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Research article

The exact solutions of the fractional-stochastic Fokas-Lenells equation in optical fiber communication

Sahar Albosaily[1], **Wael Mohammed**[1,2,*] **and Mahmoud El-Morshedy**[3,4]

[1] Department of Mathematics, Collage of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[3] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
[4] Department of Statistics and Computer Science, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

* **Correspondence:** Email: wael.mohammed@mans.edu.eg.

Abstract: The fractional-stochastic Fokas-Lenells equation (FSFLE) in the Stratonovich sense is taken into account here. The modified mapping method is used to generate new trigonometric, hyperbolic, elliptic and rational stochastic fractional solutions. Because the Fokas-Lenells equation has many implementations in telecommunication modes, complex system theory, quantum field theory, and quantum mechanics, the obtained solutions can be employed to describe a wide range of exciting physical phenomena. We plot several 2D and 3D diagrams to demonstrate how multiplicative noise and fractional derivatives affect the analytical solutions of the FSFLE. Also, we show how multiplicative noise at zero stabilizes FSFLE solutions.

Keywords: fractional Fokas-Lenells; optical solitons; multiplicative noise; modified mapping method

1. Introduction

Nonlinear evolution equations (NEEs) are used extensively in engineering and scientific fields, including wave propagation phenomena, quantum mechanics, shallow water wave propagation, chemical kinematics, solid-state physics, optical fibers, fluid mechanics, plasma physics, heat flow and so on. Recently, much research on NEEs has focused on existence, uniqueness, convergence and finding solutions: for example, [1–11], and the references therein. One of the essential physical issues for NEEs is obtaining traveling wave solutions. Therefore, the looking for mathematical techniques to generate exact solutions to NEEs has become a significant and necessary task in nonlinear sciences.

Recently, various techniques for dealing with NEEs have been established, such as the exp-function method [12], perturbation method [13, 14], sine-cosine method [15, 16], spectral methods [17], tanh-sech method [18, 19], Jacobi elliptic function [20], Hirota's method [21], $exp(-\phi(\varsigma))$-expansion method [22], extended trial equation method [23, 24], (G'/G)-expansion method [25, 26], etc.

In recent years, the use of fractional differential equations has grown due to their wide range of applications in fields such as control theory, fluid flow, finance, electrical networks, solid state physics, chemical kinematics, optical fiber, plasma physics, signal processing, and biological populations. A number of mathematicians have proposed various types of fractional derivatives. These types include, the new truncated M-fractional derivative, Caputo fractional derivative, Riemann-Liouville fractional derivative, Grunwald-Letnikov fractional derivative, He's fractional derivative, Riesz fractional derivatives, the Weyl derivative and conformable fractional definitions [27–34].

Khalil et al. [32] have developed a new derivative operator known as the conformable derivative (CD). From this point, let us define the CD for the function $u : [0, \infty) \to \mathbb{R}$ of order $\beta \in (0, 1]$ as follows:

$$\mathcal{D}_z^\beta u(z) = \lim_{h \to 0} \frac{u(z + hz^{1-\beta}) - u(z)}{h}.$$

The CD satisfies the following properties for any constants a and b:

1) $\mathcal{D}_z^\beta [au(z) + bv(z)] = a\mathcal{D}_z^\beta u(z) + b\mathcal{D}_z^\beta v(z)$, 2) $\mathcal{D}_z^\beta [a] = 0$,

3) $\mathcal{D}_z^\beta [z^a] = az^{a-\beta}$, 4) $\mathcal{D}_z^\beta u(z) = z^{1-\beta} \frac{du}{dz}$.

In contrast, stochastic partial differential equations (SPDEs) are models for spatiotemporal physical, biological and chemical systems that are sensitive to random influences. In the past few decades, these models have been the subject of extensive research. It has been emphasized how crucial it is to take stochastic effects into account when modeling complex systems. For instance, there is rising interest in employing SPDEs to represent complex phenomena mathematically in the fields of finance, mechanical and electrical engineering, biophysics, information systems, materials sciences, condensed matter physics, and climate systems [35, 36].

Therefore, it is crucial to consider NEEs with fractional derivatives and for some stochastic force. Here, we consider the fractional-stochastic Fokas-Lenells equation (FSFLE):

$$\mathcal{D}_x^\alpha \Phi_t - \gamma_1 \mathcal{D}_{xx}^\alpha \Phi - 2i\gamma_2 \mathcal{D}_x^\alpha \Phi + \vartheta |\Phi|^2 (\Phi + i\rho \mathcal{D}_x^\alpha \Phi) + \sigma \mathcal{D}_x^\alpha \Phi \circ W_t = 0, \tag{1.1}$$

where $\Phi(x, t)$ gives the complex field, γ_1, γ_2 and ρ are positive constants, $\vartheta = \pm 1$, W is the standard Wiener process, σ is the strength of the noise, and $\Phi \circ dW$ is multiplicative noise in the Stratonovich sense.

