Research article

Global dynamics of a time-delayed malaria model with asymptomatic infections and standard incidence rate

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Abstract: A time-delayed model of malaria transmission with asymptomatic infections and standard incidence rate is presented and its basic reproduction number R_0 is calculated. We focus on the global dynamics of the model with respect to R_0. If and only if $R_0 > 1$, the model exists a unique malaria-infected equilibrium E^*, whereas it always possesses the malaria-free equilibrium E_0. We first prove the local stability of the equilibria E_0 and E^* by using proof by contradiction and the properties of complex modulus. Secondly, by utilizing the Lyapunov functional method and the limiting system of the model with some novel details, we show that the equilibrium E_0 is globally asymptotically stable (GAS) when $R_0 < 1$, globally attractive (GA) when $R_0 = 1$ and unstable when $R_0 > 1$; the equilibrium E^* is GAS if and only if $R_0 > 1$. In particular, in order to obtain global attractivity of the equilibrium E^*, we demonstrate the weak persistence of the system for $R_0 > 1$. Our results imply that malaria will gradually disappear if $R_0 \leq 1$ and persistently exist if $R_0 > 1$.

Keywords: malaria transmission model; time delay; Lyapunov functional; global dynamics

1. Introduction

Malaria is one of the world's most significant infectious diseases [1]. Malaria is a life-threatening disease caused by parasites that is usually transmitted to persons through the bites of female Anopheles mosquitoes [2]. Malaria gives rise to great pressure for the global prevention and control of infectious diseases [3]. World Health Organization reported [2] that there were an estimated 247 million malaria cases, including 619,000 deaths worldwide in 2021, and the majority of cases and deaths occurred in

sub-Saharan Africa. The African region accounted for a disproportionate share of the global malaria burdens [4, 5]. In 2021, the African region was home to 95% of global malaria cases and 96% of global malaria deaths, and children under 5 years old accounted for about 80% of all malaria deaths there [2]. There are 5 kinds of parasite species that cause malaria in humans, and two of these species P. falciparum and P. vivax pose the greatest threat [6]. The first malaria symptoms such as headache, fever and chills usually appear 10–15 days after the bite of a malaria mosquito and may be mild and difficult to be recognized as malaria, which implies that malaria exists the incubation period [2]. It was reported that asymptomatic infections were more prevalent in sub-Saharan Africa, where an estimated 24 million people had asymptomatic malaria infections [7]. Thus asymptomatic infections can occur during malaria transmission.

Asymptomatic infected people have no clinical symptoms, but they are contagious and the impact of asymptomatic infections on malaria transmission is enormous [8, 9]. Bousema et al. [10] pointed out that asymptomatic carriers contributed to sustained transmission of malaria in local populations, and there was substantial evidence that an increase in the number of asymptomatic carriers at specific time intervals affected the dynamics of malaria transmission. Laishram et al. [4] concluded that asymptomatic malaria infections was a challenge for malaria control programs.

Since the emergence of malaria, scholars at home and abroad have been studying the pathogenesis and transmission dynamics of malaria. In all research methods, mathematical modeling is undoubtedly one of the most intuitive and effective methods. Many researchers have studied the dynamic evolution of malaria transmission by applying some mathematical models of malaria. In 1911, Ross [11] put forward a basic ordinary differential equations (ODEs) malaria model. Afterwards, MacDonald [12] extended Ross's model, and gave first the definition of the basic reproduction number. The extended Ross's model was said to be the Ross-Macdonald model. Subsequently, the Ross-Macdonald model has been extended to higher dimensions and more factors affecting malaria transmission have been taken into account (see, e.g., [1, 6, 13]). For example, Kingsolver [14] extended the Ross-Macdonald model and explained the greater attraction of infectious humans to mosquitoes in 1987. Safan and Ghazi [1] developed a 4D ODEs malaria transmission model with standard incidence rates, and analyzed the dynamic properties of equilibria of the malaria model. In 2020, Aguilar and Gutierrez [6] established a high-dimensional ODEs malaria model with asymptomatic carriers and standard incidence rate, and dealt with local dynamics of the disease-free equilibrium of the malaria model.

Over the years, considering the incubation period of malaria, lots of researchers established some time-delayed malaria models (see, e.g., [3, 5, 15–18]). For instance, in 2008, Ruan et al. [5] first established a class of Ross-Macdonald model with two time delays, and investigated the stability of equilibria of the model and the impact of time delays on the basic reproduction number. In 2019, Ding et al. [3] proposed a malaria model with time delay, and investigated the global stability of the uninfected equilibrium of the model as well as its uniform persistence. For the moment, there are few theoretical analysis of the model of malaria with standard incidence rate. Recently, Guo et al. [13] established a malaria transmission model with time delay and standard incidence rate, and they studied the global dynamic properties of equilibria of the model. Based on this, we extend and improve the model in [13], namely, we establish a malaria transmission model with asymptomatic infections, standard incidence rate and time delay, and then study the global dynamic properties of equilibria of the malaria model.

The remainder of this paper is organized as follows. In Section 2, we put forward a time-delayed

dynamic model of malaria with asymptomatic infections and standard incidence rate, and prove the well-posedness as well as dissipativeness of the system. In Section 3, we obtain the existence conditions of malaria-free and malaria-infected equilibria of the system, and verify the local dynamic properties of equilibria in terms of the basic reproduction number R_0. In Section 4, to obtain the global dynamic property of the malaria-infected equilibrium for $R_0 > 1$, we acquire the weak persistence of the system through some analysis techniques. In Section 5, by utilizing the Lyapunov functional method and the limiting system of the model combining stability of partial variables, we obtain the global stability results of malaria-free and malaria-infected equilibria in terms of R_0, respectively.

2. Model formulation

In order to delve into the details of malaria transmission, we develop a time-delayed model with asymptomatic infections and standard incidence rate. The population is classified into four compartments, which are denoted by S_h: susceptible individuals, A_h: asymptomatic infected individuals, I_h: symptomatic infected individuals, R_h: recovered individuals, respectively. The mosquitoes are classified into two compartments, which are denoted by S_m: susceptible mosquitoes and I_m: infected mosquitoes, respectively. Then the model of malaria transmission is proposed as follows:

$$
\begin{cases}
\dot{S}_m(t) = \lambda_m - \dfrac{\beta_1 S_m(t) A_h(t)}{N_m(t)} - \dfrac{\beta_2 S_m(t) I_h(t)}{N_m(t)} - \mu_m S_m(t), \\[2mm]
\dot{I}_m(t) = \dfrac{\beta_1 S_m(t) A_h(t)}{N_m(t)} + \dfrac{\beta_2 S_m(t) I_h(t)}{N_m(t)} - \mu_m I_m(t), \\[2mm]
\dot{S}_h(t) = \lambda_h - \dfrac{\beta_h S_h(t) I_m(t)}{N_m(t)} - \mu_h S_h(t), \\[2mm]
\dot{A}_h(t) = p\dfrac{\beta_h S_h(t-\tau) I_m(t-\tau)}{N_m(t-\tau)} - (\mu_h + \gamma_a) A_h(t), \\[2mm]
\dot{I}_h(t) = (1-p)\dfrac{\beta_h S_h(t-\tau) I_m(t-\tau)}{N_m(t-\tau)} - (\mu_h + \gamma_i) I_h(t), \\[2mm]
\dot{R}_h(t) = \gamma_a A_h(t) + \gamma_i I_h(t) - \mu_h R_h(t),
\end{cases}
\tag{2.1}
$$

where $N_m(t) = S_m(t) + I_m(t)$. Here, time delay $\tau \geq 0$, and all other parameters of system (2.1) are assumed to be positive and $p \in (0, 1)$. The description of parameters are listed in Table 1.

The phase space of system (2.1) is

$$
C_+ = \left\{\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)^T \in C = C([-\tau, 0], \mathbb{R}_+^6) : \phi_1(\theta) + \phi_2(\theta) > 0, \forall \theta \in [-\tau, 0]\right\},
$$

where C is the Banach space of continuous functions mapping from $[-\tau, 0]$ to $\mathbb{R}_+^6$ with $\mathbb{R}_+ = [0, \infty)$ and the supremum norm. In the following, the well-posedness as well as dissipativeness of system (2.1) will be investigated in C_+.

