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Research article

Normalized solutions for the mixed dispersion nonlinear Schrödinger equations with four types of potentials and mass subcritical growth

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Abstract: This paper is devoted to considering the attainability of minimizers of the L^2-constraint variational problem

$$m_{\gamma,a} = \inf\{J_\gamma(u) : u \in H^2(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 dx = a^2\},$$

where

$$J_\gamma(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx,$$

$\gamma > 0$, $a > 0$, $\sigma \in (0, \frac{2}{N})$ with $N \geq 2$. Moreover, the function $V : \mathbb{R}^N \to [0, +\infty)$ is continuous and bounded. By using the variational methods, we can prove that, when V satisfies four different assumptions, $m_{\gamma,a}$ are all achieved.

Keywords: biharmonic Schrödinger equations; normalized solution; variational method; constrained minimization technique

1. Introduction

Over the past several decades, the mixed dispersion nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} - \gamma\Delta^2\psi + \beta\Delta\psi - V(x)\psi + f(\psi) = 0 \quad \text{in } \mathbb{R}^N \tag{1.1}$$

has been studied by many researchers. Biharmonic Schrödinger equations have played an important role in considering the small biharmonic dispersion terms in the transmission of intense laser beams in a bulk medium with Kerr nonlinearity; see [1,2]. Biharmonic Schrödinger equations are also important in depicting the motion of a vortex filament in an incompressible fluid; see [3]. Since then, biharmonic Schrödinger equations have received attention due to whose applications in physics.

An interesting topic is to study the standing waves of Eq (1.1). By applying the ansatz $\psi(t,x) = e^{i\lambda t}u(x)$, Eq (1.1) yields the following equation:

$$\gamma\Delta^2 u - \beta\Delta u + V(x)u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $\gamma > 0$, $\lambda \in \mathbb{R}, \beta \in \mathbb{R}$, and $u : \mathbb{R}^N \to \mathbb{R}$ is a function which does not rely on time. Moreover, if $u(x)$ is a solution to (1.2), we can obtain that $\psi(t,x) = e^{i\lambda t}u(x)$ is a solution to (1.1).

Above all, when $\gamma = 0$, $\beta = 1$ and $V(x) \equiv 0$, we consider the existence of the L^2-constraint variational problem

$$D_\alpha = \inf\Big\{\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - \int_{\mathbb{R}^N}F(|u|)dx : u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N}|u|^2dx = \alpha^2\Big\}.$$

When $f(u) = |u|^{2\sigma}u\,(0 < \sigma < \frac{2}{N})$, by assuming H^1-precompactness of any minimizing sequences, Cazenave and Lions [4] obtained the existence of the L^2-constraint minimization problem. To this end, the subadditivity assumption

$$D_{\alpha+\beta} < D_\alpha + D_\beta \tag{1.3}$$

is very crucial. Due to the assumption (1.3), we can eliminate the dichotomy of minimizing sequences.

If only $V(x) \equiv 0$, many papers are dedicated to this equation:

$$\gamma\Delta^2 u - \beta\Delta u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

Bonheure et al. considered a mixed dispersion nonlinear Schrödinger equation in [5]. More precisely, they studied the existence of the ground states and positive solutions. They also studied the multiplicity of radial solutions and the standing waves of the related dispersive equation. Recently, Goubet and Manoubi [6] studied semilinear Schrödinger equations with a non-standard dispersion that is discontinuous at $x = 0$. They obtained both the existence and the uniqueness of standing waves for these equations. Then, they discussed the orbital stability of these standing waves in a subspace of the energy space, by using some classical methods such as the concentration-compactness method of Lions. In [7], Khiddi and Essafi obtained the existence of infinitely many solutions for a class of quasilinear Schrödinger equations without assuming the 4-superlinear at infinity on the nonlinearity. The approach is based on the fountain theorem, and the involved potential term is continuous and satisfies suitable regularities. In [8], Alotaibi et al. studied both the existence and nonexistence of global weak solutions to a class of inhomogeneous nonlinear Schrödinger equations. The main problem is related to gradient, which requires certain specific estimates to develop the precise proofs of results. The approach is based on rescaled test function arguments derived from the Mitidieri and Pokhozhaev method, and it also involves the Fujita critical exponent. In [9], Bonheure et al. studied two related constraint minimization problems: One is related to a constraint on the L^2-norm, and another one is related to a constraint on the $L^{2\sigma+2}$-norm. They also studied the attainability and the qualitative properties of minimizers, namely, their sign, symmetry, decay and so on. In [10], Fernández et al. established non-homogeneous Gagliardo-Nirenberg-type inequalities depending on the Tomas-Stein inequality. They proved the attainability of minimizers in the mass-subcritical and mass-critical cases. For more research about the biharmonic Schrödinger equations, see [11–15] and the references therein.

Usually, if $V(x) \equiv 0$, the scaling $u(sx)$ is useful, and we can show (1.3). However, when $V(x) \not\equiv 0$ the scaling $u(sx)$ does not work generally, and it is harder to show the subadditivity condition. Therefore, the L^2-constraint minimization problem is hard. Just because of this, the solutions to the problem would not be enough. For the biharmonic Schrödinger equations with a potential, see [16] and the references therein.

Although the biharmonic nonlinear Schrödinger equations are related to physics, they are far from being properly understood. The nonlinear Schrödinger equations have been studied in [17–21], but the fourth order Schrödinger equations have been studied very little. Apart from some papers already mentioned, there are actually few papers dealing with biharmonic nonlinear Schrödinger equations.

