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Research article

Multiplicity of periodic solutions for weakly coupled parametrized systems with singularities

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Abstract: We prove the existence of multiple periodic solutions for weakly coupled parametrized systems with a singularity of repulsive type at the origin and linear growth at infinity. The proof is based on a higher dimensional Poincaré-Birkhoff theorem and the phase-plane analysis of the solutions.

Keywords: periodic solutions; weakly coupled systems; systems with singularities; Poincaré-Birkhoff theorem

1. Introduction

In 2010, Fonda and Ghirardelli [1] established a multiplicity result for the parametrized equation

$$\begin{cases} x'' + g(t,x) = sw(t), \\ x(0) = x(T),\ x'(0) = x'(T). \end{cases} \tag{1.1}$$

Here, $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $w : [0,T]\times\mathbb{R} \to \mathbb{R}$ is integrable. Their result generalized the classical result of Lazer and McKenna [2], dated 1987, and extended the results of Del Pino et al. [3] and Zanini and Zanolin [4]. In 2017, Calamai and Sfecci [5] extended this multiplicity result to the weakly coupled parametrized system

$$\begin{cases} x_i'' + g_i(t,x) = sw_i(t), \\ x_i(0) = x_i(T),\ x_i'(0) = x_i'(T), \end{cases} \qquad i = 1,\cdots,N,$$

where $g_i : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $w_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, x is the vector $(x_1,\cdots,x_N)$, and s is a real parameter. The functions g_i and w_i are assumed to be T-periodic in the time variable. The proof is based on the higher dimensional Poincaré-Birkhoff theorem obtained by Fonda and Ureña [6].

Unlike the result in [1], Calamai and Sfecci [5] do not assume the Lipschitz regularity condition on the functions g_i, which is a crucial assumption in [1]. Relevant related results can be found in [7–11].

On the other hand, Boscaggin et al. [12] investigated the parametrized equation (1.1) with a suitable singularity of repulsive type at the origin and linear growth at infinity. They obtained a multiplicity of periodic solutions for Eq (1.1). In order to guarantee uniqueness of the associated Cauchy problems, they also assumed the Lipschitz regularity on the function g. For related results on periodic solutions of singularity equations, see also [9,13–20]. Furthermore, high-order nonlinear systems with nonlinear parameterization and other related stability problems have attracted some authors' attention; see, for instance, [21–26].

Motivated by the works of [12] and [5], a natural inquiry arises as to whether weakly coupled parametrized systems with a singularity of repulsive type at the origin and linear growth at infinity possess multiple periodic solutions. In this paper, we will consider the weakly coupled parametrized system

$$\begin{cases} x_i'' + \varphi_i(t, x_i) + p_i(t, x) = sw_i(t), \\ x_i(0) = x_i(T),\ x_i'(0) = x_i'(T), \end{cases} \qquad i = 1, \cdots, N. \tag{S}$$

For each index $i = 1, \cdots, N$, we assume the following hypotheses hold.

(H_0) The functions $\varphi_i : [0, T] \times (0, +\infty) \to \mathbb{R}$ and $w_i : [0, T] \to \mathbb{R}$ are continuous, and s is a real parameter.

(H_1) There exists a function $H : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x_i} H(t, x) = p_i(t, x),$$

where $p_i : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is continuous and bounded.

(H_2^i) There exists a continuous function $f : (0, \delta] \to \mathbb{R}$ satisfying

$$\varphi_i(t, x_i) \le f(x_i), \quad \text{for every } t \in [0, T] \text{ and every } x_i \in (0, \delta],$$

and

$$\lim_{x_i \to 0^+} f(x_i) = -\infty, \qquad \int_0^\delta f(x_i)dx_i = -\infty.$$

(H_3^i) There exists a function $a_i(t)$ such that

$$\lim_{x_i \to +\infty} \frac{\varphi_i(t, x_i)}{x_i} = a_i(t), \quad \text{uniformly for every } t \in [0, T].$$

(H_4^i) There exist $a_\pm^i$ and an integer $m_i \ge 0$ such that

$$\left(\frac{m_i \pi}{T}\right)^2 < a_-^i \le a_i(t) \le a_+^i < \left(\frac{(m_i + 1)\pi}{T}\right)^2, \quad \text{for every } t \in [0, T].$$

Moreover, the unique solution to

$$\begin{cases} \varsigma'' + a_i(t)\varsigma = w_i(t), \\ \varsigma(0) = \varsigma(T),\ \varsigma'(0) = \varsigma'(T), \end{cases}$$

is strictly positive.

Due to the absence of Lipschitz regularity assumptions in our system, uniqueness of the solution cannot be guaranteed for the corresponding Cauchy problem. To address this issue, we employ the higher dimensional Poincaré-Birkhoff theorem obtained by Fonda and Ureña [6]. Moreover, the solution trajectories of high-dimensional systems are inherently intricate. To characterize the twisting properties of the solution, we project the high-dimensional system's solution onto the plane and utilize sophisticated phase plane analysis methods.

Throughout the paper, we define $\left[\frac{m_i-1}{2}\right]$ as the greatest integer that is less than $\frac{m_i-1}{2}$. The main results of this paper are presented below.

Theorem 1.1. *Assume that Hamiltonian system* (S) *satisfies* $(H_0)-(H_1)$ *and* $(H_2^i)-(H_4^i)$*, for every index* $i = 1, 2, \cdots, N$*. Then, there exists* $s_0 > 0$ *such that, for every* $s \geq s_0$*, the Hamiltonian system* (S) *has at least*

$$1 + (N+1)\prod_{i=1}^{N}\left(\left[\frac{m_i-1}{2}\right]+1\right)$$

periodic solutions.

