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Research article

Existence and multiplicity of positive solutions for one-dimensional p-Laplacian problem with sign-changing weight

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Abstract: In this paper, we show the positive solutions set for one-dimensional p-Laplacian problem with sign-changing weight contains a reversed S-shaped continuum. By figuring the shape of unbounded continuum of positive solutions, we identify the interval of bifurcation parameter in which the p-Laplacian problem has one or two or three positive solutions according to the asymptotic behavior of nonlinear term at 0 and ∞. The proof of the main result is based upon bifurcation technique.

Keywords: p-Laplacian problem; positive solution; multiplicity; bifurcation; sign-changing weight

1. Introduction

Consider the following one-dimensional p-Laplacian problem

$$\begin{cases}(\varphi_p(u'))' + \lambda m(x)f(u) = 0, & x \in (0,1),\\ u(0) = u(1) = 0,\end{cases} \tag{1.1}$$

where $\varphi_p(s) := |s|^{p-2}s$, $p > 1$, $\lambda > 0$ is a parameter, $f \in C(\mathbb{R},\mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$ and $m \in C[0,1]$ changes sign.

Notice that (1.1) is the one-dimensional version of the Dirichlet problem associated with the p-Laplacian equation

$$\begin{cases}\mathrm{div}(\varphi_p(\nabla u)) + \lambda m(x)f(u) = 0, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where $\lambda > 0$ is a parameter, $m \in C(\bar{\Omega})$, $f \in C(\mathbb{R},\mathbb{R})$, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$.

Recently, the p-Laplacian problems with sign-definite weight have been studied by many authors. For example, Del Pino et al. [1] established the global bifurcation theorem for problem (1.1) in the case $m \equiv 1$. When $m(x) \geq 0$ and m may be singular at $x = 0$ and/or $x = 1$, Lee et al. [2, 3] obtained many different types of global existence results of positive solutions for problem (1.1). Dai et al. [4, 5] established a Dancer-type unilateral global bifurcation result for one-dimensional p-Laplacian problem (1.1).

On the other hand, in high-dimensional case, Del Pino and Manásevich [6] studied the global behavior of continuum of positive solutions for problem (1.2) on the general domain of $\mathbb{R}^N$. In 2022, Ye and Han [7] studied the global structure of problem (1.2) by using bifurcation technique.

To the authors' best knowledge, the study of solutions for p-Laplacian problems with sign-changing weight can be traced back to Drábek and Huang [8]. In 1997, the authors proved a Dancer-type bifurcation result for problem (1.2) with sign-changing weight. Since then, there have been many studies on problems (1.1) and (1.2) with sign-changing weight, see [9–12]. For example, Ma, Liu and Xu [9] in 2013 used bifurcation technique to show that (1.1) has a nodal solution, where the weight function m changes sign. In 2014, Dai [10] also proved a Dancer-type unilateral global bifurcation result for problem (1.2) with sign-changing weight m on the unit ball of $\mathbb{R}^N$. In 2015, Sim and Tanaka [11] proved in their Theorem 1.1 that the solution set of problem (1.1) with m changes sign has an S-shaped continuum. Here m satisfies

(F1) there exist $x_1, x_2 \in [0, 1]$ such that $x_1 < x_2$, $m(x) > 0$ on (x_1, x_2), and $m(x) \leq 0$ on $[0, 1] \setminus [x_1, x_2]$.

As applications of this bifurcation result, they determined the intervals of the parameter λ in which the problem (1.1) has one, two, or three positive solutions. In 2019, Chen and Ma [12] extended [11, Theorem 1.1] to the radial problem

$$\begin{cases} (r^{N-1}\varphi_p(u'))' + \lambda r^{N-1} m(r) f(u) = 0, \\ u'(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter, $f \in C([0, \infty), [0, \infty))$, $f(0) = 0$, $f(s) > 0$ for $s > 0$, and m is a sign-changing function satisfying $H(B) = \{m \in C(\bar{B})$ is radially symmetric$|m(x) > 0, x \in \Omega_1$ and $m(x) \leq 0, x \in \bar{B}\backslash\Omega_1\}$ with the annular domain $\Omega_1 = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\} \subset B$ for some $0 < r_1 < r_2 < 1$.