Inspired by the past work of [22–24], in this paper, we consider the attainability of minimizers of the L^2-constraint variational problem:

$$m_{\gamma,a} = \inf\{J_\gamma(u) : u \in H^2(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 dx = a^2\},$$

where

$$J_\gamma(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx,$$

$\gamma > 0$, $a > 0$, $\sigma \in (0, \frac{2}{N})$ with $N \geq 2$ and the continuous bounded function $V : \mathbb{R}^N \to [0, +\infty)$. Here, we consider four functions:

(V_1) V is a function that is 1-periodic in $x_1, x_2, \cdots, x_N$.

(V_2) V is an asymptotically periodic function. Namely there exists a function $V_q : \mathbb{R}^N \to \mathbb{R}$ which is 1-periodic in $x_1, x_2, \cdots, x_N$, and V satisfies the following conditions:

$$V_q(x) \geq V(x), \quad \text{for any } x \in \mathbb{R}^N. \tag{1.5}$$

$$|V_q(x) - V(x)| \to 0, \text{ as } |x| \to +\infty. \tag{1.6}$$

(V_3) $V \in L^\infty(\mathbb{R}^N)$, and

$$0 < \inf_{x\in\mathbb{R}^N} V(x) = V_0 < \liminf_{|x|\to+\infty} V(x) = V_\infty. \tag{1.7}$$

(V_4) Suppose that $\mu W(x) = V(x)$ and a constant $M_1 > 0$ such that

$$|\{x \in \mathbb{R}^N : M_1 < W(x)\}| < +\infty. \tag{1.8}$$

Moreover, $\Omega = \mathrm{int}(W^{-1}(0)) \neq \emptyset$.

Next, we describe the first result of this paper.

Theorem 1.1. *Let $\gamma > 0$, $\sigma \in (0, \frac{2}{N})$ and assume that (V_1) holds or (V_2) holds. There exists a constant $\delta(a) > 0$ for any $a > 0$, and if $|V|_\infty < \delta$ when V satisfies (V_1), or $|V_q|_\infty < \delta$ when V satisfies (V_2), $m_{\gamma,a} < 0$ is achieved.*

Our second result is combined with the L^2-constraint variational problem:

$$m_{\gamma,a,\varepsilon} = \inf\{J_{\gamma,\varepsilon}(u) : u \in H^2(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 dx = a^2\} \tag{1.9}$$

where

$$J_{\gamma,\varepsilon}(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx,$$

$\gamma > 0$, $a > 0$, $\varepsilon > 0$ are real numbers, and $\sigma \in (0, \frac{2}{N})$ with $N \geq 2$.

The second result is as follows.

Theorem 1.2. *Let $\gamma > 0$, $\sigma \in (0, \frac{2}{N})$ and assume that (V_3) holds. Then, there exist two constants $\delta(a)$, $\varepsilon_0 > 0$ for any $a > 0$, and if $|V|_\infty < \delta$, $m_{\gamma,a,\varepsilon} < 0$ is achieved for any $\varepsilon \in (0, \varepsilon_0)$.*

In (V_4), we choose $r > 0$ such that $B_r(x_1) \subset \Omega$. We study a constrained variational problem:

$$m_{\gamma,a,\mu} = \inf\{J_{\lambda,\mu}(u) : u \in H^2(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 dx = a^2\},$$

where

$$J_{\gamma,\mu}(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu W(x)|u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx,$$

and $\gamma > 0$, $a > 0$, $\sigma \in (0, \frac{2}{N})$ with $N \geq 2$. Finally, we describe the third main result.

Theorem 1.3. *Let $\gamma > 0, \sigma \in (0, \frac{2}{N})$ and assume that V satisfies (V_4). Then, there exist two constants $r_0(a)$, $\mu_0(a) > 0$ for any $a > 0$ such that $m_{\gamma,a,\mu} < 0$ is achieved for any $\mu \geq \mu_0$, $r \geq r_0$.*

Notation

- $C, C_1, C_2, \ldots$ represent positive constants, and they are independent of each other.
- $B_r(y)$ represents an open ball centered at $y \in \mathbb{R}^N$ with radius $r > 0$, $B_r^c(y)$ represents its complement in $\mathbb{R}^N$.
- $\|\cdot\|$ represents the common norm of the Sobolev space $H^2(\mathbb{R}^N)$, and $|\cdot|_p$ represents the common norm of the Lebesgue space $L^p(\mathbb{R}^N)$, for $p \in [1, \infty]$.
- $o_n(1)$ represents a real number sequence with $o_n(1) \to 0$ as $n \to +\infty$.

2. Variational framework and some preliminaries

In the following, we study the functional $J_\gamma : E \to \mathbb{R}$, namely,

$$J_\gamma(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx,$$

constrained on the sphere in $L^2(\mathbb{R}^N)$ given by

$$S(a) = \{u \in H^2(\mathbb{R}^N) : |u|_2 = a\},$$

where $\gamma > 0$, and the continuous function $V : \mathbb{R}^N \to [0, +\infty)$. E is described as the space

$$E = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 dx < +\infty\},$$

and the norm of E is given by

$$\|u\| = (\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + (V(x) + 1)|u|^2) dx)^{\frac{1}{2}}.$$

We can infer $E = H^2(\mathbb{R}^N)$ if $V \in L^\infty(\mathbb{R}^N)$.

