in Ω, where

$$E^* = (1-\frac{\delta_F(\tau_E+\delta_E)}{\nu\beta_F\beta_E})K = (1-\frac{1}{R_0})K, \quad F^* = \frac{\nu\beta_F}{\delta_F}(1-\frac{1}{R_0})K.$$

According to the necessity of releasing sterile mosquitoes, we will only consider the case $R_0 > 1$ in this paper.

For the convenience of illustration, the following system is introduced to uniformly express the two subsystems (2.7) and (2.8)

$$\begin{cases} \frac{dE}{dt} = \beta_E F(1-\frac{E}{K})\frac{F}{F+\gamma\tilde{p}m} - (\tau_E+\delta_E)E, \\ \frac{dF}{dt} = \nu\beta_F E - \delta_F F, \end{cases} \tag{3.2}$$

where $\tilde{p} = p+1$ or p.

To compute the jacobian matrix of the system (3.2), we obtain

$$J = \begin{pmatrix} \frac{-\beta_E F^2}{(F+\gamma\tilde{p}m)K} - (\tau_E + \delta_E) & \beta_E(1 - \frac{E}{K})(\frac{F}{\gamma\tilde{p}m+F} + \frac{\gamma\tilde{p}mF}{(\gamma\tilde{p}m+F)^2}) \\ \nu\beta_F & -\delta_F \end{pmatrix}. \tag{3.3}$$

It has non-negative extra-diagonal coefficients on $\Omega = \{(E, F)|0 \leq E \leq K, 0 \leq F\}$, indicating that the system (3.2) is monotone on Ω.

Clearly, the wild mosquito extinction equilibrium $A_0(0, 0)$ is an equilibrium of both systems (2.7) and (2.8). By using the general form (3.2) of systems (2.7) and (2.8), we can now calculate the positive equilibria by finding the positive roots of the following equations,

$$E = \frac{\delta_F}{\nu\beta_F}F, \quad \beta_E F(1 - \frac{E}{K})\frac{F}{F + \gamma\tilde{p}m} - (\tau_E + \delta_E)E = 0,$$

which can be further equivalently transformed into

$$G(F) = -R_0\frac{\delta_F F^2}{K\nu\beta_F} + (R_0 - 1)F - \gamma\tilde{p}m = 0. \tag{3.4}$$

It is easy to know that the discriminant of the above equation is

$$\Delta = (R_0 - 1)^2 - \frac{4\gamma\tilde{p}m\beta_E}{K(\tau_E + \delta_E)}, \tag{3.5}$$

then we define the following release threshold for the system (3.2)

$$m^*(\tilde{p}) \equiv \frac{K(R_0 - 1)^2(\tau_E + \delta_E)}{4\gamma\beta_E\tilde{p}}. \tag{3.6}$$

Lemma 2. Let $R_0 > 1$. The trivial equilibrium A_0 of system (3.2) is always locally asymptotically stable. Moreover,

(i) There exists $m^*(\tilde{p}) > 0$ such that the system (3.2) admits no positive equilibrium if $m > m^*(\tilde{p})$, one positive equilibrium if $m = m^*(\tilde{p})$, and two positive equilibria $A_{\cdot j}(E_{\cdot j}(\tilde{p}), F_{\cdot j}(\tilde{p})), j = 1, 2$ if $0 < m < m^*(\tilde{p})$, where 5

$$F_{\cdot j}(\tilde{p}) = \frac{(R_0 - 1) \mp \sqrt{(R_0 - 1)^2 - \frac{4\gamma\tilde{p}m\beta_E}{K(\tau_E+\delta_E)}}}{\frac{2\beta_E}{K(\tau_E+\delta_E)}}, \quad E_{\cdot j} = \frac{\delta_F F_{\cdot j}(\tilde{p})}{\nu\beta_F}, \quad j = 1, 2.$$

(ii) If $m > m^*(\tilde{p})$, then the trivial equilibrium A_0 is globally asymptotically stable in Ω. If $0 < m < m^*(\tilde{p})$, then $A_{\cdot 1}(E_{\cdot 1}(\tilde{p}), F_{\cdot 1}(\tilde{p}))$ is unstable, while $A_{\cdot 2}(E_{\cdot 2}(\tilde{p}), F_{\cdot 2}(\tilde{p}))$ is locally asymptotically stable. If $m = m^*(\tilde{p})$, then the unique positive equilibrium is a saddle node.

Proof. To calculate the Jacobian matrix of the system (3.2) at the trivial equilibrium $A_0(0, 0)$, we have

$$J_{A_0} = \begin{pmatrix} -(\tau_E + \delta_E) & 0 \\ \nu\beta_F & -\delta_F \end{pmatrix}.$$

It has two negative eigenvalues, therefore the trivial equilibrium A_0 is always locally asymptotically stable.

(i) By calculating the discriminant Δ in Eq (3.5), we can see that $\Delta = 0$ when $m = m^*(\tilde{p})$, $\Delta < 0$ when $m > m^*(\tilde{p})$, and $\Delta > 0$ when $0 < m < m^*(\tilde{p})$. Moreover, the root of the quadratic equation (3.4) must be positive roots if it exists. These observations lead to the conclusions in (i).

(ii) Since the system (3.2) is monotone on Ω, it has no closed orbits. If $m > m^*(\tilde{p})$, then the extinction equilibrium A_0 is the unique equilibrium which is stable, so it is globally asymptotically stable in Ω.

If $0 < m < m^*(\tilde{p})$, we calculate Jacobian matrix of the system (3.2) at $A_{\cdot j}$, $j = 1, 2$ by using jacobian matrix (3.3) and get

$$J_{A_{\cdot j}} = \begin{pmatrix} \frac{-\nu\beta_E\beta_F F_{\cdot j}}{\delta_F(\gamma\tilde{p}m+F_{\cdot j})} & \beta_E(1-\frac{\delta_F F_{\cdot j}}{K\nu\beta_F})(\frac{F_{\cdot j}}{\gamma\tilde{p}m+F_{\cdot j}}+\frac{\gamma\tilde{p}mF_{\cdot j}}{(\gamma\tilde{p}m+F_{\cdot j})^2}) \\ \nu\beta_F & -\delta_F \end{pmatrix}.$$