If we put $\sigma = 0$ and $\alpha = 1$, then we get the Fokas-Lenells equation [37–39]:

$$\Phi_{xt} - \gamma_1 \Phi_{xx} - 2i\gamma_2 \Phi_x + \vartheta |\Phi|^2 (\Phi + i\rho \Phi_x) = 0. \tag{1.2}$$

Equation (1.2) is one of the most significant equations, and it has many applications in telecommunication models, complex system theory, quantum field theory and quantum mechanics. Also, it appears as a pattern that identifies nonlinear pulse propagation in optical fibers. Demiray and Bulut [37] obtained the exact solutions of Eq (1.2) by utilizing the extended trial equation and generalized Kudryashov methods. Meanwhile, Xu and Fan [38] used the Riemann-Hilbert problem to obtain the long-time asymptotic behavior of the solution of Eq (1.2).

It is important to note that Stratonovich and Itô [40] are the two versions of stochastic integrals that are most frequently used. Modeling problems essentially establish which form is acceptable; nevertheless, once that form is chosen, an equivalent equation of the alternate form can be created utilizing the same solutions. The following correlation can therefore be used to switch between Stratonovich (denoted as $\int_0^t \Phi \circ dW$) and Itô (denoted as $\int_0^t \Phi dW$):

$$\int_0^t \sigma\Phi(\tau) \circ dW(\tau) = \int_0^t \sigma\Phi(\tau)dW(\tau) + \frac{\sigma^2}{2}\int_0^t \Phi(\tau)d\tau. \tag{1.3}$$

Our aim in this study is to derive the analytical fractional-stochastic solutions of the FSFLE (1.1). The modified mapping method is what we employ to obtain these solutions. Physics researchers would find the solutions very helpful in defining several major physical processes because of the stochastic term and fractional derivatives present in Eq (1.1). Additionally, by using MATLAB software, we introduce numerous graphs to investigate the effects of noise and the fractional derivative on the exact solution of the FSFLE (1.1).

The outline of this paper is as follows: In Section 2, we get the wave equation for the FSFLE (1.1). In Section 3, the modified mapping method is used to get the exact solutions of the FSFLE (1.1). In Section 4, we can see how white noise and the fractional derivative affect the acquired FSFLE solutions. At last, the conclusions of the paper are given.

2. Traveling wave equation for FSFLE

The wave equation for FSFLE (1.1) is obtained by using the wave transformation

$$\Phi(x,t) = \Psi(\eta)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}, \;\; \Theta(\mu) = \frac{\mu_1}{\alpha}x^\alpha + \mu_2 t \;\text{ and }\; \eta = \frac{\eta_1}{\alpha}x^\alpha + \eta_2 t, \tag{2.1}$$

where the function Ψ is deterministic, and $\mu_1,\ \mu_2,\ \eta_1$ and η_2 are non-zero constants. We note that

$$\begin{aligned}\Phi_t &= [\eta_2\Psi' + i\mu_2\Psi - \sigma\Psi W_t + \frac{1}{2}\sigma^2\Psi - \sigma^2\Psi]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t},\\ &= [\eta_2\Psi' + i\mu_2\Psi - \sigma\Psi W_t - \frac{1}{2}\sigma^2\Psi]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t},\\ &= [\eta_2\Psi' + i\mu_2\Psi - \sigma\Psi \circ W_t]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t},\end{aligned} \tag{2.2}$$

where we used Eq (1.3), and the term $\frac{1}{2}\sigma^2\Psi$ is the Itô correction.

$$\mathcal{D}_x^\alpha\Phi_t = [\eta_1\eta_2\Psi'' + i(\eta_1\mu_2 + \mu_1\eta_2)\Psi' - \sigma(\eta_1\Psi' + i\mu_1\Psi) \circ W_t - \mu_1\mu_2\Psi]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}, \tag{2.3}$$

and

$$\begin{aligned}\mathcal{D}_x^\alpha\Phi &= (\eta_1\Psi' + i\mu_1\Psi)e^{i\Theta(\mu)+\sigma W(t)-\sigma^2 t},\\ \mathcal{D}_{xx}^\alpha\Phi &= (\eta_1^2\Psi'' + 2i\mu_1\eta_1\Psi' - \mu_1^2\Psi)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}.\end{aligned} \tag{2.4}$$