According to the definition of E, it is obvious that the embedding $E \hookrightarrow H^2(\mathbb{R}^N)$ is continuous. The following embeddings

$$E \hookrightarrow L^{2\sigma+2}(\mathbb{R}^N), \quad \text{for} \quad 2\sigma+2 \in [2,+\infty] \quad \text{when} \quad N=2,$$

and

$$E \hookrightarrow L^{2\sigma+2}(\mathbb{R}^N), \quad \text{for} \quad 2\sigma+2 \in [2,2^*] \qquad \text{when} \quad N \geq 3,$$

are continuous, too.

In addition, we introduce two Gagliardo-Nirenberg interpolation inequalities; see [25–27]. When the function $u \in H^2(\mathbb{R}^N)$, we have

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leq C_N(\sigma)\|\Delta u\|_{L^2}^{\frac{\sigma N}{2}}\|u\|_{L^2}^{2\sigma+2-\frac{\sigma N}{2}}, \tag{2.1}$$

where

$$\begin{cases} 0 \leq \sigma & \text{for } N \leq 4, \\ 0 \leq \sigma \leq \frac{4}{N-4} & \text{for } N > 4, \end{cases}$$

and

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leq B_N(\sigma)\|\nabla u\|_{L^2}^{\sigma N}\|u\|_{L^2}^{2\sigma+2-\sigma N}, \tag{2.2}$$

where

$$\begin{cases} 0 \leq \sigma & \text{for } N \leq 2, \\ 0 \leq \sigma \leq \frac{2}{N-2} & \text{for } N > 2, \end{cases}$$

the constants $B = B_N(\sigma) > 0$ and $C = C_N(\sigma) > 0$. Therefore, we have

$$J_\gamma(u) \geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - \frac{Ba^{2\sigma+2-\sigma N}}{2\sigma+2}(\int_{\mathbb{R}^N}|\nabla u|^2dx)^{\frac{\sigma N}{2}} \tag{2.3}$$

and

$$J_\gamma(u) \geq \frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2dx - \frac{Ca^{2\sigma+2-\frac{\sigma N}{2}}}{2\sigma+2}(\int_{\mathbb{R}^N}|\Delta u|^2dx)^{\frac{\sigma N}{4}}. \tag{2.4}$$

Since $\sigma \in (0, \frac{2}{N})$, we know that $\sigma N < 2$. Hence, J_γ is bounded from below on $S(a)$ for any $a > 0$, $\gamma > 0$. Relying on the above arguments, we infer that

$$m_{\gamma,a} = \inf_{u \in S(a)} J_\gamma(u)$$

is well-defined.

Lemma 2.1. *Let $V \in L^\infty(\mathbb{R}^N)$ and $\gamma > 0$. There exists a constant $\delta(a) > 0$ for each $a > 0$ such that $m_{\gamma,a} < 0$ when $|V|_\infty < \delta$.*

Proof. We choose $u_1 \in S(a)$ for every $a > 0$, and set

$$u_k(x) = e^{\frac{Nk}{2}}u_1(e^k x), \quad \text{for all } x \in \mathbb{R}^N \text{and all } k \in \mathbb{R}.$$

By calculation, we have

$$\int_{\mathbb{R}^N}|u_k(x)|^2dx = a^2$$

and

$$\int_{\mathbb{R}^N} |u_k(x)|^{2\sigma+2} dx = e^{\sigma Nk} \int_{\mathbb{R}^N} |u_1(x)|^{2\sigma+2} dx. \tag{2.5}$$

Therefore, we infer that

$$J_\gamma(u_k) \le \frac{\gamma e^{4k}}{2} \int_{\mathbb{R}^N} |\Delta u_1|^2 dx + \frac{e^{2k}}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 dx + \frac{|V|_\infty a^2}{2} - \frac{e^{\sigma Nk}}{2\sigma+2} \int_{\mathbb{R}^N} |u_1|^{2\sigma+2} dx.$$

As $\sigma \in (0, \frac{2}{N})$, there exists a constant $k < 0$ such that

$$\frac{\gamma e^{4k}}{2} \int_{\mathbb{R}^N} |\Delta u_1|^2 dx + \frac{e^{2k}}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 dx - \frac{e^{\sigma Nk}}{2\sigma+2} \int_{\mathbb{R}^N} |u_1|^{2\sigma+2} dx = D_k < 0.$$

Now, we choose fixed $\delta = \frac{-D_k}{a^2}$ and consider $|V|_\infty < \delta$, and we have

$$J_\gamma(u_k) < D_k - \frac{D_k}{2} = \frac{D_k}{2} < 0,$$

which shows $m_{\gamma,a} < 0$.

Lemma 2.2. *Let $\gamma > 0$, and there are $x_1 \in \mathbb{R}^N$, $r > 0$ and*

$$V(x) = 0, \quad \textit{for any } x \in B_r(x_1). \tag{2.6}$$

Then, there exists a constant $r_0 > 0$ that does not rely on μ in (V_4) and such that $m_{\gamma,a} < 0$ for any $r \ge r_0$.

