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Research article

Asymptotic stability for solutions of a coupled system of quasi-linear viscoelastic Kirchhoff plate equations

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Abstract: In this manuscript, we study the asymptotic stability of solutions of two coupled quasi-linear viscoelastic Kirchhoff plate equations involving free boundary conditions, and accounting for rotational forces

$$|y_t|^\rho y_{tt} - \Delta y_{tt} + \Delta^2 y - \int_0^t h_1(t-s)\Delta^2 y(s)\, ds + f_1(y,z) = 0,$$

$$|z_t|^\rho z_{tt} - \Delta z_{tt} + \Delta^2 z - \int_0^t h_2(t-s)\Delta^2 z(s)\, ds + f_2(y,z) = 0.$$

The system under study in this contribution could be seen as a model for two stacked plates. This work is motivated by previous works about coupled quasi-linear wave equations or concerning single quasi-linear Kirchhoff plate. The existence of local weak solutions is established by the Faedo-Galerkin approach. By using the perturbed energy method, we prove a general decay rate of the energy for a wide class of relaxation functions.

Keywords: Kirchhoff plate equation; free boundary conditions; viscoelastic; Faedo-Galerkin method; asymptotic stability

1. Introduction

A coupled system of two Kirchhoff plate equations is considered:

$$\begin{cases} |y_t|^\rho y_{tt} - \Delta y_{tt} + \Delta^2 y - \displaystyle\int_0^t h_1(t-s)\Delta^2 y(s)\,ds + f_1(y,z) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ |z_t|^\rho z_{tt} - \Delta z_{tt} + \Delta^2 z - \displaystyle\int_0^t h_2(t-s)\Delta^2 z(s)\,ds + f_2(y,z) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ y = \partial_{\boldsymbol{\nu}} y = z = \partial_{\boldsymbol{\nu}} z = 0 & \text{on} \quad \Gamma_0 \times (0,\infty), \\ \mathbf{B}_1 y - \displaystyle\int_0^t h_1(t-s)\mathbf{B}_1 y(s)\,ds = \mathbf{B}_1 z - \displaystyle\int_0^t h_2(t-s)\mathbf{B}_1 z(s)\,ds = 0 & \text{on} \quad \Gamma_1 \times (0,\infty), \\ \mathbf{B}_2 y - \partial_{\boldsymbol{\nu}} y_{tt} - \displaystyle\int_0^t h_1(t-s)\mathbf{B}_2 y(s)\,ds = 0 & \text{on} \quad \Gamma_1 \times (0,\infty), \\ \mathbf{B}_2 z - \partial_{\boldsymbol{\nu}} z_{tt} - \displaystyle\int_0^t h_2(t-s)\mathbf{B}_2 z(s)\,ds = 0 & \text{on} \quad \Gamma_1 \times (0,\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$, such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, the initial data y_0, y_1, z_0 and z_1 lie in appropriate Hilbert space.

The symbols y_t and y_{tt} refer, respectively, to first order and second order derivatives (with respect to t) of y, while Δ and Δ^2 are the Laplacian and Bilaplacian operators. The functions h_i and f_i (for $i = 1,2$) verify some assumptions that will be given in the next section. ρ is a positive constant, $\boldsymbol{x} = (x_1, x_2)$ is the space variable, and the operators $\mathbf{B}_1$ and $\mathbf{B}_2$ are defined by

$$\mathbf{B}_1 y = \Delta y + (1-\mu)\left(2\nu_1\nu_2 y_{x_1x_2} - \nu_1^2 y_{x_2x_2} - \nu_2^2 y_{x_1x_1}\right),$$

and

$$\mathbf{B}_2 y = \partial_{\boldsymbol{\nu}}\Delta y + (1-\mu)\partial_{\boldsymbol{\tau}}\left((\nu_1^2 - \nu_2^2)y_{x_1x_2} + \nu_1\nu_2(y_{x_2x_2} - y_{x_1x_1})\right),$$

where the constant $0 < \mu < \frac{1}{2}$ is the Poisson coefficient. Here, $\partial_{\boldsymbol{\nu}}$ stands for normal derivative, $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is the unit outer normal vector to Γ and $\boldsymbol{\tau} = (-\nu_2, \nu_1)$ is a unit tangent vector.

Model (1.1) describes the interaction of two viscoelastic Kirchhoff plates with rotational forces, which possess a rigid surface and whose interiors are somehow permissive to slight deformations, such that the material densities vary according to the velocity [1]. Each one of these two plates is clamped along Γ_0, and without bending and twisting moments on Γ_1. The analysis of stability issues for plate models is more challenging due to free boundary conditions and the presence of rotational forces, etc. [2]. Moreover, in our case the source term competes with the dissipation induced by the viscoelastic term only. Therefore, it will be interesting to study this interaction [3].

We start off by reviewing some works related to quasi-linear wave equation and plate equation. Cavalcanti et al. [1] considered the following equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma\Delta u_t = 0, \tag{1.2}$$

and proved the global existence of weak solutions and a uniform decay rates of the energy in the presence of a strong damping, of the form $-\gamma\Delta u_t$ acting in the domain and assuming that the relaxation function decays exponentially. Messaoudi and Tatar [3] studied (1.2) but without a strong damping ($\gamma = 0$). They showed that the memory term is enough to stabilize the solution. The global existence and uniform decay for solutions of (1.2), provided that the initial data are in some stable set, are obtained in [4] with the presence of a source term and with $\gamma = 0$. Later, in [5], for $\gamma = 0$, the authors investigated the general decay result of the energy of (1.2) with nonlinear damping. In [6], the author investigated (1.2) with weakly nonlinear time-dependent dissipation and source terms, and he established an explicit and general energy decay rate results without imposing any restrictive growth assumption on the damping term at the origin. For other related results for quasi-linear wave equations, we refer the reader to [7–10]. For quasi-linear plate equations, we mention the work of Al-Gharabli et al. [11] where the authors studied the well-posedness and asymptotic stability for a quasi-linear viscoelastic plate equation with a logarithmic nonlinearity. Recently, Al-Mahdi [12] studied the same problem as in Al-Gharabli et al. [11], but with infinite memory. With the imposition of a minimal condition on the relaxation function, he obtained an explicit and general decay rate result for the energy. Very recently, in [13], the authors considered a plate equation with infinite memory, nonlinear damping, and logarithmic source. They proved explicit and general decay rate of the solution.

The stability of coupled quasi-linear systems has been discussed by many authors. Liu [14] considered two coupled quasi-linear viscoelastic wave equations. He showed that the viscoelastic terms' dissipations guarantee that the solutions decay exponentially and polynomially. Later on, with more general relaxation functions and specific initial data, He [15] extended the result of Liu [14]. Recently, Mustafa and Kafini [16] considered the same problem and improved earlier results for a wider class of relaxation functions. In [17], the authors studied the same problem, but with nonlinear damping, and showed a general decay rate estimates of energy of solutions. Very recently, Pişkin and Ekinci [18] generalized and improved earlier results by considering a degenerate damping. Finally, let's mention the recent works of Fang et al. [19] and Zhu et al. [20] that relate to our problem.

As I know, there is no work regarding quasi-linear plate equations. This paper seems to be the first that deals with this problem.

The structure of this paper is shown as follows: In Section 2, we present some presumptions that are necessary for the proof of essential results. The third section provides the proof of well-posedness of our system. The general energy decay result is stated and established in Section 4. The fifth section provides two examples that illustrate explicit formulas for the energy decay rates. A concluding section is given at the end.

2. Preliminaries

This part is devoted to give some necessary materials and assumptions for the proof of our key results. We define

$$V = \{y \in H^2(\Omega) : y = \partial_\nu y = 0 \text{ on } \Gamma_0\},$$

and

$$W = \{y \in H^1(\Omega) : y = 0 \text{ on } \Gamma_0\}.$$

Denoting $dx = dx_1 dx_2$, we define the bilinear form $b : V \times V \to \mathbb{R}$ by:

$$b(y,z) = \int_\Omega \{y_{x_1x_1} z_{x_1x_1} + y_{x_2x_2} z_{x_2x_2} + 2(1-\mu) y_{x_1x_2} z_{x_1x_2} + \mu \left(y_{x_1x_1} z_{x_2x_2} + y_{x_2x_2} z_{x_1x_1}\right)\} \, dx.$$

Firstly, we must recall Green's formula (see [2]):

$$b(y,z) = \int_\Omega \Delta^2 yz dx + \int_\Gamma (\mathcal{B}_1 y \partial_\nu z - \mathcal{B}_2 yz)\, d\Gamma, \quad \forall\, y \in H^4(\Omega), \quad z \in H^2(\Omega), \tag{2.1}$$

and a weaker version of it (see Theorem 5.6 in [21]) in the following form:

$$b(y,z) = \int_\Omega \Delta^2 yz dx + \langle \mathcal{B}_1 y, \partial_\nu z\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} - \langle \mathcal{B}_2 y, z\rangle_{H^{-\frac{3}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)}, \quad \forall\, z \in H^2(\Omega). \tag{2.2}$$

We need the following lemma.