Inserting Eqs (2.3) and (2.4) into Eq (1.1), we have the following system:

$$(\eta_1\eta_2 - \gamma_1\eta_1^2)\Psi'' + (\nu - \mu_1\mu_2 + \gamma_1^2\mu_1 + 2\gamma_2\mu_1)\Psi - \nu\rho\mu_1\Psi^3 e^{[-2\sigma W(t)-2\sigma^2 t]} = 0, \tag{2.5}$$

and

$$i[(\eta_1\mu_2 + \mu_1\eta_2 - 2\gamma_1\mu_1\eta_1 - 2\gamma_2\eta_1)\Psi' + \nu\rho\eta_1\Psi^2\Psi' e^{[-2\sigma W(t) - 2\sigma^2 t]}] = 0. \tag{2.6}$$

We have, by taking the expectation on both sides,

$$(\eta_1\eta_2 - \gamma_1\eta_1^2)\Psi'' + (\nu - \mu_1\mu_2 + \gamma_1^2\mu_1 + 2\gamma_2\mu_1)\Psi - \nu\rho\mu_1\Psi^3 e^{-2\sigma^2 t}\mathbb{E}e^{-2\sigma W(t)} = 0, \tag{2.7}$$

and

$$i[(\eta_1\mu_2 + \mu_1\eta_2 - 2\gamma_1\mu_1\eta_1 - 2\gamma_2\eta_1)\Psi' + \nu\rho\eta_1\Psi^2\Psi' e^{-2\sigma^2 t}\mathbb{E}e^{-2\sigma W(t)}] = 0. \tag{2.8}$$

Since $W(t)$ is normal distribution, then $\mathbb{E}(e^{2kW(t)}) = e^{2k^2 t}$ for any real number k. Therefore, Eqs (2.7) and (2.8) become

$$(\eta_1\eta_2 - \gamma_1\eta_1^2)\Psi'' + (\nu - \mu_1\mu_2 + \gamma_1^2\mu_1 + 2\gamma_2\mu_1)\Psi - \nu\rho\mu_1\Psi^3 = 0, \tag{2.9}$$

$$i[(\eta_1\mu_2 + \mu_1\eta_2 - 2\gamma_1\mu_1\eta_1 - 2\gamma_2\eta_1)\Psi' + \nu\rho\eta_1\Psi^2\Psi'] = 0. \tag{2.10}$$

From the imaginary part of Eq (2.10), we obtained

$$\eta_2 = \frac{1}{\mu_1}(-\eta_1\mu_2 + 2\gamma_1\mu_1\eta_1 + 2\gamma_2\eta_1 - \nu\rho\eta_1\Psi^2).$$

while the real part is given by

$$\Psi'' + A\Psi - B\Psi^3 = 0, \tag{2.11}$$

where

$$A = \frac{(\nu - \mu_1\mu_2 + \gamma_1^2\mu_1 + 2\gamma_2\mu_1)}{(\eta_1\eta_2 - \gamma_1\eta_1^2)}, \text{ and } B = \frac{\nu\rho\mu_1}{(\eta_1\eta_2 - \gamma_1\eta_1^2)}.$$

3. Exact solutions of FSFLE

In this section, we apply the modified mapping method stated in [41]. Assuming that the solutions of Eq (2.11) have the form

$$\Psi(\eta) = \sum_{i=0}^{N} \ell_i\varphi^i(\eta) + \sum_{i=1}^{N} \hbar_i\varphi^{-i}(\eta), \tag{3.1}$$

where ℓ_i and $\hbar_i$ are unknown constants to be evaluated for $i = 1, 2, ..\ell_N$, and φ satisfies the first type of the elliptic equation

$$\varphi' = \sqrt{r + q\varphi^2 + p\varphi^4}, \tag{3.2}$$

where r, q and p are real parameters.