Theorem 2.1. *The solution $u(t) = (S_m(t), I_m(t), S_h(t), A_h(t), I_h(t), R_h(t))^T$ of system (2.1) with any $\phi \in C_+$ exists uniquely, and is non-negative and ultimately bounded on $\mathbb{R}_+$. In particular, $(S_m(t), S_h(t))^T \gg \mathbf{0}$ on $(0, \infty)$, and C_+ is positively invariant for system (2.1).*

Table 1. Descriptions of parameters in the model.

Parameter	Description
μ_m	The natural death rate of mosquitoes
μ_h	The natural death rate of humans
λ_m	The natural birth rate of mosquitoes
λ_h	The natural birth rate of humans
β_1	The infection rate of susceptible mosquitoes biting asymptomatic individuals
β_2	The infection rate of susceptible mosquitoes biting symptomatic individuals
β_h	The infection rate of infected mosquitoes biting susceptible individuals
τ	The incubation period of malaria
γ_a	The recovery rate of asymptomatic infected individuals
γ_i	The recovery rate of symptomatic infected individuals
p	The transition probability of asymptomatic infected individuals

Proof. In view of the basic theory of delay differential equations (DDEs) [19, 20], the solution $u(t)$ of system (2.1) with any $\phi \in C_+$ is unique on its maximum interval $[0, T_\phi)$ of existence. Firstly, we will prove that the solution $u(t)$ is non-negative on $[0, T_\phi)$. According to the continuous dependence of solutions of DDEs on parameters [19, 20], then for any $b \in (0, T_\phi)$ and a sufficiently small $\varepsilon > 0$, the solution $u(t, \varepsilon) = (u_1(t, \varepsilon), u_2(t, \varepsilon), u_3(t, \varepsilon), u_4(t, \varepsilon), u_5(t, \varepsilon), u_6(t, \varepsilon))^T$ through ϕ of the following model:

$$
\begin{cases}
\dot{S}_m(t) = \lambda_m - \dfrac{\beta_1 S_m(t) A_h(t)}{N_m(t)} - \dfrac{\beta_2 S_m(t) I_h(t)}{N_m(t)} - \mu_m S_m(t), \\[2mm]
\dot{I}_m(t) = \dfrac{\beta_1 S_m(t) A_h(t)}{N_m(t)} + \dfrac{\beta_2 S_m(t) I_h(t)}{N_m(t)} - \mu_m I_m(t) + \varepsilon, \\[2mm]
\dot{S}_h(t) = \lambda_h - \dfrac{\beta_h S_h(t) I_m(t)}{N_m(t)} - \mu_h S_h(t), \\[2mm]
\dot{A}_h(t) = p \dfrac{\beta_h S_h(t-\tau) I_m(t-\tau)}{N_m(t-\tau)} - (\mu_h + \gamma_a) A_h(t) + \varepsilon, \\[2mm]
\dot{I}_h(t) = (1-p) \dfrac{\beta_h S_h(t-\tau) I_m(t-\tau)}{N_m(t-\tau)} - (\mu_h + \gamma_i) I_h(t) + \varepsilon, \\[2mm]
\dot{R}_h(t) = \gamma_a A_h(t) + \gamma_i I_h(t) - \mu_h R_h(t) + \varepsilon,
\end{cases}
\tag{2.2}
$$

uniformly exists on $[0, b]$. Consequently, we claim $u(t, \varepsilon) \gg \mathbf{0}$ on $[0, b)$. It is clear that $\dot{u}_i(0, \varepsilon) > 0, i \in I_6 = \{1, 2, 3, 4, 5, 6\}$ whenever $u_i(0, \varepsilon) = 0$. Next, we prove the claim by contradiction. Suppose that there exists $\bar{t} \in (0, b)$ such that $u_i(\bar{t}, \varepsilon) = 0$ for some $i \in I_6$ and $u(t, \varepsilon) \gg \mathbf{0}$ for $t \in (0, \bar{t})$, where

$$
\bar{t} = \min_{1 \leq i \leq 6} \{t_i\}, \ t_i = \sup\{t \in (0, b) : u_i(x, \varepsilon) > 0, x \in (0, t]\}.
$$

As a result, it holds

$$
\dot{u}_i(\bar{t}, \varepsilon) \leq 0.
\tag{2.3}
$$

Since

$$
u(\bar{t}, \varepsilon) \geq 0, \ N_m(\bar{t}) = N_m(\bar{t}, \varepsilon) = \frac{\lambda_m + \varepsilon}{\mu_m}(1 - e^{-\mu_m \bar{t}}) + N_m(0, \varepsilon) e^{-\mu_m \bar{t}} > 0,
$$

it follows from (2.2) that $\dot{u}_i(\bar{t}, \varepsilon) > 0$, which yields a contradiction to (2.3). Thus, we have $u(t, \varepsilon) \gg \mathbf{0}$ for $t \in (0, b)$.

Letting $\varepsilon \to 0^+$ gives that $u(t, 0) = u(t) \geq 0$ for any $t \in [0, b)$. Note that $b \in (0, T_\phi)$ is chosen arbitrarily, so that $u(t) \geq 0$ on $[0, T_\phi)$. It is obvious that $T_\phi > \tau$. Therefore, from system (2.1), we have that for any $t \geq \tau$,

$$\dot{S}_h(t - \tau) + \dot{A}_h(t) + \dot{I}_h(t) + \dot{R}_h(t) = \lambda_h - \mu_h(S_h(t - \tau) + A_h(t) + I_h(t) + R_h(t)),$$
$$\dot{S}_m(t) + \dot{I}_m(t) = \lambda_m - \mu_m(S_m(t) + I_m(t)).$$

As a consequence, by the comparison principle, we can obtain that $u(t)$ is bounded. Accordingly, from the continuation theorem of solutions of DDEs [19], it follows $T_\phi = \infty$. Consequently, we have

$$\lim_{t \to \infty}(S_h(t - \tau) + A_h(t) + I_h(t) + R_h(t)) = \lambda_h/\mu_h,$$

$$\lim_{t \to \infty} N_m(t) = \lambda_m/\mu_m. \tag{2.4}$$

Therefore, the solution $u(t)$ with any $\phi \in C_+$ uniquely exists, and is non-negative and ultimately bounded on $\mathbb{R}_+$. Moreover, it is not difficult to get that $(S_m(t), S_h(t))^T \gg \mathbf{0}$ on $\mathbb{R}_+ \backslash \{0\}$, and C_+ is a positive invariant set for system (2.1).

3. Local stability

To begin with, it follows easily the malaria-free equilibrium $E_0 = (S_m^0, 0, S_h^0, 0, 0, 0)^T$, where $S_m^0 = \lambda_m/\mu_m$ and $S_h^0 = \lambda_h/\mu_h$. By using the similar method in [21, 22], we can calculate the basic reproduction number

$$R_0 = \sqrt{\frac{p\beta_h\lambda_h\beta_1}{\mu_h\lambda_m(\mu_h + \gamma_a)} + \frac{(1 - p)\beta_h\lambda_h\beta_2}{\mu_h\lambda_m(\mu_h + \gamma_i)}}.$$

To get a malaria-infected equilibrium (i.e., positive equilibrium) $E^* = \left(S_m^*, I_m^*, S_h^*, A_h^*, I_h^*, R_h^*\right)^T$, we have the following lemma.