Remark 1.2. Similarly, we can obtain the multiplicity of periodic solutions for system (S) by the Carathéodory type of regularity. Taking $N = 1$, Theorem 1.1 leads to the existence of $1 + 2(\left[\frac{m_1-1}{2}\right] + 1)$ periodic solutions. Namely, if m_1 is odd, the Hamiltonian system (S) has at least $m_1 + 2$ periodic solutions. If m_1 is even, there exist $m_1 + 1$ periodic solutions for the system (S). Without assuming the uniqueness of solutions associated with the Cauchy problem, this result extends Theorem 1.1 in [12] to weakly coupled parametrized systems with continuous nonlinearities.

Remark 1.3. The function $\varphi_i(t, x_i)$ in system (S) is singular at the origin and merely continuous without any Lipschitz regularity assumptions. Therefore, Theorem 1.1 extends the results of [5, Theorem 1.3] to the weakly coupled parametrized systems with singularities.

The remaining sections of this paper are organized as follows. Section 2 introduces the basic concept of the i-th rotation number and provides some auxiliary lemmas for system (S). In Section 3, we present some auxiliary lemmas for system (P) below and provide a proof of Theorem 1.1.

2. Preliminaries

If the component $(x_i(t), x_i'(t))$ of $(x(t), x'(t)) \in \mathbb{R}^{2N}$ does not attain the origin, we can transform to the standard polar coordinates as

$$x_i(t) = \rho_i(t)\cos\theta_i(t), \quad x_i'(t) = \rho_i(t)\sin\theta_i(t).$$

Thus, if $(x_i(t), x_i'(t)) \neq (0, 0)$ for $t \in [\tau_0, \tau_1]$, we can define the *i-th rotation number* of $(x(t), x'(t))$ along that interval as

$$\mathrm{Rot}((x_i(t), x_i'(t)); [\tau_0, \tau_1]) = -\frac{1}{2\pi}(\theta_i(\tau_1) - \theta_i(\tau_0)) = \frac{1}{2\pi}\int_{\tau_0}^{\tau_1}\frac{x_i'(t)^2 - x_i(t)x_i''(t)}{x_i(t)^2 + x_i'(t)^2}dt.$$

Here, $\mathrm{Rot}((x_i(t), x_i'(t)); [\tau_0, \tau_1])$ describes clockwise rotations performed by the path of $(x(t), x'(t))$ around the origin in the time interval $[\tau_0, \tau_1]$ and (x_i, x_i') phase-plane. The modified version of the

i-th rotation number of $(x(t), x'(t))$ on $[\tau_0, \tau_1]$ is defined as

$$\mathrm{Rot}_{\sqrt{a_+^i}}((x_i(t), x_i'(t)); [\tau_0, \tau_1]) = \frac{\sqrt{a_+^i}}{2\pi}\int_{\tau_0}^{\tau_1} \frac{x_i'(t)^2 - x_i(t)x_i''(t)}{a_+^i x_i(t)^2 + x_i'(t)^2}dt.$$

For more details about the modified rotation numbers, please see [11,27]. For $i \in \{1, \cdots, N\}$, let

$$\mathcal{N}_i(x, y) = \sqrt{\frac{1}{x_i^2} + x_i^2 + y_i^2},\ x_i > 0,\ y_i \in \mathbb{R},$$

which is a function similar to the "norm" in the phase-plane. For more details on the function, see [12, page 4461].

Remark 2.1. Similar to [11, Theorem 4 and Remark 1], for every integer j, we have

$$\mathrm{Rot}((x_i(t), x_i'(t)); [0, T]) < j \Longleftrightarrow \mathrm{Rot}_{\sqrt{a_+^i}}((x_i(t), x_i'(t)); [0, T]) < j;$$

$$\mathrm{Rot}((x_i(t), x_i'(t)); [0, T]) > j \Longleftrightarrow \mathrm{Rot}_{\sqrt{a_+^i}}((x_i(t), x_i'(t)); [0, T]) > j.$$

Lemma 2.2. *There exist $\tilde{s} > 0$ and two positive constant $c_0 < C_0$ such that, for every $s \geq \tilde{s}$, system* (S) *has a solution $\hat{x} = \hat{x}(s, t)$ whose components satisfy*

$$c_0 \leq \frac{\hat{x}_i(s, t)}{s} \leq C_0, \tag{2.1}$$

for every $t \in [0, T]$ and $i \in \{1, \cdots, N\}$.

Proof. Consider the truncated function

$$\tilde{g}_i(t, x) = \begin{cases} \varphi_i(t, x_i) + p_i(t, x), & \text{if } x_i > 1, \\ \varphi_i(t, 1) + p_i(t, x), & \text{if } x_i \leq 1, \end{cases}$$

for $t \in [0, T]$ and $x \in \mathbb{R}^N$. By (H_1), there exists a constant $M > 0$ such that

$$|p_i(t, x)| \leq M,$$

for every $(t, x) \in [0, T] \times \mathbb{R}^N$ and $i = 1, \cdots, N$. Consequently, by (H_3^i), we have

$$\lim_{x_i \to +\infty} \frac{\tilde{g}_i(t, x)}{x_i} = a_i(t) \quad \text{and} \quad \lim_{x_i \to -\infty} \frac{\tilde{g}_i(t, x)}{x_i} = 0,$$

uniformly for every $t \in [0, T]$. Consider the system

$$\begin{cases} z_i'' + \frac{\tilde{g}_i(t, sz)}{s} = w_i(t), \\ z_i(0) = z_i(T),\ z_i'(0) = z_i'(T), \end{cases} \quad i = 1, \cdots, N, \tag{2.2}$$

