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Research article

Barycentric rational interpolation method for solving fractional cable equation

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Abstract: A fractional cable (FC) equation is solved by the barycentric rational interpolation method (BRIM). As the fractional derivative is a nonlocal operator, we develop a spectral method to solve the FC equation to get the coefficient matrix as the full matrix. First, the fractional derivative of the FC equation is changed to a nonsingular integral from the singular kernel to the density function. Second, an efficient quadrature of a new Gauss formula is constructed to compute it simply. Third, a matrix equation of the discrete FC equation is obtained by the unknown function replaced by a barycentric rational interpolation basis function. Then, convergence rate for FC equation of the BRIM is derived. At last, a numerical example is given to illustrate our results.

Keywords: barycentric rational interpolation; collocation method; Fractional Cable equation; fractional derivative

1. Introduction

Lots of physical phenomena can be expressed by the FC equation, including, inter alia, dissipative and dispersive partial differential equations (PDEs). In this paper, we consider the FC equation

$$\frac{\partial \phi(t,s)}{\partial t} = -\mu_0\, {}_0C_t^{1-\alpha_1}\phi(t,s) + {}_0C_t^{1-\alpha_2}\frac{\partial^2 \phi(t,s)}{\partial s^2} + f(t,s), 0 \le s \le 1, 0 \le t \le T, \tag{1.1}$$

$$\phi(0,s) = 0, \phi(1,s) = 0, s \in [0,T], \tag{1.2}$$

$$\phi(t,0) = \varphi(t), t \in [0,1] \tag{1.3}$$

where $\mu_0 \in R, 0 < \alpha_1, \alpha_2 < 1$ are constants. There are some definitions of fractional derivatives, such as the Caputo type, Riemann-Liouville type and so on. In the following, we adopt the Caputo type time fractional-order partial derivative as

$${}_0C_t^{\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\phi'(t)}{(t-\tau)^{\alpha}}d\tau, \tag{1.4}$$

and $\Gamma(\alpha)$ is the Γ function.

In [1], a scheme combining the finite difference method in the time direction and a spectral method in the space direction was proposed. In [2], two implicit compact difference schemes for the FC equation were studied, this scheme was proved to be stable, and the convergence order $O(\tau + h^4)$ was given. In [3], a two-dimensional FC equation was solved by orthogonal spline collocation (OSC) methods for space discretization and finite difference method for time, which was proved to be unconditionally stable. In [4], the FC equation with two time Riemann-Liouville derivatives was solved by an explicit numerical method; and the accuracy, stability and convergence of this method were studied. In [5], FC equation with two fractional time derivatives were considered, and two new implicit numerical methods for the FC equation were proposed, respectively. The stability and convergence of these methods were also investigated. In [6], nonlinear FC equation was solved by a two-grid algorithm with the finite element (FE) method. A time second-order fully discrete two-grid FE scheme and the space direction were approximated. In [7], the discrete Crank-Nicolson (CN) finite element method was obtained by the finite difference in time and the finite element in space to approximate the FC equation, the stability and error estimate were analyzed in detail and the optimal convergence rate was obtained. In [8], the FC equation involving two integro-differential operators was solved by semi-discrete finite difference approximation, and the scheme was proved unconditionally stable. In reference [9], numerical integration with the reproducing kernel gradient smoothing integration are constructed. In reference [10], recursive moving least squares (MLS) approximation was constructed.

Like the above methods to solve the FC equation by finite difference approach or finite element method, the time direction and space direction were solved separatively. In the following, we presented the BRIM to solve the time direction and space direction of FC equation at the same time. Lagrange interpolation has been presented by mathematician Lagrange to fitting data to be a certain function. When the number n increases, there are Runge phenomenon that the interpolation result deviates from the original function. In order to avoid the Runge phenomenon, among them, barycentric interpolation was developed in 1960s to overcome it. In recent years, linear rational interpolation (LRI) was proposed by Floater [14–16] and error of linear rational interpolation [11–13] is also proved. The barycentric interpolation collocation method (BICM) has developed by Wang et al. [25, 26] and the algorithm of BICM has used for linear/non-linear problems [27, 28]. In recent research, Volterra integro-differential equation (VIDE) [17, 21], heat equation (HE) [18], biharmonic equation (BE) [19], telegraph equation (TE) [20], generalized Poisson equations [22], fractional reaction-diffusion equation [23] and KPP equation [24] have been studied by the linear barycentric rational interpolation method (LBRIM) and their convergence rate are also proved.

In this paper, BRIM has been used to solve the FC equation. As the fractional derivative is the nonlocal operator, the spectral method is developed to solve the FC equation and the coefficient matrix is the full matrix. The fractional derivative of the FC equation is changed to nonsingular integral by the order of density function plus one. New Gauss formula is constructed to compute it simply and matrix equation of discrete FC equation is obtained by the unknown function replaced by barycentric rational interpolation basis function. Then, the convergence rate of BRIM is proved.

2. Matrix equation of FC equation

As there is singularity in Eq (1.1), the numerical methods cannot get high accuracy, by fractional integration to second part of (1.1) to overcome the difficulty of singularity. We get

$$
\begin{aligned}
&{}_0C_t^{\alpha}\phi(t,s)\\
&=\frac{1}{\Gamma(\xi-\alpha)}\int_0^t\frac{\partial^{\xi}\phi(\tau,s)}{\partial\tau^{\xi}}\frac{d\tau}{(\tau-t)^{\alpha+1-\xi}}\\
&=\frac{1}{(\xi-\alpha)\Gamma(\xi-\alpha)}\left[\frac{\partial^{\xi}\phi(0,s)}{\partial t^{\xi}}t^{\xi-\alpha}+\int_0^t\frac{\partial^{\xi+1}\phi(\tau,s)}{\partial\tau^{\xi+1}}\frac{d\tau}{(t-\tau)^{\alpha-\xi}}\right]\\
&=\Gamma_{\alpha}^{\xi}\left[\frac{\partial^{\xi}\phi(0,s)}{\partial t^{\xi}}t^{\xi-\alpha}+\int_0^t\frac{\partial^{\xi+1}\phi(\tau,s)}{\partial\tau^{\xi+1}}\frac{d\tau}{(t-\tau)^{\alpha-\xi}}\right],
\end{aligned}
\tag{2.1}
$$

where $\Gamma_{\alpha}^{\xi}=\frac{1}{(\xi-\alpha)\Gamma(\xi-\alpha)}$.