Lemma 3.1. *System (2.1) exists a unique $E^* \gg \mathbf{0}$ when and only when $R_0 > 1$.*

Proof. First of all, the malaria-infected equilibrium equations can be obtained as follows:

$$\begin{cases} 0 = \lambda_m - \dfrac{\beta_1 S_m^* A_h^* + \beta_2 S_m^* I_h^*}{S_m^* + I_m^*} - \mu_m S_m^*, \\[2mm] 0 = \dfrac{\beta_1 S_m^* A_h^* + \beta_2 S_m^* I_h^*}{S_m^* + I_m^*} - \mu_m I_m^*, \\[2mm] 0 = \lambda_h - \dfrac{\beta_h S_h^* I_m^*}{S_m^* + I_m^*} - \mu_h S_h^*, \\[2mm] 0 = p\dfrac{\beta_h S_h^* I_m^*}{S_m^* + I_m^*} - (\mu_h + \gamma_a)A_h^*, \\[2mm] 0 = (1 - p)\dfrac{\beta_h S_h^* I_m^*}{S_m^* + I_m^*} - (\mu_h + \gamma_i)I_h^*, \\[2mm] 0 = \gamma_a A_h^* + \gamma_i I_h^* - \mu_h R_h^*. \end{cases} \tag{3.1}$$

Note that $S_m^* + I_m^* = \lambda_m/\mu_m$, it follows from (3.1) that

$$
\begin{cases}
I_m^* = \dfrac{\beta_1 S_m^* A_h^* + \beta_2 S_m^* I_h^*}{\lambda_m}, \\[2mm]
S_h^* = \dfrac{\lambda_h \lambda_m}{\mu_h \lambda_m + \mu_m \beta_h I_m^*}, \\[2mm]
A_h^* = \dfrac{p \beta_h \lambda_h \mu_m I_m^*}{(\mu_h + \gamma_a)(\lambda_m \mu_h + \mu_m \beta_h I_m^*)}, \\[2mm]
I_h^* = \dfrac{(1-p)\beta_h \lambda_h \mu_m I_m^*}{(\mu_h + \gamma_i)(\lambda_m \mu_h + \mu_m \beta_h I_m^*)}.
\end{cases}
\tag{3.2}
$$

Substituting the third and the fourth equations in (3.2) and $S_m^* = \lambda_m/\mu_m - I_m^*$ into the first equation in (3.2), there holds

$$
I_m^* = \frac{\lambda_m \mu_h I_m^* - \mu_h \mu_m (I_m^*)^2}{\mu_h \lambda_m + \mu_m \beta_h I_m^*} R_0^2.
\tag{3.3}
$$

In consequence, (3.3) possesses a unique positive root

$$
I_m^* = \frac{\lambda_m \mu_h (R_0^2 - 1)}{\mu_m (\mu_h R_0^2 + \beta_h)} > 0
$$

if and only if $R_0 > 1$. Thus, we can conclude that E^* is a unique malaria-infected equilibrium of system (2.1) if and only if $R_0 > 1$, where E^* satisfies

$$
\begin{cases}
S_m^* = \dfrac{\lambda_m (\mu_h + \beta_h)}{\mu_m (\mu_h R_0^2 + \beta_h)}, \\[2mm]
I_m^* = \dfrac{\lambda_m \mu_h (R_0^2 - 1)}{\mu_h \mu_m R_0^2 + \mu_m \beta_h}, \\[2mm]
S_h^* = \dfrac{\lambda_h (\mu_h R_0^2 + \beta_h)}{(\mu_h + \beta_h)\mu_h R_0^2}, \\[2mm]
A_h^* = \dfrac{p \beta_h \lambda_h (R_0^2 - 1)}{(\mu_h + \beta_h)(\mu_h + \gamma_a) R_0^2}, \\[2mm]
I_h^* = \dfrac{(1-p)\beta_h \lambda_h (R_0^2 - 1)}{(\mu_h + \beta_h)(\mu_h + \gamma_i) R_0^2}, \\[2mm]
R_h^* = \dfrac{\gamma_a A_h^* + \gamma_i I_h^*}{\mu_h}.
\end{cases}
\tag{3.4}
$$

Next, by adopting similar techniques in [23–25], we will discuss the local dynamic properties of the malaria-free equilibrium E_0 and the the malaria-infected equilibrium E^* with respect to R_0. First of all, for the local stability of the equilibrium E_0, we have the theorem as follows.

Theorem 3.1. *For any $\tau \geq 0$, the malaria-free equilibrium E_0 is locally asymptotically stable (LAS) when $R_0 < 1$, and unstable when $R_0 > 1$.*

Proof. With some calculations, the characteristic equation of the linear system of system (2.1) at E_0 can be obtained as follows:

$$
\begin{vmatrix}
\lambda + \mu_m & 0 & 0 & \beta_1 & \beta_2 & 0 \\
0 & \lambda + \mu_m & 0 & -\beta_1 & -\beta_2 & 0 \\
0 & \frac{\beta_h \lambda_h \mu_m}{\lambda_m \mu_h} & \lambda + \mu_h & 0 & 0 & 0 \\
0 & -\frac{p\beta_h \lambda_h \mu_m}{\lambda_m \mu_h}e^{-\lambda\tau} & 0 & \lambda + (\mu_h + \gamma_a) & 0 & 0 \\
0 & -\frac{(1-p)\beta_h \lambda_h \mu_m}{\lambda_m \mu_h}e^{-\lambda\tau} & 0 & 0 & \lambda + (\mu_h + \gamma_i) & 0 \\
0 & 0 & 0 & -\gamma_a & -\gamma_i & \lambda + \mu_h
\end{vmatrix}
$$

$$
= (\lambda + \mu_m)(\lambda + \mu_h)^2 \mathcal{H}(\lambda) = 0, \tag{3.5}
$$

where

$$
\mathcal{H}(\lambda) = (\lambda + \mu_m)(\lambda + \mu_h + \gamma_a)(\lambda + \mu_h + \gamma_i)
$$
$$
- \left[\frac{p\beta_1\beta_h\lambda_h\mu_m}{\lambda_m\mu_h}(\lambda + \mu_h + \gamma_i) + \frac{(1-p)\beta_2\beta_h\lambda_h\mu_m}{\lambda_m\mu_h}(\lambda + \mu_h + \gamma_a) \right]e^{-\lambda\tau}. \tag{3.6}
$$

Clearly, Eq (3.5) possesses three negative real roots: $-\mu_h$ (double) and $-\mu_m$. The other roots of Eq (3.5) satisty $\mathcal{H}(\lambda) = 0$. Next, we will prove that any root λ of $\mathcal{H}(\lambda) = 0$ has negative real part. Suppose, by contradiction, λ has the nonegative real part. Then it follows from $\mathcal{H}(\lambda) = 0$ that

$$
\lambda + \mu_m = \frac{p\beta_1\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\lambda + \mu_h + \gamma_a)}e^{-\lambda\tau} + \frac{(1-p)\beta_2\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\lambda + \mu_h + \gamma_i)}e^{-\lambda\tau}. \tag{3.7}
$$

Taking the modulus of both sides in (3.7), we have

$$
|\lambda + \mu_m| \geq \mu_m
$$

and

$$
\left\| \left[\frac{p\beta_1\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\lambda + \mu_h + \gamma_a)} + \frac{(1-p)\beta_2\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\lambda + \mu_h + \gamma_i)} \right]e^{-\lambda\tau} \right\|
$$
$$
\leq \left| \frac{p\beta_1\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\lambda + \mu_h + \gamma_a)} \right| + \left| \frac{(1-p)\beta_2\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\lambda + \mu_h + \gamma_i)} \right|
$$
$$
\leq \frac{p\beta_1\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\mu_h + \gamma_a)} + \frac{(1-p)\beta_2\beta_h\lambda_h\mu_m}{\lambda_m\mu_h(\mu_h + \gamma_i)} = \mu_m R_0^2 < \mu_m
$$

for $R_0 < 1$ and $\tau \geq 0$, which leads to a contradiction. Therefore, the real part of each root of the Eq (3.5) is negative. Accordingly, E_0 is LAS for $R_0 < 1$ and $\tau \geq 0$.