where $z_i(t) = x_i(t)/s$. By using [5, Lemma 2.3], we have that there are three positive constants $\bar{s}, c_0$ and C_0 such that, for every $s \geq \bar{s}$, system (2.2) has a solution $z = z(s, t)$ whose components satisfy $c_0 \leq z_i(s, t) \leq C_0$ for every $t \in [0, T]$ and $i \in \{1, \cdots, N\}$. Hence, there exists a solution $\hat{x}(s, t)$ of

$$\begin{cases} x_i'' + \tilde{g}_i(t, x) = sw_i(t), \\ x_i(0) = x_i(T),\ x_i'(0) = x_i'(T), \end{cases} \quad i = 1, \cdots, N,$$

whose components satisfy (2.1) for every $t \in [0,T]$ and $i \in \{1,\cdots,N\}$. Clearly, (2.1) implies that $\hat{x}_i(s,t) \to +\infty$ as $s \to +\infty$ for every $t \in [0,T]$ and $i \in \{1,\cdots,N\}$. Then, there exists $\tilde{s} > \bar{s}$ such that, $\hat{x}_i(s,t) > 1$ for every $t \in [0,T]$ and $i \in \{1,\cdots,N\}$, and hence $\hat{x}_i(s,t)$ is also a solution of system (S).

Remark 2.3. Note that Remark 2.2 in [5] remains valid even when $\nu_1^i(t) = \nu_2^i(t) = 0$. Therefore, the conclusion of Lemma 2.3 in [5] is also valid when

$$\lim_{x_i \to -\infty} \frac{g_i(t,x)}{x_i} = 0$$

uniformly for every $t \in [0,T]$. Our proof of Lemma 2.2 relies on this fact.

To simplify the proof below, we define $g_i(t,x) = \varphi_i(t,x_i) + p_i(t,x)$, where φ_i and p_i are defined as in $(H_0) - (H_1)$ and $(H_2^i) - (H_4^i)$. Then, the system (S) can be written as

$$\begin{cases} x_i'' + g_i(t,x) = sw_i(t), \\ x_i(0) = x_i(T),\ x_i'(0) = x_i'(T), \end{cases} \quad i = 1,\cdots,N. \tag{S'}$$

We can verify that $g_i(t,x)$ satisfies the hypotheses $(H_0)'$, $(H_2^i)'$ and $(H_3^i)'$ as follows.

$(H_0)'$ The function $g_i(t,x) : [0,T] \times (0,+\infty)^N \to \mathbb{R}$ is continuous.

$(H_2^i)'$ There is a continuous function $\tilde{f} : (0,\delta] \to \mathbb{R}$ satisfying

$$g_i(t,x) \le \tilde{f}(x_i), \quad \text{for every } t \in [0,T] \text{ and every } x_i \in (0,\delta],$$

and

$$\lim_{x_i \to 0^+} \tilde{f}(x_i) = -\infty, \quad \int_0^\delta \tilde{f}(x_i)dx_i = -\infty.$$

$(H_3^i)'$ $\lim_{x_i \to +\infty} \frac{g_i(t,x)}{x_i} = a_i(t)$, uniformly for every $t \in [0,T]$.

Lemma 2.4. *For every $s \in \mathbb{R}$, the solution to the Cauchy problem*

$$\begin{cases} x_i'' + g_i(t,x) = sw_i(t), \\ x_i(0) = \bar{x}_i > 0,\ x_i'(0) = \bar{y}_i, \end{cases} \quad i = 1,\cdots,N, \tag{2.3}$$

is globally defined on $[0,T]$.

Proof. Let $x(t)$ be a solution of the system (2.3). Assume the contrary, that is, there is a component $x_{i_0}(t)$ of $x(t)$ whose maximal interval of definition is $[0,\tau)$ for $\tau < T$. By standard arguments in the theory of initial value problems, we have

$$\limsup_{\sigma \to \tau^-} \mathcal{N}_{i_0}(x(\sigma), x'(\sigma)) = +\infty.$$

By the arguments in [28, Lemma 1], we have

$$\lim_{\sigma \to \tau^-} \mathrm{Rot}((x_{i_0}(t) - 1, x_{i_0}'(t)); [0,\sigma]) = +\infty. \tag{2.4}$$

On the other hand, by using the computation in [28, Lemma 2], we find that $\mathrm{Rot}((x_{i_0}(t) - 1, x_{i_0}'(t)); [0,\tau])$ is bounded, which is a contradiction with (2.4).

Lemma 2.5. *There exists $\hat{R}_s > 0$ such that, if $x : [0, T] \to \mathbb{R}^N$ is a solution of (S') with $\mathcal{N}_i(x(t), x'(t)) \geq \hat{R}_s$ for a certain index i and every $t \in [0, T]$, then*

$$\mathrm{Rot}((x_i(t) - \hat{x}_i(s, 0), x_i'(t) - \hat{x}_i'(s, 0)); [0, T]) > m_i.$$

Proof. For simplicity, we assume $m_i = 1$. Take $\hat{R}_s$ such that

$$\mathcal{N}_i(\hat{x}_i(s, 0), \hat{x}_i'(s, 0)) < \hat{R}_s.$$

Let $x(t)$ be a solution of (S') with $\mathcal{N}_i(x(t), x'(t)) \geq \hat{R}_s$ for a certain index i and every $t \in [0, T]$. We will show that

$$\mathrm{Rot}((x_i(t) - \hat{x}_i(s, 0), x_i'(t) - \hat{x}_i'(s, 0)); [0, T]) > 1. \tag{2.5}$$

That is, in the $(x_i(t), x_i'(t))$-phase plane, $(x(t), x'(t))$ performs more than one turn around $(\hat{x}_i(s, 0), \hat{x}_i'(s, 0))$ in the time interval $[0, T]$.