Proof. We choose $u_1 \in S(a) \cap C_0^\infty(\mathbb{R}^N)$, $x_1 \in \mathbb{R}^N$ with $V(x) = 0$ for any $x \in B_r(x_1)$ and set

$$u_k(x) = e^{\frac{Nk}{2}} u_1(e^k(x - x_1)), \text{ for any } x \in \mathbb{R}^N \text{and any } k \in \mathbb{R}.$$

By calculation we have

$$\int_{\mathbb{R}^N} |u_k(x)|^2 dx = a^2$$

and

$$\int_{\mathbb{R}^N} |u_k(x)|^{2\sigma+2} dx = e^{\sigma Nk} \int_{\mathbb{R}^N} |u_1(x)|^{2\sigma+2} dx, \tag{2.7}$$

which lead to

$$J_\gamma(u_k) = \frac{\gamma e^{4k}}{2} \int_{\mathbb{R}^N} |\Delta u_1|^2 dx + \frac{e^{2k}}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 dx + \frac{1}{2} \int_{\mathrm{supp}(u_1)} V(e^{-k}x + x_1)|u_1|^2 dx - \frac{e^{\sigma Nk}}{2\sigma+2} \int_{\mathbb{R}^N} |u_1|^{2\sigma+2} dx.$$

As $\sigma \in (0, \frac{2}{N})$, there exists a constant $k < 0$ such that

$$\frac{\gamma e^{4k}}{2} \int_{\mathbb{R}^N} |\Delta u_1|^2 dx + \frac{e^{2k}}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 dx - \frac{e^{\sigma Nk}}{2\sigma+2} \int_{\mathbb{R}^N} |u_1|^{2\sigma+2} dx = D_k < 0.$$

Now, we can choose $M_1 = \sup\{|x| : x \in \mathrm{supp}(u_1)\}$, $r_0 = e^{-k}M_1$ for each $r \geq r_0 > 0$. Then, it is easy to deduce that

$$V(e^{-k}x + x_1) = 0, \text{ for all } x \in \mathrm{supp}(u_1).$$

Hence,

$$J_\gamma(u_k) < 0,$$

which shows $m_{\gamma,a} < 0$.

Lemma 2.3. *Let the conditions of Lemma 2.1 hold or Lemma 2.2 hold. When* $0 < a_2 < a_1$*, then* $\frac{a_2^2}{a_1^2} m_{\gamma,a_1} < m_{\gamma,a_2} < 0$.

Proof. We set $\omega > 1$ such that $a_1 = \omega a_2$ and choose a minimizing sequence $(u_n) \subset S(a_2)$ with respect to m_{γ,a_2}, namely,

$$J_\gamma(u_n) \to m_{\gamma,a_2}, \text{ as } n \to +\infty.$$

Let $\tilde{u}_n = \omega u_n$, and it is easy to see $\tilde{u}_n \in S(a_1)$. Then

$$m_{\gamma,a_1} \leq J(\tilde{u}_n) = \omega^2 J(u_n) + \frac{(\omega^2 - \omega^{2\sigma+2})}{2\sigma + 2} \int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx.$$

Claim 2.4. *There exist two constants* $C_1 > 0$ *and* $n_1 \in \mathbb{N}$ *such that* $\int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx \geq C_1$ *for any* $n \geq n_1$.

If not, we can infer that

$$\int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx \to 0, \text{ as } n \to +\infty,$$

and if necessary we can choose a subsequence. Let us recall that

$$-\frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx \leq J_\gamma(u_n) = m_{\gamma,a_2} + o_n(1) < 0, \quad \text{for } n \in \mathbb{N},$$

which is a contradiction. Hence, the proof of Claim 2.4 is completed.

Applying Claim 2.4, $\omega^2 - \omega^{2\sigma+2} < 0$, we can get that for $n \in \mathbb{N}$ big enough

$$m_{\gamma,a_1} \leq \omega^2 J(u_n) + \frac{(\omega^2 - \omega^{2\sigma+2})C_1}{2\sigma + 2}.$$

Taking the limit $n \to +\infty$, we have

$$m_{\gamma,a_1} \leq \omega^2 m_{\gamma,a_2} + \frac{(\omega^2 - \omega^{2\sigma+2})C_1}{2\sigma + 2} < \omega^2 m_{\gamma,a_2},$$

namely,

$$\frac{a_2^2}{a_1^2} m_{\gamma,a_1} < m_{\gamma,a_2},$$

and the proof of the Lemma 2.3 is completed.

Lemma 2.5. *Suppose that there exists a minimizing sequence* $(u_n) \subset S(a)$ *with respect to* $m_{\gamma,a}$ *such that* $u \neq 0$*,* $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^N$*, and* $u_n \rightharpoonup u$ *in* $H^2(\mathbb{R}^N)$*. Then,* $u_n \to u$ *in* $H^2(\mathbb{R}^N)$*,* $J_\gamma(u) = m_{\gamma,a}$ *and* $u \in S(a)$*.*

Proof. Actually, if not, we can obtain that $|u|_2 = c \neq a$. Relying on $u \neq 0$ and Fatou's lemma, it is easy to infer that $c \in (0, a)$. Applying Brezis-Lieb lemma (see [28, Lemma 1.32]), we can obtain

$$|u_n - u|_2^2 = |u_n|_2^2 - |u|_2^2 + o_n(1)$$

and

$$|u_n - u|_{2\sigma+2}^{2\sigma+2} = |u_n|_{2\sigma+2}^{2\sigma+2} - |u|_{2\sigma+2}^{2\sigma+2} + o_n(1).$$