Now, we prove that the E_0 is unstable for $R_0 > 1$ and $\tau \geq 0$ by the zero theorem. Clearly, for $R_0 > 1$ and $\tau \geq 0$, we can get

$$
\mathcal{H}(0) = \mu_m(\mu_h + \gamma_a)(\mu_h + \gamma_i)(1 - R_0^2) < 0, \lim_{\lambda\to\infty} \mathcal{H}(\lambda) = \infty.
$$

According to the zero theorem, there must exsit a positive real root in Eq (3.6). Thus, E_0 is unstable for $R_0 > 1$ and $\tau \geq 0$.

For the local stability of the equilibrium E^*, we can obtain the theorem as follows.

Theorem 3.2. *For any $\tau \geq 0$, the malaria-infected equilibrium E^* is LAS if and only if $R_0 > 1$.*

Proof. By Lemma 3.1, we just require to demonstrate the sufficiency. Let

$$G = \frac{(R_0^2 - 1)^2 \mu_m \mu_h^2}{(\mu_h + \beta_h)(\mu_h R_0^2 + \beta_h)}, \quad H = \frac{\beta_h \lambda_h \mu_m (R_0^2 - 1)}{(\mu_h + \beta_h)\lambda_m R_0^2}, \quad K = \frac{\mu_h + \beta_h}{\mu_h R_0^2 + \beta_h},$$

$$T = \frac{\beta_h \mu_h (R_0^2 - 1)}{\mu_h R_0^2 + \beta_h}, \quad M = \frac{\beta_h \lambda_h \mu_m}{\mu_h \lambda_m R_0^2}, \quad J = \frac{\mu_m \mu_h (R_0^2 - 1)}{\mu_h R_0^2 + \beta_h}.$$

With direct calculation, the characteristic equation of the linear system of system (2.1) at E^* can be got as follows:

$$\begin{vmatrix} \lambda + G + \mu_m & -J & 0 & \beta_1 K & \beta_2 K & 0 \\ -G & \lambda + J + \mu_m & 0 & -\beta_1 K & -\beta_2 K & 0 \\ -H & M & \lambda + T + \mu_h & 0 & 0 & 0 \\ pHe^{-\lambda\tau} & -pMe^{-\lambda\tau} & -pTe^{-\lambda\tau} & \lambda + (\mu_h + \gamma_a) & 0 & 0 \\ (1-p)He^{-\lambda\tau} & (p-1)Me^{-\lambda\tau} & (p-1)Te^{-\lambda\tau} & 0 & \lambda + (\mu_h + \gamma_i) & 0 \\ 0 & 0 & 0 & -\gamma_a & -\gamma_i & \lambda + \mu_h \end{vmatrix}$$

$$= (\lambda + \mu_h)(\lambda + \mu_m) g(\lambda) = 0, \tag{3.8}$$

where

$$g(\lambda) = (\lambda + \mu_h + \gamma_i)(\lambda + \mu_h + \gamma_a)(\lambda + G + J + \mu_m)(\lambda + T + \mu_h)$$

$$- [(1-p)\beta_2 (\lambda + \mu_h + \gamma_a) + p\beta_1 (\lambda + \mu_h + \gamma_i)](\lambda + \mu_h) \frac{\beta_h \lambda_h \mu_m}{\mu_h \lambda_m R_0^2} e^{-\lambda\tau}.$$

Clearly, Eq (3.8) has two negative roots: $-\mu_h$ and $-\mu_m$. The other roots of Eq (3.8) satisfy $g(\lambda) = 0$. Then, we will prove that any root λ of $g(\lambda) = 0$ has negative real part by contradiction. Assume that λ has the non-negative real part. By $g(\lambda) = 0$, we can get

$$\frac{\lambda + T + \mu_h}{\lambda + \mu_h} = \left[\frac{(1-p)\beta_2}{(\lambda + G + J + \mu_m)(\lambda + \mu_h + \gamma_i)} + \frac{p\beta_1}{(\lambda + G + J + \mu_m)(\lambda + \mu_h + \gamma_a)} \right] \frac{\beta_h \lambda_h \mu_m}{\mu_h \lambda_m R_0^2} e^{-\lambda\tau}. \tag{3.9}$$

Taking the modulus of both sides in (3.9), for $R_0 > 1$ any $\tau \geq 0$, it follows

$$\left| \frac{\lambda + T + \mu_h}{\lambda + \mu_h} \right| > 1$$

and

$$\left\| \left[\frac{(1-p)\beta_2}{(\lambda + G + J + \mu_m)(\lambda + \mu_h + \gamma_i)} + \frac{p\beta_1}{(\lambda + G + J + \mu_m)(\lambda + \mu_h + \gamma_a)} \right] \frac{\beta_h \lambda_h \mu_m}{\mu_h \lambda_m R_0^2} e^{-\lambda\tau} \right\|$$

$$\leq \left[\left| \frac{(1-p)\beta_2}{(\lambda + G + J + \mu_m)(\lambda + \mu_h + \gamma_i)} \right| + \left| \frac{p\beta_1}{(\lambda + G + J + \mu_m)(\lambda + \mu_h + \gamma_a)} \right| \right] \left| \frac{\beta_h \lambda_h \mu_m}{\mu_h \lambda_m R_0^2} e^{-\lambda\tau} \right|$$

$$\leq \left[\frac{(1-p)\beta_2 \beta_h \lambda_h}{\lambda_m (G + J + \mu_m) \mu_h (\mu_h + \gamma_i)} + \frac{p\beta_1 \beta_h \lambda_h}{\lambda_m (G + J + \mu_m) \mu_h (\mu_h + \gamma_a)} \right] \frac{\mu_m}{R_0^2}$$

$$< \left[\frac{(1-p)\beta_2 \beta_h \lambda_h}{\lambda_m \mu_m \mu_h (\mu_h + \gamma_i)} + \frac{p\beta_1 \beta_h \lambda_h}{\lambda_m \mu_m \mu_h (\mu_h + \gamma_a)} \right] \frac{\mu_m}{R_0^2} = 1.$$

Obviously, this is a contradiction. Hence, the real part of each root of the Eq (3.8) is negative for $R_0 > 1$ and $\tau \geq 0$, which ensures the local stability of the equilibrium E^*.

4. Weak persistence

Generally, to obtain the global stability of the equilibrium E^*, we need to prove the strong persistence or uniform persistence of system (2.1). However, we study the weak persistence of system (2.1), which can ensure the global stability of the equilibrium E^*. Of course, the weak persistence of system (2.1) is more accessible than its strong or uniform persistence. Now, we define

$$F = \{\phi \in C_+ : \phi_2(0) > 0\},$$

and let

$$u(t) = (S_m(t), I_m(t), S_h(t), A_h(t), I_h(t), R_h(t))^T$$

be the solution of system (2.1) with any $\phi \in F$. It follows easily that F is a positive invariant set of system (2.1), and $u(t) \gg \mathbf{0}$ for $t > 0$. Hence, we discuss the weak persistence of system (2.1) in F.

According to [26], system (2.1) is said to be weakly persistent if

$$\limsup_{t\to\infty} \varrho(t) > 0, \ \varrho = S_m, I_m, S_h, A_h, I_h, R_h.$$

We define $u_t = (S_{mt}, I_{mt}, S_{ht}, A_{ht}, I_{ht}, R_{ht})^T \in C_+$ to be $u_t(\theta) = u(t + \theta)$, $\theta \in [-\tau, 0]$ for $t \geq 0$, and u_t is the solution of system (2.1) with ϕ. Inspired by the work in [13], we study the weak persistence of system (2.1). First, we have the following lemma.