Lemma 2.1. *([22]) For any $y \in C^1(0,T; H^2(\Omega))$, we get*

$$\begin{aligned} b\left(\int_0^t h_1(t-s)y(s)ds, y_t\right) &= -\frac{1}{2}h_1(t)b(y,y) - \frac{1}{2}\frac{d}{dt}\left\{(h_1 \Box y)(t) - \left(\int_0^t h_1(s)ds\right)b(y,y)\right\} \\ &\quad + \frac{1}{2}(h_1' \Box y)(t), \end{aligned} \tag{2.3}$$

where

$$(h_1 \Box y)(t) = \int_0^t h_1(t-s) b\,(y(t) - y(s), y(t) - y(s))\, ds.$$

In this paper, we suppose that:
(A1): The two non-increasing C^1 functions $h_i : [0, +\infty) \to (0, +\infty)$ (for $i = 1, 2$) such that

$$1 - \int_0^\infty h_i(s)\, ds = l_i > 0. \tag{2.4}$$

(A2): There are a positive C^1 functions $G_i : (0, +\infty) \to (0, +\infty)$, that are linear or strictly increasing and strictly convex C^2 on $(0, r], (r < 1)$, with $G_i(0) = G_i'(0) = 0$, satisfying for all $t \geq 0$

$$h_i'(t) \leq -\xi_i(t) G_i(h_i(t)), \text{ for } i = 1, 2, \tag{2.5}$$

where ξ_1 and ξ_2 are positive non-increasing differentiable functions.
(A3): $f_i : \mathbb{R}^2 \to \mathbb{R}$ (for $i = 1, 2$) are C^1 functions and there exists a positive function F, such that

$$f_1(x_1, x_2) = \frac{\partial F}{\partial x_1}, \quad f_2(x_1, x_2) = \frac{\partial F}{\partial x_2}, \; x_1 f_1(x_1, x_2) + x_2 f_2(x_1, x_2) - F(x_1, x_2) \geq 0,$$

and

$$\left|\frac{\partial f_i}{\partial x_1}(x_1, x_2)\right| + \left|\frac{\partial f_i}{\partial x_2}(x_1, x_2)\right| \leq d(1 + |x_1|^{\beta_{i1}-1} + |x_2|^{\beta_{i2}-1}), \;\forall\, (x_1, x_2) \in \mathbb{R}^2, \tag{2.6}$$

for some constant $d > 0$ and $\beta_{ij} \geq 1$ for $i, j = 1, 2$.

Remark 2.1. *1. The condition* (**A1**) *guarantees the hyperbolicity of the first two equations in the system* (1.1).

2. By (2.6) *and the mean value theorem, we have for some positive constant* d_1

$$|f_i(x_1, x_2)| \leq d_1(|x_1| + |x_2| + |x_1|^{\beta_{i1}} + |x_2|^{\beta_{i2}}), \tag{2.7}$$

and

$$|f_i(x_1, x_2) - f_i(u_1, u_2)| \leq d_1(1 + |x_1|^{\beta_{i1}} + |x_2|^{\beta_{i2}} + |u_1|^{\beta_{i1}} + |u_2|^{\beta_{i2}})(|x_1 - u_1| + |x_2 - u_2|), \tag{2.8}$$

for all $(x_1, x_2), (u_1, u_2) \in \mathbb{R}^2$ *and* $i = 1, 2$.

The energy functional is defined by

$$\begin{aligned} E(t) &= \underbrace{\frac{1}{\rho+2}\int_\Omega |y_t|^{\rho+2}\,dx + \frac{1}{2}\|\nabla \boldsymbol{y_t}\|^2}_{\mathcal{K}_y(t)} + \underbrace{\frac{1}{2}\left(1 - \int_0^t h_1(s)ds\right) b(y,y) + \frac{1}{2}(h_1 \Box y)(t)}_{\mathcal{P}_y(t)} \\ &+ \underbrace{\frac{1}{\rho+2}\int_\Omega |z_t|^{\rho+2}\,dx + \frac{1}{2}\|\nabla \boldsymbol{z_t}\|^2}_{\mathcal{K}_z(t)} + \underbrace{\frac{1}{2}\left(1 - \int_0^t h_2(s)ds\right) b(z,z) + \frac{1}{2}(h_2 \Box z)(t)}_{\mathcal{P}_z(t)} \\ &+ \int_\Omega F(y,z)\,dx. \end{aligned} \tag{2.9}$$

Here,

$$\mathcal{K}(t) = \mathcal{K}_y(t) + \mathcal{K}_z(t) \ \text{ and } \ \mathcal{P}(t) = \mathcal{P}_y(t) + \mathcal{P}_z(t) + \int_\Omega F(y,z)\,dx$$

represent, respectively, the kinetic and the elastic potential energy of the model.

We have the following dissipation identity:

Proposition 2.1.

$$E'(t) = \frac{1}{2}(h_1' \Box y)(t) - \frac{1}{2}h_1(t)b(y,y) + \frac{1}{2}(h_2' \Box z)(t) - \frac{1}{2}h_2(t)b(z,z) \leq 0. \tag{2.10}$$

Proof. Multiplying $(1.1)_1$ by y_t and $(1.1)_2$ by z_t, summing the resultant equations and integrating over Ω to get

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{\rho+2}\int_\Omega |y_t|^{\rho+2}\,dx + \frac{1}{2}\|\nabla \boldsymbol{y_t}\|^2 + \frac{1}{2}b(y,y) + \frac{1}{\rho+2}\int_\Omega |z_t|^{\rho+2}\,dx + \frac{1}{2}\|\nabla \boldsymbol{z_t}\|^2 \\ &+\frac{1}{2}b(z,z) + \int_\Omega F(y,z)\,dx\Big\} - \int_0^t h_1(t-s)b\,(y(s), y_t)\,ds - \int_0^t h_2(t-s)b\,(z(s), z_t)\,ds = 0. \end{aligned} \tag{2.11}$$

Inserting (2.3) in (2.11), we get the desired result. □

Throughout this paper, c denotes a generic positive constant, and not necessarily the same at different occurrences.

3. Global existence

We begin this part by defining a weak solution of the system (1.1).

Definition 3.1. *A couple of functions* (y,z) *defined on* $[0,T]$ *is a weak solution of the problem* (1.1) *if* $y \in C([0,T],V) \cap C^1([0,T],W),\ z \in C([0,T],V) \cap C^1([0,T],W)$, *and satisfies*

$$\begin{aligned}&\int_\Omega |y_t|^\rho y_{tt} u\,dx + \int_\Omega \nabla y_{tt} \nabla u\,dx + b(y,u) - \int_0^t h_1(t-s)b(y(s),u)\,ds + \int_\Omega f_1(y,z)u\,dx = 0,\\ &y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x),\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega |z_t|^\rho z_{tt} v\,dx + \int_\Omega \nabla z_{tt} \nabla v\,dx + b(z,v) - \int_0^t h_2(t-s)b(z(s),v)\,ds + \int_\Omega f_2(y,z)v\,dx = 0,\\ &z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x),\end{aligned}$$

for a.e. $t \in [0,T]$ *and all test functions* $u, v \in V$.

Theorem 3.1. *Let* $(y_0,y_1),(z_0,z_1) \in V \times W$. *Assume that assumptions* **(A1)–(A3)** *are true. Then, the system* (1.1) *has at least a local weak solution. Moreover, this solution is global and bounded.*

Proof. With the help of the Faedo-Galerkin approach, the existence is demonstrated. In order to achieve this, let $\{w_j\}_{j=1}^\infty$ be a basis of V. Define $E_m = span\{w_1, w_2, ..., w_m\}$. On the finite dimensional subspaces E_m, the initial data are projected as follows:

$$y_0^m(x) = \sum_{k=1}^m a_k w_k, \quad y_1^m(x) = \sum_{k=1}^m b_k w_k, \quad z_0^m(x) = \sum_{k=1}^m c_k w_k, \quad z_1^m(x) = \sum_{k=1}^m d_k w_k,$$

such that

$$(y_0^m, z_0^m) \to (y_0, z_0) \ \text{ in } \ V^2, \quad \text{and} \ \ (y_1^m, z_m^1) \to (y_1, z_1) \ \text{ in } \ W^2. \tag{3.1}$$

Considering the following solution

$$y^m(x,t) = \sum_{k=1}^m p_k(t) w_k(x), \qquad z^m(x,t) = \sum_{k=1}^m q_k(t) w_k(x),$$

which satisfies the following approximate problem in E_m:

$$\begin{aligned}&\int_\Omega |y_t^m|^\rho y_{tt}^m w\,dx + \int_\Omega \nabla y_{tt}^m \nabla w\,dx + b(y^m,w) - \int_0^t h_1(t-s)b(y^m(s),w)\,ds + \int_\Omega f_1(y^m,z^m)w\,dx = 0,\\ &\int_\Omega |z_t^m|^\rho z_{tt}^m w\,dx + \int_\Omega \nabla z_{tt}^m \nabla w\,dx + b(z^m,w) - \int_0^t h_2(t-s)b(z^m(s),w)\,ds + \int_\Omega f_2(y^m,z^m)w\,dx = 0,\\ &y^m(0) = y_0^m, y_t^m(0) = y_1^m, z^m(0) = z_0^m, z_t^m(0) = z_1^m.\end{aligned} \tag{3.2}$$

This leads to a system of ordinary differential equations (ODEs) for unknown functions p_k and q_k. Hence, from the standard theory of system of ODEs, a solution (y^m, z^m) of (3.2) exists, for all $m \geq 1$, on $[0, t_m)$, with $0 < t_m \le T,\ \forall\, m \ge 1$.