First, let us balance Ψ'' with Ψ^3 in Eq (2.11) to find the parameter N as

$$N + 2 = 3N \Longrightarrow N = 1.$$

With $N = 1$, Eq (3.2) takes the form

$$\Psi(\eta) = \ell_0 + \ell_1\varphi(\eta) + \frac{\hbar_1}{\varphi(\eta)}. \tag{3.3}$$

Differentiating Eq (3.3) twice and using (3.2), we get

$$\Psi'' = \ell_1(q\varphi + 2p\varphi^3) + \hbar_1(q\varphi^{-1} + 2r\varphi^{-3}). \tag{3.4}$$

Putting Eqs (3.3) and (3.4) into Eq (2.11) we have

$$\begin{aligned}&(2\ell_1 p - B\ell_1^3)\varphi^3 - 3B\ell_0\ell_1^2\varphi^2 + (\ell_1 q - 3B\ell_0^2\ell_1 - 3B\hbar_1\ell_1^2 + \ell_1 A)\varphi\\ &+(A\ell_0 - B\ell_0^3 - 6B\ell_0\ell_1\hbar_1) + (A\hbar_1 + \hbar_1 q - 3B\ell_0^2\hbar_1 - 3B\ell_1\hbar_1^2)\varphi^{-1}\\ &-3B\hbar_1^2\varphi^{-2} + (2r\hbar_1 - B\hbar_1^3)\varphi^{-3} = 0.\end{aligned}$$

Comparing each coefficient of φ^k and φ^{-k} with zero for $k = 3, 2, 1, 0$, we attain

$$2\ell_1 p - B\ell_1^3 = 0,$$

$$\begin{gathered}-3B\ell_0\ell_1^2 = 0,\\ \ell_1 q - 3B\ell_0^2\ell_1 - 3B\hbar_1\ell_1^2 + \ell_1 A = 0,\\ A\ell_0 - B\ell_0^3 - 6B\ell_0\ell_1\hbar_1 = 0,\\ A\hbar_1 + \hbar_1 q - 3B\ell_0^2\hbar_1 - 3B\ell_1\hbar_1^2 = 0,\\ -3B\ell_0\hbar_1^2 = 0\end{gathered}$$

and

$$2r\hbar_1 - B\hbar_1^3 = 0.$$

When we solve these equations, we get three different families:

First family:

$$\ell_0 = 0, \quad \ell_1 = \pm\sqrt{\frac{2p}{B}}, \ \hbar_1 = 0, \quad q = -A. \tag{3.5}$$

Second family:

$$\ell_0 = 0, \quad \ell_1 = 0, \ \hbar_1 = \pm\sqrt{\frac{2r}{B}}, \quad q = -A. \tag{3.6}$$

Third family:

$$\ell_0 = 0, \quad \ell_1 = \pm\sqrt{\frac{2p}{B}}, \ \hbar_1 = \pm\sqrt{\frac{2r}{B}}, \quad q = 6\sqrt{pr} - A. \tag{3.7}$$

First family: The solution of Eq (2.11), by utilizing Eqs (3.3) and (3.5), takes the form

$$\Phi(x,t) = \pm\sqrt{\frac{2p}{B}}\varphi(\eta)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.8}$$

There are many cases depending on $p > 0$:

Case 1-1: If $p = \kappa^2$, $q = -(1+\kappa^2)$ and $r = 1$, then $\varphi(\eta) = sn(\eta)$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.8), is

$$\Phi(x,t) = \pm\kappa\sqrt{\frac{2}{B}}sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.9}$$

If $\kappa \to 1$, then Eq (3.9) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\tanh(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.10}$$

Case 1-2: If $p = 1,\ q = 2\kappa^2 - 1$ and $r = -\kappa^2(1-\kappa^2)$, then $\varphi(\eta) = ds(\eta)$. In this case the solution of FSFLE (1.1), by using Eq (3.8), is

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}ds(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.11}$$

If $\kappa \to 1$, then Eq (3.11) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\operatorname{csch}(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.12}$$

If $\kappa \to 0$, then Eq (3.11) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\csc(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.13}$$

Case 1-3: If $p = 1,\ q = 2 - \kappa^2$ and $r = (1-\kappa^2)$, then $\varphi(\eta) = cs(\eta)$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.8), is

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}cs(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.14}$$

If $\kappa \to 1$, then Eq (3.14) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\operatorname{csch}(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.15}$$

When $\kappa \to 0$, then Eq (3.14) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\cot(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.16}$$

Case 1-4: If $p = \frac{\kappa^2}{4},\ q = \frac{(\kappa^2-2)}{2}$ and $r = \frac{1}{4}$, then $\varphi(\eta) = \frac{sn(\eta)}{1+dn(\eta)}$. In this case the solution of FSFLE (1.1), by using Eq (3.8), is

$$\Phi(x,t) = \pm\kappa\sqrt{\frac{1}{2B}}\frac{sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{1 + dn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.17}$$

If $\kappa \to 1$, then Eq (3.17) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}\frac{\tanh(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{1 + \operatorname{sech}(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.18}$$