Lemma 4.1. *Assume that $R_0 > 1$, $\theta \in (0, 1)$ and $\limsup_{t\to\infty} I_m(t) \leq \theta I_m^*$. Then*

$$\liminf_{t\to\infty} S_m(t) \geq \bar{S}_m \equiv \frac{\lambda_m - \theta \mu_m I_m^*}{\mu_m} = \frac{\lambda_m[\mu_h R_0^2 (1-\theta) + \beta_h + \theta \mu_h]}{\mu_m \left(\mu_h R_0^2 + \beta_h\right)} > S_m^*,$$

$$\liminf_{t\to\infty} S_h(t) \geq \bar{S}_h \equiv \frac{\lambda_h}{\theta \beta_h I_m^*/S_m^0 + \mu_h} = \frac{S_h^0}{\theta \beta_h(\mu_h + \beta_h)/\mu_h(\mu_h R_0^2 + \beta_h) + 1} > S_h^*.$$

Proof. It follows from (2.4) that

$$\liminf_{t\to\infty} S_m(t) = \liminf_{t\to\infty}(N_m(t) - I_m(t)) = S_m^0 - \limsup_{t\to\infty} I_m(t) \geq S_m^0 - \theta I_m^* = \bar{S}_m.$$

For any $\epsilon > 1$, there can be found $\varrho = \varrho(\phi, \epsilon) \geq 0$ such that

$$\frac{I_m(t)}{N_m(t)} \leq \frac{\epsilon \theta I_m^*}{S_m^0}, \ t > \varrho,$$

and then

$$\dot{S}_h(t) = \lambda_h - \frac{\beta_h S_h(t) I_m(t)}{N_m(t)} - \mu_h S_h(t) > \lambda_h - \left(\frac{\epsilon \theta \beta_h I_m^*}{S_m^0} + \mu_h\right) S_h(t).$$

Consequently,

$$\liminf_{t\to\infty} S_h(t) \geq \frac{\lambda_h}{\epsilon \theta \beta_h I_m^*/S_m^0 + \mu_h}.$$

Letting $\epsilon \to 1^+$, it holds

$$\liminf_{t\to\infty} S_h(t) \geq \bar{S}_h.$$

The malaria-infected equilibrium equations imply that

$$\bar{S}_m = \frac{\lambda_m\left(\mu_h R_0^2 (1-\theta) + \beta_h + \theta\mu_h\right)}{\mu_m\left(\mu_h R_0^2 + \beta_h\right)} > S_m^*, \ \bar{S}_h = \frac{S_h^0}{\theta\beta_h(\mu_h + \beta_h)/\mu_h(\mu_h R_0^2 + \beta_h) + 1} > S_h^*.$$

Theorem 4.1. *Let $R_0 > 1$. Then $\limsup_{t\to\infty} I_m(t) \geq I_m^*$.*

Proof. We will use the proof by contradiction to verify this result. Provided that $\limsup_{t\to\infty} I_m(t) < I_m^*$. Whereupon, one can find a $\theta \in (0,1)$ such that $\limsup_{t\to\infty} I_m(t) \leq \theta I_m^*$. Using Lemma 4.1, we can get that there is an $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$\frac{\bar{S}_h}{S_m^0 + \epsilon} > \frac{S_h^*}{S_m^0}, \frac{\bar{S}_m}{S_m^0 + \epsilon} > \frac{S_m^*}{S_m^0}. \tag{4.1}$$

Thanks to Lemma 4.1, it follows that for any $\epsilon \in (0, \epsilon_0)$, there can be found $\mathcal{T} \equiv \mathcal{T}(\epsilon, \phi) > 0$ such that

$$\frac{S_h(t)}{N_m(t)} > \frac{\bar{S}_h}{S_m^0 + \epsilon}, \frac{S_m(t)}{N_m(t)} > \frac{\bar{S}_m}{S_m^0 + \epsilon}, S_m(t) > S_m^*, \ t \geq \mathcal{T}.$$

Now, we define the functional on F as follows,

$$L(\phi) = S_m^0 \phi_2(0) + \frac{\beta_1 S_m^*}{\mu_h + \gamma_a}\phi_4(0) + \frac{\beta_2 S_m^*}{\mu_h + \gamma_i}\phi_5(0) + \frac{\lambda_m R_0^2 S_m^*}{S_h^0}\int_{-\tau}^{0}\frac{\phi_3(\theta)\phi_2(\theta)}{\phi_1(\theta) + \phi_2(\theta)}d\theta.$$

Obviously, $L(u_t)$ is bounded since L is continous on F. Then for $t \geq \mathcal{T}$, the derivative of L along the solution u_t is given by

$$\dot{L}(u_t) \geq \left(\frac{\lambda_m R_0^2 S_m^* S_h(t)}{S_h^0 N_m(t)} - \lambda_m\right)I_m(t)$$

$$> \left(\frac{\lambda_m R_0^2 S_m^* \bar{S}_h}{S_h^0 (S_m^0 + \epsilon)} - \lambda_m\right)I_m(t).$$

Denote

$$\bar{I}_m = \min_{\theta\in[-\tau,0]} I_m(\mathcal{T} + \tau + \theta),$$

$$c = \min\left\{\bar{I}_m, \frac{(\mu_h + \gamma_a)(S_m^0 + \epsilon)A_h(\mathcal{T} + \tau)}{p\beta_h\bar{S}_h}, \frac{(\mu_h + \gamma_i)(S_m^0 + \epsilon)I_h(\mathcal{T} + \tau)}{(1-p)\beta_h\bar{S}_h}\right\}.$$

Next, we will prove that $I_m(t) \geq c$ for $t \geq \mathcal{T}$. If not, there exists a $\mathcal{T}_0 \geq 0$ such that $I_m(t) \geq c$ for $t \in [\mathcal{T}, \mathcal{T} + \tau + \mathcal{T}_0]$, $I_m(\mathcal{T} + \tau + \mathcal{T}_0) = c$ and $\dot{I}_m(\mathcal{T} + \tau + \mathcal{T}_0) \leq 0$. Then it follows that for $t \in [\mathcal{T}, \mathcal{T} + \tau + \mathcal{T}_0]$,

$$\dot{A}_h(t) = p\frac{\beta_h S_h(t-\tau)I_m(t-\tau)}{N_m(t-\tau)} - (\mu_h + \gamma_a)A_h(t) \geq \frac{p\beta_h\bar{S}_h c}{S_m^0 + \epsilon} - (\mu_h + \gamma_a)A_h(t). \tag{4.2}$$

It easily follows from Eq (4.2) that for $t \in [\mathcal{T}, \mathcal{T} + \tau + \mathcal{T}_0]$,

$$A_h(t) \geq \frac{p\beta_h\bar{S}_h c}{(S_m^0 + \epsilon)(\mu_h + \gamma_a)} + \left(A_h(\mathcal{T}) - \frac{p\beta_h\bar{S}_h c}{(S_m^0 + \epsilon)(\mu_h + \gamma_a)}\right)e^{\mu\mathcal{T}-\mu t} \geq \frac{p\beta_h\bar{S}_h c}{(S_m^0 + \epsilon)(\mu_h + \gamma_a)}.$$

Analogously, one can get

$$I_h(t) \geq \frac{(1-p)\beta_h \bar{S}_h c}{(S_m^0 + \epsilon)(\mu_h + \gamma_i)},$$

for $t \in [\mathscr{T}, \mathscr{T} + \tau + \mathscr{T}_0]$. By $R_0^2 = S_h^0 S_m^0 / S_h^* S_m^*$ and (4.1), we have

$$\frac{\lambda_m R_0^2 S_m^* \bar{S}_h}{S_h^0 (S_m^0 + \epsilon)} - \lambda_m > \frac{\lambda_m R_0^2 S_m^* S_h^*}{S_h^0 S_m^0} - \lambda_m = 0.$$

Accordingly, we conclude that

$$
\begin{aligned}
&\dot{I}_m(\mathscr{T} + \tau + \mathscr{T}_0) \\
&= \left(\frac{\beta_1 S_m(\mathscr{T} + \tau + \mathscr{T}_0) A_h(\mathscr{T} + \tau + \mathscr{T}_0) + \beta_2 S_m(\mathscr{T} + \tau + \mathscr{T}_0) I_h(\mathscr{T} + \tau + \mathscr{T}_0)}{N_m(\mathscr{T} + \tau + \mathscr{T}_0)} - \lambda_m I_m(\mathscr{T} + \tau + \mathscr{T}_0) \right) \\
&> \beta_1 \bar{S}_m \frac{p\beta_h \bar{S}_h c}{(S_m^0 + \epsilon)^2 (\mu_h + \gamma_a)} + \beta_2 \bar{S}_m \frac{(1-p)\beta_h \bar{S}_h c}{(S_m^0 + \epsilon)^2 (\mu_h + \gamma_i)} - \lambda_m c \\
&= c \left(\frac{\lambda_m R_0^2 \bar{S}_m \bar{S}_h}{S_h^0 (S_m^0 + \epsilon)^2} - \lambda_m \right) > c \left(\frac{\lambda_m R_0^2 S_m^* S_h^*}{S_h^0 S_m^0} - \lambda_m \right) = 0.
\end{aligned}
$$