A priori estimate 1: Let $w = y_t^m$ in $(3.2)_1$ and $w = z_t^m$ in $(3.2)_2$. Combining the resultant equations and integrating on Ω to obtain

$$\frac{d}{dt}E^m(t) = \frac{1}{2}\Big\{(h_1{}'\Box y^m)(t) - h_1(t)b(y^m, y^m) + (h_2{}'\Box z^m)(t) - h_2(t)b(z^m, z^m)\Big\}, \tag{3.3}$$

where

$$\begin{aligned} E^m(t) \;=\; & \frac{1}{\rho+2}\int_\Omega |y_t^m|^{\rho+2}\,dx + \frac{1}{2}\left(1 - \int_0^t h_1(s)ds\right)b(y^m, y^m) + \frac{1}{2}\|\nabla \boldsymbol{y_t^m}\|^2 + \frac{1}{2}(h_1\Box y^m)(t) \\ & + \frac{1}{\rho+2}\int_\Omega |z_t^m|^{\rho+2}\,dx + \frac{1}{2}\left(1 - \int_0^t h_2(s)ds\right)b(z^m, z^m) + \frac{1}{2}\|\nabla \boldsymbol{z_t^m}\|^2 \\ & + \frac{1}{2}(h_2\Box z^m)(t) + \int_\Omega F(y^m, z^m)\,dx. \end{aligned}$$

Noting, by (3.1), that

$$\|(y_0^m, z_m^0)\|_{V^2},\quad \|(y_1^m, z_1^m)\|_{W^2} \le c.$$

Then, by integrating (3.3) over $(0, t)$, $0 < t < t_m$, we get a constant $M_1 > 0$ that doesn't depend on t and m, satisfying

$$E^m(t) \le E^m(0) \le M_1. \tag{3.4}$$

Hence, t_m can be replaced by some $T > 0$, for all $m \ge 1$.

A priori estimate 2: Let $w = y_{tt}^m$ in $(3.2)_1$ and $w = z_{tt}^m$ in $(3.2)_2$, adding the resultant equations, integrating on Ω , and using Young's inequality to obtain for all $\eta > 0$

$$\begin{aligned} & \int_\Omega |y_t^m|^\rho |y_{tt}^m|^2\,dx + \int_\Omega |z_t^m|^\rho |z_{tt}^m|^2\,dx + \int_\Omega |\nabla \boldsymbol{y_{tt}^m}|^2\,dx + \int_\Omega |\nabla \boldsymbol{z_{tt}^m}|^2\,dx \\ = \;& -b(y^m, y_{tt}^m) + \int_0^t h_1(t-s)b(y^m(s), y_{tt}^m)\,ds \\ & -b(z^m, z_{tt}^m) + \int_0^t h_2(t-s)b(z^m(s), z_{tt}^m)\,ds \\ & - \int_\Omega f_1(y^m, z^m)y_{tt}^m\,dx - \int_\Omega f_2(y^m, z^m)z_{tt}^m\,dx \\ \le \;& 2\eta\left(b(y_{tt}^m, y_{tt}^m) + b(z_{tt}^m, z_{tt}^m)\right) + \frac{1}{4\eta}\left(b(y^m, y^m) + b(z^m, z^m)\right) \\ & + \frac{(1-l_1)h_1(0) + (1-l_2)h_2(0)}{4\eta}\int_0^t \left(b(y^m(s), y^m(s)) + b(z^m(s), z^m(s))\right)\,ds \\ & - \int_\Omega f_1(y^m, z^m)y_{tt}^m\,dx - \int_\Omega f_2(y^m, z^m)z_{tt}^m\,dx. \end{aligned} \tag{3.5}$$

Using Hölder's inequality, Sobolev's embedding, (2.7) and (3.4), one has for some $M_2 > 0$,

$$\begin{aligned}
\Big|\int_\Omega f_1(y^m,z^m)y^m_{tt}\,dx\Big| &\le d\int_\Omega\left(|y^m|+|z^m|+|y^m|^{\beta_{11}}+|z^m|^{\beta_{12}}\right)|y^m_{tt}|\,dx\\
&\le c\left(\|y^m\|_2+\|y^m\|_2+\|y^m\|^{\beta_{11}}_{2\beta_{11}}+\|z^m\|^{\beta_{12}}_{2\beta_{12}}\right)\|y^m_{tt}\|_2\\
&\le c\left(\{b(y^m,y^m)\}^{\frac12}+\{b(z^m,z^m)\}^{\frac12}+\{b(y^m,y^m)\}^{\frac{\beta_{11}}{2}}+\{b(z^m,z^m)\}^{\frac{\beta_{12}}{2}}\right)\{b(y^m_{tt},y^m_{tt})\}^{\frac12}\\
&\le M_2\{b(y^m_{tt},y^m_{tt})\}^{\frac12}. \qquad (3.6)
\end{aligned}$$

Similarly, we obtain that

$$\Big|\int_\Omega f_2(y^m,z^m)z^m_{tt}\,dx\Big| \le M_2\{b(z^m_{tt},z^m_{tt})\}^{\frac12}. \qquad (3.7)$$

From (3.5)–(3.7), we infer that

$$\begin{aligned}
&\int_\Omega |y^m_t|^\rho|y^m_{tt}|^2\,dx+\int_\Omega |z^m_t|^\rho|z^m_{tt}|^2\,dx+\left(\int_\Omega|\nabla \boldsymbol{y^m_{tt}}|^2\,dx-2\eta b(y^m_{tt},y^m_{tt})\right)+\left(\int_\Omega|\nabla \boldsymbol{z^m_{tt}}|^2\,dx-2\eta b(z^m_{tt},z^m_{tt})\right)\\
\le\ & \frac{1}{4\eta}\left(b(y^m,y^m)+a(z^m,z^m)\right)+M_2\left(\{b(y^m_{tt},y^m_{tt})\}^{\frac12}+\{b(z^m_{tt},z^m_{tt})\}^{\frac12}\right)\\
+\ & \frac{(1-l_1)h_1(0)+(1-l_2)h_2(0)}{4\eta}\int_0^t\left(b(y^m(s),y^m(s))+b(z^m(s),z^m(s))\right)\,ds. \qquad (3.8)
\end{aligned}$$

Integrating (3.8) on $(0,T)$, and using (3.4) gives us

$$\begin{aligned}
&\int_0^T\int_\Omega |y^m_t|^\rho|y^m_{tt}|^2\,dx\,dt+\int_0^T\int_\Omega |z^m_t|^\rho|z^m_{tt}|^2\,dx\,dt+\int_0^T\left(\int_\Omega|\nabla \boldsymbol{y^m_{tt}}|^2\,dx-3\eta b(y^m_{tt},y^m_{tt})\right)dt+\int_0^T\left(\int_\Omega|\nabla \boldsymbol{z^m_{tt}}|^2\,dx-3\eta b(z^m_{tt},z^m_{tt})\right)dt\\
\le\ & \frac{T}{4\eta}\Big\{M_1\left(1+T[(1-l_1)h_1(0)+(1-l_2)h_2(0)]\right)+M_2^2\Big\}. \qquad (3.9)
\end{aligned}$$

Choosing η small enough, such that

$$\frac12\|\nabla \boldsymbol{u}\|_2^2-3\eta b(u,u)>0,\ \forall\ u\in V,$$

and so that

$$\|\nabla \boldsymbol{u}\|_2^2-3\eta b(u,u)>\frac12\|\nabla \boldsymbol{u}\|_2^2,\ \forall\ u\in V.$$

Consequently, (3.9) becomes

$$\begin{aligned}
&\int_0^T\int_\Omega |y^m_t|^\rho|y^m_{tt}|^2\,dx\,dt+\int_0^T\int_\Omega |z^m_t|^\rho|z^m_{tt}|^2\,dx\,dt+\frac12\int_0^T\int_\Omega\left(|\nabla \boldsymbol{y^m_{tt}}|^2+|\nabla \boldsymbol{z^m_{tt}}|^2\right)dx\,dt\\
\le\ & \frac{T}{4\eta}\Big\{M_1\left(1+T[(1-l_1)h_1(0)+(1-l_2)h_2(0)]\right)+M_2^2\Big\}.
\end{aligned}$$

Then, we have

$$\int_0^T\int_\Omega\left(|\nabla \boldsymbol{y^m_{tt}}|^2+|\nabla \boldsymbol{z^m_{tt}}|^2\right)dx\,dt\le M_3, \qquad (3.10)$$

for some constant $M_3 > 0$.