Let $\tilde{u}_n = u_n - u$, $b_n = |\tilde{u}_n|_2$, and assume $|\tilde{u}_n|_2 \to b$. We infer $a^2 = c^2 + b^2$ and $b_n \in (0, a)$ for n big enough. Moreover, by using Lemma 2.3, we have

$$\begin{aligned} m_{\gamma,a} + o_n(1) = J_\gamma(u_n) &= J_\gamma(u) + J_\gamma(\tilde{u}_n) + o_n(1) \\ &\geq m_{\gamma,b_n} + m_{\gamma,c} + o_n(1) \geq \frac{b_n^2}{a^2} m_{\gamma,a} + m_{\gamma,c} + o_n(1). \end{aligned}$$

Taking the limit $n \to +\infty$, we obtain that

$$m_{\gamma,a} \geq \frac{b^2}{a^2} m_{\gamma,a} + m_{\gamma,c}. \tag{2.8}$$

As $c \in (0, a)$, we can apply the Lemma 2.3 in (2.8), and it is easy to obtain this inequality:

$$m_{\gamma,a} > \frac{b^2}{a^2} m_{\gamma,a} + \frac{c^2}{a^2} m_{\gamma,a} = (\frac{b^2}{a^2} + \frac{c^2}{a^2}) m_{\gamma,a} = m_{\gamma,a}.$$

We get a contradiction, which implies $|u|_2 = a$, namely, $u \in S(a)$.
Since $|u_n|_2 = |u|_2 = a$, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^N)$,

$$u_n \to u \quad \text{in} \quad L^2(\mathbb{R}^N).$$

Using the interpolation theorem in the Lebesgue space, it is easy to obtain that

$$u_n \to u \quad \text{in} \quad L^{2\sigma+2}(\mathbb{R}^N).$$

Moreover, as $\int_{\mathbb{R}^N} \gamma|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2 dx$ is convex and continuous in $H^2(\mathbb{R}^N)$, this functional is weakly lower semicontinuous, namely,

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \gamma|\Delta u_n|^2 + |\nabla u_n|^2 + V(x)|u_n|^2 dx \geq \int_{\mathbb{R}^N} \gamma|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2 dx.$$

The above limit together with $m_{\gamma,a} = \lim_{n\to+\infty} J_\gamma(u_n)$ shows that

$$m_{\gamma,a} \geq J_\gamma(u).$$

Since $u \in S(a)$, $J_\gamma(u) = m_{\gamma,a}$, and $J_\gamma(u_n) \to J_\gamma(u)$. Since $u_n \to u$ in $L^{2\sigma+2}(\mathbb{R}^N)$, we infer $u_n \to u$ in $H^2(\mathbb{R}^N)$.

3. Periodic case

We suppose that V satisfies V_1 and set a minimizing sequence $u_n \subset S(a)$ with respect to $m_{\gamma,a}$, namely,

$$J_\gamma(u_n) \to m_{\gamma,a}, \quad \text{as } n \to +\infty.$$

As $\sigma \in (0, \frac{2}{N})$, the above limit combined with (2.3) and (2.4) ensures $(|\Delta u_n|_2)$ and $(|\nabla u_n|_2)$ are bounded sequences. Hence, (u_n) is a bounded sequence in $H^2(\mathbb{R}^N)$. Furthermore, there exist a subsequence of (u_n), still represented by (u_n), and $u \in H^2(\mathbb{R}^N)$ such that

$$u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N,$$

and

$$u_n \rightharpoonup u \text{ in } H^2(\mathbb{R}^N).$$

As the discussion of Claim 2.4, there is a constant $C_1 > 0$, and the following inequality is established

$$\int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx \geq C_1, \quad \text{for } n \in \mathbb{N} \text{ big enough.} \tag{3.1}$$

Lemma 3.1. *If $(u_n) \subset S(a)$ is a minimizing sequence, then (u_n) can be chosen to be a new minimizing sequence $\tilde{u}_n \subset S(a)$ such that $\tilde{u}_n \rightharpoonup \tilde{u}$ and $\tilde{u} \neq 0$.*

Proof. Above all, there exist $\beta > 0$, $R > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |u_n|^2 dx \geq \beta, \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

If not, we can infer that $u_n \to 0$ in $L^{2\sigma+2}(\mathbb{R}^N)$, which is a contradiction. A brief discussion implies that we can suppose $y_n \in \mathbb{R}^N$ and $R > 0$ big enough in (3.2). Then, setting $\tilde{u}_n(x) = u(x + y_n)$, we infer $(\tilde{u}_n) \subset S(a)$ and $(\tilde{u}_n)$ is also a minimizing sequence with respect to $m_{\gamma,a}$. Furthermore, there exists $\tilde{u} \in H^2(\mathbb{R}^N)\backslash\{0\}$ such that

$$\tilde{u}_n(x) \to \tilde{u}(x) \quad \text{a.e. in} \quad \mathbb{R}^N,$$

and

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in} \quad H^2(\mathbb{R}^N).$$

The proof is completed.