Case 1-5: If $p = \frac{(1-\kappa^2)^2}{4}$, $q = \frac{(1-\kappa^2)^2}{2}$ and $r = \frac{1}{4}$, then $\varphi(\eta) = \frac{sn(\eta)}{dn(\eta)+cn(\eta)}$. In this case the solution of FSFLE (1.1), by using Eq (3.8), is

$$\Phi(x,t) = \pm(1-\kappa^2)\sqrt{\frac{1}{2B}}[\frac{sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{dn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) + cn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.19}$$

If $\kappa \to 0$, then Eq (3.19) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}[\frac{\sin(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{1+\cos(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.20}$$

Case 1-6: If $p = \frac{1-\kappa^2}{4}$, $q = \frac{(1-\kappa^2)}{2}$ and $r = \frac{1-\kappa^2}{4}$, then $\varphi(\eta) = \frac{cn(\eta)}{1+sn(\eta)}$. In this case the solution of FSFLE (1.1), by using Eq (3.8), is

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\frac{cn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{1+sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.21}$$

When $\kappa \to 0$, then Eq (3.21) transfers to

$$\Phi(x,t) = \pm\frac{1}{2}\sqrt{\frac{2}{B}}\frac{\cos(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{1+\sin(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.22}$$

Case 1-7: If $p = 1$, $q = 0$ and $r = 0$, then $\varphi(\eta) = \frac{c}{\eta}$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.8), is

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\frac{c}{(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.23}$$

Second family: The solution of Eq (2.11), by using Eqs (3.3) and (3.6), takes the form

$$\Phi(x,t) = \pm\sqrt{\frac{2r}{B}}\frac{1}{\varphi(\eta)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.24}$$

There are many cases depending on $r > 0$:

Case 2-1: If $p = \kappa^2$, $q = -(1+\kappa^2)$ and $r = 1$, then $\varphi(\eta) = sn(\eta)$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.24), is

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\frac{1}{sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.25}$$

If $\kappa \to 1$, then Eq (3.25) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\coth(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.26}$$

Case 2-2: If $p = 1$, $q = 2-\kappa^2$ and $r = (1-\kappa^2)$, then $\varphi(\eta) = cs(\eta)$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.24), is

$$\Phi(x,t) = \pm\sqrt{\frac{2(1-\kappa^2)}{B}}\frac{1}{cs(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.27}$$

When $\kappa \to 0$, then Eq (3.27) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\tan(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.28}$$

Case 2-3: If $p = -\kappa^2$, $q = 2\kappa^2 - 1$ and $r = (1-\kappa^2)$, then $\varphi(\eta) = cn(\mu)$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.24), is

$$\Phi(x,t) = \pm\sqrt{\frac{2(1-\kappa^2)}{B}}\frac{1}{cn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.29}$$

If $\kappa \to 0$, then Eq (3.31) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\sec(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.30}$$

Case 2-4: If $p = \frac{\kappa^2}{4}$, $q = \frac{(\kappa^2-2)}{2}$ and $r = \frac{1}{4}$, then $\varphi(\eta) = \frac{sn(\eta)}{1+dn(\eta)}$. In this case the solution of FSFLE (1.1), by using Eq (3.24), is

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}\frac{1+dn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.31}$$

If $\kappa \to 1$, then Eq (3.31) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}[\coth(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) + \mathrm{csch}(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.32}$$

Case 2-5: If $p = \frac{1-\kappa^2}{4}$, $q = \frac{(1-\kappa^2)}{2}$ and $r = \frac{1-\kappa^2}{4}$, then $\varphi(\eta) = \frac{cn(\eta)}{1+sn(\eta)}$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.8), is

$$\Phi(x,t) = \pm\sqrt{\frac{1-\kappa^2}{2B}}\frac{1+sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{cn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.33}$$

When $\kappa \to 0$, then Eq (3.33) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}[\sec(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) \pm \tan cn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.34}$$

Case 2-6: If $p = \frac{(1-\kappa^2)^2}{4}$, $q = \frac{(1-\kappa^2)^2}{2}$ and $r = \frac{1}{4}$, then $\varphi(\eta) = \frac{sn(\eta)}{dn(\eta)+cn(\eta)}$. In this case the solution of FSFLE (1.1), by using Eq (3.24), is

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}[\frac{dn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) + cn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}{sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.35}$$

If $\kappa \to 0$, then Eq (3.35) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}[\csc(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) + \cot(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.36}$$