We can easily get $trJ_{A_{\cdot j}} = \frac{-\nu\beta_E\beta_F F_{\cdot j}}{\delta_F(m_1+F_{\cdot j})} - \delta_F < 0$ and

$$\begin{aligned} |J_{A_{\cdot j}}| &= \frac{\nu\beta_E\beta_F F_{\cdot j}}{\gamma\tilde{p}m+F_{\cdot j}} - [\frac{\nu\beta_E\beta_F F_{\cdot j}}{\gamma\tilde{p}m+F_{\cdot j}} + \frac{\nu\beta_E\beta_F\gamma\tilde{p}mF_{\cdot j}}{(\gamma\tilde{p}m+F_{\cdot j})^2} - \frac{\beta_E\delta_F F_{\cdot j}^2}{K(\gamma\tilde{p}m+F_{\cdot j})} - \frac{\beta_E\delta_F\gamma\tilde{p}mF_{\cdot j}^2}{K(\gamma\tilde{p}m+F_{\cdot j})^2}] \\ &= \frac{\beta_E\delta_F\gamma\tilde{p}mF_{\cdot j}^2}{K(\gamma\tilde{p}m+F_{\cdot j})^2} + \frac{\beta_E\delta_F F_{\cdot j}^2}{K(\gamma\tilde{p}m+F_{\cdot j})} - \frac{\nu\beta_E\beta_F\gamma\tilde{p}mF_{\cdot j}}{(\gamma\tilde{p}m1+F_{\cdot j})^2} \\ &= -\frac{\nu\beta_E\beta_F}{R_0}F_{\cdot j}\Phi'(F_{\cdot j}), \end{aligned} \tag{3.7}$$

where $\Phi(F) = \frac{-G(F)}{\gamma\tilde{p}m+F}$.

Since $-\Phi'(F) = -\frac{G(F)}{(\gamma\tilde{p}m+F)^2} + \frac{G'(F)}{(\gamma\tilde{p}m+F)}$, $G(F_{\cdot 1}) = G(F_{\cdot 2}) = 0$ and $G'(F_{\cdot 1}) > 0, G(F_{\cdot 2}) < 0$, we can obtain $|J(A_{\cdot 1})| < 0$ and $|J(A_{\cdot 2})| > 0$. Therefore, $A_{\cdot 1}(E_{\cdot 1}, F_{\cdot 1})$ is an unstable saddle, while $A_{\cdot 2}(E_{\cdot 2}, F_{\cdot 2})$ is locally asymptotically stable.

If the release amount m increases to $m = m^*(\tilde{p})$, then the two positive equilibria $A_{\cdot 1}$ and $A_{\cdot 2}$ merge into a single semi-stable saddle node. The proof is completed.

Remark 1. According to Lemma 2, the constant continuous release system (3.2) has a threshold $m(\tilde{p})$ for the release amount m. When the release amount $m > m(\tilde{p})$, the wild mosquito population in the field will be successfully suppressed.

Referring to the two subsystems (2.7) and (2.8), we denote their corresponding release thresholds as m_1^* and m_2^*, respectively, given by:

$$m_1^* = m^*(p+1) < m_2^* = m^*(p). \tag{3.8}$$

When there are two positive equilibria, the positive equilibria corresponding to systems (2.7) and (2.8) are denoted by $A_{1j}(E_{1j}, F_{1j})$ and $A_{2j}(E_{2j}, F_{2j})$, $j = 1, 2$, respectively, where

$$E_{1j} = E_{\cdot j}(p+1), \quad F_{1j} = F_{\cdot j}(p+1), \quad E_{2j} = E_{\cdot j}(p), \quad F_{2j} = F_{\cdot j}(p), \quad j = 1, 2.$$

Lemma 2 provides insights into the behavior of systems (2.7) and (2.8). We can now state the following result:

Lemma 3. Let $R_0 > 1$. The trivial equilibrium A_0 of system (2.7) (or (2.8)) is always locally asymptotically stable. In addition,

(*i*) If $0 < m < m_1^*$, system (2.7) (or (2.8)) has two positive equilibria $A_{11}(E_{11}, F_{11})$ and $A_{12}(E_{12}, F_{12})$ (or $A_{21}(E_{21}, F_{21})$ and $A_{22}(E_{22}, F_{22})$), where $A_{11}(E_{11}, F_{11})$ (or $A_{21}(E_{21}, F_{21})$) is unstable, and $A_{12}(E_{12}, F_{12})$ (or $A_{22}(E_{22}, F_{22})$) is locally asymptotically stable.

(*ii*) If $m_1^* < m < m_2^*$, system (2.8) has two positive equilibria $A_{21}(E_{21}, F_{21})$ and $A_{22}(E_{22}, F_{22})$, where $A_{21}(E_{21}, F_{21})$ is unstable, and $A_{22}(E_{22}, F_{22})$ is locally asymptotically stable. However, system (2.7) has no positive equilibrium.

(*iii*) If $m > m_2^*$, both systems (2.7) and (2.8) have no positive equilibrium, and the unique equilibrium $A_0(0, 0)$ is globally asymptotically stable in Ω.

3.2. Periodic and impulsive releases

When the release period of sterile mosquitoes is shorter than their sexual lifespan, that is, $T < \bar{T}$, the number of sterile mosquitoes that are sexual active will continually switch between two levels, as shown in function (2.6). The population dynamics of wild ones will then depend on the switched systems (2.7) and (2.8).

With the appearance of continual switching, the extinction equilibrium $A_0(0, 0)$ becomes the unique equilibrium of the systems (2.7) and (2.8). We will first discuss its stability.

Since $R_0 = \frac{\nu\beta_E\beta_F}{\delta_F(\tau_E+\delta_E)} > 1, \beta_F = \frac{\tau_P\tau_L\tau_E}{(\tau_P+\delta_P)(\tau_L+\delta_L)}$ and $\nu \in (0, 1)$, we get $\nu\beta_E\beta_F > \delta_F(\tau_E + \delta_E)$ and $\beta_F < \tau_E$, and then $\tau_E + \delta_E > \nu\beta_F$. Let

$$\alpha := \min\{(\tau_E + \delta_E) - \nu\beta_F, \delta_F\},$$

then for any given $0 < \sigma < 1$, denote

$$\ell := \frac{\gamma(p + 1)m\alpha\sigma}{\beta_E}, \quad \ell_1 := \frac{\gamma p m\alpha\sigma}{\beta_E}.$$

Theorem 1. Assuming $R_0 > 1$ and $\bar{T} = pT + q$, for any solution $(E(t), F(t))$ of systems (2.7) and (2.8) with $0 < E(0) + F(0) < \ell_1$, the inequality

$$0 \le E(t) + F(t) \le (E(0) + F(0))e^{-rt}, \ \ t > 0, \tag{3.9}$$

holds, where $r = (1 - \sigma)\alpha > 0$. Thus, the extinction equilibrium $A_0(0, 0)$ of systems (2.7) and (2.8) is exponentially asymptotically stable.