From (3.4) and (3.10), we conclude that

$$y^m,\ z^m \text{ are uniformly bounded in } L^\infty(0,T;V), \tag{3.11}$$

$$y_t^m,\ z_t^m \text{ are uniformly bounded in } L^\infty(0,T;W), \tag{3.12}$$

and

$$y_{tt}^m,\ z_{tt}^m \text{ are uniformly bounded in } L^2(0,T;W). \tag{3.13}$$

Hence, we can extract subsequence of (y^m) and (z^m), still denoted by (y^m) and (z^m) respectively, such that

$$y^m \overset{*}{\rightharpoonup} y,\ z^m \overset{*}{\rightharpoonup} z \text{ in } L^\infty(0,T;V) \text{ and } y^m \rightharpoonup y,\ z^m \rightharpoonup z \text{ in } L^2(0,T;V), \tag{3.14}$$

$$y_t^m \overset{*}{\rightharpoonup} y_t,\ z_t^m \overset{*}{\rightharpoonup} z_t \text{ in } L^\infty(0,T;W) \text{ and } y_t^m \rightharpoonup y_t,\ z_t^m \rightharpoonup z_t \text{ in } L^2(0,T;W), \tag{3.15}$$

and

$$y_{tt}^m \rightharpoonup y_{tt},\ z_{tt}^m \rightharpoonup z_{tt} \text{ weakly in } L^2(0,T;W). \tag{3.16}$$

Analysis of the non-linear terms:

1. Term $f_i(y^m, z^m)$: We have that (y^m) and (z^m) are bounded in $L^\infty(0,T;V)$. This shows, by the use of the embedding of $V \subset L^\infty(\Omega)(\Omega \subset \mathbb{R}^2)$, the boundedness of (y^m) and (z^m) in $L^2(\Omega \times (0,T))$. Likewise, (y_t^m) and (z_t^m) are bounded in $L^2(\Omega \times (0,T))$. Hence, by the use of the Aubin-Lions Theorem, we get, up to a subsequence, that

$$y^m \to y \text{ and } z^m \to z \text{ strongly in } L^2(\Omega \times (0,T)).$$

Then,

$$y^m \to y \text{ and } z^m \to z \text{ a.e in } \Omega \times (0,T),$$

and, therefore, from (**A3**),

$$f_i(y^m, z^m) \to f_i(y,z) \text{ a.e in } \Omega \times (0,T), \text{ for } i = 1,2. \tag{3.17}$$

On the other hand, we have (y^m) and (z^m) that are bounded in $L^\infty(0,T;L^2(\Omega))$, then, by using (2.7) and (3.4), we get that $f_i(y^m, z^m)$ is bounded in $L^\infty(0,T;L^2(\Omega))$. This fact and (3.17) leads to

$$f_i(y^m, z^m) \rightharpoonup f_i(y,z) \text{ in } L^2(0,T;L^2(\Omega)), \text{ for } i = 1,2.$$

2. Terms $|y_t^m|^\rho y_t^m$ and $|z_t^m|^\rho z_t^m$: We recall that (y_t^m) and (z_t^m) are bounded in $L^\infty(0,T;W)$, which gives that (y_t^m) and (z_t^m) are bounded in $L^\infty(\Omega\times(0,T))$, and so in $L^2(\Omega\times(0,T))$. By the same, we deduce that (y_{tt}^m) and (z_{tt}^m) are bounded in $L^2(\Omega \times (0,T))$. Now, using Aubin-Lions theorem, we conclude, up to a subsequence, that

$$y_t^m \to y_t \text{ and } z_t^m \to z_t \text{ strongly in } L^2(\Omega \times (0,T)),$$

and

$$|y_t^m|^\rho y_t^m \to |y_t|^\rho y_t \text{ and } |z_t^m|^\rho z_t^m \to |z_t|^\rho z_t \text{ a.e in } \Omega \times (0,T). \tag{3.18}$$

Using (3.4), we see that

$$\left\| \left| y_t^m \right|^\rho y_t^m \right\|_{L^2(0,T;L^2(\Omega))}^2 \leq C_*^{2(\rho+1)} \int_0^T \|\nabla \boldsymbol{y_t^m}\|_2^{2(\rho+1)}\, dt \leq C_*^{2(\rho+1)} M_1^{\rho+1} T, \tag{3.19}$$

and similarly

$$\left\| \left| z_t^m \right|^\rho z_t^m \right\|_{L^2(0,T;L^2(\Omega))}^2 \leq C_*^{2(\rho+1)} M_1^{\rho+1} T, \tag{3.20}$$

where C_* is a positive constant satisfying $\|u\|_2 \leq C_* \|\nabla \boldsymbol{u}\|_2$, for all $u \in W$.

Then, the sequences $(|y_t^m|^\rho y_t^m)$ and $(|z_t^m|^\rho z_t^m)$ are bounded in $L^2(\Omega \times (0,T))$. Combining (3.18), (3.19) and (3.20) and using Lion's lemma [23], one derives

$$|y_t^m|^\rho y_t^m \rightharpoonup |y_t|^\rho y_t \;\text{ and }\; |z_t^m|^\rho z_t^m \rightharpoonup |z_t|^\rho z_t \;\text{ in }\; L^2(0,T;L^2(\Omega)). \tag{3.21}$$

Next, by integrating (3.2) on $(0,t)$, one obtains

$$\begin{aligned}
&\frac{1}{\rho+1}\int_\Omega |y_t^m|^\rho y_t^m w\, dx + \int_\Omega \nabla \boldsymbol{y_t^m} \nabla \boldsymbol{w}\, dx + \int_0^t b(y^m, w)\, ds - \int_0^t \int_0^s h_1(s-\zeta) b(y^m(\zeta), w)\, d\zeta\, ds \\
&+ \int_0^t \int_\Omega f_1(y^m, z^m) w\, dx\, ds = \frac{1}{\rho+1}\int_\Omega |y_1^m|^\rho y_1^m w\, dx + \int_\Omega \nabla \boldsymbol{y_1^m} \nabla \boldsymbol{w}\, dx, \qquad (3.22)\\
&\frac{1}{\rho+1}\int_\Omega |z_t^m|^\rho z_t^m w\, dx + \int_\Omega \nabla \boldsymbol{z_t^m} \nabla \boldsymbol{w}\, dx + \int_0^t b(z^m, w)\, ds - \int_0^t \int_0^s h_2(s-\zeta) b(z^m(\zeta), w)\, d\zeta\, ds \\
&+ \int_0^t \int_\Omega f_2(y^m, z^m) w\, dx\, ds = \frac{1}{\rho+1}\int_\Omega |z_1^m|^\rho z_1^m w\, dx + \int_\Omega \nabla \boldsymbol{z_1^m} \nabla \boldsymbol{w}\, dx.
\end{aligned}$$

Letting $m \to +\infty$, the aforementioned convergence results give that

$$\begin{aligned}
&\frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t w\, dx + \int_\Omega \nabla \boldsymbol{y_t} \nabla \boldsymbol{w}\, dx - \frac{1}{\rho+1}\int_\Omega |y_1|^\rho y_1 w\, dx - \int_\Omega \nabla \boldsymbol{y_1} \nabla \boldsymbol{w}\, dx \\
=\;& -\int_0^t b(y,w) ds + \int_0^t \int_0^s h_1(s-\zeta) b(y(\zeta), w) d\zeta ds - \int_0^t \int_\Omega f_1(y,z) w\, dx\, ds, \qquad (3.23)\\
&\frac{1}{\rho+1}\int_\Omega |z_t|^\rho z_t w\, dx + \int_\Omega \nabla \boldsymbol{z_t} \nabla \boldsymbol{w}\, dx - \frac{1}{\rho+1}\int_\Omega |z_1|^\rho z_1 w\, dx - \int_\Omega \nabla \boldsymbol{z_1} \nabla \boldsymbol{w}\, dx \\
=\;& -\int_0^t b(z,w) ds + \int_0^t \int_0^s h_2(s-\zeta) b(z(\zeta), w) d\zeta ds - \int_0^t \int_\Omega f_2(y,z) w\, dx\, ds,
\end{aligned}$$

for all $w \in V$.

Since the terms in the right hand side of $(3.23)_1$ and $(3.23)_2$ are absolutely continuous, then (3.23) is differentiable for a.e. $t \geq 0$, and, therefore, one has for all $w \in V$

$$\begin{aligned}
&\int_\Omega |y_t|^\rho y_{tt} w dx + b(y,w) + \int_\Omega \nabla \boldsymbol{y_{tt}} \nabla \boldsymbol{w}\, dx - \int_0^t h_1(t-s) b(y(s), w)\, ds + \int_\Omega f_1(y,z) w\, dx = 0, \\
&\int_\Omega |z_t|^\rho z_{tt} w dx + b(z,w) + \int_\Omega \nabla \boldsymbol{z_{tt}} \nabla \boldsymbol{w}\, dx - \int_0^t h_2(t-s) b(z(s), w)\, ds + \int_\Omega f_2(y,z) w\, dx = 0.
\end{aligned}$$

Regarding the initial conditions, we recall that

$$\begin{cases} y^m \rightharpoonup y, \quad z^m \rightharpoonup z \ \text{ in } L^2(0,T;V) \\ \\ y_t^m \rightharpoonup y_t, \quad z_t^m \rightharpoonup z_t \ \text{ in } L^2(0,T;W). \end{cases} \tag{3.24}$$

Consequently, the use of Lion's Lemma [23] leads to

$$y^m \to y, \quad z^m \to z \ \text{ in } \ C([0,T), L^2(\Omega)). \tag{3.25}$$

Hence, $y^m(x,0)$ and $z^m(x,0)$ make sense and $y^m(x,0) \to y(x,0)$, $z^m(x,0) \to z(x,0)$ in $L^2(\Omega)$. Recalling that

$$y^m(x,0) = y_0^m(x) \to y_0(x), \quad z^m(x,0) = z_0^m(x) \to z_0(x) \ \text{ in } \ V,$$

we obtain that

$$y(x,0) = y_0(x) \ \text{ and } \ z(x,0) = z_0(x). \tag{3.26}$$

Besides, multiplying (3.2) by $\phi \in C_0^\infty(0,T)$ [24] and integrating on $(0,T)$, to get

$$\begin{aligned} -\frac{1}{\rho+1}\int_0^T (|y_t^m|^\rho y_t^m, w)_{L^2(\Omega)}\phi'(t)\,dt = \; & - \int_0^T \int_\Omega \boldsymbol{\nabla y_{tt}^m \nabla w}\phi(t)\,dx\,dt - \int_0^T b(y^m,w)\phi(t)\,dt \\ & + \int_0^T \int_0^t h_1(t-s)b(y^m(s),w)\phi(t)\,ds\,dt \\ & - \int_0^T \int_\Omega f_1(y^m,z^m)w\phi(t)\,dx\,dt, \end{aligned}$$

and

$$\begin{aligned} -\frac{1}{\rho+1}\int_0^T (|z_t^m|^\rho z_t^m, w)_{L^2(\Omega)}\phi'(t)\,dt = \; & - \int_0^T \int_\Omega \boldsymbol{\nabla z_{tt}^m \nabla w}\phi(t)\,dx\,dt - \int_0^T b(z^m,w)\phi(t)\,dt \\ & + \int_0^T \int_0^t h_2(t-s)b(z^m(s),w)\phi(t)\,ds\,dt \\ & - \int_0^T \int_\Omega f_2(y^m,z^m)w\phi(t)\,dx\,dt. \end{aligned}$$