Combining (2.1) and (1.1), we have

$$
\begin{aligned}
&\frac{\partial\phi}{\partial t}+\mu_0\Gamma_{\alpha_1}^{\xi}\left[\frac{\partial^{\xi}\phi(0,s)}{\partial t^{\xi}}t^{\xi-\alpha_1}+\int_0^t\frac{\partial^{\xi+1}\phi(\tau,s)}{\partial\tau^{\xi+1}}\frac{d\tau}{(t-\tau)^{\alpha_1-\xi}}\right]\\
=&\Gamma_{\alpha_2}^{\xi}\left[\frac{\partial^{\xi+2}\phi(0,s)}{\partial t^{\xi}\partial s^2}s^{\xi-\alpha_2}+\int_0^t\frac{\partial^{\xi+3}\phi(\tau,s)}{\partial\tau^{\xi+1}\partial s^2}\frac{d\tau}{(s-\tau)^{\alpha_2-\xi}}\right]+f(t,s).
\end{aligned}
\tag{2.2}
$$

In the following, we give the discrete formula of FC equation and to get the matrix equation from BRIM.

Let

$$\phi(t,s)=\sum_{j=1}^{m}R_j(t)\phi_j(s) \tag{2.3}$$

where

$$\phi(t_i,s)=\phi_i(s), i=1,2,\cdots,m$$

and

$$R_j(t)=\frac{\dfrac{\lambda_j}{t-t_j}}{\displaystyle\sum_{k=1}^{n}\frac{\lambda_k}{t-t_k}} \tag{2.4}$$

where

$$\lambda_k=\sum_{j\in J_k}(-1)^j\prod_{i=j,j\neq k}^{j+d_t}\frac{1}{t_k-t_i},\quad J_k=\{j\in\{0,1,\cdots,l-d_t\}:k-d_t\leq j\leq k\}$$

is the basis function [18]. Taking (2.3) into Eq (2.2),

$$\sum_{j=1}^{m}R_j'(t)\phi_j(s)+\mu_0\Gamma_{\alpha_1}^{\xi}\sum_{j=1}^{m}\left[R_j^{(\xi)}(0)\phi_j(s)t^{\xi-\alpha_1}+\int_0^t\phi_j(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t-\tau)^{\alpha_1-\xi}}\right]$$

$$= \Gamma_{\alpha_2}^{\xi} \sum_{j=1}^{m} \left[R_j^{(\xi)}(0)\phi_j^{(2)}(s)t^{\xi-\alpha_2} + \int_0^t \phi_j^{(2)}(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t-\tau)^{\alpha_2-\xi}} \right] + f(t,s). \tag{2.5}$$

By taking $0 = t_1 < t_2 < \cdots < t_m = T, a = s_1 < s_2 < \cdots < s_n = b$ with $h_t = T/m, h_s = (b-s)/n$ or uninform as Chebychev point $s = \cos((0:m)'\pi/m), t = \cos((0:n)'\pi/n)$, we get

$$\sum_{j=1}^{m} R_j'(t_i)\phi_j(s) + \mu_0\Gamma_{\alpha_1}^{\xi} \sum_{j=1}^{m} \left[R_j^{(\xi)}(0)\phi_j(s)t_i^{\xi-\alpha_1} + \int_0^{t_i} \phi_j(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_1-\xi}} \right]$$
$$= \Gamma_{\alpha_2}^{\xi} \sum_{j=1}^{m} \left[R_j^{(\xi)}(0)\phi_j^{(2)}(s)t_i^{\xi-\alpha_2} + \int_0^{t_i} \phi_j^{(2)}(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_2-\xi}} \right] + f(t_i,s), \tag{2.6}$$

by noting the notation, $R_j(t_i) = \delta_{ij}, R_j'(t_i) = R_{ij}^{(1,0)}$, where $R_{ij}^{(1,0)}$ is the first order derivative of barycentric matrix. Equation (2.6) can be written as

$$\sum_{j=1}^{m} R_{ij}^{(1,0)}\phi_j(s) + \mu_0\Gamma_{\alpha_1}^{\xi} \sum_{j=1}^{m} \left[R_j^{(\xi)}(0)\phi_j(s)t_i^{\xi-\alpha_1} + \int_0^{t_i} \phi_j(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_1-\xi}} \right]$$
$$= \Gamma_{\alpha_2}^{\xi} \sum_{j=1}^{m} \left[R_j^{(\xi)}(0)\phi_j^{(2)}(s)t_i^{\xi-\alpha_2} + \int_0^{t_i} \phi_j^{(2)}(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_2-\xi}} \right] + f(t_i,s). \tag{2.7}$$

Similarly as the discrete t for s, we get

$$\phi_j(s) = \sum_{k=1}^{n} R_k(s)\phi_{ik} \tag{2.8}$$

where $\phi_i(s_j) = \phi(t_i, s_j) = \phi_{ij}, i = 1, \cdots, m; j = 1, \cdots, n$ and

$$R_i(s) = \frac{\dfrac{w_i}{s-s_i}}{\displaystyle\sum_{k=1}^{m} \frac{w_k}{s-s_k}} \tag{2.9}$$

where

$$w_i = \sum_{j \in J_i} (-1)^j \prod_{k=j, j\neq i}^{j+d_s} \frac{1}{s_i - s_k}, \; J_i = \{j \in \{0, 1, \cdots, m-d_s\} : i - d_s \leq j \leq i\},$$

is the basis function [18].