Clearly, this contradicts $\dot{I}_m(\mathscr{T} + \tau + \mathscr{T}_0) \leq 0$. As a result, $I_m(\mathscr{T}) \geq c$ for $t \geq \mathscr{T}$. Hence, for $t \geq \tau$,

$$\dot{L}(u_t) \geq \left(\frac{\lambda_m R_0^2 \bar{S}_m \bar{S}_h}{S_h^0 (S_m^0 + \epsilon)} - \lambda_m \right) c > 0,$$

which hints $L(u_t) \to \infty$ as $t \to \infty$. Accordingly, this contradicts the boundedness of $L(u_t)$.

According to Theorem 4.1, we have the following result.

Corollary 4.1. *If $R_0 > 1$, then for any $\tau \geq 0$, system (2.1) is weakly persistent.*

5. Global stability

We will study the global asymptotic stability of the equilibria E_0 and E^* with respect to R_0. For this purpose, we get from (2.4) the following limiting system of system (2.1):

$$
\begin{cases}
\dot{S}_m(t) = \lambda_m - \dfrac{\beta_1 S_m(t) A_h(t) + \beta_2 S_m(t) I_h(t)}{S_m^0} - \mu_m S_m(t), \\[2mm]
\dot{I}_m(t) = \dfrac{\beta_1 S_m(t) A_h(t) + \beta_2 S_m(t) I_h(t)}{S_m^0} - \mu_m I_m(t), \\[2mm]
\dot{S}_h(t) = \lambda_h - \dfrac{\beta_h S_h(t) I_m(t)}{S_m^0} - \mu_h S_h(t), \\[2mm]
\dot{A}_h(t) = \dfrac{p\beta_h S_h(t-\tau) I_m(t-\tau)}{S_m^0} - (\mu_h + \gamma_a) A_h(t), \\[2mm]
\dot{I}_h(t) = \dfrac{(1-p)\beta_h S_h(t-\tau) I_m(t-\tau)}{S_m^0} - (\mu_h + \gamma_i) I_h(t), \\[2mm]
\dot{R}_h(t) = \gamma_a A_h(t) + \gamma_i I_h(t) - \mu_h R_h(t).
\end{cases}
\tag{5.1}
$$

Adopting a similar argument as in the proof of Theorem 2.1, it follows that the solution

$$z(t) = (S_m(t), I_m(t), S_h(t), A_h(t), I_h(t), R_h(t))^T$$

of system (5.1) through any $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)^T \in C_+$ uniquely exists, and is non-negative and ultimately bounded on $[0, \infty)$. Setting

$$z_t(\theta) = z(t + \theta), \quad \theta \in [-\tau, 0]$$

gives that $z_t = (S_{mt}, I_{mt}, S_{ht}, A_{ht}, I_{ht}, R_{ht})^T \in C_+$ is also the solution of system (5.1) through φ for $t \ge 0$. We can find easily that E^* and E_0 are also the equilibria of system (5.1), and C_+ is a positive invariant set of system (5.1). By the way, $(S_m(t), S_h(t))^T \gg \mathbf{0}$ for $t > 0$. Define $H(v) = v - 1 - \ln v, \ v > 0$. Thereupon, for the global dynamic property of the equilibrium E_0 of system (2.1), we have the theorem as follows.

Theorem 5.1. *For any $\tau \ge 0$, the malaria-free equilibrium E_0 is GAS when $R_0 < 1$ and GA when $R_0 = 1$ in C_+.*

Proof. By Theorem 3.1, it follows that for $R_0 < 1$, E_0 is LAS. Thus, we only need to prove that for $R_0 \le 1$, E_0 is GA. Let u_t be the solution of system (2.1) with any $\phi \in C_+$ and z_t be the solution of system (5.1) though any $\varphi \in C_+$. Let $\omega(\phi)$ be the ω-limit set of ϕ with respect to system (2.1). In order to prove the global attractivity of E_0, we just need to show that $\omega(\phi) = \{E_0\}$. By Theorem 2.1, we know that u_t is bounded on C_+. Hence, it follows from (2.4) that $\omega(\phi)$ is a compact set, and is also a subset of C_+.

Let us define the following functional $\mathcal{V}$ on $L_1 = \{\varphi \in C_+ : \varphi_1(0) > 0, \varphi_3(0) > 0\} \subseteq C_+$

$$\mathcal{V}(\varphi) = \mathcal{V}_1(\varphi(0)) + \frac{\mu_h \lambda_m}{\lambda_h} \int_{-\tau}^{0} \varphi_3(\theta)\varphi_2(\theta)d\theta, \tag{5.2}$$

where

$$\mathcal{V}_1(\varphi(0)) = (S_m^0)^2 H\left(\frac{\varphi_1(0)}{S_m^0}\right) + S_m^0 \varphi_2(0) + \frac{S_m^0 \lambda_m}{\beta_h} H\left(\frac{\varphi_3(0)}{S_h^0}\right) + \frac{\beta_1 S_m^0}{\mu_h + \gamma_a}\varphi_4(0) + \frac{\beta_2 S_m^0}{\mu_h + \gamma_i}\varphi_5(0).$$

Obviously, $\mathcal{V}_1$ is continuous on L_1. Since $z_t \in L_1$ for $t \ge 1$, the derivative of $\mathcal{V}$ along z_t $(t \ge 1)$ is given by

$$\dot{\mathcal{V}}(z_t) = \frac{\lambda_m}{\mu_m}\left(1 - \frac{S_m^0}{S_m(t)}\right)\left(\lambda_m - \frac{\beta_1 S_m(t)A_h(t) + \beta_2 S_m(t)I_h(t)}{S_m^0} - \mu_m S_m(t)\right)$$

$$+ \frac{\lambda_m}{\mu_m}\left(\frac{\beta_1 S_m(t)A_h(t) + \beta_2 S_m(t)I_h(t)}{S_m^0} - \mu_m I_m(t)\right) + \frac{\lambda_m}{hS_h^0}\left(1 - \frac{S_h^0}{S_h(t)}\right)(\lambda_h - hS_h(t)I_m(t) - \mu_h S_h(t))$$

$$+ \frac{\beta_1 S_m^0}{\mu_h + \gamma_a}\left(phS_h(t - \tau)I_m(t - \tau) - (\mu_h + \gamma_a)A_h(t)\right)$$

$$+ \frac{\beta_2 S_m^0}{\mu_h + \gamma_i}\left((1 - p)hS_h(t - \tau)I_m(t - \tau) - (\mu_h + \gamma_i)I_h(t)\right) + \frac{\mu_h \lambda_m}{\lambda_h}(S_h(t)I_m(t) - S_h(t - \tau)I_m(t - \tau))$$

$$= -\frac{\lambda_m(S_m(t) - S_m^0)^2}{S_m(t)} - \frac{\mu_h \lambda_m(S_h(t) - S_h^0)^2}{hS_h^0 S_h(t)} + \frac{\lambda_m \mu_h}{\lambda_h}S_h(t - \tau)I_m(t - \tau)(R_0^2 - 1) \le 0, \tag{5.3}$$

where $h = \beta_h \mu_m / \lambda_m$. Considering (5.2) and (5.3), we can conclude that both $S_h(t)$ and $S_m(t)$ are persistent. In other words, there exists a $\sigma = \sigma(\varphi) > 0$ such that $\liminf_{t\to\infty} S_h(t) > \sigma$ and $\liminf_{t\to\infty} S_m(t) > \sigma$. As a result, $\omega(\varphi) \subseteq L_1$, where $\omega(\varphi)$ is the ω-limit set of φ with respect to system (5.1). It is evident that $\mathcal{V}$ is a Lyapunov functional on $\{z_t : t \geq 1\} \subseteq L_1$. Then it follows from [27, Corollry 2.1] that $\dot{\mathcal{V}}(\psi) = 0, \forall \psi \in \omega(\varphi)$.