Note that the solution of (1.3) is the radially symmetric solutions of the N-dimensional Dirichlet problem (1.2), where $\Omega = B$ is the unit ball of $\mathbb{R}^N, N \geq 2$.

It is worth remarking that [11, 12] only studied the case of $f_0 \in (0, \infty)$ and $f_\infty \in (0, \infty)$, which means that there is a constant $C > 0$ such that $f(s) \leq Cs^{p-1}$ for all $s \geq 0$, where $f_0 := \lim\limits_{s\to 0} \frac{f(s)}{s^{p-1}}$, $f_\infty := \lim\limits_{s\to\infty} \frac{f(s)}{s^{p-1}}$. However, if f is superlinear near ∞ (i.e., $f_\infty = \lim\limits_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$), similar results have not been studied. One possible reason is that in this case we cannot use standard bifurcation techniques by linearization. Another reason is that we cannot prove the direction of bifurcation using the method in [11].

Naturally, a new question is, can we study the existence and multiplicity of positive solutions to problem (1.1) when $f_0 \in (0, \infty)$, $f_\infty = \infty$ and the weight function m changes sign?

Motivated by the interesting studies of [11, 12] and some earlier works in the literature, in present paper, we prove that an unbounded subcontinuum of positive solutions of (1.1) bifurcates from the trivial solution and grows to the left from the initial point, to the right at some point and to the left near $\lambda = 0$. Roughly speaking, we obtain that there exists a *reversed S-shaped continuum* in the positive solution set of problem (1.1).

Throughout this paper, we assume that

(H1) $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$;

(H2) $m \in C[0,1]$ changes sign and meas$\{x \in [0,1] | m(x) = 0\} = 0$;
(H3) there exist constants $\alpha > 0$, $f_0 > 0$, and $f_1 > 0$ such that

$$\lim_{s \to 0^+} \frac{f(s) - f_0 s^{p-1}}{s^{p-1+\alpha}} = f_1;$$

(H4) $f_\infty = \lim_{s \to \infty} \frac{f(s)}{s^{p-1}} = \infty$;
(H5) there exists $s_0 > 0$ such that $0 \le s \le s_0$ implies that

$$f(s) \le \frac{f_0}{\mu_1 \int_0^1 |m(x)| dx} s_0^{p-1},$$

where μ_1 is the first eigenvalue for the following linear eigenvalue problem

$$\begin{cases} (\varphi_p(u'(x)))' + \mu m(x) \varphi_p(u(x)) = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.4}$$

It is well-known that μ_1 is simple, isolated and the associated eigenfunction ϕ_1 has fixed sign in $[0,1)$ (see for example [11, 13]).

Arguing the shape of bifurcation, we have the following main result:

Theorem 1.1 (see Figure 1). Assume that (H1)–(H5) hold. Then there exist $\lambda_* \in (0, \frac{\mu_1}{f_0})$ and $\lambda^* > \frac{\mu_1}{f_0}$ such that
(i) (1.1) has at least one positive solution if $0 < \lambda < \lambda_*$;
(ii) (1.1) has at least two positive solutions if $\lambda = \lambda_*$;
(iii) (1.1) has at least three positive solutions if $\lambda_* < \lambda < \mu_1/f_0$;
(iv) (1.1) has at least two positive solutions if $\mu_1/f_0 < \lambda \le \lambda^*$;
(v) (1.1) has at least one positive solution if $\lambda = \lambda^*$;
(vi) (1.1) has no positive solution if $\lambda > \lambda^*$.