Writing $(x_i(t) - \hat{x}_i(s, 0), x_i'(t) - \hat{x}_i'(s, 0)$ in polar coordinates,

$$x_i(t) = \hat{x}_i(s, 0) + \rho_i(t) \cos\theta_i(t), \qquad x_i'(t) = \hat{x}_i'(s, 0) + \rho_i(t) \sin\theta_i(t),$$

we can deduce that

$$-\theta_i'(t) = \frac{x_i'(t)(x_i'(t) - \hat{x}_i'(s, 0)) + (g_i(t, x) - sw_i(t))(x_i(t) - \hat{x}_i(s, 0))}{(x_i(t) - \hat{x}_i(s, 0))^2 + (x_i'(t) - \hat{x}_i'(s, 0))^2} \tag{2.6}$$

for every $t \in [0, T]$. With fixed $\alpha \in ((\pi/T)^2, a_-^i)$, by $(H_3^i)'$ and (H_4^i), we can choose $d > \hat{x}_i(s, 0)$ such that

$$g_i(t, x) - sw_i(t) \geq \alpha(x_i(t) - \hat{x}_i(s, 0)) \tag{2.7}$$

for every $t \in [0, T]$ and every $x_i(t) \geq d$.

We first consider the case when $x_i(0) > d$. The proof will be divided into three steps.

Step 1. We claim that there exists $t_1 \in (0, T]$ such that $x_i(t_1) = d$ and $x_i(t) > d$ for every $t \in [0, t_1)$ (see Figure 1).

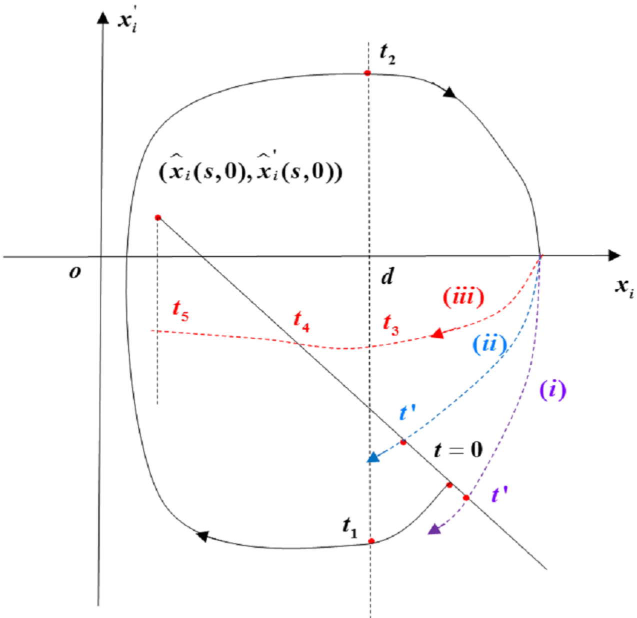

Figure 1. The possible trajectories for the solution in (x_i, x_i')-phase plane.

Suppose, contrary to our claim, that $x_i(t) > d$ for every $t \in [0, T]$. Note that, enlarging $\hat{R}_s$, by $\mathcal{N}_i(x(t), x'(t)) \geq \hat{R}_s$, we have

$$\left|\frac{\hat{x}_i'(s,0)x_i'(t) - \hat{x}_i'(s,0)^2}{(x_i(t) - \hat{x}_i(s,0))^2 + (x_i'(t) - \hat{x}_i'(s,0))^2}\right| \leq \frac{\eta}{2T}\min\{\alpha, 1\} \tag{2.8}$$

for $x_i(t) > d$, where $\eta > 0$ is sufficiently small such that

$$\frac{\pi}{\sqrt{\alpha}} + 4\eta < T. \tag{2.9}$$

Combining (2.6), (2.7) with (2.8), we have

$$\begin{aligned}
-\theta_i'(t) &\geq \sin^2\theta_i(t) + \alpha\cos^2\theta_i(t) + \frac{\hat{x}_i'(s,0)x_i'(t) - \hat{x}_i'(s,0)^2}{(x_i(t) - \hat{x}_i(s,0))^2 + (x_i'(t) - \hat{x}_i'(s,0))^2} \\
&\geq \sin^2\theta_i(t) + \alpha\cos^2\theta_i(t) - \frac{\eta}{2T}\min\{\alpha, 1\},
\end{aligned} \tag{2.10}$$

that is,

$$\min\{\alpha, 1\}\frac{\eta}{2T} - \theta_i'(t) \geq \sin^2\theta_i(t) + \alpha\cos^2\theta_i(t) \tag{2.11}$$

for every $t \in [0, T]$. Notice that

$$\frac{\min\{\alpha, 1\}}{\sin^2\theta_i(t) + \alpha\cos^2\theta_i(t)} \leq 1$$

for every $t \in [0, T]$ and $\alpha \in ((\pi/T)^2, a_-^i)$. Indeed, if $1 \leq \alpha < a_-^i$, for every $t \in [0, T]$, one has

$$\frac{\min\{\alpha, 1\}}{\sin^2\theta_i(t) + \alpha\cos^2\theta_i(t)} = \frac{1}{1 + (\alpha - 1)\cos^2\theta_i(t)} \leq 1.$$