Taking (2.8) into Eq (2.7), we get

$$\sum_{j=1}^{m}\sum_{k=1}^{n} R_{ij}^{(1,0)}R_k(s)\phi_{ik} + \mu_0\Gamma_{\alpha_1}^{\xi} \sum_{j=1}^{m}\sum_{k=1}^{n} \left[R_j^{(\xi)}(0)R_k(s)t_i^{\xi-\alpha_1} + \int_0^{t_i} R_k(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_1-\xi}} \right]\phi_{ik}$$
$$= \Gamma_{\alpha_2}^{\xi} \sum_{j=1}^{m}\sum_{k=1}^{n} \left[R_j^{(\xi)}(0)R_k^{(2)}(s)t_i^{\xi-\alpha_2} + \int_0^{t_i} R_k^{(2)}(s)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_2-\xi}} \right]\phi_{ik} + f(t_i,s). \tag{2.10}$$

By taking $s_1, s_2, \cdots, s_n$ at the mesh-point, we get

$$\sum_{j=1}^{m}\sum_{k=1}^{n} R_{ij}^{(1,0)} R_k(s_l)\phi_{ik} + \mu_0\Gamma_{\alpha_1}^{\xi} \sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_j^{(\xi)}(0)R_k(s_l)t_i^{\xi-\alpha_1} + \int_0^{t_i} R_k(s_l)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_1-\xi}}\right]\phi_{ik}$$
$$= \Gamma_{\alpha_2}^{\xi}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_j^{(\xi)}(0)R_k^{(2)}(s_l)t_i^{\xi-\alpha_2} + \int_0^{t_i} R_k^{(2)}(s_l)\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_2-\xi}}\right]\phi_{ik} + f(t_i, s_l). \tag{2.11}$$

By noting the notation, $R_k(s_l) = \delta_{kl}, R_k''(s_l) = R_{ij}^{(0,2)}$, where $R_{ij}^{(0,2)}$ is the second order derivative of barycentrix matrix.

$$\sum_{j=1}^{m}\sum_{k=1}^{n} R_{ij}^{(1,0)}\delta_{kl}\phi_{ik} + \mu_0\Gamma_{\alpha_1}^{\xi}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_j^{(\xi)}(0)\delta_{kl}t_i^{\xi-\alpha_1} + \delta_{kl}\int_0^{t_i}\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_1-\xi}}\right]\phi_{ik}$$
$$= \Gamma_{\alpha_2}^{\xi}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_j^{(\xi)}(0)R_{ij}^{(0,2)}t_i^{\xi-\alpha_2} + R_{ij}^{(0,2)}\int_0^{t_i}\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_2-\xi}}\right]\phi_{ik} + f(t_i, s_l), \tag{2.12}$$

where

$$R_k(\tau) = \frac{\dfrac{\lambda_k}{\tau-\tau_k}}{\displaystyle\sum_{k=0}^{n}\frac{\lambda_k}{\tau-\tau_k}}$$

and

$$\begin{cases} R_i^{'}(\tau) = R_i(\tau)\left[-\dfrac{1}{\tau-\tau_k} + \dfrac{\sum_{s=0}^{l}\frac{\lambda_k}{(\tau-\tau_k)^2}}{\sum_{s=0}^{l}\frac{\lambda_k}{\tau-\tau_k}}\right], \\ \vdots \\ R_i^{(\xi+1)}(\tau) = [R_i^{(\xi)}(\tau)]', \xi \in \mathbf{N}^+. \end{cases}$$

The integral term of (2.12) can be written as

$$\int_0^{t_i}\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_1-\xi}} = Q_j^{\alpha_1}(t_i) = Q_{ji}^{\alpha_1}, \tag{2.13}$$

$$\int_0^{t_i}\frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_2-\xi}} = Q_j^{\alpha_2}(t_i) = Q_{ji}^{\alpha_2}, \tag{2.14}$$

then we get

$$\sum_{j=1}^{m}\sum_{k=1}^{n} R_{ij}^{(1,0)}\delta_{kl}\phi_{ik} + \mu_0\Gamma_{\alpha_1}^{\xi}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_j^{(\xi)}(0)\delta_{kl}t_i^{\xi-\alpha_1} + \delta_{kl}Q_j^{\alpha_2}(t_i)\right]\phi_{ik}$$
$$= \Gamma_{\alpha_2}^{\xi}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_j^{(\xi)}(0)R_{ij}^{(0,2)}t_i^{\xi-\alpha_2} + R_{ij}^{(0,2)}Q_j^{\alpha_1}(t_i)\right]\phi_{ik} + f(t_i, s_l). \tag{2.15}$$

The integral (2.12) is calculated by

$$Q_j^{\alpha_1}(t_i) = \int_0^{t_i} \frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_1-\xi}} := \sum_{i=1}^{g} R_i^{(\xi+1)}(\tau_i^{\theta,\alpha_1})G_i^{\theta,\alpha_1}, \tag{2.16}$$

and

$$Q_j^{\alpha_2}(t_i) = \int_0^{t_i} \frac{R_j^{(\xi+1)}(\tau)d\tau}{(t_i-\tau)^{\alpha_2-\xi}} := \sum_{i=1}^{g} R_i^{(\xi+1)}(\tau_i^{\theta,\alpha_2})G_i^{\theta,\alpha_2}, \tag{2.17}$$

where $G_i^{\theta,\alpha_1}, G_i^{\theta,\alpha_2}$ are Gauss weights and $\tau_i^{\theta,\alpha_1}, \tau_i^{\theta,\alpha_2}$ are Gauss points with weights $(t_i-\tau)^{\xi-\alpha_1}, (t_i-\tau)^{\xi-\alpha_2}$, see reference [22].

Equation systems (2.15) can be written as

$$\left[R^{(01)} \otimes I_n + \Gamma_{\alpha_2}^{\xi}\left(M_1^{(\xi 0)} \otimes I_n + I_m \otimes Q^{\alpha_2}\right)\right]\begin{bmatrix}\phi_{11}\\ \vdots \\ \phi_{1n}\\ \phi_{n1}\\ \vdots\\ \phi_{mn}\end{bmatrix}$$

$$-\left[\mu_0\Gamma_{\alpha_1}^{\xi}\left(M_1^{(\xi 0)} \otimes I_n + I_m \otimes Q^{\alpha_1}\right)\right]\begin{bmatrix}\phi_{11}\\ \vdots \\ \phi_{1n}\\ \phi_{n1}\\ \vdots\\ \phi_{mn}\end{bmatrix} = \begin{bmatrix}f_{11}\\ \vdots \\ f_{1n}\\ f_{n1}\\ \vdots\\ f_{mn}\end{bmatrix}, \tag{2.18}$$

I_m and I_n are identity matrices, $\otimes$ is Kronecker product.