Proof. Suppose $(E(t), F(t))$ is a solution of the systems (2.7) and (2.8) in Ω with $0 < E(0) + F(0) < \ell_1$ for fixed $\sigma \in (0, 1)$. Let $P(t) := E(t) + F(t)$ for $t \ge 0$. According to the first subsystem (2.7), we have for $t \in [kT, kT + q]$, $k = 0, 1, 2, \cdots$,

$$\begin{aligned}
P'(t) &= \beta_E F(1 - \frac{E}{K})\frac{F}{F + \gamma(p + 1)m} - (\tau_E + \delta_E)E + \nu\beta_F E - \delta_F F \\
&\le \frac{\beta_E F^2}{\gamma(p + 1)m} - [(\tau_E + \delta_E) - \nu\beta_F]E - \delta_F F \\
&\le \frac{\beta_E}{\gamma(p + 1)m}P^2(t) - \alpha P(t) \\
&= [\frac{\beta_E}{\gamma(p + 1)m}P(t) - \alpha]P(t).
\end{aligned} \tag{3.10}$$

We claim that if $P(kT) < \ell_1 < \ell$, then $P(t) < \ell_1$ for all $t \in [kT, kT + q]$, $k = 0, 1, 2, \cdots$. Let's prove this by contradiction. Suppose it does not holds, then there must exist $\bar{t} \in (kT, kT + q]$ such that $P(t) < \ell_1 < \ell$ for $t \in [kT, \bar{t})$ and $P(\bar{t}) = \ell_1$, then $P'(\bar{t}) \geq 0$. However, from inequality (Eq 3.10), we deduce

$$P'(\bar{t}) \leq (\frac{\beta_E}{\gamma(p+1)m}\ell_1 - \alpha)\ell_1 = -(1 - \frac{p}{p+1}\sigma)\alpha\ell_1 < 0,$$

which leads a contradiction. Therefore, we have $P(t) < \ell_1$ for all $t \in [kT, kT+q]$ and $P'(t) \leq (\frac{\beta_E}{\gamma(p+1)m}\ell_1 - \alpha)P(t) = -(1 - \frac{p}{p+1}\sigma)\alpha P(t) < 0$. Then we can further deduce

$$P(t) \leq P(kT)e^{-(1-\frac{p}{p+1}\sigma)\alpha(t-kT)} < P(kT)e^{-(1-\frac{p}{p+1}\sigma)\alpha q}, \ t \in [kT, kT + q). \tag{3.11}$$

Similarly, for the second subsystem (2.8), we can obtain

$$P'(t) \leq [\frac{\beta_E}{\gamma p m}P(t) - \alpha]P(t), \ t \in [kT + q, (k+1)T).$$

Then if $P(kT + q) < \ell_1$, then $P(t) < \ell_1$ for all $t \in [kT + q, (k+1)T]$, $k = 0, 1, 2, \cdots$. In addition,

$$P(t) \leq P(kT + q)e^{-(1-\sigma)\alpha(t-kT)} < P(kT + q)e^{-(1-\sigma)\alpha(T-q)}, \ t \in [kT + q, (k+1)T). \tag{3.12}$$

Combining with inequalities (Eq 3.11) and (Eq 3.12), we know that

$$P(t) \leq P(kT)e^{-(1-\frac{p}{p+1}\sigma)\alpha q}e^{-(1-\sigma)\alpha(T-q)} = \xi P(kT), \ t \in [kT + q, (k+1)T),$$

where $\xi := e^{-(1-\sigma)\alpha T} < 1$.

Thus,

$$P((k+1)T) \leq \xi P(kT) \text{ and } P(kT) < \xi^k P(0), \ k = 0, 1, 2, \cdots.$$

If $t \in [kT, kT + q]$, then from inequality (Eq 3.11) we get

$$P(t) \leq P(0)\xi^k e^{-(1-\frac{p}{p+1}\sigma)\alpha(t-kT)} \leq P(0)e^{-((1-\sigma)\alpha)t} = P(0)e^{-rt}, \ t \in [kT, kT + q]. \tag{3.13}$$

If $t \in [kT + \bar{T}, (k+1)T]$, then from inequality (Eq 3.12) we get

$$P(t) \leq P(0)\xi^k e^{-(1-\frac{p}{p+1}\sigma)\alpha q}e^{-(1-\sigma)\alpha(t-(kT+q))} \leq P(0)e^{-rt}, \ t \in [kT + q, (k+1)T]. \tag{3.14}$$

By combining inequalities (Eq 3.13) and (Eq 3.14), we can see that the inequality (Eq 3.9) holds and $A_0(0, 0)$ is exponentially asymptotically stable. The proof is completed.

In Theorem 1, we show the exponential asymptotic stability of $A_0(0, 0)$. However, this stability is only a local feature. Due to the periodic release of sterile mosquitoes, we are concerned about the possibility of a periodic change in the wild mosquito population, specifically, whether the systems (2.7) and (2.8) has a positive periodic solution. In the following discussion, we focus on the existence of positive periodic solutions.

Based on Lemma 3, when $R_0 > 1$, $T < \bar{T}$, and $0 < m < m_1^*$, both subsystems (2.7) and (2.8) have a pair of positive equilibria, namely $A_{11}(E_{11}, F_{11})$, $A_{12}(E_{12}, F_{12})$ and $A_{21}(E_{21}, F_{21})$, $A_{22}(E_{22}, F_{22})$. It is easy to verify that $F_{22} > F_{12} > F_{11} > F_{21} > 0$ and $K > E_{22} > E_{12} > E_{11} > E_{21} > 0$. Let $\Omega_0 := [E_{12}, E_{22}] \times [F_{12}, F_{22}]$ be a rectangle in the EF-phase plane. Then we can get the following result.

Theorem 2. If $R_0 > 1$ and $0 < m < m_1^*$, then the systems (2.7) and (2.8) has a T-periodic solution in Ω_0. Suppose $(E(t), F(t))$ is a solution of the systems (2.7) and (2.8) with $E(t_0) \geq E_{11}$ and $F(t_0) \geq F_{11}$ for some $t_0 \geq 0$, then $E(t) \geq E_{11}$ and $F(t) \geq F_{11}$ for all $t > t_0$, and $dist((E(t), F(t)), \Omega_0) \to 0$ as $t \to \infty$. Moreover, if such a solution is T-periodic, then $(E(t), F(t)) \in \Omega_0$ for all $t \geq 0$.