If $(\pi/T)^2 < \alpha < 1$, we have

$$\frac{\min\{\alpha, 1\}}{\sin^2\theta_i(t) + \alpha\cos^2\theta_i(t)} = \frac{1}{\frac{1}{\alpha}\sin^2\theta_i(t) + \cos^2\theta_i(t)} = \frac{1}{1 + \left(\frac{1}{\alpha} - 1\right)\sin^2\theta_i(t)} \leq 1$$

for every $t \in [0, T]$. Hence,

$$\begin{aligned}
\int_{\theta_i(T)}^{\theta_i(0)} \frac{d\theta_i}{\min\{\alpha, 1\}\frac{\eta}{2T} - \theta_i'(t)} &= \int_0^T \frac{\theta_i'(t)}{\theta_i'(t) - \min\{\alpha, 1\}\frac{\eta}{2T}} dt \\
&= \int_0^T \left(1 - \frac{\eta}{2T} \cdot \frac{\min\{\alpha, 1\}}{\min\{\alpha, 1\}\frac{\eta}{2T} - \theta_i'(t)}\right) dt \\
&= \int_0^T \left(1 - \frac{\eta}{2T} \cdot \frac{\min\{\alpha, 1\}}{\sin^2\theta_i(t) + \alpha\cos^2\theta_i(t)}\right) dt \\
&\geq T\left(1 - \frac{\eta}{2T}\right).
\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have

$$T\left(1 - \frac{\eta}{2T}\right) \leq \int_{\theta_i(T)}^{\theta_i(0)} \frac{d\theta_i}{\sin^2\theta_i(t) + \alpha\cos^2\theta_i(t)}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\alpha}} \arctan(\frac{1}{\sqrt{\alpha}} \tan \theta_i(t))|_T^0 \\
&\le \frac{\pi}{\sqrt{\alpha}},
\end{aligned} \tag{2.13}
$$

which contradicts (2.9).

Step 2. If $x_i(t) \ge d$, by the computation in Step 1, then

$$\theta_i'(t) < 0, \quad \text{for every } t \in [0, T]. \tag{2.14}$$

On the other hand, if $x_i(t) \in (0, d)$, since $\mathcal{N}_i(x(t), x'(t))$ is large for every $t \in [0, T]$, either $x_i(t)$ is near the singularity, or $|x_i'(t)|$ is large. By $(H_2^i)'$, we have

$$\lim_{x_i \to 0^+} (g_i(t, x) - sw_i(t)) = -\infty$$

uniformly for every $t \in [0, T]$. Arguing as in [28, Lemma 2], inequality (2.14) holds. Thus, up to enlarging $\hat{R}_s$, we can find $t_2 \in (t_1, T]$ such that $x_i(t_2) = d$, $x_i'(t_2) > 0$ and $x_i(t) \in (0, d)$ for every $t \in (t_1, t_2)$ (see Figure 1). Moreover, by a similar argument as that in [28, Lemma 2], we have

$$t_2 - t_1 < \eta. \tag{2.15}$$

Step 3. Note that there are three possible trajectories for the solution when $t > t_2$ (see Figure 1). First, we will prove that there exists $t' \in (t_2, T)$ such that

$$\theta_i(0) - \theta_i(t') = 2\pi \tag{2.16}$$

for the trajectories (i) and (ii). If not, it holds that

$$\theta_i(0) - \theta_i(t) < 2\pi \quad \text{for every } t \in (t_2, T).$$

Similar to (2.13), we have

$$
\begin{aligned}
t_1 - \eta < t_1 \left(1 - \frac{\eta}{2T}\right) &\le \int_{\theta_i(t_1)}^{\theta_i(0)} \frac{d\theta_i}{\sin^2 \theta_i(t) + \alpha \cos^2 \theta_i(t)} \\
&= \frac{1}{\sqrt{\alpha}} \arctan(\frac{1}{\sqrt{\alpha}} \tan \theta_i(t))|_{t_1}^0
\end{aligned} \tag{2.17}
$$

and

$$
\begin{aligned}
T - t_2 - \eta < (T - t_2)\left(1 - \frac{\eta}{2T}\right) &\le \int_{\theta_i(T)}^{\theta_i(t_2)} \frac{d\theta_i}{\sin^2 \theta_i(t) + \alpha \cos^2 \theta_i(t)} \\
&= \frac{1}{\sqrt{\alpha}} \arctan(\frac{1}{\sqrt{\alpha}} \tan \theta_i(t))|_T^{t_2}.
\end{aligned} \tag{2.18}
$$

Combining (2.17), (2.18) with (2.15), we have

$$T - \eta < T - (t_2 - t_1) = t_1 + (T - t_2) < \frac{\pi}{\sqrt{\alpha}} + 2\eta, \tag{2.19}$$

which contradicts (2.9).

Second, for the trajectory (*iii*), we claim that there exists $t_3 \in (t_2, T)$ such that $x_i(t_3) = d$, and $x_i(t) > d$ when $t \in (t_2, t_3)$ (see Figure 1). Suppose, contrary to our claim, that $x_i(t) > d$ for every $t \in (t_2, T]$. In this case, the inequalities (2.17)–(2.19) are still valid. That is a contradiction. Moreover, we have

$$\begin{aligned} t_1 + (t_3 - t_2) - 2\eta &< \int_{\theta_i(t_3)}^{\theta_i(t_2)} \frac{d\theta_i}{\sin^2\theta_i(t) + \alpha\cos^2\theta_i(t)} + \int_{\theta_i(t_1)}^{\theta_i(0)} \frac{d\theta_i}{\sin^2\theta_i(t) + \alpha\cos^2\theta_i(t)} \\ &= \frac{1}{\sqrt{\alpha}}\left(\arctan(\frac{1}{\sqrt{\alpha}}\tan\theta_i(t))\Big|_{t_3}^{t_2} + \arctan(\frac{1}{\sqrt{\alpha}}\tan\theta_i(t))\Big|_{t_1}^{0}\right) \\ &\le \frac{\pi}{\sqrt{\alpha}}. \end{aligned} \tag{2.20}$$