Then Eq (2.18) can be noted as

$$\left[R^{(01)} \otimes I_n + \Gamma_{\alpha_2}^{\xi}\left(M_1^{(\xi 0)} \otimes I_n + I_m \otimes Q^{\alpha_2}\right) - \mu_0\Gamma_{\alpha_1}^{\xi}\left(M_1^{(\xi 0)} \otimes I_n + I_m \otimes Q^{\alpha_1}\right)\right]\Phi = F \tag{2.19}$$

and

$$R\Phi = F, \tag{2.20}$$

with $R = R^{(01)} \otimes I_n + \Gamma_{\alpha_2}^{\xi}\left(M_1^{(\xi 0)} \otimes I_n + I_m \otimes Q^{\alpha_2}\right) - \mu_0\Gamma_{\alpha_1}^{\xi}\left(M_1^{(\xi 0)} \otimes I_n + I_m \otimes Q^{\alpha_1}\right)$ and $\Phi = [\phi_{11}\ldots\phi_{1n}\ldots\phi_{n1}\ldots\phi_{mn}]^T, F = [f_{11}\ldots f_{1n}\ldots f_{n1}\ldots f_{mn}]^T$.

The boundary condition can be solved by substitution method, additional method or elimination method, see [26]. We adopt substitution method and additional method to deal with boundary condition.

3. Convergence rate of FC equation

In this part, error estimate of the FC equation is given with $r_n(s) = \sum_{i=1}^{n} r_i(s)\phi_i$ to replace $\phi(s)$, where $r_i(s)$ is defined as (2.9) and $\phi_i = \phi(s_i)$. We also define

$$e(s) := \phi(s) - r_n(s) = (s-s_i)\cdots(s-s_{i+d})\phi\left[s_i, s_{i+1}, \ldots, s_{i+d}, s\right], \tag{3.1}$$

see reference [18].

Then we have

Lemma 1. *For $e(s)$ be defined by (3.1) and $\phi(s) \in C^{d+2}[a,b], d = 1,2,\cdots$, there*

$$\left|e^{(k)}(s)\right| \le Ch^{d-k+1}, k = 0,1,\cdots. \tag{3.2}$$

For the FC equation, rational interpolation function of $\phi(t,s)$ is defined as $r_{mn}(t,s)$

$$r_{mn}(t,s) = \frac{\sum_{i=1}^{m+d_s}\sum_{j=1}^{n+d_t}\frac{w_{i,j}}{(s-s_i)\left(t-t_j\right)}\phi_{i,j}}{\sum_{i=1}^{m+d_s}\sum_{j=1}^{n+d_t}\frac{w_{i,j}}{(s-s_i)\left(t-t_j\right)}} \tag{3.3}$$

where

$$w_{i,j} = (-1)^{i-d_s+j-d_t}\sum_{k_1\in J_i}\prod_{h_1=k_1,h_1\neq j}^{k_1+d_s}\frac{1}{\left|s_i-s_{h_1}\right|}\sum_{k_2\in J_i}\prod_{h_2=k_2,h_2\neq j}^{k_2+d_t}\frac{1}{\left|t_j-t_{h_2}\right|}. \tag{3.4}$$

We define $e(t,s)$ be the error of $\phi(t,s)$ as

$$\begin{aligned} e(t,s):&= \phi(t,s) - r_{mn}(t,s) \\ &= (s-s_i)\cdots(s-s_{i+d_s})\phi\left[s_i,s_{i+1},\ldots,s_{i+d_1},s;t\right] \\ &+ \left(t-t_j\right)\cdots\left(t-t_{j+d_t}\right)\phi\left[s;t_j,t_{j+1},\ldots,t_{j+d_2},t\right] \\ &- (s-s_i)\cdots(s-s_{i+d_s})\left(t-t_j\right)\cdots\left(t-t_{j+d_t}\right)\phi\left[s_i,s_{i+1},\ldots,s_{i+d_1},s;t_j,t_{j+1},\ldots,t_{j+d_2},t\right]. \end{aligned} \tag{3.5}$$

With similar analysis of Lemma 1, we have

Theorem 1. *For $e(t,s)$ defined as (3.5) and $\phi(t,s) \in C^{d_s+2}[a,b]\times C^{d_t+2}[0,T]$, then we have*

$$\left|e^{(k_1,k_2)}(s,t)\right| \le C(h_s^{d_s-k_1+1} + h_t^{d_t-k_2+1}), k_1,k_2 = 0,1,\cdots. \tag{3.6}$$

Let $\phi(s_m,t_n)$ be the approximate function of $\phi(t,s)$ and L to be bounded operator, there holds

$$L\phi(t_m,s_n) = f(t_m,s_n) \tag{3.7}$$

and

$$\lim_{m,n\to\infty} L\phi(t_m,s_n) = \phi(t,s). \tag{3.8}$$

Then we get

Theorem 2. *For $\phi(t_m,s_n) : L\phi(t_m,s_n) = \phi(t,s)$ and L defined as (3.7), there*

$$|\phi(t,s) - \phi(t_m,s_n)| \le C(h^{d_s-1} + \tau^{d_t-1}).$$

Proof. By

$$
\begin{aligned}
&L\phi(t,s) - L\phi(t_m,s_n)\\
&= \frac{\partial \phi(t,s)}{\partial t} - {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi(t,s)}{\partial s^2} + \mu_0\, {}_0C_t^{1-\alpha_2}\phi(t,s) - f(t,s)\\
&-\left[\frac{\partial \phi(t_m,s_n)}{\partial t} - {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi(t_m,s_n)}{\partial s^2} + \mu_0\, {}_0C_t^{1-\alpha_2}\phi(t_m,s_n) - f(t_m,s_n)\right]\\
&= \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial t}(t_m,s_n) - \left[{}_0C_t^{1-\alpha_1}\frac{\partial^2\phi}{\partial s^2} - {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi}{\partial s^2}(s_m,t_n)\right]\\
&+\mu_0\left[{}_0C_t^{1-\alpha_2}\phi(t,s) - {}_0C_t^{1-\alpha_2}(t_m,s_n))\right]\\
&-[f(t,s) - f(t_m,s_n)]\\
&:= E_1(t,s) + E_2(t,s) + E_3(t,s) + E_4(t,s),
\end{aligned}
\tag{3.9}
$$