Assume that z_t is the solution of system (5.1) through any $\psi \in \omega(\varphi)$. Then the invariance of $\omega(\varphi)$ gives that $z_t \in \omega(\varphi)$ for $t \in \mathbb{R}$. According to (5.3), we have $S_m(t) = S_m^0$ and $S_h(t) = S_h^0$ for $t \in \mathbb{R}$. From system (5.1) and the invariance of $\omega(\varphi)$, it follows that $I_m(t) = A_h(t) = I_h(t) = R_h(t) = 0$ for $t \in \mathbb{R}$. Thus for $R_0 \leq 1$, it holds that $\omega(\varphi) = \{E_0\}$, which implies that $W^s(E_0) = C_+$, where $W^s(E_0)$ is the stable set of E_0 with respect to system (5.1).

Now, we prove that the equilibrium E_0 of system (5.1) is uniformly stable for $R_0 \leq 1$ by using the similar approach in [28, 29]. Observe that the first five equations of system (5.1) can constitute an independent subsystem

$$
\begin{cases}
\dot{S}_m(t) = \lambda_m - \dfrac{\beta_1 S_m(t) A_h(t) + \beta_2 S_m(t) I_h(t)}{S_m^0} - \mu_m S_m(t), \\[2ex]
\dot{I}_m(t) = \dfrac{\beta_1 S_m(t) A_h(t) + \beta_2 S_m(t) I_h(t)}{S_m^0} - \mu_m I_m(t), \\[2ex]
\dot{S}_h(t) = \lambda_h - \dfrac{\beta_h S_h(t) I_m(t)}{S_m^0} - \mu_h S_h(t), \\[2ex]
\dot{A}_h(t) = \dfrac{p \beta_h S_h(t-\tau) I_m(t-\tau)}{S_m^0} - (\mu_h + \gamma_a) A_h(t), \\[2ex]
\dot{I}_h(t) = \dfrac{(1-p) \beta_h S_h(t-\tau) I_m(t-\tau)}{S_m^0} - (\mu_h + \gamma_i) I_h(t).
\end{cases}
\tag{5.4}
$$

It is not difficult to find that

$$
C_+ = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)^T \in C([-\tau, 0], \mathbb{R}_+^5) : \xi_1(\theta) + \xi_2(\theta) > 0, \theta \in [-\tau, 0] \right\}
$$

is a positive invariant set with respect to system (5.4). Clearly, system (5.4) has a malaria-free equilibrium $X^0 = \left(S_m^0, 0, S_h^0, 0, 0 \right)^T$. According to (5.2), (5.3), [27, Corollary 3.3] hints that X^0 is uniformly stable. Define

$$
k := \min \left\{ \frac{\mu_h}{\gamma_a}, \frac{\mu_h}{\gamma_i}, 1 \right\}.
$$

By the definition of uniform stability of X^0, it follows that for any $\epsilon > 0$, there is $\delta \leq 2\epsilon/3$ such that for any $\xi \in C_+$ and $\|\xi - X^0\| < \delta$, there holds

$$
\|X_t - X^0\| < \frac{\epsilon k}{3}, \ \forall t \geq 0,
$$

where X_t is the solution of system (5.4) with ξ. Then considering the sixth equation of system (5.1), we can get

$$
R_h(t) = \phi_6(0) e^{-\mu_h t} + \gamma_a e^{-\mu_h t} \int_0^t A_h(s) e^{\mu_h s} ds + \gamma_i e^{-\mu_h t} \int_0^t I_h(s) e^{\mu_h s} ds.
\tag{5.5}
$$

Consequently, for any $\phi \in C_+$ and $\|\phi - E_0\| < \delta$, it follows that for any $t \geq 0$,

$$R_h(t) < \frac{2\epsilon}{3} e^{-\mu_h t} + \gamma_a \frac{\mu_h}{\gamma_a} \frac{\epsilon}{3} \frac{1 - e^{-\mu_h t}}{\mu_h} + \gamma_i \frac{\mu_h}{\gamma_i} \frac{\epsilon}{3} \frac{1 - e^{-\mu_h t}}{\mu_h} = \frac{2\epsilon}{3},$$

and then

$$\|u_t - E_0\| \leq \|X_t - X^0\| + \|R_{ht}\| < \frac{\epsilon k}{3} + \frac{2\epsilon}{3} \leq \epsilon.$$

Thus, the equilibrium E_0 is uniformly stable for system (5.1).

Next, we claim that $\omega(\phi) = \{E_0\}$ for $R_0 \leq 1$. We first have $E_0 \in \omega(\phi)$ since $\omega(\phi) \subseteq C_+ = W^s(E_0)$. Assume that there exists $\psi \in \omega(\phi)$ such that $\psi \neq E_0$. Let $\alpha(\psi)$ be the α-limit set of ψ for system (5.1). Then it follows from the invariance and the compactness of $\omega(\phi)$ that $\alpha(\psi) \subseteq \omega(\phi)$. The invariance of $\alpha(\psi)$ and the stable set C_+ of E_0 yield that $E_0 \in \alpha(\psi)$. Obviously, this contradicts to the stability of E_0 for system (5.1). Therefore, $\omega(\phi) = \{E_0\}$.

Remark 5.1. *In fact, the stability of the malaria-free equilibrium E_0 of system* (5.1) *can be acquired for $R_0 = 1$ in the proof of Theorem 5.1. But using the proof of Theorem 3.1, we can not obtain the stability of the equilibrium E_0 for $R_0 = 1$.*

For the global dynamic property of the equilibrium E^* of system (2.1), we can draw the following theorem.

Theorem 5.2. *For any $\tau \geq 0$, the malaria-infected equilibrium E^* is globally asymptotically stable if and only if $R_0 > 1$ in F.*

Proof. From Lemma 3.1 and Theorem 3.2, we just require to prove that E^* is GA for $R_0 > 1$. Let u_t be the solution of system (2.1) with any $\phi \in F$ and z_t be the solution of system (5.1) through any $\varphi \in F$. We can obtain that F is positively invariant for system (5.1), and $z(t) \gg \mathbf{0}$ for $t \geq 0$. In order to show that E^* is GA, we only need to show that $\omega(\phi) = \{E^*\}$. It follows from Theorem 2.1 that u_t is bounded on F. Thus, it holds that $\omega(\phi)$ is compact.

Let us define a functional $\mathcal{V}$ on $L_2 = \{\varphi \in C_+ : \varphi_i(0) > 0, i = 1, 2, 3, 4, 5\} \subseteq F$ as follows

$$\mathcal{V}(\varphi) = \mathcal{V}_2(\varphi(0)) + \lambda_m I_m^* \int_{-\tau}^0 H\left(\frac{\varphi_3(\theta)\varphi_2(\theta)}{S_h^* I_m^*}\right) d\theta, \tag{5.6}$$

where

$$\mathcal{V}_2(\varphi(0)) = \frac{\lambda_m}{\mu_m} S_m^* H\left(\frac{\varphi_1(0)}{S_m^*}\right) + \frac{\lambda_m}{\mu_m} I_m^* H\left(\frac{\varphi_2(0)}{I_m^*}\right) + \frac{S_m^0 \lambda_m}{\beta_h} H\left(\frac{\varphi_3(0)}{S_h^*}\right)$$

$$+ \frac{\beta_1 S_m^*}{\mu_h + \gamma_a} H\left(\frac{\varphi_4(0)}{A_h^*}\right) + \frac{\beta_2 S_m^*}{\mu_h + \gamma_i} H\left(\frac{\varphi_5(0)}{I_h^*}\right).$$