As $m \to +\infty$, we have for any $w \in V$ and any $\phi \in C_0^\infty(0,T)$

$$\begin{aligned} -\frac{1}{\rho+1}\int_0^T (|y_t|^\rho y_t, w)_{L^2(\Omega)}\phi'(t)\,dt = \; & - \int_0^T \int_\Omega \boldsymbol{\nabla y_{tt} \nabla w}\phi(t)\,dx\,dt - \int_0^T b(y,w)\phi(t)\,dt \\ & + \int_0^T \int_0^t h_1(t-s)b(y(s),w)\phi(t)\,ds\,dt \\ & - \int_0^T \int_\Omega f_1(y,z)w\phi(t)\,dx\,dt, \end{aligned}$$

and

$$-\frac{1}{\rho+1}\int_0^T (|z_t|^\rho z_t, w)_{L^2(\Omega)}\phi'(t)\,dt = -\int_0^T\int_\Omega \nabla \boldsymbol{z_{tt}} \nabla \boldsymbol{w}\phi(t)\,dx\,dt - \int_0^T b(z,w)\phi(t)\,dt + \int_0^T\int_0^t h_2(t-s)b(z(s),w)\phi(t)\,ds\,dt - \int_0^T\int_\Omega f_2(y,z)w\phi(t)\,dx\,dt.$$

This means that (see [24])

$$y_{tt} \text{ and } z_{tt} \in L^2(0,T;V).$$

Since y_t and $z_t \in L^2(0,T;L^2(\Omega))$, we deduce that y_t and $z_t \in C(0,T;V)$.

So, $y_t^m(x,0)$ and $z_t^m(x,0)$ make sense and

$$y_t^m(x,0) \to y_t(x,0), \quad z_t^m(x,0) \to z_t(x,0) \text{ in } V.$$

But

$$y_t^m(x,0) = y_1^m(x) \to y_1(x), \quad z_t^m(x,0) = z_1^m(x) \to z_1(x) \text{ in } W.$$

Hence,

$$y_t(x,0) = y_1(x) \text{ and } z_t(x,0) = z_1(x).$$

Consequently, the proof of local existence of weak solutions is complete. Besides, it is easy to see that

$$l_1 b(y,y) + \|\nabla \boldsymbol{y_t}\|^2 + l_2 b(z,z) + \|\nabla \boldsymbol{z_t}\|^2 \le 2E(t) \le 2E(0), \tag{3.27}$$

which gives the globalness and boundedness of the solution of problem (1.1). □

4. General decay

We denote by $\xi(t) = \min\{\xi_1(t),\xi_2(t)\}$, $h(t) = \max\{h_1(t),h_2(t)\}$ and $G(t) = \min\{G_1(t),G_2(t)\}$.

Theorem 4.1. *Let $(u_0,u_1),(v_0,v_1) \in V \times W$. Suppose that* **(A1)–(A3)** *hold. Thus, the energy $E(t)$ satisfies*

$$E(t) \le \beta_2 G_0^{-1}\left(\beta_1 \int_{h^{-1}(r)}^t \xi(s)ds\right), \ \forall\, t > h^{-1}(r), \text{ with } G_0(t) = \int_t^r \frac{1}{sG'(s)}ds, \tag{4.1}$$

for some positive constants β_1 and β_2.

Remark 4.1. *([16])*

1. *We recall the Jensen's inequality: Assume F is a concave function on $[a,b]$, $f:\Omega\to[a,b]$ and g are in $L^1(\Omega)$, with $g(x)\ge 0$ and $\int_\Omega g(x)\,dx = m > 0$, then*

$$\frac{1}{m}\int_\Omega F[f(x)]g(x)\,dx \le F\Big[\frac{1}{m}\int_\Omega f(x)g(x)\,dx\Big].$$

2. *From* **(A2)**, *one has* $\lim_{t\to+\infty} h_i(t) = 0$. *Hence,* $\exists\ t_1 \geq 0$ *is large enough, verifying*

$$h_i(t_1) = r \Rightarrow h_i(t) \leq r, \quad \forall\ t \geq t_1. \tag{4.2}$$

One can easily check, for $i = 1, 2$, *that*

$$a_i \leq \xi_i(t) G_i(h_i(t)) \leq b_i,$$

for some constants $a_i > 0$ *and* $b_i > 0$. *This implies that*

$$h_i'(t) \leq -\xi_i(t) G_i(h_i(t)) \leq -\frac{a_i}{h_i(0)} h_i(0) \leq -\frac{a_i}{h_i(0)} h_i(t), \ \ \forall\ t \in [0, t_1]. \tag{4.3}$$

Proof of Theorem (4.1)**:** The proof is divided into three steps.

Step 1: In this step, we give estimates for the derivatives (with respect to t) of the functionals $\varphi(t)$ and $\psi(t)$ defined below by:

$$\varphi(t) = \varphi_1(t) + \varphi_2(t), \tag{4.4}$$

with

$$\varphi_1(t) = \frac{1}{\rho+1}\int_\Omega y|y_t|^\rho y_t\, dx + \int_\Omega \nabla y_t \nabla y\, dx,$$
$$\varphi_2(t) = \frac{1}{\rho+1}\int_\Omega z|z_t|^\rho z_t\, dx + \int_\Omega \nabla z_t \nabla z\, dx,$$

and

$$\psi(t) = \psi_1(t) + \psi_2(t), \tag{4.5}$$

with

$$\begin{aligned}\psi_1(t) &= -\frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t \int_0^t h_1(t-s)(y(t)-y(s))\, ds\, dx \\ &\quad -\int_\Omega \nabla y_t \int_0^t h_1(t-s)\nabla(y(t)-y(s))\, ds\, dx,\end{aligned} \tag{4.6}$$

$$\psi_2(t) = -\frac{1}{\rho+1}\int_\Omega |z_t|^\rho z_t \int_0^t h_2(t-s)(z(t)-z(s))\, ds\, dx - \int_\Omega \nabla z_t \int_0^t h_2(t-s)\nabla(z(t)-z(s))\, ds\, dx.$$

Lemma 4.1. *If* **(A1)**–**(A3)** *hold. The functional* $\varphi(t)$ *defined in (4.4) verifies, along the solution of* (1.1),

$$\begin{aligned}\varphi'(t) &\leq -\frac{l_1}{2} b(y,y) + \int_\Omega |\nabla y_t|^2\, dx + \frac{1}{\rho+1}\int_\Omega |y_t|^{\rho+2}\, dx + c(h_1 \square y)(t) \\ &- \frac{l_2}{2} b(z,z) + \int_\Omega |\nabla z_t|^2\, dx + \frac{1}{\rho+1}\int_\Omega |z_t|^{\rho+2}\, dx + c(h_2 \square z)(t) - \int_\Omega F(y,z)\, dx.\end{aligned} \tag{4.7}$$

Proof. We have $\varphi'(t) = \varphi_1'(t) + \varphi_2'(t)$. By using (1.1), we obtain

$$\begin{aligned}\varphi_1'(t) &= \int_\Omega |y_t|^\rho y_{tt} y\, dx + \frac{1}{\rho+1}\int_\Omega |y_t|^{\rho+2}\, dx + \int_\Omega |\nabla \boldsymbol{y_t}|^2\, dx + \int_\Omega \nabla \boldsymbol{y} \nabla \boldsymbol{y_{tt}}\, dx \\ &= -b(y,y) + \int_0^t h_1(t-s) b(y(s), y(t))\, ds + \int_\Omega |\nabla \boldsymbol{y_t}|^2\, dx + \frac{1}{\rho+1}\int_\Omega |y_t|^{\rho+2}\, dx \\ &\quad - \int_\Omega f_1(y,z) y\, dx. \end{aligned} \tag{4.8}$$

Since $\int_0^t h_1(s)ds \le \int_0^{+\infty} h_1(s)ds = 1 - l_1$, then, by the use of Cauchy-Schwarz's inequality and Young's inequality, we derive

$$\begin{aligned}&\int_0^t h_1(t-s) b(y(t), y(s))\, ds \\ &= \int_0^t h_1(t-s) b(y(s)-y(t), y(t))\, ds + \int_0^t h_1(t-s) b(y(t), y(t))\, ds \\ &\le \int_0^t h_1(t-s)\{b(y(s)-y(t), y(s)-y(t))\}^{\frac12}\{b(y(t),y(t))\}^{\frac12}\, ds + \left(\int_0^t h_1(s)ds\right) b(y(t),y(t)) \\ &\le \frac{l_1}{2} b(y(t),y(t)) + \frac{1}{2l_1}\left(\int_0^t \sqrt{h_1(t-s)}\Big\{h_1(t-s) b(y(s)-y(t), y(s)-y(t))\Big\}^{\frac12}\, ds\right)^2 \\ &\quad +(1-l_1) b(y(t),y(t)) \\ &\le \left(1 - \frac{l_1}{2}\right) b(y(t),y(t)) + c(h_1 \square y)(t). \end{aligned} \tag{4.9}$$