If $\kappa \to 1$, then Eq (3.35) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\text{csch}(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.37}$$

Third family: The solution of Eq (2.11), using Eqs (3.3) and (3.7), takes the form

$$\Phi(x,t) = [\pm\sqrt{\frac{2p}{B}}\varphi(\eta) \pm \sqrt{\frac{2r}{B}}\frac{1}{\varphi(\eta)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.38}$$

There are many cases depending on $r > 0$:

Case 3-1: If $p = \kappa^2,\ q = -(1+\kappa^2)$ and $r = 1$, then $\varphi(\eta) = sn(\eta)$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.38), is

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}[\kappa sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) + \frac{1}{sn(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.39}$$

If $\kappa \to 1$, then Eq (3.39) transfers to

$$\Phi(x,t) = \pm[\sqrt{\frac{2}{B}}\tanh(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) + \sqrt{\frac{2}{B}}\coth(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.40}$$

Case 3-2: If $p = 1,\ q = 2-\kappa^2$ and $r = (1-\kappa^2)$, then $\varphi(\eta) = cs(\eta)$. In this case the solution of FSFLE (1.1), by using Eq (3.38), is

$$\Phi(x,t) = \pm[\sqrt{\frac{2}{B}}cs(\eta) + \sqrt{\frac{2(1-\kappa^2)}{B}}\frac{1}{cs(\eta)}e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}, \tag{3.41}$$

where $\eta = \frac{\eta_1}{\alpha}x^\alpha + \eta_2 t$. When $\kappa \to 0$, then Eq (3.41) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}[\cot(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t) + \tan(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.42}$$

Case 3-3: If $p = \frac{\kappa^2}{4},\ q = \frac{(\kappa^2-2)}{2}$ and $r = \frac{1}{4}$, then $\varphi(\eta) = \frac{sn(\eta)}{1\pm dn(\eta)}$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.38), is

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}[\kappa\frac{sn(\eta}{1+dn(\eta)} + \frac{1+dn(\eta)}{sn(\eta)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}, \tag{3.43}$$

where $\eta = \frac{\eta_1}{\alpha}x^\alpha + \eta_2 t$. If $\kappa \to 1$, then Eq (3.43) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\coth(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.44}$$

Case 3-4: If $p = \frac{1-\kappa^2}{4},\ q = \frac{(1-\kappa^2)}{2}$ and $r = \frac{1-\kappa^2}{4}$, then $\varphi(\eta) = \frac{cn(\eta)}{1+sn(\eta)}$. In this case the solution of FSFLE (1.1), by utilizing Eq (3.38), is

$$\Phi(x,t) = \pm\sqrt{\frac{1-\kappa^2}{2B}}[\frac{cn(\eta)}{1+sn(\eta)} + \frac{1+sn(\eta)}{cn(\eta)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}, \tag{3.45}$$

where $\eta = \frac{\eta_1}{\alpha} x^\alpha + \eta_2 t$. When $\kappa \to 0$, then Eq (3.45) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\sec(\eta)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.46}$$

Case 3-5: If $p = \frac{(1-\kappa^2)^2}{4}$, $q = \frac{(1-\kappa^2)^2}{2}$ and $r = \frac{1}{4}$, then $\varphi(\eta) = \frac{sn(\eta)}{dn(\eta)+cn(\eta)}$. In this case the solution of the FSFLE (1.1), by using Eq (3.38), is

$$\Phi(x,t) = \pm\sqrt{\frac{1}{2B}}[\frac{(1-\kappa^2)sn(\eta)}{dn(\eta)+cn(\eta)} + \frac{dn(\eta)+cn(\eta)}{sn(\eta)}]e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}, \tag{3.47}$$

where $\eta = \frac{\eta_1}{\alpha} x^\alpha + \eta_2 t$. If $\kappa \to 0$, then Eq (3.35) transfers to

$$\Phi(x,t) = \pm\sqrt{\frac{2}{B}}\csc(\frac{\eta_1}{\alpha}x^\alpha + \eta_2 t)e^{i\Theta(\mu)-\sigma W(t)-\sigma^2 t}. \tag{3.48}$$