Clearly, $\mathcal{V}_2$ is continuous on L_2. In as much as $z_t \in L_2$ for $t \geq \tau + 1$, the derivative of the functional

$\mathcal{V}$ along z_t in $t \geq \tau + 1$ is given by

$$
\begin{aligned}
\dot{\mathcal{V}}(z_t) = {} & \frac{\lambda_m}{\mu_m}\left(1 - \frac{S_m^*}{S_m(t)}\right)\left(\lambda_m - \frac{\beta_1 S_m(t)A_h(t) + \beta_2 S_m(t)I_h(t)}{S_m^0} - \mu_m S_m(t)\right) \\
& + \frac{\lambda_m}{\mu_m}\left(1 - \frac{I_m^*}{I_m(t)}\right)\left(\frac{\beta_1 S_m(t)A_h(t) + \beta_2 S_m(t)I_h(t)}{S_m^0} - \mu_m I_m(t)\right) \\
& + \frac{\lambda_m}{hS_h^*}\left(1 - \frac{S_h^*}{S_h(t)}\right)(\lambda_h - hS_h(t)I_m(t) - \mu_h S_h(t)) \\
& + \frac{\beta_1 S_m^*}{(\mu_h + \gamma_a)}\left(1 - \frac{A_h^*}{A_h(t)}\right)(phS_h(t-\tau)I_m(t-\tau) - (\mu_h + \gamma_a)A_h(t)) \\
& + \frac{\beta_2 S_m^*}{(\mu_h + \gamma_i)}\left(1 - \frac{I_h^*}{I_h(t)}\right)((1-p)hS_h(t-\tau)I_m(t-\tau) - (\mu_h + \gamma_i)I_h(t)) \\
& + \frac{\lambda_m}{S_h^*}S_h(t)I_m(t) - \frac{\lambda_m}{S_h^*}S_h(t-\tau)I_m(t-\tau) + \lambda_m I_m^* \ln\frac{S_h(t-\tau)I_m(t-\tau)}{S_h(t)I_m(t)}.
\end{aligned}
$$

where $h = \beta_h \mu_m / \lambda_m$. Further, it follows from the equilibrium equations that

$$
\begin{aligned}
\dot{\mathcal{V}}(z_t) = {} & -\frac{\lambda_m(S_m(t) - S_m^*)^2}{S_m(t)} - \lambda_m I_m^* H\left(\frac{S_m^*}{S_m(t)}\right) - \beta_1 S_m^* A_h^* H\left(\frac{S_m(t)A_h(t)I_m^*}{S_m^* A_h^* I_m(t)}\right) - \beta_2 S_m^* I_h^* H\left(\frac{S_m(t)I_h(t)I_m^*}{S_m^* I_h^* I_m(t)}\right) \\
& -\frac{\lambda_m \mu_h(S_h(t) - S_h^*)^2}{hS_h(t)S_h^*} - \lambda_m I_m^* H\left(\frac{S_h^*}{S_h(t)}\right) - \beta_1 S_m^* A_h^* H\left(\frac{S_h(t-\tau)I_m(t-\tau)A_h^*}{S_h^* I_m^* A_h(t)}\right) \\
& -\beta_2 S_m^* I_h^* H\left(\frac{S_h(t-\tau)I_m(t-\tau)I_h^*}{S_h^* I_m^* I_h(t)}\right) \leq 0. \quad (5.7)
\end{aligned}
$$

By (5.6) and (5.7), it follows that $\omega(\varphi) \subseteq L_2$. It is clear that $\mathcal{V}$ is a Lyapunov functional on $\{z_t : t \geq \tau + 1\} \subseteq L_2$. As a consequence, [27, Corollary2.1] implies that $\dot{\mathcal{V}}(\psi) = 0$ for any $\psi \in \omega(\varphi)$.

Let z_t be the solution of system (5.1) for any $\psi \in \omega(\varphi)$. Then the invariance of $\omega(\varphi)$ indicates that $z_t \in \omega(\varphi)$ for any $t \in \mathbb{R}$. Thus, from (5.7), it follows that for any $t \in \mathbb{R}$,

$$
S_m(t) = S_m^*, \; S_h(t) = S_h^*, \; A_h(t)I_m^* = A_h^* I_m(t), \; I_h(t)I_m^* = I_h^* I_m(t). \quad (5.8)
$$

By (5.8) and the third equation of system (5.1), we get that for any $t \in \mathbb{R}$, $I_m(t) = I_m^*$, $A_h(t) = A_h^*$ and $I_h(t) = I_h^*$. Consequently, by the invariance of $\omega(\varphi)$ and system (5.1), it holds that $R_h(t) = R_h^*$ for any $t \in \mathbb{R}$. Therefore, it follows that $z_0 = \psi = E^*$, and then $\omega(\varphi) = \{E^*\}$, which implies $W^s(E^*) = F$, where $W^s(E^*)$ is the stable set of E^* with respect to system (5.1).

Now, we prove that the equilibrium E^* of system (5.1) is uniformly stable by using the similar argument in [28, 29]. Note that system (5.4) has a unique malaria-infected equilibrium $X^* = \left(S_m^*, I_m^*, S_h^*, A_h^*, I_h^*\right)$. It follows from (5.6), (5.7) and [27, Corollary 3.3] that X^* is uniformly stable. By the definition of uniform stability of X^*, it follows that for any $\epsilon > 0$, there is $\delta \leq 2\epsilon/3$ such that for any $\xi \in C_+$ and $\|\xi - X^*\| < \delta$, there holds

$$
\|X_t - X^*\| < \frac{\epsilon k}{3}, \; \forall t \geq 0,
$$

where X_t is the solution of system (5.4) through ξ. Hence, for any $\phi \in L_2$ and $\|\phi - E^*\| < \delta$, it follow from (5.5) that

$$\|R_h(t) - R_h^*\| < \frac{2\epsilon}{3}e^{-\mu_h t} + \gamma_a \frac{\mu_h}{\gamma_a} \frac{\epsilon}{3} \frac{1 - e^{-\mu_h t}}{\mu_h} + \gamma_i \frac{\mu_h}{\gamma_i} \frac{\epsilon}{3} \frac{1 - e^{-\mu_h t}}{\mu_h} = \frac{2\epsilon}{3}$$

for any $t \geq 0$, where

$$R_h^* = R_h^* e^{-\mu_h t} + \gamma_a e^{-\mu_h t} \int_0^t A_h^* e^{\mu_h s} ds + \gamma_i e^{-\mu_h t} \int_0^t I_h^* e^{\mu_h s} ds$$

is used. Thus, we have

$$\|u_t - E^*\| \leq \|X_t - X^*\| + \|R_{ht} - R_h^*\| < \frac{\epsilon k}{3} + \frac{2\epsilon}{3} \leq \epsilon,$$

which gives that the equilibrium E^* is uniformly stable with respect to system (5.1).

Next, we claim that $\omega(\phi) = \{E^*\}$. From Theorem 4.1, it follows that $\omega(\phi) \cap F \neq \emptyset$, which gives that $E^* \in \omega(\phi)$. Assume that there is $\psi \in \omega(\phi)$ such that $\psi \neq E^*$. Then the invariance and the compactness of $\omega(\phi)$ implies that $\alpha(\psi) \subseteq \omega(\phi)$. By Theorem 3.2, we have that $E^* \in \alpha(\psi)$. Obviously, this contradicts to the stability of E^* with respect to system (5.1). Therefore, $\omega(\phi) = \{E^*\}$.

Remark 5.2. *Indeed, the proof of Theorem 5.2 can be simplified, i.e., the stability of system* (5.1) *is not a must, because $E^* \in \omega(\phi)$ and Theorem 3.2 can imply that $\omega(\phi) = \{E^*\}$. But if we use [30, Theorem 4.1] to prove that $\omega(\phi) = \{E^*\}$, then the stability of system* (5.1) *is required.*

Conflict of interest

The authors declare there is no conflict of interest.

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