Inserting (4.9) in (4.8), we get that

$$\varphi_1'(t) \le -\frac{l_1}{2} b(y,y) + \int_\Omega |\nabla \boldsymbol{y_t}|^2\, dx + \frac{1}{\rho+1}\int_\Omega |y_t|^{\rho+2}\, dx + \frac{1-l_1}{2l_1}(h_1 \square y)(t) - \int_\Omega f_1(y,z) y\, dx.$$

Similarly, we infer that

$$\varphi_2'(t) \le -\frac{l_2}{2} b(z,z) + \int_\Omega |\nabla \boldsymbol{z_t}|^2\, dx + \frac{1}{\rho+1}\int_\Omega |z_t|^{\rho+2}\, dx + \frac{1-l_2}{2l_2}(h_2 \square z)(t) - \int_\Omega f_2(y,z) z\, dx.$$

Summing the last two inequalities, we get the desired inequality (4.7). □

Lemma 4.2. *If* **(A1)–(A3)** *hold. The functional defined in (4.6) verifies, for any* $0 < \delta < 1$ *and for all* $t \ge t_1$*, along the solution of* (1.1)*,*

$$\psi_1'(t) \le -\frac{h_0}{\rho+1}\int_\Omega |y_t|^{\rho+2} dx - \frac{h_0}{2}\|\nabla \boldsymbol{y_t}\|^2 + c\delta(b(y,y) + b(z,z)) + \frac{c}{\delta}(h_1 \square y)(t) - c(h_1' \square y)(t). \tag{4.10}$$

Here $h_0 = \min\left\{\int_0^t h_1(s)ds, \int_0^t h_2(s)ds\right\}$.

Proof. Differentiating $\psi_1(t)$ with respect to t and using $(1.1)_1$, we get

$$\begin{aligned}
\psi_1'(t) &= -\int_\Omega |y_t|^\rho y_{tt}\int_0^t h_1(t-s)(y(t)-y(s))\,ds\,dx - \frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t \int_0^t h_1'(t-s)(y(t)-y(s))ds\,dx \\
&- \frac{1}{\rho+1}\left(\int_0^t h_1(s)ds\right)\int_\Omega |y_t|^{\rho+2}dx - \int_\Omega \nabla \boldsymbol{y_{tt}}\int_0^t h_1(t-s)\nabla(\boldsymbol{y(t)}-\boldsymbol{y(s)})ds\,dx \\
&- \int_\Omega \nabla y_t\int_0^t h_1'(t-s)\nabla(\boldsymbol{y(t)}-\boldsymbol{y(s)})ds\,dx - \left(\int_0^t h_1(s)ds\right)\int_\Omega |\nabla \boldsymbol{y_t}|^2dx \\
&= \int_0^t h_1(t-s)b\,(y, y(t)-y(s))\,ds - \int_0^t h_1(t-\zeta)\int_0^t h_1(t-s)b\,(y(s), y(t)-y(\zeta))\,ds\,d\zeta \\
&+ \int_\Omega f_1(y,z)\int_0^t h_1(t-s)(y(t)-y(s))ds\,dx - \frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t\int_0^t h_1'(t-s)(y(t)-y(s))ds\,dx \\
&- \frac{1}{\rho+1}\left(\int_0^t h_1(s)ds\right)\int_\Omega |y_t|^{\rho+2}dx - \int_\Omega \nabla\boldsymbol{y_t}\int_0^t h_1'(t-s)\nabla(\boldsymbol{y(t)}-\boldsymbol{y(s)})ds\,dx \\
&- \left(\int_0^t h_1(s)ds\right)\int_\Omega |\nabla\boldsymbol{y_t}|^2dx. \qquad (4.11)
\end{aligned}$$

Now, we estimate the terms in the right-hand side of (4.11) as follows:

• **Estimation of the term** $\int_0^t h_1(t-s)b\,(y, y(t)-y(s))\,ds$.

Cauchy Schwarz's inequality and Young's inequality are used to get, for any $\delta > 0$,

$$\begin{aligned}
&\int_0^t h_1(t-s)b\,(y, y(t)-y(s))\,ds \\
&\le \int_0^t h_1(t-s)\,[b(y(t),y(t))]^{\frac12}\,[b\,(y(t)-y(s), y(t)-y(s))]^{\frac12}\,ds \\
&\le \delta b(y,y) + \frac{1}{4\delta}\left\{\int_0^t h_1(t-s)\,[b\,(y(t)-y(s), y(t)-y(s))]^{\frac12}\,ds\right\}^2 \\
&\le \delta b(y,y) + \frac{c}{\delta}(h_1\Box y)(t). \qquad (4.12)
\end{aligned}$$

• **Estimation of the term** $-\int_0^t h_1(t-\zeta)\int_0^t h_1(t-s)b\,(y(s), y(t)-y(\zeta))\,ds\,d\zeta$.

We have

$$\begin{aligned}
&- \int_0^t h_1(t-\zeta)\int_0^t h_1(t-s)b\,(y(s), y(t)-y(\zeta))\,ds\,d\zeta \\
&\le \int_0^t h_1(t-\zeta)\int_0^t h_1(t-s)b\,(y(s)-y(t), y(t)-y(\zeta))\,ds\,d\zeta + \int_0^t h_1(t-\zeta)\int_0^t h_1(t-s)b\,(y(t), y(t)-y(\zeta))\,ds\,d\zeta \\
&\le \int_0^t h_1(t-\zeta)\int_0^t h_1(t-s)\Big[\delta b\,(y(t)-y(s), y(t)-y(s)) + \frac{1}{4\delta}b\,(y(t)-y(\zeta), y(t)-y(\zeta))\Big]ds\,d\zeta \\
&+\left(\int_0^t h_1(\zeta)d\zeta\right)\int_0^t h_1(t-\zeta)b\,(y, y(t)-y(\zeta))\,d\zeta \\
&\le c\delta b(y,y) + c\left(\delta + \frac{1}{\delta}\right)(h_1\Box y)(t). \qquad (4.13)
\end{aligned}$$

• **Estimation of the term** $-\int_\Omega \nabla \boldsymbol{y_t} \int_0^t h_1'(t-s)(y(t)-y(s))dsdx.$

One has

$$
\begin{aligned}
-\int_\Omega \nabla \boldsymbol{y_t} \int_0^t h_1'(t-s)(y(t)-y(s))dsdx &\leq \delta_1 \int_\Omega |\nabla \boldsymbol{y_t}|^2 + \frac{1}{4\delta_1}\int_\Omega \left(\int_0^t h_1'(t-s)(y(t)-y(s))ds\right)^2 dx \\
&\leq \delta_1 \int_\Omega |\nabla \boldsymbol{y_t}|^2 - \frac{h_1(0)}{4\delta_1}\int_0^t h_1'(t-s)\int_\Omega |y(t)-y(s)|^2 dxds \\
&\leq \delta_1 \int_\Omega |\nabla \boldsymbol{y_t}|^2 - \frac{c}{\delta_1}(h_1'\Box y)(t). \qquad (4.14)
\end{aligned}
$$

• **Estimation of the term** $\int_\Omega f_1(y,z)\int_0^t h_1(t-s)(y(t)-y(s))ds\, dx.$

By using (2.7) and (3.27), we derive that

$$
\begin{aligned}
&\int_\Omega f_1(y,z)\int_0^t h_1(t-s)(y(t)-y(s))ds\, dx \\
\leq\ & c\delta \int_\Omega \left(|y|^2+|z|^2+|y|^{2\beta_{11}}+|z|^{2\beta_{12}}\right)dx + \frac{c}{\delta}\int_\Omega\left(\int_0^t h_1(t-s)(y(t)-y(s))ds\right)^2 dx \\
\leq\ & c\delta\left(b(y,y)+b(z,z)+(b(y,y))^{\beta_{11}}+(b(z,z))^{\beta_{12}}\right)+\frac{c}{\delta}(h_1\Box y)(t) \\
=\ & c\delta\left(b(y,y)+b(z,z)+b(y,y)(b(y,y))^{\beta_{11}-1}+b(z,z)(b(z,z))^{\beta_{12}-1}\right)+\frac{c}{\delta}(h_1\Box y)(t) \\
\leq\ & c\delta\left(b(y,y)+b(z,z)+b(y,y)\Big(\frac{2E(0)}{l_1}\Big)^{\beta_{11}-1}+b(z,z)\Big(\frac{2E(0)}{l_2}\Big)^{\beta_{12}-1}\right)+\frac{c}{\delta}(h_1\Box y)(t) \\
\leq\ & c\delta\left(b(y,y)+b(z,z)\right)+\frac{c}{\delta}(h_1\Box y)(t). \qquad (4.15)
\end{aligned}
$$

• **Estimation of the term** $-\frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t \int_0^t h_1'(t-s)(y(t)-y(s))ds\, dx.$

Using (3.27) again, we infer that

$$
\begin{aligned}
-\frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t \int_0^t h_1'(t-s)(y(t)-y(s))ds\, dx &\leq c\delta_1\int_\Omega |y_t|^{2(\rho+1)}dx - \frac{c}{\delta_1}(h_1'\Box y)(t) \\
&\leq c\delta_1\|\nabla \boldsymbol{y_t}\|^{2(\rho+1)} - \frac{c}{\delta_1}(h_1'\Box y)(t) \\
&\leq c\delta_1\big(2E(0)\big)^\rho\|\nabla \boldsymbol{y_t}\|^2 - \frac{c}{\delta_1}(h_1'\Box y)(t). \qquad (4.16)
\end{aligned}
$$

By combining (4.12)–(4.16), using the fact that $-\left(\int_0^t h_1(s)ds\right) \leq -h_0$ for all $t \geq t_1$ and choosing δ_1 small enough, we derive the estimate (4.10). □

Repeating the calculations above with $\psi_2(t)$ yields

$$
\psi_2'(t) \leq -\frac{h_0}{\rho+1}\int_\Omega |z_t|^{\rho+2}dx - \frac{h_0}{2}\|\nabla \boldsymbol{z_t}\|^2 + c\delta(b(y,y)+b(z,z)) + \frac{c}{\delta}(h_2\Box y)(t) - c(h_2'\Box z)(t). \qquad (4.17)
$$

Combining (4.10) and (4.17), we obtain the following result.