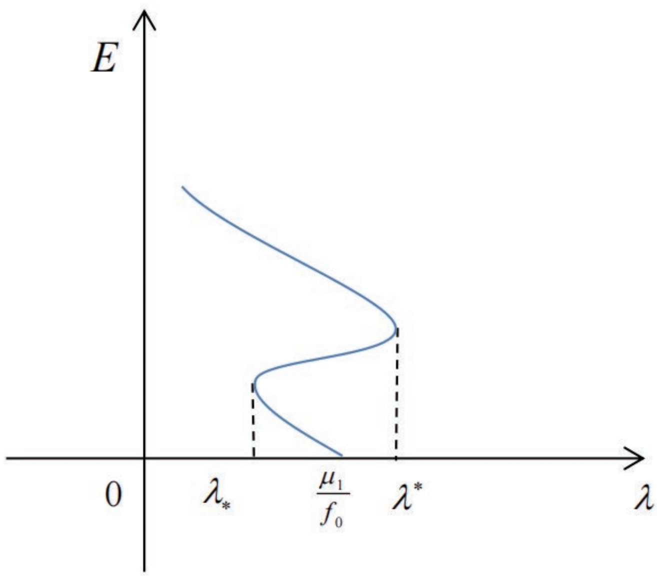

Figure 1. reversed S-shaped continuum.

Remark 1.1. Let us consider the function

$$f(s) = ks^{p-1} + s^{p-1}\ln(1+s),\ k > 0,\ s \in [0,\infty).$$

Obviously, f satisfies (H3) and (H4) with $\alpha = 1, f_0 = k, f_1 = 1$. It is easy to see that if $k > 0$ is sufficiently large, then the function f satisfies (H5).

Remark 1.2. We note that, in [11], condition (F1) is the key condition to obtain the *S-shaped continuum* in the positive solutions set of problem (1.1). In other words, if the condition (F1) is replaced by (H1), the method in [11] cannot prove whether there is *S-shaped continuum* in the solution set of problem (1.1), even if $f_0, f_\infty \in (0,+\infty)$.

Remark 1.3. Recently, many scholars have studied other problems with p-Laplacian operator, such as about characterization of solutions, see [14–17]; Extensions of a p-Laplacian operator to higher order operator, see [18, 19]; Aplication of p-Laplacian operator to a physical phenomena, see [20].

The rest of this paper is arranged as follows. In Section 2, we show global bifurcation phenomena from the trivial branch. Section 3 is devoted to showing that there are at least two direction turns of the continuum and completing the proof of Theorem 1.1.

2. Existence of unbounded continuum

Let $X = \{u \in C[0,1] : u(0) = u(1) = 0\}$ with the norm $\|u\|_\infty = \sup_{x\in[0,1]} u(x)$. Let E be the Banach space $C_0^1[0,1]$ with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$. Let $Y = L^1(0,1)$ with its usual normal $\|\cdot\|_{L^1}$.

Consider the following boundary value problem

$$\begin{cases}(\varphi_p(u'(x)))' + h(x) = 0, & x \in (0,1),\\ u(0) = u(1) = 0,\end{cases} \tag{2.1}$$

where $h \in L^1(0,1)$. Problem (2.1) is equivalently

$$u(x) = G_p(h)(x) := \int_0^x \varphi_p^{-1}(a(h) + \int_0^s h(\tau)d\tau)ds,$$

where $a : Y \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 \varphi_p^{-1}(a(h) + \int_0^s h(\tau)d\tau)ds = 0.$$

It is known that $G_p : Y \to E$ is continuous and maps equi-integrable sets of E into relatively compacts of E (see [2]).

Lemma 2.1 ([11, Lemma 2.1]). Assume that (H1)–(H3) hold. Then there exists an unbounded subcontinuum C of the set of solution of problem (1.1) in $\mathbb{R}\times X$ bifurcating from $(\mu_1/f_0, 0)$ such that

$$C \subseteq \Big((\mathbb{R}^+ \times \mathrm{int}P) \cup \{(\frac{\mu_1}{f_0}, 0)\}\Big).$$

Here $P = \{u \in C[0,1] : u(t) \geq 0\}$ is the positive cone in X.