Proof. For the conditions $R_0 > 1$, $T < \bar{T}$, and $0 < m < m_1^*$, we obtain the existence of A_{ij}, $i, j = 1, 2$. Let $\eta(E) := \frac{KE(\tau_E + \tau_E)}{K - E}$, $E \in [0, K]$. The vertical isoclines $E'(t) = 0$ of systems (2.7) and (2.8) in the first quadrant of the EF-phase plane are given by

$$F = L_1(E) = \frac{1}{2\beta_E}[\eta(E) + \sqrt{\eta^2(E) + 4\gamma m(p+1)\beta_E \eta(E)}],$$
$$F = L_2(E) = \frac{1}{2\beta_E}[\eta(E) + \sqrt{\eta^2(E) + 4\gamma m p \beta_E \eta(E)}],$$

respectively (see Figure 1). Since $L_1(E) > L_2(E)$, the curve of $F = L_1(E)$ is strictly above the curve of $F = L_2(E)$ for $E > 0$. Additionally, $F = G(E) := \frac{\nu\beta_E E}{\delta_F}$ is the shared horizontal isocline of systems (2.7) and (2.8).

Select any two positive numbers E_m and E_M satisfying $E_{11} < E_m < E_{12} < E_{22} < E_M < K$, then $F_m = G(E_m)$ and $F_M = G(E_M)$. Due to the strict monotonic increase property with respect to E of $G(E)$, it follows that $F_m < F_M$. Let Ω_1 be the closed rectangle with the vertices P_1, P_2, Q_1 and Q_2, where $P_1 = (E_m, F_m)$, $P_2 = (E_M, F_m)$, $Q_1 = (E_m, F_M)$, and $Q_2 = (E_M, F_M)$.

Clearly, $F_{11} < F_m < F_{12} < F_{22} < F_M$ and $\Omega_0 \subset \Omega_1$. Since $G(E)$ is strictly monotone increasing with respect to E, the segment $\overline{P_1P_2}$ lies strictly below the isocline $F = G(E)$ except for the point P_1, while the segment $\overline{Q_1Q_2}$ lies strictly above the isocline $F = G(E)$ except for the point Q_2. According to the second equation of systems (2.7) and (2.8), we have

$$\frac{dF}{dt} = \delta_F(G(E) - F) > \delta_F(G(E_m) - F_m) = 0, \quad E \in [E_m, E_M],$$
$$\frac{dF}{dt} = \delta_F(G(E) - F) < \delta_F(G(E_M) - F_M) = 0, \quad E \in [E_m, E_M).$$

Thus, the vector fields of (2.7) and (2.8) point into the closed rectangle Ω_1 on $\overline{P_1P_2}$ and $\overline{Q_1Q_2}$.

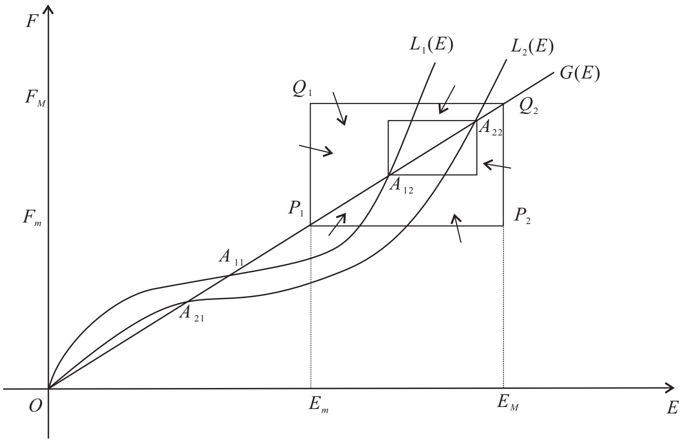

Figure 1. The phase-plane analysis in Theorem 2.

In addition, since $L_1(E)$ and $L_2(E)$ are both strictly monotone increasing with respect to E on $[0, K]$, we can see that the entire segment $\overline{P_1Q_1}$ lies strictly above the isocline $F = L_1(E)$, and the entire segment $\overline{P_2Q_2}$ lies strictly below the isocline $F = L_2(E)$. From system (2.7), we obtain

$$\begin{aligned}\frac{dE}{dt} &= \frac{\beta_E}{F + \gamma(p+1)m}(1 - \frac{E}{K})(F^2 - \frac{\gamma(p+1)m + F}{\beta_E}\eta(E))\\ &= \frac{\beta_E}{\gamma(p+1)m}(1 - \frac{E}{K})(F - L_1(E))(F + L_1^-(E)),\end{aligned}$$

where $L_1^-(E) = \frac{1}{2\beta_E}[-\eta(E) + \sqrt{\eta(E)^2 + 4\gamma(p+1)m\beta_E\eta(E)}] > 0$.

From the system (2.8), we can obtain

$$\begin{aligned}\frac{dE}{dt} &= \frac{\beta_E}{F + \gamma pm}(1 - \frac{E}{K})(F^2 - \frac{\gamma pm + F}{\beta_E}\eta(E))\\ &= \frac{\beta_E}{\gamma pm}(1 - \frac{E}{K})(F - L_2(E))(F + L_2^-(E))\end{aligned}$$

where $L_2^-(E) = \frac{1}{2\beta_E}[-\eta(E) + \sqrt{\eta(E)^2 + 4\gamma pm\beta_E\eta(E)}] > 0$.

Clearly, on the segment $\overline{P_1Q_1}$, we have $E \equiv E_m$, so $F > L_1(E_m)$ and $F > L_2(E_m)$. As a result, $\frac{dE}{dt} > 0$ holds for both systems (2.7) and (2.8) on $\overline{P_1Q_1}$. Similarly, on the segment $\overline{P_2Q_2}$, we have $F < L_1(E_M)$ and $F < L_2(E_M)$, leading to $\frac{dE}{dt} < 0$ for both systems (2.7) and (2.8) on this segment. Therefore, the vector fields of systems (2.7) and (2.8) point towards the inside of Ω_1 on both $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$.

In summary, Ω_1 represents a positive invariant set of the switched systems (2.7) and (2.8), and any point $A \in \Omega_1$ will have the trajectory of the systems (2.7) and (2.8) starting from it stay within Ω_1.