Combining (2.15) with (2.20), we have

$$t_3 < \frac{\pi}{\sqrt{\alpha}} + 3\eta < T - \eta.$$

Now, we claim that there exists $t_4 \in (t_3, T)$ such that

$$\theta_i(0) - \theta_i(t_4) = 2\pi \tag{2.21}$$

for the trajectory (*iii*). Suppose, contrary to our claim, that

$$\theta_i(0) - \theta_i(t) < 2\pi \quad \text{for every } t \in (t_3, T). \tag{2.22}$$

Therefore, using a similar argument as in [28, Lemma 2], there exists $t_5 \in (t_3, t_3 + \eta)$ such that $x_i(t_5) = \hat{x}_i(s, 0)$ (see Figure 1). So,

$$\theta_i(0) - \theta_i(t_5) > 2\pi,$$

which is a contradiction with (2.22).

By (2.14), (2.16) and (2.21), we have

$$\mathrm{Rot}((x_i(t) - \hat{x}_i(s,0), x_i'(t) - \hat{x}_i'(s,0)); [0,T]) > 1, \quad \text{for } x_i(0) > d.$$

Using a similar argument as given above, inequality (2.5) is valid for $x_i(0) \le d$. Moreover, for any positive integer $m_i > 1$, the conclusion can be proved by similar arguments.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we introduce the variable $u = (u_1, \cdots, u_N)$ as defined by

$$u = \frac{x - \hat{x}(s,t)}{s}, \tag{3.1}$$

and we transform system (S') into

$$\begin{cases} u_i'' + h_i(s,t,u) = 0, \\ u_i(0) = u_i(T),\ u_i'(0) = u_i'(T), \end{cases} \quad i = 1, \cdots, N, \tag{P}$$

where, for every index i,

$$h_i(s,t,u) = \frac{g_i(t, su + \hat{x}(s,t)) - g_i(t, \hat{x}(s,t))}{s}.$$

It is clear that $h_i(s,t,u)$ is well defined for every $t \in [0,T]$ and $u_i > -\hat{x}_i(s,t)/s$, and $h_i(s,t,0) \equiv 0$. Consider the Cauchy problem

$$\begin{cases} u_i'' + h_i(s,t,u) = 0, \\ u_i(0) = \bar{u}_i > -\frac{\hat{x}_i(s,0)}{s}, \ u_i'(0) = \bar{v}_i, \end{cases} \qquad i = 1, \cdots, N. \tag{3.2}$$

By (3.1), we find that $x(t)$ is a solution of (2.3) if and only if $u(t)$ solves (3.2), so $u(t)$ is globally defined on $[0,T]$ by Lemma 2.4. If the component $(u_i(t), u_i'(t))$ of $(u(t), u'(t))$ does not attain the origin, we can pass to the standard polar coordinates as

$$u_i(t) = r_i(t)\cos\theta_i(t), \quad u_i'(t) = r_i(t)\sin\theta_i(t).$$

Throughout the rest of the proof, $D(\Gamma_s^i)$ denotes the open bounded region delimited by a Jordan curve Γ_s^i, $\overline{B}_i(0,\tilde{r})$ denotes the closed ball of radius $\tilde{r}$ centered at the origin, and $\mathcal{D}_s^i$ denotes the set $\{(u,v) \in \mathbb{R}^N | u_i > -\hat{x}_i(s,0)/s\}$.

Lemma 3.1. *For $i \in \{1, \cdots, N\}$, it holds that*

$$\lim_{s \to +\infty} h_i(s,t,u) = a_i(t)u_i \tag{3.3}$$

uniformly for every $t \in [0,T]$ and $u \in \mathbb{R}^N$ with $|u_i| \le \frac{1}{2}c_0$, where c_0 is as defined in (2.1).

Proof. From the definition, we have $u_i > -\hat{x}_i(s,t)/s$. So, by (2.1), $h_i(s,t,u)$ is well defined for every $t \in [0,T]$ and $u \in \mathbb{R}^N$ with $|u_i| \le \frac{1}{2}c_0$.

By (2.1), we have, for $|u_i| \le \frac{1}{2}c_0$ and $t \in [0,T]$,

$$\begin{aligned} |h_i(s,t,u) - a_i(t)u_i| \le& \left|\frac{g_i(t, su + \hat{x}(s,t)) - a_i(t)(su_i + \hat{x}_i(s,t))}{s}\right| \\ &+ \left|\frac{a_i(t)\hat{x}_i(s,t) - g_i(t, \hat{x}_i(s,t))}{s}\right| \\ \le& \left|u_i + \frac{\hat{x}_i(s,t)}{s}\right| \left|\frac{g_i(t, su + \hat{x}(s,t)) - a_i(t)(su_i + \hat{x}_i(s,t))}{su_i + \hat{x}_i(s,t)}\right| \\ &+ \frac{\hat{x}_i(s,t)}{s}\left|\frac{a_i(t)\hat{x}_i(s,t) - g_i(t, \hat{x}(s,t))}{\hat{x}_i(s,t)}\right| \\ \le& (c_0 + C_0)\left|\frac{g_i(t, su + \hat{x}(s,t)) - a_i(t)(su_i + \hat{x}_i(s,t))}{su_i + \hat{x}_i(s,t)}\right| \\ &+ C_0\left|\frac{a_i(t)\hat{x}_i(s,t) - g_i(t, \hat{x}(s,t))}{\hat{x}_i(s,t)}\right|. \end{aligned} \tag{3.4}$$

Since $\hat{x}_i(s,t) \to +\infty$ as $s \to +\infty$, $su_i + \hat{x}_i(s,t) \to +\infty$ as $s \to +\infty$ uniformly for every $t \in [0,T]$ and $|u_i| \le \frac{1}{2}c_0$. By $(H_3^i)'$ and (3.4), the conclusion is thus achieved.