here

$$
\begin{aligned}
E_1(t,s) &= \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial t}(t_m,s_n),\\
E_2(t,s) &= {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi}{\partial s^2} - {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi}{\partial s^2}(t_m,s_n),\\
E_3(t,s) &= \mu_0\left[{}_0C_t^{1-\alpha_2}\phi(t,s) - {}_0C_t^{1-\alpha_2}(t_m,s_n))\right],\\
E_4(t,s) &= f(t,s) - f(t_m,s_n).
\end{aligned}
$$

As for $E_1(t,s)$, we get

$$
\begin{aligned}
E_1(t,s) &= \left|\frac{\partial \phi}{\partial t}(t,s) - \frac{\partial \phi}{\partial t}(t_m,s_n)\right|\\
&= \left|\frac{\partial \phi}{\partial t}(t,s) - \frac{\partial \phi}{\partial t}(t_m,s) + \frac{\partial \phi}{\partial t}(t_m,s) - \frac{\partial \phi}{\partial t}(t_m,s_n)\right|\\
&\le \left|\frac{\partial \phi}{\partial t}(t,s) - \frac{\partial \phi}{\partial t}(t_m,s)\right| + \left|\frac{\partial \phi}{\partial t}(t_m,s) - \frac{\partial \phi}{\partial t}(t_m,s_n)\right|\\
&= \left|\frac{\sum_{i=1}^{m-d_s}(-1)^i\frac{\partial \phi}{\partial t}[s_i,s_{i+1},\ldots,s_{i+d_1},s_n,t]}{\sum_{i=1}^{m-d_s}\lambda_i(s)}\right|\\
&+ \left|\frac{\sum_{j=1}^{n-d_t}(-1)^j\frac{\partial \phi}{\partial t}[t_j,t_{j+1},\ldots,t_{j+d_2},s_n,t_m]}{\sum_{j=1}^{n-d_t}\lambda_j(t)}\right|\\
&= \left|\frac{\partial e}{\partial t}(t_m,s)\right| + \left|\frac{\partial e}{\partial t}(t_m,s_n)\right|,
\end{aligned}
$$

we get

$$|E_1(t,s)| \le C(h^{d_s} + \tau^{d_t}). \tag{3.10}$$

As $E_2(t,s)$, we have

$$\begin{aligned}
&E_2(t,s) = {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi}{\partial s^2} - {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi}{\partial s^2}(t_m,s_n)\\
&= \Gamma_{\alpha_2}^{\xi}\left[\frac{\partial^{\xi+2}\phi(0,s)}{\partial t^{\xi}\partial s^2}s^{\xi-\alpha_2} + \int_0^t \frac{\partial^{\xi+3}\phi(\tau,s)}{\partial\tau^{\xi+1}\partial s^2}\frac{d\tau}{(t-\tau)^{\alpha_2-\xi}}\right]\\
&-\Gamma_{\alpha_2}^{\xi}\left[\frac{\partial^{\xi+2}\phi(0,s_n)}{\partial t^{\xi}\partial s^2}s_n^{\xi-\alpha_2} + \int_0^{t_m} \frac{\partial^{\xi+3}\phi(\tau,s_n)}{\partial\tau^{\xi+1}\partial s^2}\frac{d\tau}{(t_m-\tau)^{\alpha_2-\xi}}\right]\\
&= \Gamma_{\alpha_2}^{\xi}\left[\frac{\partial^{\xi+2}\phi(0,s)}{\partial t^{\xi}\partial s^2}s^{\xi-\alpha_2} - \frac{\partial^{\xi+2}\phi(0,s_n)}{\partial t^{\xi}\partial s^2}s_n^{\xi-\alpha_2}\right]\\
&+\Gamma_{\alpha_2}^{\xi}\left[\int_0^t \frac{\partial^{\xi+3}\phi(\tau,s)}{\partial\tau^{\xi+1}\partial s^2}\frac{d\tau}{(t-\tau)^{\alpha_2-\xi}} - \int_0^{t_m} \frac{\partial^{\xi+3}\phi(\tau,s_n)}{\partial\tau^{\xi+1}\partial s^2}\frac{d\tau}{(t_m-\tau)^{\alpha_2-\xi}}\right]
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&|E_2(t,s)|\\
&\le \left|\Gamma_{\alpha_2}^{\xi}\left[\frac{\partial^{\xi+2}\phi(0,s)}{\partial t^{\xi}\partial s^2}s^{\xi-\alpha_2} - \frac{\partial^{\xi+2}\phi(0,s_n)}{\partial t^{\xi}\partial s^2}s_n^{\xi-\alpha_2}\right]\right|\\
&+\left|\Gamma_{\alpha_2}^{\xi}\left[\int_0^t \frac{\partial^{\xi+3}\phi(\tau,s)}{\partial\tau^{\xi+1}\partial s^2}\frac{d\tau}{(t-\tau)^{\alpha_2-\xi}} - \int_0^{t_m} \frac{\partial^{\xi+3}\phi(\tau,s_n)}{\partial\tau^{\xi+1}\partial s^2}\frac{d\tau}{(t_m-\tau)^{\alpha_2-\xi}}\right]\right|\\
&\le |\Gamma_{\alpha_2}^{\xi}|\left|\frac{\partial^{\xi+2}\phi}{\partial t^{\xi}\partial s^2}(0,s) - \frac{\partial^{\xi+2}\phi}{\partial t^{\xi}\partial s^2}(0,s_n)\right| + |\Gamma_{\alpha_2}^{\xi}|\left|\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t,s) - \frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s_n)\right|\\
&:= E_{21}(t,s) + E_{22}(t,s)
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
E_{21}(t,s) &= |\Gamma_{\alpha_2}^{\xi}|\left|\frac{\partial^{\xi+2}\phi}{\partial t^{\xi}\partial s^2}(0,s) - \frac{\partial^{\xi+2}\phi}{\partial t^{\xi}\partial s^2}(0,s_n)\right|,\\
E_{22}(t,s) &= |\Gamma_{\alpha_2}^{\xi}|\left|\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t,s) - \frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s_n)\right|.
\end{aligned} \tag{3.13}$$