We define a map $\Phi : \Omega_1 \to \Omega_1$ as given in [30]:

$$\Phi(A) := (\psi_{T-q} \circ \phi_q)(A) = \psi_{T-q}(\phi_q(A)) \tag{3.15}$$

Here, $\phi_t(A)$ and $\psi_t(A)$ are the solutions of systems (2.7) and (2.8) starting from the point A, respectively. Since Φ is continuous and maps Ω_1 to Ω_1, by Brouwer's fixed point theorem, we know that Φ has a fixed point $\bar{A} \in \Omega_1$. Then, the solution of systems (2.7) and (2.8) passing through the point $\bar{A}$ is a continuous T-periodic solution whose trajectory lies entirely in Ω_1.

It is worth noting the arbitrariness of E_m and E_M. If $E_m \to E_{12}$ and $E_M \to E_{22}$, then $\Omega_1 \to \Omega_0$, and the continuous T-periodic solution of systems (2.7) and (2.8) in Ω_1 lies in Ω_0. If $E_m \to E_{11}$ and $E_M \to \infty$, then $\Omega_1 \to [E_{11}, \infty) \times [F_{11}, \infty)$. Since Ω_1 is an attraction domain, $[E_{11}, \infty) \times [F_{11}, \infty)$ is also a positively invariant set of systems (2.7) and (2.8). Then, for any solution of systems (2.7) and (2.8) with $E(t_0) \geq E_{11}$ and $F(t_0) \geq F_{11}$, it follows that $E(t) \geq E_{11}$ and $F(t) \geq F_{11}$ for all $t > t_0$. Moreover, $dist((E(t), F(t)), \Omega_0) \to 0$ as $t \to \infty$. Furthermore, if this solution is periodic, then $(E(t), F(t)) \in \Omega_0$. Thus, the proof is completed.

When we increase the release amount m to exceed the smaller threshold m_1^*, that is, $m_1^* \leq m < m_2^*$, we obtain the following result:

Theorem 3. Suppose $R_0 > 1$ and $m_1^* < m < m_2^*$. Then there exist positive constants $T_0 > 0$ and $\delta_0 > 0$ such that the systems (2.7) and (2.8) has a continuous and positive T-periodic solution when

$$q < T_0, \quad T - q < \delta_0. \tag{3.16}$$

Proof. According to Lemma 3, when $R_0 > 1$, $T < \bar{T}$ and $m_1^* < m < m_2^*$, the subsystem (2.8) has two positive equilibria A_{21} and A_{22}, while (2.7) has only the extinction equilibrium.

Similar to Theorem 2, we first select two arbitrary positive numbers E_m and E_M such that $0 < E_m < E_{22} < E_M$ and $G(E_M) < L_1(E_m)$ ($G(E)$ and $L_i(E), i = 1, 2$ are defined in the proof of Theorem 2). Let $F_m = G(E_m)$ and $F_M = G(E_M)$. Then we construct a closed rectangle Ω_1 with vertices at $P_1 = (E_m, F_m)$, $P_2 = (E_M, F_m)$, $Q_1 = (E_m, F_M)$, and $Q_2 = (E_M, F_M)$. In addition, let P_3 be the intersection point of the segment P_1P_2 and the isocline $F = L_2(E)$, and for any $E_0 \in (E_m, E_3)$, denote points $P_0 := (E_0, F_m)$ and $Q_0 := (E_0, F_M)$ (see Figure 2). Then Ω_1 can be regarded as a rectangle composed of two parts, namely, $\Omega_1 = \Omega_1^1 \bigcup \Omega_1^2$, where Ω_1^1 is the rectangle with P_1, P_0, Q_1, and Q_0 as its vertices, and Ω_1^2 is the rectangle with P_0, P_2, Q_0, and Q_2 as its vertices.

We define ω_1 and ω_2 as follows:

$$\begin{aligned} \omega_1 &:= \min\{G_1(E, F) : (E, F) \in \Omega_1^1\} < 0, \\ \omega_2 &:= \min\{G_2(E, F) : (E, F) \in \Omega_1^1\} > 0, \end{aligned} \tag{3.17}$$

where $G_1(E, F)$ and $G_2(E, F)$ are the right-hand side functions of the first equations of systems (2.7) and (2.8), respectively.

We assume that

$$q < T_0, \quad T - q > \delta_0, \tag{3.18}$$

where $T_0 := \frac{E_0 - E_m}{-\omega_1}$, $\delta_0 := \frac{E_0 - E_m}{\omega_2}$.

In the following, we will prove that the Poincare map Φ defined in Theorem 2 is continuous and maps Ω_1^2 to itself. Then, similarly to the proof of Theorem 2, we can deduce that the systems (2.7) and (2.8) has a continuous and positive T-periodic solution in Ω_1^2.

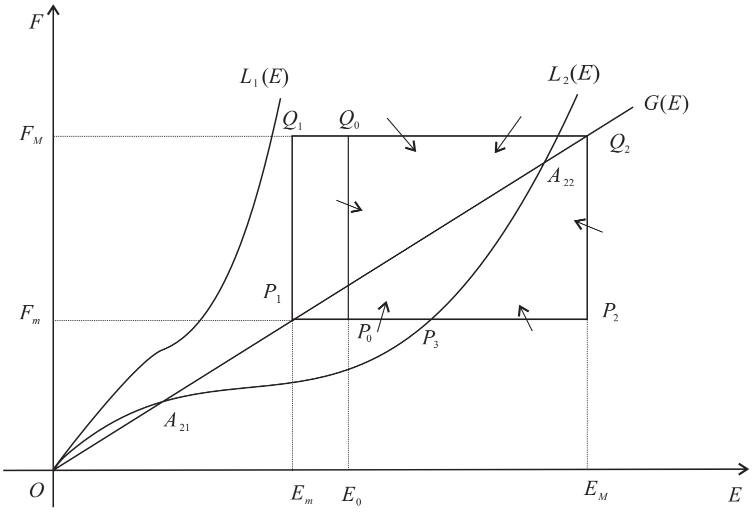

Figure 2. The phase-plane analysis in Theorem 3.

The vector fields of system (2.7) point towards the interior of Ω_1 along the segments $\overline{P_1P_2}$, $\overline{Q_1Q_2}$, and $\overline{P_2Q_2}$. Therefore, for any solution of system (2.7) with initial value $(E(0), F(0)) \in \Omega_1^2$, it can only cross the segment $\overline{P_0Q_0}$ if it leaves Ω_1. We will now show that this is impossible by contradiction.