Lemma 2.2. Assume that (H1) and (H2) hold. Let u be a positive solution of (1.1). If there is a constant $f_* \in (0,\infty)$ such that $0 \le f(s) \le f_* s^{p-1}$. Then

$$|u'(x)| \le \lambda^{\frac{1}{p-1}}\Big(f_* \int_0^1 |m(x)|dx\Big)^{\frac{1}{p-1}} \|u\|_\infty,\ x \in [0,1].$$

Proof. By Rolle's theorem, there exists $\tau \in (0,1)$ such that $u'(\tau) = 0$. Integrating the equation of (1.1) over $[x,\tau]$, we have

$$\varphi_p(u'(x)) = \lambda \int_x^\tau m(t) f(u(t))dt,\quad x \in [0,1].$$

By (H1) and (H2), we get

$$|u'(x)|^{p-1} = \lambda\Big|\int_x^\tau m(t)f(u(t))dt\Big| \le \lambda f_* \|u\|_\infty^{p-1}\Big|\int_x^\tau |m(t)|dt\Big| \le \lambda f_* \|u\|_\infty^{p-1}\int_0^1 |m(x)|dx,\ x \in [0,1].$$

The proof is complete. □

Lemma 2.3 ([9, Lemma 10]). Let (H2) holds. Let $I = [a,b]$ be such that $I \subset I^+ := \{x \in [0,1] | m(x) > 0\}$ and meas $I > 0$. Let $g_n : [0,1] \to (0,\infty)$ be such that

$$\lim_{n\to\infty} g_n(x) = \infty, \text{ uniformly on } I.$$

Let $y_n \in E$ be a solution of the equation

$$(\varphi_p(y_n'))' + m(x)g_n(x)\varphi_p(y_n) = 0,\ x \in (0,1).$$

Then the number of zeros of $y_n|_I$ goes to infinity as $n \to \infty$.

Lemma 2.4. Assume that (H1)–(H3) hold. If $f_0 \in (0,\infty)$ and $f_\infty \in (0,\infty)$, then for any $\lambda \in (\mu_1/f_\infty, \mu_1/f_0) \cup (\mu_1/f_0, \mu_1/f_\infty)$, problem (1.1) has one solution u such that u is positive on $(0,1)$.

Proof. The arguments are quite similar to those from the proof of Theorem 11 in [9]. However, for the sake of completeness, we give a sketch of the proof below.

Let $\zeta \in C(\mathbb{R})$ such that $f(s) = f_0\varphi_p(s) + \zeta(s)$ with $\lim_{s\to 0}\zeta(s)/\varphi_p(s) = 0$. By Lemma 2.1, there exists an unbounded subcontinuum C, such that

$$C \subseteq \Big((\mathbb{R}^+ \times \text{int}P) \cup \{(\frac{\mu_1}{f_0}, 0)\}\Big).$$

Let $(\lambda_n, u_n) \in C$ satisfy $\lambda_n + \|u_n\| \to \infty$. We note that $\lambda_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of problem (1.1) for $\lambda = 0$ and $C \cap (\{0\} \times E) = \emptyset$.

We divide the rest proofs into two steps.

Step 1. We show that there exists a constant M such that $\lambda_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim_{n\to\infty}\lambda_n = \infty$. Note that

$$-(\varphi_p(u_n'(x)))' = \lambda_n m \tilde{f}_n(x)\varphi_p(u_n(x)),$$

where

$$\tilde{f}_n(x) = \begin{cases} \dfrac{f(u_n)}{\varphi_p(u_n)}, & \text{if } u_n \neq 0,\\ f_0, & \text{if } u_n = 0.\end{cases}$$

Conditions (H1), (H3) and the fact that $f_\infty \in (0,\infty)$ imply that there exists a positive constant ρ such that $\tilde{f}_n(x) \geq \rho$ for any $x \in (0,1)$. By Lemma 2.3, we get that u_n must change its sign in $(0,1)$ for n large enough, and this contradicts the fact that $u_n \in C$.