Now we estimate $E_{21}(t,s)$ and $E_{22}(t,s)$ part by part, for the second part we have

$$
\begin{aligned}
E_{22}(t,s) &= |\Gamma^{\xi}_{\alpha_2}|\left|\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t,s)-\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s_n)\right|\\
&= |\Gamma^{\xi}_{\alpha_2}|\left|\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t,s)-\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s)+\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s)-\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s_n)\right|\\
&\le |\Gamma^{\xi}_{\alpha_2}|\left|\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t,s)-\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s)\right|+|\Gamma^{\xi}_{\alpha_2}|\left|\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s)-\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}(t_m,s_n)\right|
\end{aligned}
$$

$$
= |\Gamma^{\xi}_{\alpha_2}|\left|\frac{\sum_{i=1}^{m-d_s}(-1)^i\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}[s_i,s_{i+1},\ldots,s_{i+d_1},s_n,t]}{\sum_{i=1}^{m-d_s}\lambda_i(s)}\right|
$$

$$
+|\Gamma^{\xi}_{\alpha_2}|\left|\frac{\sum_{j=1}^{n-d_t}(-1)^j\frac{\partial^{\xi+3}\phi}{\partial t^{\xi+1}\partial s^2}[t_j,t_{j+1},\ldots,t_{j+d_2},s_n,t_m]}{\sum_{j=1}^{n-d_t}\lambda_j(t)}\right|
$$

$$
= |\Gamma^{\xi}_{\alpha_2}|\left|\frac{\partial^{\xi+3}e}{\partial t^{\xi+1}\partial s^2}(t_m,s)\right|+|\Gamma^{\xi}_{\alpha_2}|\left|\frac{\partial^{\xi+3}e}{\partial t^{\xi+1}\partial s^2}(t_m,s_n)\right|,
$$

then we have

$$
|E_{22}(t,s)|\le\left|\frac{\partial^{\xi+3}e}{\partial t^{\xi+1}\partial s^2}(t_m,s)\right|+\left|\frac{\partial^{\xi+3}e}{\partial t^{\xi+1}\partial s^2}(t_m,s_n)\right|\le C(h^{d_s-\xi}+\tau^{d_t-1}). \tag{3.14}
$$

For $E_{21}(t,s)$, we get

$$
|E_{21}(t,s)|\le C(h^{d_s+1-\xi}+\tau^{d_t-1}). \tag{3.15}
$$

Similarly as $E_2(t,s)$, for $E_3(t,s)$ we have

$$
|E_3(t,s)|\le C(h^{d_s}+\tau^{d_t}). \tag{3.16}
$$

Combining (3.9), (3.15), (3.16) together, proof of Theorem 2 is completed.

4. Numerical examples

In this part, one example is presented to test the theorem. The nonuniform partition in this experiment defined as second kind of Chybechev point $s=\cos((0:m)'\pi/m), t=\cos((0:n)'\pi/n)$.

Example 1. *Consider the FC equation*

$$
\frac{\partial\phi}{\partial t} = {}_0C_t^{1-\alpha_1}\frac{\partial^2\phi}{\partial s^2}\phi(t,s)-\mu_0\,{}_0C_t^{1-\alpha_2}\phi(t,s)+f(t,s), 0\le s\le 1, 0\le t\le T,
$$