Lemma 3.2. *There exist b, $\tilde{r}$ with $0 < b < \tilde{r} < c_0/2$ and $s_1 \ge \tilde{s}$ such that, for every $s \ge s_1$, if $u : [0,T] \to \mathbb{R}^N$ is a solution of* (P) *with $u_i(0)^2 + u_i'(0)^2 = \tilde{r}^2$ for a certain index i, then*

$$b < r_i(t) < \frac{1}{2}c_0, \quad \textit{for every } t \in [0,T].$$

Proof. We first prove that $r_i(t) < \frac{1}{2}c_0$ for every $t \in [0, T]$. On the contrary, suppose that there exists $\bar{t} \in [0, T]$ satisfying

$$r_i(\bar{t}) = \frac{1}{2}c_0 \text{ and } r_i(t) < \frac{1}{2}c_0 \text{ for every } t \in [0, \bar{t}). \tag{3.5}$$

Set

$$\tilde{r} = \frac{1}{8}c_0 \exp\left(-\left(\frac{1 + a^i_+}{2}\right)T\right), \quad b = \frac{\tilde{r}}{4}\exp\left(-\left(\frac{1 + a^i_+}{2}\right)T\right) \text{ and } \varepsilon = \frac{\tilde{r}}{T}.$$

It is clear that $0 < b < \tilde{r} < c_0/2$. By Lemma 3.1, since $r_i(t) \leq c_0/2$ for every $t \in [0, \bar{t}]$, there exists $s_1 \geq \tilde{s}$ such that

$$|h_i(s, t, u) - a_i(t)u_i| \leq \varepsilon$$

for every $s \geq s_1$ and $t \in [0, \bar{t}]$. From (P), we get

$$\begin{aligned} |r_i'(t)| &= \left|\frac{u_i'(t)(u_i(t) - h_i(s, t, u))}{\sqrt{u_i(t)^2 + u_i'(t)^2}}\right| \\ &\leq \frac{|u_i'(t)||(a^i_+ + 1)u_i(t) + \varepsilon|}{r_i(t)} \\ &\leq \frac{a^i_+ + 1}{2}r_i(t) + \varepsilon. \end{aligned}$$

By a Gronwall argument we have

$$r_i(\bar{t}) \leq (r_i(0) + \varepsilon\bar{t})\exp\left(\frac{a^i_+ + 1}{2}\bar{t}\right) \leq \frac{1}{4}c_0,$$

which contradicts (3.5). By a similar argument as above, we can see that $r_i(t) > b$ for every $t \in [0, T]$.

Lemma 3.3. *There exists $s_2 \geq s_1$ such that, for every $s \geq s_2$, if $u : [0, T] \to \mathbb{R}^N$ is a solution of* (P) *with $u_i(0)^2 + u_i'(0)^2 = \tilde{r}^2$ for a certain index i, then*

$$\mathrm{Rot}_{\sqrt{a^i_+}}((u_i(t), u_i'(t)); [0, T]) < \frac{m_i + 1}{2}.$$

Proof. By (H^i_4), we can fix $\varepsilon > 0$ small such that

$$\frac{T\sqrt{a^i_+}}{2\pi}\left(1 + \frac{c_0\varepsilon}{2b^2\min\{a^i_+, 1\}}\right) < \frac{m_i + 1}{2}.$$

By Lemma 3.2, we have $b < \sqrt{u_i(t)^2 + u_i'(t)^2} < \frac{1}{2}c_0$ for every $t \in [0, T]$. From (3.3), there exists $s_2 \geq s_1$ such that, for every $s \geq s_2$, one has

$$|h_i(s, t, u) - a_i(t)u_i| \leq \varepsilon.$$

Therefore,

$$\mathrm{Rot}_{\sqrt{a^i_+}}((u_i(t), u_i'(t)); [0, T]) = \frac{\sqrt{a^i_+}}{2\pi}\int_0^T \frac{u_i'(t)^2 + h_i(s, t, u)u_i(t)}{a^i_+u_i(t)^2 + u_i'(t)^2}dt$$

$$\leq \frac{\sqrt{a_+^i}}{2\pi}\left(\int_0^T \frac{u_i'(t)^2 + a_i(t)u_i(t)^2}{a_+^i u_i(t)^2 + u_i'(t)^2}dt + \int_0^T \frac{(h_i(s,t,u) - a_i(t)u_i(t))u_i(t)}{a_+^i u_i(t)^2 + u_i'(t)^2}dt\right)$$
$$\leq \frac{T\sqrt{a_+^i}}{2\pi}\left(1 + \frac{c_0\varepsilon}{2b^2\min\{a_+^i, 1\}}\right) < \frac{m_i+1}{2}.$$