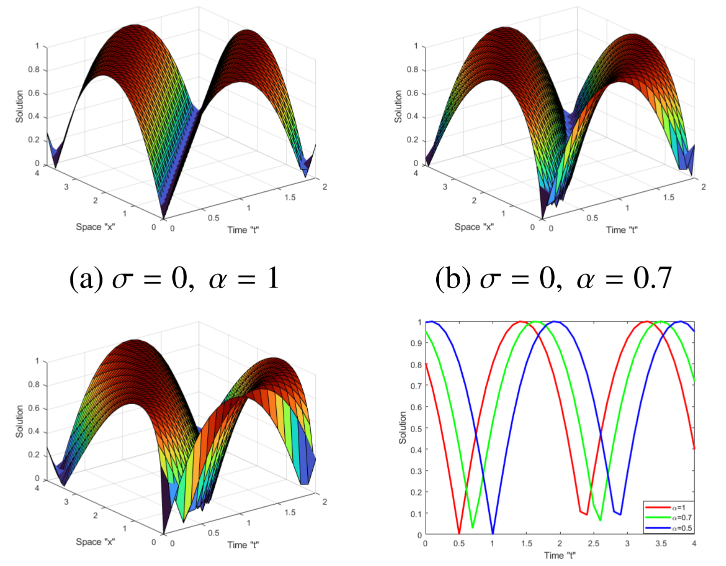
4. Effects of noise and fractional derivative

In deterministic systems, the stabilizing and destabilizing consequences of noisy terms are well known at this time, based on the research done on the issue [42, 43]. There is no longer any doubt that these effects are critical to comprehending the long-term behavior of real systems. Recently, there have been studies on the stabilization problem of stochastic nonlinear delay systems; see, for instance [44, 45]. Now, we examine the effect of white noise and fractional derivatives on the exact solution of the FSFLE (1.1). To describe the behavior of these solutions, we present a number of diagrams. For certain obtained solutions such as Eqs (3.9) and (3.10), let us fix the parameters $\rho = \gamma_1 = \mu_1 = \eta_1 = 1$, $\eta_2 = 2$, $\mu_2 = -2$, $x \in [0,4]$ and $t \in [0,2]$ to simulate these diagrams.

First, the fractional derivative effects: In Figures 1 and 2, if $\sigma = 0$, we can see that the graph's shape is compressed as the value of β decreases:

We deduced from Figures 1 and 2 that there is no overlap between the curves of the solutions. Furthermore, as the order of the fractional derivative decreases, the surface moves to the right.

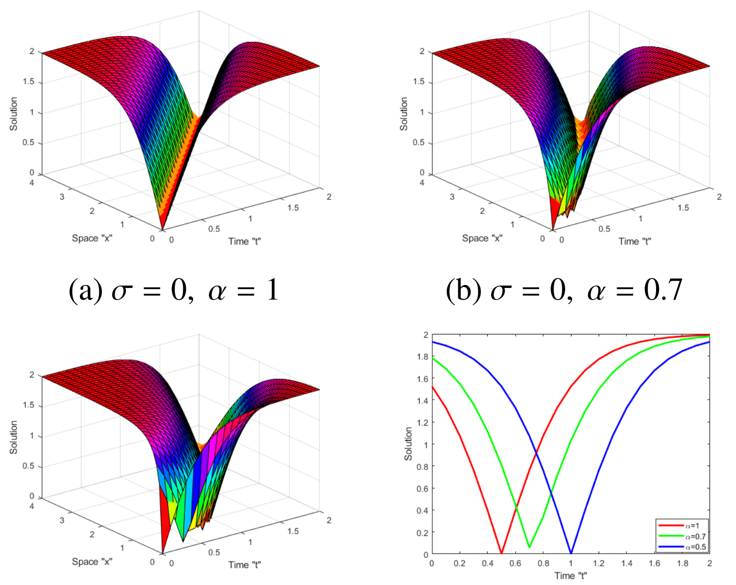
Second, the noise effects: In Figure 3, the surface is not flat and contains various fluctuations when $\sigma = 0$ (i.e., there is no noise).



(a) $\sigma = 0,\ \alpha = 1$ (b) $\sigma = 0,\ \alpha = 0.7$

(c) $\sigma = 0,\ \alpha = 0.5$ (d) $\sigma = 0,\ \alpha = 1,\ 0.7,\ 0.5$

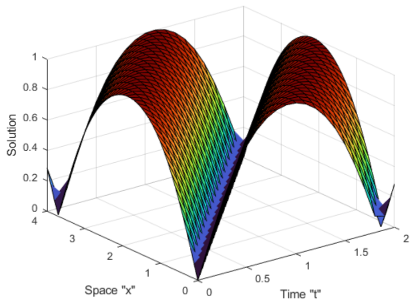
Figure 1. (a–c) 3D graph of solution $|\Phi(x,t)|$ in Eq (3.9) with $\sigma = 0$ and different values of $\alpha = 1,\ 0.7,\ 0.5$ (d) 2D graph of Eq (3.9) with different values of $\alpha = 1,\ 0.7,\ 0.5$.