with the analysis solutions is

$$\phi(t, s) = t^2 \sin(\pi s),$$

with the initial condition

$$\phi(s, 0) = 0,$$

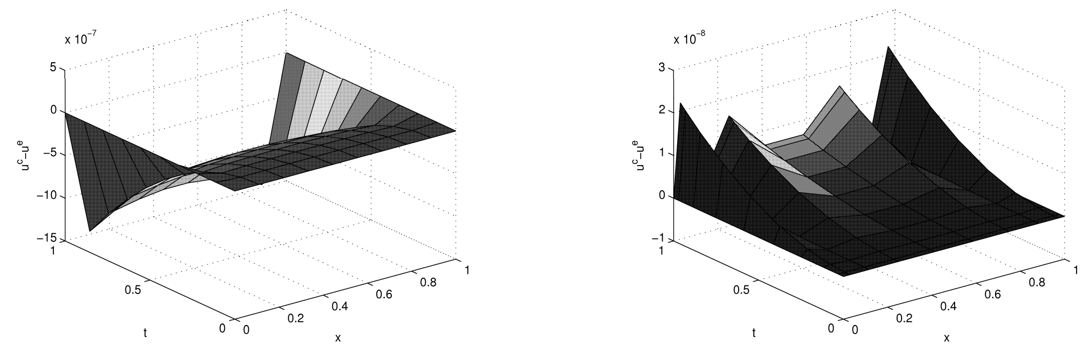
and boundary condition

$$\phi(0, t) = \phi(1, t) = 0,$$

and

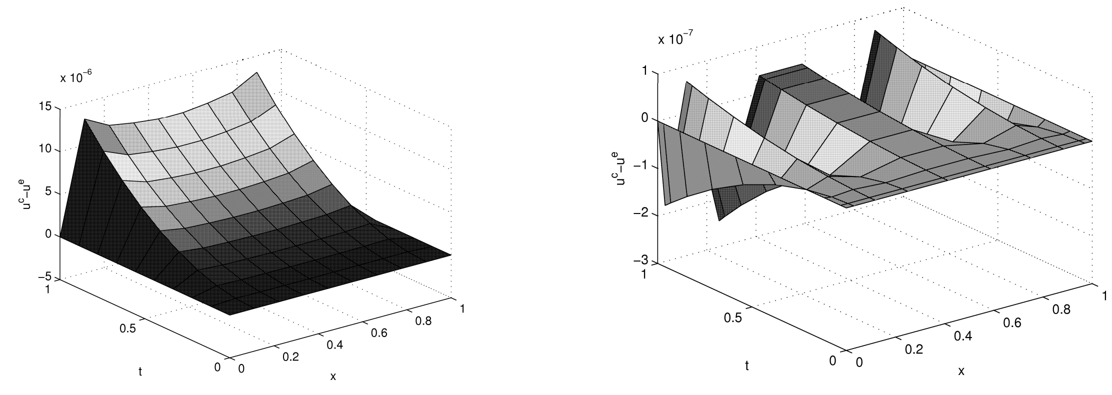
$$f(t, s) = 2\left(t + \frac{\pi^2 t^{1+\alpha_1}}{\Gamma(2+\alpha_1)} + \frac{t^{1+\alpha_2}}{\Gamma(2+\alpha_2)}\right)\sin(\pi s).$$

In Figures 1 and 2, errors of $m = n = 10$, $[a, b] = [0, 1]$ and $m = n = 10$, $d_t = d_s = 7$, $[a, b] = [0, 1]$ in Example 1. (a) uniform; (b) nonuniform for FC equation by rational interpolation collocation methods are presented, respectively. From the figure, we know that the precision can reach to 10^{-6} for both uniform and nonuniform partition.



(a) uniform (b) nonuniform

Figure 1. Errors of $m = n = 10$, $[a, b] = [0, 1]$ in Example 1 (a) uniform; (b) nonuniform.



(a) uniform

(b) nonuniform

Figure 2. Errors of $m = n = 10, d_t = d_s = 7$, $[a, b] = [0, 1]$ in Example 1 (a) uniform; (b) nonuniform.

Table 1. Errors of FC equation with $m = n = 10, \alpha_1 = \alpha_2 = 0.2$.

	method of substitution		method of additional	
	uniform	nonuniform	uniform	nonuniform
Larange	1.4662e-06	2.1919e-08	2.7900e-07	1.4310e-07
Rational	1.3038e-05	2.4541e-07	4.9788e-06	1.4310e-07

In Table 1, errors of the FC equation with $m = n = 10, \alpha_1 = \alpha_2 = 0.2$ for substitution methods and additional methods are presented, there are nearly no difference for the two methods. Additional method is more simple than substitution methods to add the boundary condition. In the following, we choosing the substitution method to deal with the boundary condition.

Table 2. Errors of FC equation for $\alpha_1 = 0.4, \alpha_2 = 0.6, d_t = d_s = 5$.

	uniform	nonuniform	uniform	nonuniform
t	$(12, 12)$	$(12, 12)$	$(12, 12) d_t = d_s = 5$	$(12, 12) d_t = d_s = 5$
0.5	2.1021e-11	3.8250e-09	6.8506e-06	1.6436e-06
1	9.0394e-13	4.4206e-10	4.6667e-06	7.8141e-07
5	6.1833e-12	5.6655e-08	2.3777e-04	4.2230e-05
10	1.0094e-12	8.5622e-07	1.9813e-04	1.5634e-05
15	3.5397e-12	1.8827e-05	8.5498e-04	8.2551e-05

Errors of the FC equation for $\alpha_1 = 0.4, \alpha_2 = 0.6, d_t = d_s = 5$ with $t = 0.1, 0.9, 1, 5, 10, 15$ are presented under the uniform and nonuniform in Table 2. As the time variable become from 0.5 to 15, there are high accuracy for our methods. We can improve the accuracy by increasing m, n or choosing the parameter d_t, d_s approximately which means our methods is useful.

In Table 3, errors of $\alpha_1 = 0.01, 0.1, 0.3, 0.5, 0.9, 0.99$ under uniform with $m = n = 10, d_t = 5, d_s = 5$ with $\alpha_2 = 0.1, 0.4, 0.6, 0.8, 0.99$ are presented under the uniform partition. From the table, we know that for different α_1, α_2 our methods have high accuracy with little number m and n. In the following table, numerical results are presented to test our theorem. From Tables 4 and 5, error of uniform for $\alpha_1 = \alpha_2 = 0.2, d_s = 5$ with different d_t are given, the convergence rate is $O(h^{d_t})$. From Table 5, with space variable uniform for $\alpha_1 = \alpha_2 = 0.2, d_t = 5$, the convergence rate is $O(h^7)$, we will investigate in future paper. For Tables 6 and 7, the errors of Chebyshev partition for s and t are presented. For $d_t = 5$, the convergence rate is $O(h^{d_s})$ in Table 6, while in Table 7, the convergence rate is $O(h^{d_t})$ which agrees with our theorem.

Table 3. Errors of α_1 under uniform with $m = n = 10, d_t = 5, d_s = 5$.

α_1	$\alpha_2 = 0.1$	$\alpha_2 = 0.4$	$\alpha_2 = 0.6$	$\alpha_2 = 0.8$	$\alpha_2 = 0.99$
0.01	1.0153e-04	1.0246e-04	1.0300e-04	1.0346e-04	1.0384e-04
0.1	1.2753e-05	1.2865e-05	1.2930e-05	1.2987e-05	1.3033e-05
0.3	2.7464e-05	2.7704e-05	2.7845e-05	2.7971e-05	2.8074e-05
0.5	4.5746e-06	4.6152e-06	4.6399e-06	4.6609e-06	4.6794e-06
0.9	9.0295e-06	9.1193e-06	9.1240e-06	9.2142e-06	9.2479e-06
0.99	1.8981e-06	1.8247e-06	1.5293e-06	1.9193e-06	2.0670e-06

Table 4. Errors of uniform for $\alpha_1 = \alpha_2 = 0.2, d_s = 5$.

m, n	$d_t = 2$		$d_t = 3$		$d_t = 4$		$d_t = 5$	
8	1.3626e-02		6.9619e-03		2.0708e-03		9.8232e-04	
10	9.6780e-03	1.5332	3.4354e-03	3.1653	6.9542e-04	4.8900	3.2829e-04	4.9117
12	7.0485e-03	1.7389	1.9408e-03	3.1320	2.9186e-04	4.7621	1.3132e-04	5.0255
14	5.4466e-03	1.6725	1.2017e-03	3.1097	1.4211e-04	4.6686	6.0148e-05	5.0654