Lemma 3.4. *For every $s \geq s_2$, there exists a strictly star-shaped Jordan curve Γ_s^i around the origin such that, if $u : [0,T] \to \mathbb{R}^N$ is a solution of* (P) *with $(u_i(0), u_i'(0)) \in \Gamma_s^i$ for a certain index i, then*

$$\mathrm{Rot}((u_i(t), u_i'(t)); [0,T]) > m_i.$$

Proof. Let $x(t)$ be a solution of (S'). In the i-th half plane $\{x_i > 0\}$, choose a suitable closed rectangle $\mathcal{K}_s$ such that $(\hat{x}_i(s,t), \hat{x}_i'(s,t)) \in \mathcal{K}_s$ for every $t \in [0,T]$. Without loss of generality, we assume that if $\mathcal{N}_i(x(t), x'(t)) \geq \hat{R}_s$ for large enough $\hat{R}_s$, then $(x(t), x'(t)) \notin \mathcal{K}_s$. Since $x(t)$ is globally defined on $[0,T]$, by the elastic property in [11, Lemma 6], there exists $\tilde{R}_s \geq \hat{R}_s$ such that, for any solution $x(t)$ of (S'),

$$\mathcal{N}_i(x(0), x'(0)) \geq \tilde{R}_s \Rightarrow \mathcal{N}_i(x(t), x'(t)) \geq \hat{R}_s \text{ for every } t \in [0,T]. \tag{3.6}$$

Let $\Gamma_s^i = \{(u_i, u_i') \in D_s^i : \mathcal{N}_i(su + \hat{x}(s,0), su' + \hat{x}'(s,0)) = \tilde{R}_s\}$. Suppose that $u : [0,T] \to \mathbb{R}^N$ is a solution of (P) with $(u_i(0), u_i'(0)) \in \Gamma_s^i$ for some index i. Based on (3.1), we have $\mathcal{N}_i(x(0), x'(0)) = \tilde{R}_s$, which implies that $\mathcal{N}_i(x(t), x'(t)) \geq \hat{R}_s$ for all $t \in [0,T]$.

Using Proposition 2.2 in [12], we can obtain the independence of

$$\mathrm{Rot}((x_i(t) - (\lambda\hat{x}_i(s,t) + (1-\lambda)\hat{x}_i(s,0)), x_i'(t) - (\lambda\hat{x}_i'(s,t) + (1-\lambda)\hat{x}_i'(s,0))); [0,T])$$

with respect to $\lambda \in [0,1]$ in the (x_i, x_i') phase-plane. By applying (3.1), Lemma 2.5 and the definition of i-th rotation number, we have

$$\begin{aligned}
\mathrm{Rot}((u_i(t), u_i'(t)); [0,T]) &= \mathrm{Rot}((su_i(t), su_i'(t)); [0,T]) \\
&= \mathrm{Rot}((x_i(t) - \hat{x}_i(s,t), x_i'(t) - \hat{x}_i'(s,t)); [0,T]) \\
&= \mathrm{Rot}((x_i(t) - \hat{x}_i(s,0), x_i'(t) - \hat{x}_i'(s,0)); [0,T]) \\
&> m_i.
\end{aligned}$$

Proof of Theorem 1.1. By (2.1), there exists $\hat{r}$ such that

$$\frac{c_0}{2} < \hat{r} < \frac{\hat{x}_i(s,0)}{s}.$$

For any $(u_i, u_i') \in \Gamma_s^i$, if $u_i \geq -\hat{r}$, by the definition of $\mathcal{N}_i(su + \hat{x}_i(s,0), su' + \hat{x}_i'(s,0))$, we have

$$\begin{aligned}
2s^2(u_i^2 + u_i'^2) &\geq \tilde{R}_s^2 - 2(\hat{x}_i(s,0)^2 + \hat{x}_i'(s,0)^2) - \frac{1}{(su_i + \hat{x}_i(s,0))^2} \\
&\geq \tilde{R}_s^2 - 2(\hat{x}_i(s,0)^2 + \hat{x}_i'(s,0)^2) - \frac{1}{(-s\hat{r} + \hat{x}_i(s,0))^2}.
\end{aligned}$$

So,

$$\sqrt{u_i^2 + u_i'^2} > \hat{r} > \frac{c_0}{2} \tag{3.7}$$

for large enough $\tilde{R}_s$ as in (3.6). On the other hand, if $u_i < -\hat{r}$, inequality (3.7) clearly holds for any $(u_i, u_i') \in \Gamma_s^i$. Recall $\tilde{r} < c_0/2$, so $\overline{B}_i(0, \tilde{r}) \subset D(\Gamma_s^i)$.

We will prove the multiplicity of periodic solutions by the Poincaré-Birkhoff theorem stated in [5, Theorem 3.1], which is a simplified version of [6, Theorem 1.2]. Note that there exist $[\frac{m_i-1}{2}]+1$ integers in the interval $[\frac{m_i-1}{2}, m_i]$. Choose $s_0 = s_2$ and fix $s \geq s_0$. Taking $\Omega = (\overline{D}(\Gamma_s^i)\backslash B(0, \tilde{r}))^N$, the number of possible choices of the values $(l_1, l_2, \cdots, l_N)$ in [5, Theorem 3.1] is $\prod_{i=1}^N \left(\left[\frac{m_i-1}{2}\right] + 1\right)$ by Remark 2.1 and Lemmas 3.3 and 3.4. Applying [5, Theorem 3.1], there exist $N + 1$ periodic solutions for system (P) for every $(l_1, l_2, \cdots, l_N)$. Coming back to the system (S'), by (3.1), there exist $(N + 1) \prod_{i=1}^N \left(\left[\frac{m_i-1}{2}\right] + 1\right)$ periodic solutions for system (S'). The pivot solution to (S') is provided by Lemma 2.2. The proof is thus concluded.

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Conflict of interest

The authors declare there is no conflict of interest.

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