3.1. Proof of Theorem 1.1 (Part I)

Proof. Using Lemma 3.1, we can get a bounded minimizing sequence $(u_n) \subset S(a)$ with respect to $m_{\gamma,a}$ and its weak limit $u \neq 0$. Now, by Lemma 2.5, it is easy to infer $u \in S(a)$, $J_\gamma(u) = m_{\gamma,a}$ and $u_n \to u$ in $H^2(\mathbb{R}^N)$. Hence, by the Lagrange multiplier, there exists a constant $\lambda(a) \in \mathbb{R}$ such that

$$J'_\gamma(u) = \lambda(a)\Psi'(u) \quad \text{in} \quad (H^2(\mathbb{R}^N))', \tag{3.3}$$

where $\Psi : H^2(\mathbb{R}^N) \to \mathbb{R}$, and

$$\Psi(u) = \int_{\mathbb{R}^N} |u|^2 dx, \quad u \in H^2(\mathbb{R}^N).$$

According to (3.3),

$$\gamma\Delta^2 u - \Delta u + V(x)u = \lambda(a)u + |u|^{2\sigma}u \quad \text{in } \mathbb{R}^N.$$

Therefore, the proof is completed when V satisfies (V_1).

4. Asymptotically periodic case

In this section, let $V \not\equiv V_q$. Therefore, there exists a measurable set $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$ and

$$V_q(x) > V(x), \quad \text{for any } x \in \Omega. \tag{4.1}$$

Then, let us represent by $J_{\gamma,q} : H^2(\mathbb{R}^N) \to \mathbb{R}$ the functional

$$J_{\gamma,q}(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_q(x)|u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx,$$

and the constant

$$m_{\gamma,a,q} = \inf_{u\in S(a)} J_{\gamma,q}(u).$$

Depending on the conditions of Theorem 1.1, as the discussion in Section 3, there exists $u_p \in S(a)$ such that $J_{\gamma,p}(u_p) = m_{\gamma,a,p}$. Furthermore, by (4.1), we can infer

$$m_{\gamma,a} < m_{\gamma,a,q}. \tag{4.2}$$

Now, we choose a minimizing sequence $(u_n) \subset S(a)$ with respect to $m_{\gamma,a}$, namely,

$$J_\gamma(u_n) \to m_{\gamma,a}, \quad \text{as} \quad n \to +\infty.$$

Since $\sigma \in (0, \frac{2}{N})$, as the discussion in Section 2, sequence (u_n) is bounded in $H^2(\mathbb{R}^N)$. Therefore, there exist $u \in H^2(\mathbb{R}^N)$ and a subsequence of (u_n), still represented by (u_n), such that

$$u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N,$$

and

$$u_n \rightharpoonup u \quad \text{in} \quad H^2(\mathbb{R}^N).$$

Lemma 4.1. *If u is the weak limit of $(u_n) \subset S(a)$, then $u \neq 0$.*

Proof. If not, we have that $u_n \rightharpoonup 0$ in $H^2(\mathbb{R}^N)$. Then,

$$J_{\gamma,q}(u_n) + \int_{\mathbb{R}^N} (V(x) - V_q(x))|u_n|^2 dx + o_n(1) = J_\gamma(u_n) = m_{\gamma,a} + o_n(1),$$

which leads to

$$m_{\gamma,a,p} + \int_{\mathbb{R}^N} (V(x) - V_q(x))|u_n|^2 dx + o_n(1) \leq J_\gamma(u_n) = m_{\gamma,a} + o_n(1). \tag{4.3}$$

Since $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$, the condition (1.6) shows

$$\int_{\mathbb{R}^N} (V(x) - V_q(x))|u_n|^2 dx \to 0.$$

Let $n \to +\infty$ in (4.3), and applying the above limit, we get that

$$m_{\gamma,a} \geq m_{\gamma,a,q}.$$

We get a conclusion that contradicts (4.2). Hence, u is nontrivial.

4.1. Proof of Theorem 1.1 (Part II)

Relying on Lemma 4.1, first of all, we have a minimizing sequence $(u_n) \subset S(a)$ with respect to $m_{\gamma,a}$ and its weak limit $u \neq 0$. By using Lemma 2.5, we can obtain $u_n \to u$ in $H^2(\mathbb{R}^N)$, $u \in S(a)$, $J_\gamma(u) = m_{\gamma,a}$. The rest is similar to Theorem 1.1 when V satisfies (V_1), and we do not repeat it.

5. Proof of Theorem 1.2

In this section, let us represent by $J_{\gamma,\infty}, J_{\gamma,0}, : H^2(\mathbb{R}^N) \to \mathbb{R}$ these functionals:

$$J_{\gamma,\infty}(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty |u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx$$

and

$$J_{\gamma,0}(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_0 |u|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx.$$

Furthermore, let us represent by $m_{\gamma,a,\infty}$, $m_{\gamma,a,0}$ these constants:

$$m_{\gamma,a,\infty} = \inf_{u\in S(a)} J_{\gamma,\infty}(u)$$

and

$$m_{\gamma,a,0} = \inf_{u\in S(a)} J_{\gamma,0}(u).$$

According to the conditions of Theorem 1.2, we can get a constant $\delta(a) > 0$ for any $a > 0$, and $|V|_\infty < \delta$. Hence, by (V_3), we have $V_0 < \delta$ and $V_\infty < \delta$. Depending on Section 3, we can get two functions $u_0, u_\infty \in S(a)$ and $J_{\gamma,\infty}(u_\infty) = m_{\gamma,a,\infty}$, $J_{\gamma,0}(u_0) = m_{\gamma,a,0}$. Moreover, from Lemma 2.1 and (1.7),

$$m_{\gamma,a,0} < m_{\gamma,a,\infty} < 0. \tag{5.1}$$

Lemma 5.1. $m_{\gamma,a,0} \geq \limsup_{\varepsilon\to 0^+} m_{\gamma,a,\varepsilon}$.