Suppose there exists a time $t_0 \in [0, q)$ such that $(E(t_0), F(t_0)) \in \overline{P_0Q_0}$ and $(E(t), F(t)) \in \Omega_1^2$ for $t \in [0, t_0)$. Then, we have $E'(t_0) < 0$, which means that the solution immediately enters Ω_1^1 after t_0. When the solution leaves Ω_1^1, it can only go through the segment $\overline{P_1Q_1}$.

Now, assume there exists a time $\tilde{t} \in (t_0, q]$ such that $(E(\tilde{t}), F(\tilde{t})) \in \overline{P_1Q_1}$ and $(E(t), F(t)) \in \Omega_1^1$ for $t \in [t_0, \tilde{t})$. Then, we have $E'(t) = G_1(E(t), F(t)) \geq \omega_1$ for $t \in [t_0, \tilde{t}]$, and $E(\tilde{t}) = E_m \geq E(t_0) + \omega_1(\tilde{t} - t_0) \geq E_0 + \omega_1 q$. Since $\omega_1 < 0$ and $q < T_0$, we have $E_m < E_0 + \omega_1 T_0$, which is a contradiction. Therefore, any solution $(E(t), F(t))$ of system (2.7) with $(E(0), F(0)) \in \Omega_1^2$ will remain in Ω_1 for $t \in [0, q]$.

Next, we show that for any solution of system (2.8) with $(E(q), F(q)) \in \Omega_1$, the solution will stay in Ω_1 for $t \in [q, T]$ with $(E(T), F(T)) \in \Omega_1^2$. Noting that the vector fields of system (2.8) point to the inside of both Ω_1 and Ω_1^2, we see that both the rectangles Ω_1 and Ω_1^2 are positive invariant sets of system (2.8). Therefore, $(E(q), F(q)) \in \Omega_1$ implies $(E(t), F(t)) \in \Omega_1$ for all $t \in [q, T]$. Furthermore, we can claim that $(E(T), F(T)) \in \Omega_1^2$. If this does not hold, then $(E(t), F(t)) \in \Omega_1^1$ for all $t \in [q, T]$ due to the properties of the invariant set Ω_1^2. This implies that $E'(t) \geq \omega_2$ for $t \in [q, T]$ and $E(T) \geq E(q) + \omega_2(T - q)$. We further deduce that $E_0 - E_m \geq E(T) - E(q) \geq \omega_2(T - q)$, which contradicts the inequality (3.18).

In summary, we have shown that for any solution $(E(t), F(t))$ of systems (2.7) and (2.8) starting from the initial point $(E(0), F(0)) \in \Omega_1^2$, the solution will stay in Ω_1 for $t \in [0, T]$ and $(E(T), F(T)) \in \Omega_1^2$. Thus, the Poincare map $\Phi : \Omega_1^2 \to \Omega_1^2$ is well-defined and has at least one fixed point A^* in Ω_1^2. Then the solution of systems (2.7) and (2.8) through the point A^* is a continuous and positive T-periodic solution. The proof is completed.

Remark 2. If $m = m_1^*$, then the two equilibria $A_{11}(E_{11}, F_{11})$ and $A_{12}(E_{12}, F_{12})$ of (2.7) coincide into one, denoted by $A_{1c}^*(E_{1c}^*, F_{1c}^*)$. By slightly modifying the proof of Theorem 3 and selecting values of E_m and E_M such that $E_{1c}^* < E_m < E_{22} < E_M$ and $G(E_M) < L_1(E_m)$, we can similarly prove that the systems (2.7) and (2.8) has a positive T-periodic solution.

Next, we consider the case where the release amount m is further increased until $m \geq m_2^*$, and we present the following result.

Theorem 4. If $R_0 > 1$ and $m > m_2^*$, then the unique extinction equilibrium A_0 of systems (2.7) and (2.8) is globally asymptotically stable.

Proof. According to Theorem 1, if $R_0 > 1$ and $m > m_2^*$, the extinction equilibrium $A_0(0, 0)$ of systems (2.7) and (2.8) is locally asymptotically stable. We will now prove that it is also globally attractive.

To this end, we construct the following system:

$$\begin{cases} \frac{d\tilde{E}}{dt} = \beta_E \tilde{F}(1 - \frac{\tilde{E}}{K})\frac{\tilde{F}}{\tilde{F}+\gamma pm} - (\tau_E + \delta_E)\tilde{E}, \\ \frac{d\tilde{F}}{dt} = \nu\beta_F \tilde{E} - \delta_F \tilde{F}, \end{cases} \quad t \geq 0. \tag{3.19}$$

Similar to the discussion about system (3.1), we can show that system (3.19) is monotone on Ω. Furthermore, we have

$$\frac{dE}{dt} \leq \frac{d\tilde{E}}{dt}, \quad \frac{dF}{dt} \leq \frac{d\tilde{F}}{dt}, \quad t \in (kT, (k+1)T], k = 0, 1, 2, \cdots.$$

Thus, system (3.19) can be regarded as a comparison system for systems (2.7) and (2.8). With the same initial values $\tilde{E}(0) = E(0)$ and $\tilde{F}(0) = F(0)$, we obtain

$$0 \le E(t) \le \tilde{E}(t), \quad 0 \le F(t) \le \tilde{F}(t), \quad t \ge 0.$$

From Lemma 2, we know that if $R_0 > 1$ and $m > m_2^*$, the trivial equilibrium $(0,0)$ of system (3.19) is globally asymptotically stable. That is, for any non-negative initial value $(\tilde{E}(0), \tilde{F}(0))$, the solution of system (3.19) from it satisfies

$$\lim_{t\to\infty} \tilde{E}(t) = 0, \quad \lim_{t\to\infty} \tilde{F}(t) = 0.$$

By the comparison theorem, we can easily see that for any non-negative initial value $(E(0), F(0))$, the solution of systems (2.7) and (2.8) from it satisfies

$$\lim_{t\to\infty} E(t) = 0, \quad \lim_{t\to\infty} F(t) = 0.$$

Thus, the extinction equilibrium $A_0(0,0)$ of systems (2.7) and (2.8) is also globally attractive. The proof is completed.