Corollary 4.1. *Assume that* **(A1)–(A3)** *hold. Then, the functional* ψ *satisfies, along the solution, the estimate*

$$\begin{aligned}\psi^{'}(t) \;\leq\; & -\frac{h_0}{\rho+1}\int_\Omega |y_t|^{\rho+2}dx - \frac{h_0}{2}\|\nabla \boldsymbol{y_t}\|^2 + c\delta b(y,y) + \frac{c}{\delta}(h_1\square y)(t) - c(h_1^{'}\square y)(t) \\ & - \frac{h_0}{\rho+1}\int_\Omega |z_t|^{\rho+2}dx - \frac{h_0}{2}\|\nabla \boldsymbol{z_t}\|^2 + c\delta b(z,z) + \frac{c}{\delta}(h_2\square z)(t) - c(h_2^{'}\square z)(t), \quad \forall\, t \geq t_1, \end{aligned} \tag{4.18}$$

for any $0 < \delta < 1$.

Step 2: The aim of this step is to establish the inequality (4.26).

Let's define the functional

$$F(t) = NE(t) + \varphi(t) + \frac{4}{h_0}\psi(t), \tag{4.19}$$

where $N > 0$. For N sufficiently large, one has that $F \sim E$, i. e.

$$c_1 E(t) \leq F(t) \leq c_2 E(t), \tag{4.20}$$

for some $c_1, c_2 > 0$.

Let $l = \min\{l_1, l_2\}$. By using (2.10), (4.7), (4.18), and taking $\delta = \frac{lh_0}{16c}$, we get for any $t \geq t_1$

$$\begin{aligned}F'(t) \;\leq\; & -\frac{l}{4}\left(b(y,y) + b(z,z)\right) - \frac{3}{\rho+1}\int_\Omega \left(|y_t|^{\rho+2} + |z_t|^{\rho+2}\right)dx - \|\nabla \boldsymbol{y_t}\|^2 - \|\nabla \boldsymbol{z_t}\|^2 - \int_\Omega F(y,z)dx \\ & + \left(c + \frac{64c^2}{lh_0^2}\right)\Big((h_1\square y)(t) + (h_2\square z)(t)\Big) + \left(\frac{N}{2} - \frac{4c}{h_0}\right)\Big((h_1^{'}\square y)(t) + (h_2^{'}\square z)(t)\Big).\end{aligned}$$

Taking N, such that

$$\frac{N}{2} - \frac{4c}{h_0} > 0,$$

to obtain that

$$\begin{aligned}F'(t) \;\leq\; & -\frac{l}{4}\left(b(y,y) + b(z,z)\right) - \frac{3}{\rho+1}\int_\Omega \left(|y_t|^{\rho+2} + |z_t|^{\rho+2}\right)dx - \|\nabla \boldsymbol{y_t}\|^2 - \|\nabla \boldsymbol{z_t}\|^2 - \int_\Omega F(y,z)dx \\ & + c\Big((h_1\square y)(t) + (h_2\square z)(t)\Big), \quad \forall\, t \geq t_1.\end{aligned} \tag{4.21}$$

By the virtue of (2.10) and (4.3), we infer that for any $t \geq t_1$,

$$\begin{aligned}&\int_0^{t_1} h_1(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \\ &\leq -\frac{h_1(0)}{a_1}\int_0^{t_1} h_1'(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \leq -cE'(t),\end{aligned}$$

and similarly

$$\int_0^{t_1} h_2(s)b(z(t)-z(t-s), z(t)-z(t-s))ds \leq -cE'(t).$$

Hence, (4.21) becomes

$$\begin{aligned} F'(t) &\le -\alpha E(t) + c(h_1\Box y)(t) + c(h_2\Box z)(t) \\ &\le -\alpha E(t) - cE'(t) + c\int_{t_1}^{t} h_1(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \\ &\quad +c\int_{t_1}^{t} h_2(s)b(z(t)-z(t-s), z(t)-z(t-s))ds, \quad \forall\ t \ge t_1, \end{aligned} \tag{4.22}$$

where $\alpha > 0$. Define $\mathcal{H}(t) = F(t) + cE(t)$. It is easy to see that $\mathcal{H}(t) \sim E(t)$. Using (4.22), we get

$$\begin{aligned} \mathcal{H}'(t) &\le -\alpha E(t) + c\int_{t_1}^{t} h_1(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \\ &\quad +c\int_{t_1}^{t} h_2(s)b(z(t)-z(t-s), z(t)-z(t-s))ds. \end{aligned} \tag{4.23}$$

The following two situations are then distinguished.

First Case: $G_1(t)$ and $G_2(t)$ are linear.
By multiplying (4.23) by $\xi(t)$ and using **(A2)** and (2.10) to obtain

$$\begin{aligned} \xi(t)\mathcal{H}'(t) &\le -\alpha\xi(t)E(t) + c\xi(t)\int_{t_1}^{t} h_1(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \\ &\quad +c\xi(t)\int_{t_1}^{t} h_2(s)b(z(t)-z(t-s), z(t)-z(t-s))ds \\ &\le -\alpha\xi(t)E(t) + c\int_{t_1}^{t} \xi_1(s)h_1(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \\ &\quad +c\int_{t_1}^{t} \xi_2(s)h_2(s)b(z(t)-z(t-s), z(t)-z(t-s))ds \\ &\le -\alpha\xi(t)E(t) - c\int_{t_1}^{t} h_1'(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \\ &\quad -c\int_{t_1}^{t} h_2'(s)b(z(t)-z(t-s), z(t)-z(t-s))ds \\ &\le -\alpha\xi(t)E(t) - cE'(t). \end{aligned} \tag{4.24}$$

Since ξ is non-increasing, then by using (4.24), the functional $\mathcal{F}(t) = \xi(t)\mathcal{H}(t) + cE(t)$ satisfies for any $t \ge t_1$,

$$\mathcal{F}'(t) \le -\alpha\xi(t)E(t).$$

It is obvious that $\mathcal{F} \sim E$, and then we get the existence of some positive constant m_1, such that

$$\mathcal{F}'(t) \le -m_1\xi(t)\mathcal{F}(t).$$

By applying Gronwall's Lemma, there exists a constant $m_2 > 0$, such that

$$\mathcal{F}(t) \le m_2 e^{-m_1\int_{t_1}^{t} \xi(s)\, ds},$$

and then we have

$$E(t) \leq m_3 e^{-m_1 \int_{t_1}^{t} \xi(s)\, ds},$$

where $m_3 > 0$.

Second Case: $G_1(t)$ or $G_2(t)$ is nonlinear. Defining J_1 and J_2 by

$$J_1(t) = \frac{\lambda}{t} \int_0^t b(y(t) - y(t-s), y(t) - y(t-s))ds, \quad t > 0,$$

and

$$J_2(t) = \frac{\lambda}{t} \int_0^t b(z(t) - z(t-s), z(t) - z(t-s))ds, \quad t > 0.$$

Since $b(y(t), y(t)) + b(y(t-s), y(t-s)) \leq \frac{2}{l}(E(t) + E(t-s)) \leq \frac{4}{l}E(0)$, for all $0 < s < t$, we infer that

$$J_1(t) \leq \frac{8\lambda}{lt} \int_0^t E(0)ds = \frac{8\lambda}{l} E(0) < +\infty,$$

and similarly

$$J_2(t) \leq \frac{8\lambda}{l} E(0) < +\infty.$$

By taking $0 < \lambda < 1$ sufficiently small, we get, for all $t > 0$,

$$J_1(t) < 1 \text{ and } J_2(t) < 1. \tag{4.25}$$

Now, defining $K_1(t)$ and $K_2(t)$ by

$$K_1(t) = -\int_0^t h_1'(s) b(y(t) - y(t-s), y(t) - y(t-s))ds,$$

and

$$K_2(t) = -\int_0^t h_2'(s) b(z(t) - z(t-s), z(t) - z(t-s))ds.$$

One can easily check that $K_i(t) \leq -cE'(t)$, for $i = 1, 2$.