Proof. In the following, let $x_1 \in \mathbb{R}^N$ such that

$$V(x_1) = \inf_{x\in\mathbb{R}^N} V(x),$$

and $v_\varepsilon(x) = u_1(x - \frac{x_1}{\varepsilon})$. So, $v_\varepsilon \in S(a)$, and

$$m_{\gamma,a,\varepsilon} \leq J_{\gamma,\varepsilon}(v_\varepsilon) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u_1|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x + x_1)|u_1|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u_1|^{2\sigma+2} dx.$$

Taking the limit $\varepsilon \to 0^+$, we have

$$\limsup_{\varepsilon\to 0^+} m_{\gamma,a,\varepsilon} \le \lim_{\varepsilon\to 0^+} J_{\gamma,\varepsilon}(v_\varepsilon) = J_{\gamma,0}(u_0) = m_{\gamma,a,0}.$$

From Lemma 5.1 and (5.1), there exists $\varepsilon_0 > 0$ such that

$$m_{\gamma,a,\varepsilon} < m_{\gamma,a,\infty}, \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0). \tag{5.2}$$

We choose a minimizing sequence $(u_n) \subset S(a)$ with respect to $m_{\gamma,a,\varepsilon}$, namely,

$$J_{\gamma,\varepsilon}(u_n) \to m_{\gamma,a,\varepsilon}, \quad \text{as} \quad n \to +\infty.$$

As $\sigma \in (0, \frac{2}{N})$, the above limit combined with (2.3) and (2.4) ensures $(|\Delta u_n|_2)$ and $(|\nabla u_n|_2)$ are bounded sequences, from where we infer that (u_n) is bounded in $H^2(\mathbb{R}^N)$. Therefore, there exist a function $u \in H^2(\mathbb{R}^N)$ and a subsequence of (u_n), still represented by (u_n), such that

$$u_n(x) \to u(x) \quad \text{a.e. in} \quad \mathbb{R}^N,$$

and

$$u_n \rightharpoonup u \quad \text{in} \quad H^2(\mathbb{R}^N).$$

Lemma 5.2. *When $\varepsilon \in (0, \varepsilon_0)$, $u \neq 0$, where u is the weak limit of (u_n).*

Proof. If not, we have $u = 0$. Then,

$$m_{\gamma,a,\varepsilon} + o_n(1) = J_{\gamma,\varepsilon}(u_n) = J_{\gamma,\infty}(u_n) + \int_{\mathbb{R}^N} (V(\varepsilon x) - V_\infty)|u_n|^2 dx.$$

Relying on (V_3), for each given $\eta > 0$, there exist a constant $R > 0$ and

$$V(x) \ge V_\infty - \eta, \quad \text{for any} \quad |x| \ge R.$$

Therefore,

$$m_{\gamma,a,\varepsilon} + o_n(1) = J_{\gamma,\varepsilon}(u_n) \ge J_{\gamma,\infty}(u_n) + \int_{B_{R/\varepsilon}(0)} (V(\varepsilon x) - V_\infty)|u_n|^2 dx - \eta \int_{B^c_{R/\varepsilon}(0)} |u_n|^2 dx.$$

We recall that (u_n) is bounded in $H^2(\mathbb{R}^N)$, $u_n \to 0$ in $L^2(B_{R/\varepsilon}(0))$. Hence, can infer that

$$m_{\gamma,a,\varepsilon} + o_n(1) \ge J_{\gamma,\infty}(u_n) - \eta C_1 \ge m_{\gamma,a,\infty} - \eta C_1$$

where $C_1 > 0$. As $\eta > 0$ is arbitrary, we have

$$m_{\gamma,a,\infty} \le m_{\gamma,a,\varepsilon},$$

and we get a conclusion that contradicts (5.2). Hence, $u \neq 0$ when $\varepsilon \in (0, \varepsilon_0)$.

5.1. Proof of Theorem 1.2

Depending on Lemma 5.2, we can choose $(u_n) \subset S(a)$ which is a minimizing sequence with respect to $m_{\gamma,a,\varepsilon}$ such that $u_n \rightharpoonup u$ and $u \neq 0$. Moreover, using Lemma 2.5, it follows that $u_n \to u$ in $H^2(\mathbb{R}^N)$, $u \in S(a)$, and $J_{\gamma,\varepsilon}(u) = m_{\gamma,a,\varepsilon}$.