Remark 3. If $m = m_2^*$, then the extinction equilibrium $A_0(0,0)$ of system (2.7) is globally asymptotically stable, meaning that every solution of system (2.7) will tend to the extinction equilibrium $A_0(0,0)$. Moreover, the two equilibria $A_{21}(E_{21}, F_{21})$ and $A_{22}(E_{22}, F_{22})$ of system (2.8) coincide into one, denoted by $A_{2c}^*(E_{2c}^*, F_{2c}^*)$. According to Lemma 2, this case presents a possible semi-stability, where the basin of attraction of $A_0(0,0)$ contains the interval $[0, A_{2c}^*)$, denoted as $\{(E,F) \in R_2^+ : 0 \le E < E_{2c}^*, 0 \le F < F_{2c}^*\}$, and the basin of attraction of $A_{2c}^*(E_{2c}^*, F_{2c}^*)$ contains the interval (A_{2c}^*, ∞), denoted as $\{(E,F) \in R_2^+ : E > E_{2c}^*, F > F_{2c}^*\}$. Due to the continual switching between two subsystems (2.7) and (2.8), any solution of systems (2.7) and (2.8) must enter $[0, A_{2c}^*)$ after some time and will eventually tend to the extinction equilibrium $A_0(0,0)$. Therefore, the extinction equilibrium $A_0(0,0)$ of systems (2.7) and (2.8) is globally asymptotically stable.

Remark 4. According to Theorems 1–4 and Remark 3, the periodic and impulsive release systems (2.7) and (2.8) has a threshold m_2^* for the release amount m, and only when the release amount $m \ge m_2^*$, can the wild mosquito population in the field be successfully suppressed.

4. Numerical simulation

This section aims to illustrate the theoretical results developed in the previous section through several numerical examples. We use the values of most model parameters from [31, 34], which are summarized in Table 1.

Another important parameter to be determined is the carrying capacity, denoted by K. We consider an island with an area of 74 hectares, as in [31, 34], with an estimated male population of about 69 males per hectare. If there are no sterile mosquitoes in the field and the wild population reaches a stable state, the number of adult wild males is assumed to be $M^* = F^* = 69 \times 74 = 5106$. Then, the number of eggs at the stable state is estimated as $E^* = \frac{\delta_F F^*}{\nu\beta_F} = 40848$, and the carrying capacity can be calculated as

$$K = \frac{E^*}{1 - \frac{(\tau_E+\delta_E)\delta_F}{\nu\beta_E\beta_F}} \approx 43641.$$

Table 1. Model parameter values from [31, 34].

Parameters	Value	Unit	Parameters	Value	Unit
β_E	10	day^{-1}	γ	1	-
τ_E	0.05	-	δ_E	0.03	day^{-1}
δ_F	0.04	day^{-1}	β_F	0.01	day^{-1}
ν	0.5	-			

For the sexual lifespan of sterile mosquitoes, we take the value from [17], and set $\bar{T} = 14$. Through direct calculation, we can get

$$R_0 := \frac{\nu\beta_E\beta_F}{\delta_F(\tau_E+\delta_E)} = 15.625 > 1,\ m^*(\tilde{p}) = \frac{K(1-\aleph_0)^2(\tau_E+\delta_E)}{4\beta_E\gamma\tilde{p}} \approx \frac{18669}{\tilde{p}}.$$

When no sterile ones are released, the unique positive equilibrium $A^*(E^*, F^*)$ is globally and uniformly asymptotically stable (see Figure 3(a)), which is consistent with the result in Lemma 1.

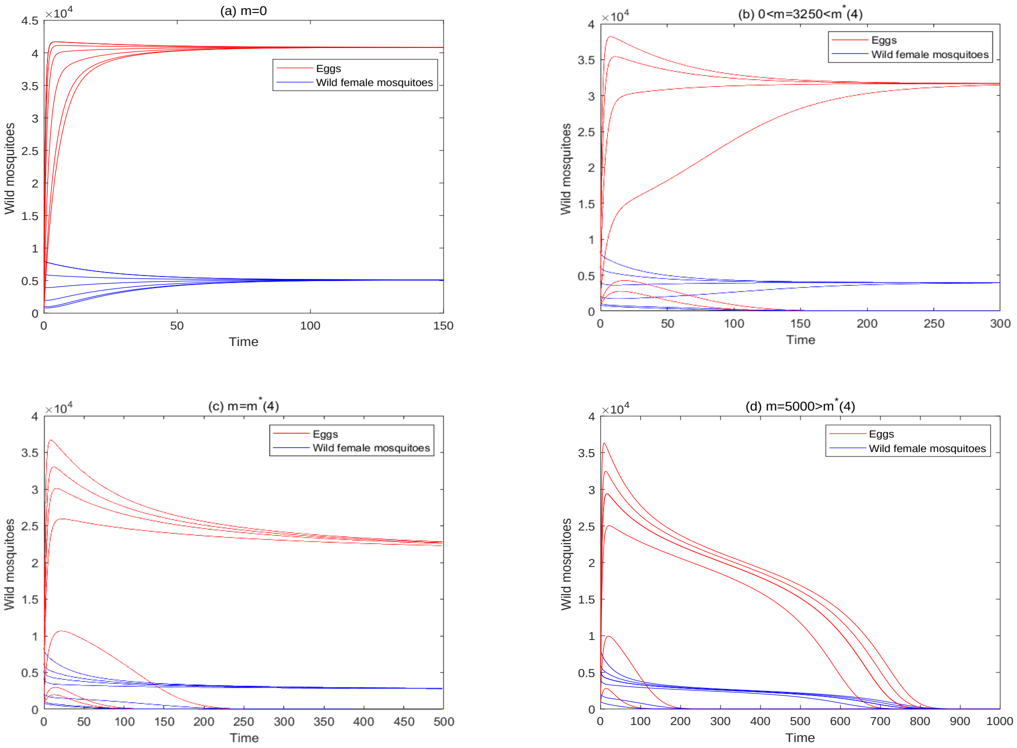

Figure 3. Dynamic phenomena of system (3.2). (a) Global stability of the positive equilibrium A^* with $m = 0$; (b) Bistability phenomenon with $m = 3250 \in (0, m^*(4))$; (c) Semi-stability phenomenon with $m = 4667 = m^*(4)$; (d) Global stability of the extinction equilibrium A_0 with $m = 5000 > m^*(4)$.

For the constant release system (3.2), we can take $\tilde{p} = 4$ without loss of generality, yielding $m^*(4) \approx 4667$. When $0 < m < m^*(4)$, the system (3.2) displays bistability, as shown in Figure 3(b), where the extinction equilibrium A_0 and a positive equilibrium are locally stable. Increasing the release amount to $m = m^*(4)$ results in a unique positive equilibrium that is semi-stable (see Figure 3(c)). However, if $m > m^*(4)$, then the unique equilibrium A_0 becomes globally stable (see Figure 3(d)).