Table 5. Errors of uniform for $\alpha_1 = \alpha_2 = 0.2, d_t = 5$.

m, n	$d_s = 2$		$d_s = 3$		$d_s = 4$	
8	4.9495e-04		4.9492e-04		4.9486e-04	
10	1.0051e-04	7.1443	1.0053e-04	7.1431	1.0053e-04	7.1426
12	2.7700e-05	7.0690	2.7711e-05	7.0679	2.7714e-05	7.0673
14	9.4272e-06	6.9921	9.4315e-06	6.9917	9.4314e-06	6.9925

Table 6. Errors of non-uniform partition with $\alpha_1 = \alpha_2 = 0.2, d_t = 5$.

m,n	$d_s = 2$		$d_s = 3$		$d_s = 4$	
8	2.8113e-05		2.8110e-05		2.8108e-05	
10	2.1197e-05	1.2654	2.1196e-05	1.2652	2.1195e-05	1.2651
12	6.6990e-06	6.3180	6.6989e-06	6.3178	6.6988e-06	6.3176
14	1.6712e-06	9.0069	1.6712e-06	9.0068	1.6712e-06	9.0067

Table 7. Errors of non-uniform partition $\alpha_1 = \alpha_2 = 0.2, d_s = 5$.

m,n	$d_t = 2$		$d_t = 3$		$d_t = 4$		$d_t = 5$	
8	3.1539e-02		8.7995e-03		2.1930e-03		3.3004e-04	
10	2.4329e-02	1.1632	4.0288e-03	3.5010	2.7133e-04	9.3648	2.2278e-04	1.7613
12	1.5223e-02	2.5716	1.9127e-03	4.0859	9.5194e-05	5.7449	5.1702e-05	8.0116
14	1.1407e-02	1.8721	1.1143e-03	3.5049	3.5772e-05	6.3493	1.1369e-05	9.8255

Table 8. Errors of uniform with $\alpha_1 = 0.4, \alpha_2 = 0.6, d_t = 5$.

m,n	$d_s = 2$		$d_s = 3$		$d_s = 4$	
8	4.9427e-04		4.9426e-04		4.9414e-04	
10	1.0035e-04	7.1455	1.0041e-04	7.1427	1.0041e-04	7.1413
12	2.7639e-05	7.0720	2.7674e-05	7.0684	2.7684e-05	7.0669
14	9.3984e-06	6.9977	9.4153e-06	6.9942	9.4254e-06	6.9895

Table 9. Errors of uniform with $\alpha_1 = 0.4, \alpha_2 = 0.6, d_s = 5$.

m,n	$d_t = 1$		$d_t = 2$		$d_t = 3$		$d_t = 4$	
8	1.3587e-02		6.9513e-03		2.0677e-03		9.8084e-04	
10	9.6497e-03	1.5334	3.4314e-03	3.1637	6.9462e-04	4.8884	3.2791e-04	4.9102
12	7.0259e-03	1.7404	1.9389e-03	3.1311	2.9157e-04	4.7613	1.3118e-04	5.0249
14	5.4269e-03	1.6752	1.2005e-03	3.1096	1.4198e-04	4.6682	6.0090e-05	5.0648

In the following table, $\alpha_1 = 0.4, \alpha_2 = 0.6$ is chosen to present numerical results. From Tables 8 and 9, error of uniform partition $d_t = 5$ with different d_s are given, the convergence rate is $O(h^7)$. From Table 8, with space variable $s, d_s = 5$, the convergence rate is $O(h^{d_t})$ which agrees with our theorem.

For Tables 10 and 11, the errors of Chebyshev partition for non-uniform with $\alpha_1 = 0.4, \alpha_2 = 0.6$ are presented. For $d_t = 5$, the convergence rate is $O(h^7)$ in Table 11, while in Table 10, the convergence rate is $O(h^{d_t})$ which agrees with our theorem.

Table 10. Errors of non-uniform with $\alpha_1 = 0.4, \alpha_2 = 0.6, d_s = 5$.

m, n	$d_t = 1$		$d_t = 2$		$d_t = 3$		$d_t = 4$	
8	3.1481e-02		8.7825e-03		2.1876e-03		3.2930e-04	
10	2.4263e-02	1.1671	4.0219e-03	3.5000	2.7124e-04	9.3553	2.2231e-04	1.7606
12	1.5185e-02	2.5704	1.9076e-03	4.0912	9.5106e-05	5.7481	5.1649e-05	8.0057
14	1.1373e-02	1.8751	1.1117e-03	3.5026	3.5733e-05	6.3504	1.1365e-05	9.8211

Table 11. Errors of non-uniform with $\alpha_1 = 0.4, \alpha_2 = 0.6, d_t = 5$.

m, n	$d_s = 2$		$d_s = 3$		$d_s = 4$	
8	2.8065e-05		2.8059e-05		2.8056e-05	
10	2.1156e-05	1.2665	2.1154e-05	1.2660	2.1153e-05	1.2656
12	6.6875e-06	6.3168	6.6874e-06	6.3164	6.6873e-06	6.3161
14	1.6693e-06	9.0033	1.6693e-06	9.0031	1.6693e-06	9.0030

5. Concluding remarks

In this paper, BRIM was used to solve the (1+1) dimensional FC equation that is presented. For fractional-order PDEs, the convergence order is seriously affected by the orders of fractional derivatives. By fractional integration, the singularity of the fractional derivative of the FC equation can be changed to nonsingular integral, with adding one order to the derivatives of density function. So there are no effects on the orders of fractional derivatives. The singularity of fractional derivative is overcome by the integral to density function from the singular kernel. For the arbitrary fractional derivative, the new Gauss formula is constructed to calculated it simply. For the Diriclet boundary condition, the FC equation is changed to the discrete FC equation and the matrix equation of it is given. In the future, the FC equation with Nuemann condition can be solved by BRIM, and high dimensional FC equation can also be studied by our methods.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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