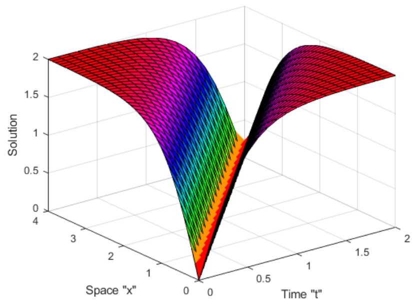
(a) $\sigma = 0,\ \alpha = 1$ (b) $\sigma = 0,\ \alpha = 0.7$

(c) $\sigma = 0,\ \alpha = 0.5$ (d) $\sigma = 0,\ \alpha = 1,\ 0.7,\ 0.5$

Figure 2. (a–c) indicate 3D-graph of solution $|\Phi(x,t)|$ in Eq (3.10) with $\sigma = 0$ and different values of $\alpha = 1,\ 0.7,\ 0.5$ (d) denotes 2D-graph of Eq (3.10) for different values of $\alpha = 1,\ 0.7,$ and 0.5.



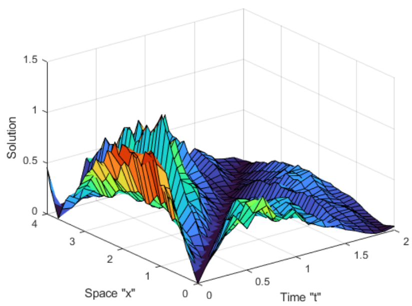
$\sigma = 0$



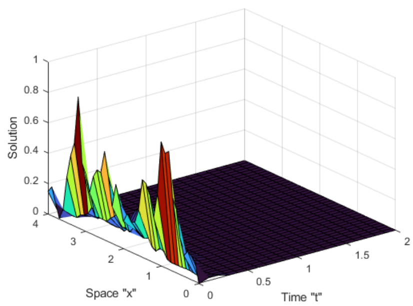
$\sigma = 0$

Figure 3. 3D diagram of solution $|\Phi(x, t)|$ in Eqs (3.9) and (3.10).

While we can see in Figures 4 and 5, after small transit behaviors, the surface has become more planar:

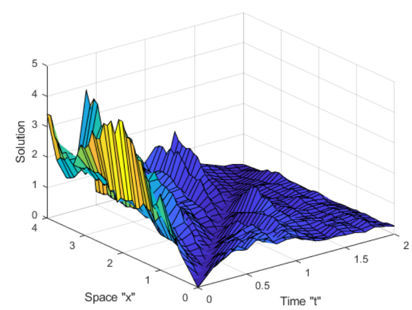


$\sigma = 1$

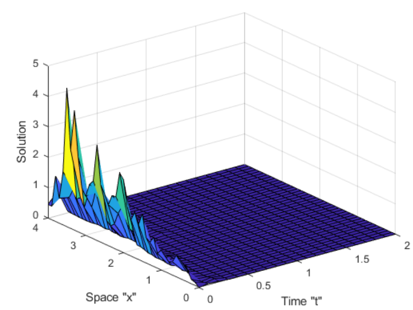


$\sigma = 2$

Figure 4. 3D graph of solution $|\Phi(x, t)|$ in Eq (3.9) for $\sigma = 1, 2$.



$\sigma = 1$



$\sigma = 2$

Figure 5. 3D graph of solution $|\Phi(x, t)|$ in Eq (3.10) for $\sigma = 1, 2$.

In the end, we can deduce from Figures 4 and 5 that, when the noise is ignored (i.e., at $\sigma = 0$), there are some different types of solutions, such as a periodic solution, kink solution, etc. After minor transit patterns, the surface becomes significantly flatter when noise is included and its strength is increased by $\sigma = 1, 2$. This demonstrates that the multiplicative white noise has an effect on the FSFLE solutions and stabilizes them around zero.

5. Conclusions

We looked at FSFLE derived in the Itô sense by multiplicative white noise in this paper. By using a modified mapping technique, we were able to acquire the exact fractional-stochastic solutions. These solutions play a crucial role in the explanation of a wide range of exciting and complex physical phenomena. In addition, the fractional derivative and multiplicative white noise effects on the analytical solution of FSFLE (1.1) were demonstrated using MATLAB software. We came to the conclusion that the multiplicative white noise stabilized the solutions around zero and the fractional-derivative pushed the surface to the right when the fractional-order derivative declined.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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