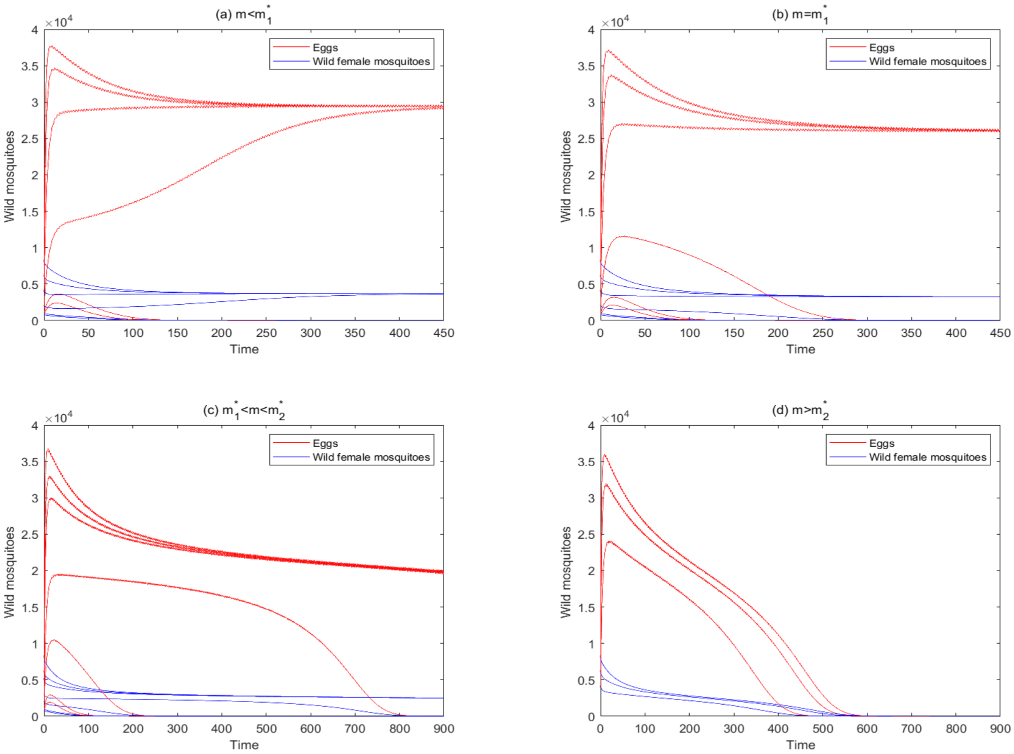

Figure 4. Dynamic phenomena of systems (2.7) and (2.8). (a)–(c) show the bistability of systems (2.7) and (2.8), which corresponds to the case of $0 < m = 3250 < m_1^*$, $m = m_1^*$ and $m_1^* < m = 4100 < m_2^*$, respectively. In these three cases, systems (2.7) and (2.8) has a stable extinction equilibrium and a stable positive periodic solution. (d) shows the global stability of the extinction equilibrium A_0 with $m = 4670 > m_2^*$.

For the switched systems (2.7) and (2.8), textcolorredwe choose the release period $T = 3$ and set $\bar{T} = pT + q$ with $p = 4$ and $q = 2$. After a simple calculation, we obtain $m_1^* = m^*(p + 1) \approx 3735$ and $m_2^* = m^*(p) \approx 4667$.

Firstly, let $m = 3250 < m_1^*$ and we note that the systems (2.7) and (2.8) displays bistability. In addition to the locally stable extinction equilibrium, there also exists a stable positive periodic solution (see Figure 4 (a)). This is consistent with the conclusions of Theorems 1 and 2. As shown in Figures 4(b,c), when the release amount increases to $m_1^* < m < m_2^*$, the systems (2.7) and (2.8) continues to exhibit bistability, which is consistent with the conclusions of Theorems 1 and 3. Furthermore, we can see that the attraction region of the extinction equilibrium increases with the release amount m until the release amount is not less than m_2^*. If $m < m_2^*$, the extinction equilibrium becomes globally attractive (see Figure 4(d)).

5. Conclusions

In this work, we constructed and analyzed a two-dimensional switched system of the wild mosquito population with stage structure and periodic releases of sterile mosquitoes. To consider the role of sterile mosquitoes in the interaction between the two mosquito populations, we referred to some previous studies and only included sexually active sterile males in the system. Existing research data suggests that the sexual lifespan $\bar{T}$ of sterile mosquitoes can be shorter or longer than their release period T, and in [30], the authors studied the case $\bar{T} \leq T$. Therefore, we focused on investigating the case $\bar{T} > T$ based on the switched systems (2.7) and (2.8).

We first investigated the non-release system (3.1) and the general form (3.2) of the two subsystems. We gave conditions for the existence and stability of their equilibria (Lemmas 2 and 3), which prepared for the later study of the switched system. Two important release amount thresholds (m_1^* and m_2^*) were determined. Then, we proved that the extinction equilibrium $A_0(0,0)$ of systems (2.7) and (2.8) must be exponentially asymptotically stable (Theorem 1). This differs from the conclusion in [30] when $\bar{T} < T$ since, in the latter case, the extinction equilibrium may be unstable. As systems (2.7) and (2.8) has no positive equilibrium, we further discussed the existence of positive periodic solutions. In Theorems 2 and 3, through the analysis of vector fields and using the fixed-point theorem, we show that systems (2.7) and (2.8) has a positive periodic solution when $0 < m < m_1^*$ and $m_1^* \leq m < m_2^*$. Further increasing the release of sterile mosquitoes, and when $m \geq m_2^*$, we confirmed that the extinction equilibrium $A_0(0,0)$ of systems (2.7) and (2.8) becomes globally asymptotically stable. This is a previously unconfirmed dynamic phenomenon when studying the case of $\bar{T} < T$.

Regarding the periodic switched systems (2.7) and (2.8), due to the complexity of its model construction, we only prove the existence of positive periodic solutions using the fixed-point theorem. However, neither the number nor the stability of the positive periodic solutions are determined. In future research, we will try more methods to solve these problems, such as constructing Lyapunov functions to study the stability of periodic solutions [35–37]. In addition, we mainly focus on the theoretical study and the validation of the related results in this work. For the numerical analysis, we used data from a mosquito population on a 74-hectare island. Due to the relatively small number of wild mosquitoes on the island, the amplitude of the periodic fluctuation in the mosquito population is indeed small. In future research, we will collect and organize relevant data from other scenarios with large mosquito populations, and apply our research results to the development of management strategies for wild mosquito populations.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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