Given that $G_1(0) = 0$ and the strict convexity of G_1 on $(0, r]$, one has then $G_1(\kappa x) \leq \kappa G_1(x)$, $\forall\ 0 \leq \kappa \leq 1$ and $x \in (0, r]$. Now, using **(A1)**, (4.25) and Jensen's inequality, we obtain

$$\begin{aligned}
K_1(t) &= \frac{1}{\lambda J_1(t)} \int_0^t J_1(t)(-h_1'(s))\lambda b(y(t) - y(t-s), y(t) - y(t-s))ds \\
&\geq \frac{1}{\lambda J_1(t)} \int_0^t J_1(t)\xi_1(s) G_1(h_1(s))\lambda b(y(t) - y(t-s), y(t) - y(t-s))ds \\
&\geq \frac{\xi_1(t)}{\lambda J_1(t)} \int_0^t G_1(J_1(t)h_1(s))\lambda b(y(t) - y(t-s), y(t) - y(t-s))ds \\
&\geq \frac{\xi_1(t)}{\lambda} G_1\left(\frac{1}{J_1(t)} \int_0^t J_1(t)h_1(s)\lambda b(y(t) - y(t-s), y(t) - y(t-s))ds\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\xi_1(t)}{\lambda} G_1\left(\lambda \int_0^t h_1(s) b(y(t)-y(t-s), y(t)-y(t-s)) ds\right)\\
&= \frac{\xi_1(t)}{\lambda} \overline{G}_1\left(\lambda \int_0^t h_1(s) b(y(t)-y(t-s), y(t)-y(t-s)) ds\right).
\end{aligned}
$$

Note that $\overline{G}_1$ is an extension of G_1, satisfying $\overline{G}_1$ as strictly convex and strictly increasing on $(0, +\infty)$. Thus, we have

$$
\int_0^t h_1(s) b(y(t)-y(t-s), y(t)-y(t-s)) ds \leq \frac{1}{\lambda} \overline{G}_1^{-1}\left(\frac{\lambda K_1(t)}{\xi_1(t)}\right).
$$

Similarly, we have

$$
\int_0^t h_2(s) b(z(t)-z(t-s), z(t)-z(t-s)) ds \leq \frac{1}{\lambda} \overline{G}_2^{-1}\left(\frac{\lambda K_2(t)}{\xi_2(t)}\right),
$$

where $\overline{G}_2$ is an extension of G_2.

We infer from (4.23) that

$$
\mathcal{H}'(t) \leq -\alpha E(t) + c\overline{G}_1^{-1}\left(\frac{\lambda K_1(t)}{\xi_1(t)}\right) + c\overline{G}_2^{-1}\left(\frac{\lambda K_2(t)}{\xi_2(t)}\right), \quad \forall\ t \geq t_1. \tag{4.26}
$$

Step 3: Here, we shall prove the desired inequality (4.1).

We set $G = \min\{\overline{G}_1, \overline{G}_2\}$. For $\varepsilon_0 < r$, using (4.26) and since $E' \leq 0$, $\overline{G}_i' > 0$, $\overline{G}_i'' > 0$, $i = 1, 2$, we claim that the functional $\mathcal{G}$, defined by

$$
\mathcal{G}(t) = G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{H}(t) + E(t),
$$

is equivalent to $E(t)$ and satisfies

$$
\begin{aligned}
\mathcal{G}'(t) &= E'(t) + \varepsilon_0 \frac{E'(t)}{E(0)} G''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{H}(t) + G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{H}'(t)\\
&\leq -\alpha E(t) G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cG'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\overline{G}_1^{-1}\left(\frac{\lambda K_1(t)}{\xi_1(t)}\right)\\
&\quad + cG'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\overline{G}_2^{-1}\left(\frac{\lambda K_2(t)}{\xi_2(t)}\right).
\end{aligned} \tag{4.27}
$$

The convex conjugate of G in the Young's sense (see [25]) is denoted by G^* and satisfies

$$
G^*(t) = t(G')^{-1}(t) - G((G')^{-1}(t)). \tag{4.28}
$$

The following inequality holds true:

$$
AB_i \leq G^*(A) + G(B_i), \ \ i = 1, 2, \tag{4.29}
$$

with $A = G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B_i = \overline{G}_i^{-1}\left(\frac{\lambda K_i(t)}{\xi_i(t)}\right),\ i = 1, 2.$
Using (4.27), (4.28) and (4.29), we obtain

$$\mathcal{G}'(t) \leq -\alpha E(t) G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)} G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\lambda\left(\frac{K_1(t)}{\xi_1(t)} + \frac{K_2(t)}{\xi_2(t)}\right).$$

Since $K_i(t) \leq -cE'(t)$ (for $i = 1, 2$), we infer that

$$\xi(t)\mathcal{G}'(t) \leq -\alpha E(t)\xi(t) G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)} \xi(t) G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t). \tag{4.30}$$

Consequently, letting $\mathcal{G}_1 = \xi\mathcal{G} + cE$, we have: $\alpha_1 \mathcal{G}_1(t) \leq E(t) \leq \alpha_2 \mathcal{G}_1(t)$, for some $\alpha_1, \alpha_2 > 0$.

Thus, we get

$$\mathcal{G}_1'(t) \leq -\beta_1 \xi(t) \frac{E(t)}{E(0)} G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) := -\beta_1 \xi(t) \mathcal{G}_2\left(\frac{E(t)}{E(0)}\right),\ \forall\, t \geq t_1, \tag{4.31}$$

where $\beta_1 > 0$ and $\mathcal{G}_2(t) = tG'(\varepsilon_0 t)$. Since $\mathcal{G}_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 t G''(\varepsilon_0 t)$, then using the strict convexity of $G_i (i = 1, 2)$ on $(0, r]$, we have $\mathcal{G}_2'(t), \mathcal{G}_2(t) > 0$ on $(0, 1]$. Since $\mathcal{G}_1 \sim E$ and using (4.31), one derives that

$$\mathcal{R}(t) \sim E(t),\ \text{where } \mathcal{R}(t) = \frac{\alpha_1 \mathcal{G}_1(t)}{E(0)}, \tag{4.32}$$

and

$$\mathcal{R}'(t) \leq -\beta_2 \xi(t) \mathcal{G}_2(\mathcal{R}(t)), \forall\, t \geq t_1,$$

with $\beta_2 > 0$. Integrating the last inequality over (t_1, t) yields

$$\int_{t_1}^{t} \frac{-\mathcal{R}'(s)}{\mathcal{G}_2(\mathcal{R}(s))} ds \geq \beta_2 \int_{t_1}^{t} \xi(s) ds \Rightarrow \int_{\varepsilon_0 \mathcal{R}(t)}^{\varepsilon_0 \mathcal{R}(t_1)} \frac{1}{sG'(s)} ds \geq \beta_2 \int_{t_1}^{t} \xi(s) ds.$$

Now, the function G_0 defined by $G_0(t) = \displaystyle\int_t^r \frac{1}{sG'(s)} ds$, is strictly decreasing on $(0, r]$ and satisfies $\lim\limits_{t \to 0} G_0(t) = +\infty$. Thus, we deduce that

$$\mathcal{R}(t) \leq \frac{1}{\varepsilon_0} G_0^{-1}\left(\beta_1 \int_{t_1}^{t} \xi(s) ds\right).$$

This inequality together with (4.32) yields to (4.1). This ends the proof of Theorem (4.1). □

5. Examples

In this section, we give two examples that illustrate explicit formulas for the decay rates of the energy.

1. Let $h_1(t) = h_2(t) = pe^{-k(1+t)^q}$, $t \geq 0$, where $p > 0$, $0 < q \leq 1$ and $p > 0$ is chosen so that h_i satisfies (2.4). We can see, for $i = 1, 2$, that

$$h_i'(t) = -pqk(1+t)^{q-1}e^{-k(1+t)^q} = -\xi_i(t)G_i(h_i(t)),$$

where $\xi_i(t) = qk(1+t)^{q-1}$ and $G_i(t) = t$. From (4.1), it holds that

$$E(t) \leq \beta_2 e^{-\beta_1 k(1+t)^q}, \ \forall\, t \geq 0.$$

2. Let $h_i(t) = \frac{p_i}{(1+t)^{q_i}}$, $i = 1, 2$, where $q_i > 0$ and $p_i > 0$ is chosen such that, (2.4) holds true. One has, for $i = 1, 2$,

$$h_i'(t) = \frac{-p_i q_i}{(1+t)^{q_i+1}} = -\frac{q_i}{p_i^{\frac{1}{q_i}}}\left(\frac{p_i}{(1+t)^{q_i}}\right)^{\frac{q_i+1}{q_i}} = -\xi_i(t)G_i(h_i(t)),$$

where $\xi_i(t) = \frac{q_i}{p_i^{\frac{1}{q_i}}}$ and $G_i(t) = t^{\frac{q_i+1}{q_i}}$.

Putting $q_3 = \min\{q_1, q_2\}$. Therefore, it follows from (4.1) that

$$E(t) \leq \frac{c}{(1+t)^{q_3}}, \ \forall\, t \geq 0.$$

6. Conclusions

This paper focuses on the existence and the asymptotic stability of solutions for a system of two coupled quasi-linear Kirchhoff plate equations in a bounded domain of $\mathbb{R}^2$, subject only to viscoelasticity dissipative terms and with the presence of rotational forces and source terms. Each one of these two equations describes the motion of a plate, which is clamped along one portion of its boundary and has free vibrations on the other portion of the boundary. This work is motivated by previous results concerning coupled quasi-linear wave equations [14–16] and single quasi-linear plate equation [12, 13].

As future works, we can change the type of damping by considering, for example, weak damping (of the form y_t), Balakrishnan-Taylor damping (of the form $(\nabla y, \nabla y_t)\Delta y$) or strong damping (of the form $\Delta^2 y_t$).

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Conflict of interest

The author declares that there are no conflicts of interest.

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