6. Proof of Theorem 1.3

The following part is dedicated to studying the attainability of minimizers when V satisfies (V_4). We can choose a minimizing sequence $(u_n) \subset S(a)$ with respect to $m_{\gamma,a,\mu}$, namely,

$$J_{\gamma,\mu} \to m_{\gamma,a,\mu}, \quad \text{as} \quad n \to +\infty.$$

Since $\sigma \in (0, \frac{2}{N})$, arguing as the discussion in Section 2, we infer that (u_n) is bounded in E. Therefore, we can get $u \in E$ and a subsequence of (u_n), still represented by (u_n), such that

$$u_n(x) \rightharpoonup u(x) \quad \text{a.e. in} \quad \mathbb{R}^N,$$

and

$$u_n \rightharpoonup u \quad \text{in} \quad E.$$

Lemma 6.1. *There exist two constants C_1, $r_0 > 0$,which do not rely on $\mu > 0$, and*

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx \geq C_1, \ \textit{for all} \quad r \geq r_0. \tag{6.1}$$

Proof. Set $u_1 \in S(a) \cap C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(u_1) \subset \Omega$, where $\Omega = \operatorname{int}(W^{-1}(0)) \neq \emptyset$, $x_1 \in \Omega$, and

$$u_k(x) = e^{\frac{Nk}{2}} u_1(e^k(x - x_1)), \text{ for any } x \in \mathbb{R}^N \text{and any } k \in \mathbb{R}.$$

By calculation, we have

$$\int_{\mathbb{R}^N} |u_k(x)|^2 dx = a^2,$$

and

$$\int_{\mathbb{R}^N} |u_k(x)|^{2\sigma+2} dx = e^{\sigma Nk} \int_{\mathbb{R}^N} |u_1(x)|^{2\sigma+2} dx. \tag{6.2}$$

Therefore,

$$\begin{aligned} J_{\gamma,\mu}(u_k) = &\frac{\gamma e^{4k}}{2} \int_{\mathbb{R}^N} |\Delta u_1|^2 dx + \frac{e^{2k}}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 dx + \frac{\mu}{2} \int_{\operatorname{supp}(u_1)} W(e^{-k}x + x_1)|u_1|^2 dx \\ &- \frac{e^{\sigma Nk}}{2\sigma + 2} \int_{\mathbb{R}^N} |u_1|^{2\sigma+2} dx. \end{aligned}$$

As $\sigma \in (0, \frac{2}{N})$, there exists a constant $k < 0$ such that

$$\frac{\gamma e^{4k}}{2} \int_{\mathbb{R}^N} |\Delta u_1|^2 dx + \frac{e^{2k}}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 dx - \frac{e^{\sigma Nk}}{2\sigma + 2} \int_{\mathbb{R}^N} |u_1|^{2\sigma+2} dx = D_k < 0.$$

Now, we choose $M_1 = \sup\{|x| : x \in \mathrm{supp}(u_1)\}$ and $r_0 = e^{-k}M_1$. If $r \geq r_0 > 0$, and then

$$W(e^{-k}x + x_1) = 0, \quad \text{for all } x \in \mathrm{supp}(u_1).$$

Hence,

$$J_{\gamma,\mu}(u_k) < 0.$$

It shows there is a constant $C_1 > 0$ that does not rely on μ and $m_{\gamma,a,\mu} < -C_1$ when $\mu > 0$. Let us recall

$$\begin{aligned} m_{\gamma,a,\mu} + o_n(1) &= J_{\gamma,\mu}(u_n) \\ &= \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\mu}{2}\int_{\mathbb{R}^N} W(x)|u_n|^2 dx \\ &- \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx. \end{aligned}$$

Then, we have

$$-C_1 + o_n(1) \geq -\frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx$$

and

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx \geq C_1 > 0,$$

where C_1 is independent of μ.

Lemma 6.2. *There exist two constants $\mu_0 > 0$, $R > 0$, and when $\mu \geq \mu_0 > 0$, this inequality is established:*

$$\limsup_{n\to+\infty} \int_{B_R^c(0)} |u_n|^{2\sigma+2} dx \leq \frac{C_1}{2},$$

where $C_1 > 0$ is given in Lemma 6.1.

Proof. Actually, as the discussion in [29, Lemma 2.5], for each $\varepsilon > 0$, there exist $R > 0$ and $\mu_0 > 0$ such that

$$\limsup_{n\to+\infty} \int_{B_R^c(0)} |u_n|^2 dx \leq \varepsilon, \quad \text{for all } \mu \geq \mu_0.$$

Now, apply that the result (u_n) is bounded in $L^{2^*}(\mathbb{R}^N)$ by a constant, and the constant is independent of μ. In the following, we can use interpolation theorem of Lebesgue spaces and the proper $\varepsilon > 0$ to get the expected result.

Lemma 6.3. *When $\mu \geq \mu_0 > 0$, $u \neq 0$, where u is the weak limit of (u_n).*

Proof. If not, we have that $u = 0$ for some $\mu \geq \mu_0$. As $u_n \to 0$ in $L^{2\sigma+2}(B_R(0))$ for each $R > 0$, according to Lemmas 6.1 and 6.2 we have

$$C_1 \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{2\sigma+2} dx = \liminf_{n\to+\infty} \int_{B_R^c(0)} |u_n|^{2\sigma+2} dx \leq \limsup_{n\to+\infty} \int_{B_R^c(0)} |u_n|^{2\sigma+2} dx \leq \frac{C_1}{2}.$$

We get a contradiction, which implies that there exists a constant $\mu_0 > 0$ such that $u \neq 0$ when $\mu \geq \mu_0$.

6.1. Proof of Theorem 1.3

According to Lemma 6.3, we can get $(u_n) \subset S(a)$ which is a minimizing sequence with respect to $m_{\gamma,a,\mu}$ such that $u_n \rightharpoonup u$ and $u \neq 0$. Moreover, we can use Lemma 2.5, and then, $u_n \to u$ in E, $u \in S(a)$ and $J_{\gamma,\mu}(u) = m_{\gamma,a,\mu}$.

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Conflict of interest

The author declares there is no conflict of interest.

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