Step 2. We show that C joins $(\mu_1/f_0, 0)$ to $(\mu_1/f_\infty, \infty)$

It follows from Step 1 that $\|u_n\| \to \infty$. Let $\xi \in C(\mathbb{R})$ be such that $f(s) = f_\infty \varphi_p(s) + \xi(s)$. Then $\lim_{s\to\infty} \xi(s)/\varphi_p(s) = 0$. Let $\tilde{\xi}(u) = \max_{u\leq s\leq 2u} |\xi(s)|$. Then $\tilde{\xi}$ is nondecreasing and

$$\lim_{u\to\infty} \frac{\tilde{\xi}(u)}{\varphi_p(u)} = 0. \tag{2.2}$$

We divide the equation

$$-(\varphi_p(u_n'(x)))' = \lambda_n m(x)(f_\infty \varphi_p(u_n(x)) + \xi(u_n))$$

by $\|u_n\|$ and set $\bar{u}_n = u_n/\|u_n\|$. Since $\bar{u}_n$ is bounded in E, after taking a subsequence if necessary, we have $\bar{u}_n \rightharpoonup \bar{u}$ for some $\bar{u} \in E$ and $\bar{u}_n \to \bar{u}$ in Y with $\|\bar{u}\| = 1$. Moreover, from (2.2) and the fact that $\tilde{\xi}$ is nondecreasing, we have

$$\lim_{n\to\infty} \frac{\xi(u_n(x))}{\|u_n\|^{p-1}} = 0,$$

since

$$\frac{\xi(u_n(x))}{\|u_n\|^{p-1}} \leq \frac{\tilde{\xi}(|u_n(x)|)}{\|u_n\|^{p-1}} \leq \frac{\tilde{\xi}(\|u_n(x)\|)}{\|u_n\|^{p-1}}.$$

By the continuity and compactness of G_p, it follows that

$$\bar{u} = G_p(\bar{\mu} m(x) f_\infty \varphi_p(\bar{u})),$$

where $\bar{\mu} = \lim_{n\to\infty} \lambda_n$, again choosing a subsequence and relabeling if necessary.

It is clear that $\|\bar{u}\| = 1$ and $\bar{u} \in C$ since C is closed in $\mathbb{R} \times E$. Thus, $\bar{\mu} f_\infty = \mu_1$, i.e., $\bar{\mu} = \mu_1/f_\infty$. Therefore, C joins $(\mu_1/f_0, 0)$ to $(\mu_1/f_\infty, \infty)$. □

Lemma 2.5. ([21]) Let X be a Banach space and let C_n be a family of closed connected subsets of X. Assume that:

(i) there exist $z_n \in C_n$, $n = 1, 2, \cdots$, and $z^* \in X$, such that $z_n \to z^*$;

(ii) $r_n = \sup\{\|x\| | x \in C_n\} = \infty$;

(iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap \bar{B}_R(0)$ is a relatively compact set of X.

Then $D := \limsup_{n\to\infty} C_n$ is unbounded, closed and connected.

Lemma 2.6. If $f_0 \in (0,\infty)$ and $f_\infty = \infty$, then the unbounded subcontinuum C of positive solutions for (1.1) joins $(\frac{\mu_1}{f_0}, 0)$ to $(0,\infty)$.

Proof. Note that Lemma 2.6 cannot be proved using standard bifurcation techniques by linearization. To overcome this difficulty we shall employ a limiting procedure. Let us define a function f_n as the following

$$f_n(s) = \begin{cases} f(s), & s \in [-n, n], \\ \dfrac{n\varphi_p(2n) - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ \dfrac{n\varphi_p(2n) + f(-n)}{n}(s+n) + f(-n), & s \in [-2n, -n), \\ n\varphi_p(s), & s \in (-\infty, -2n] \cup [2n, \infty). \end{cases}$$

Next, we consider the following problem

$$\begin{cases}(\varphi_p(u'(x)))' + \lambda m(x) f_n(u(x)) = 0, & x \in (0,1),\\ u(0) = u(1) = 0.\end{cases} \tag{2.3}$$

Clearly, $\lim_{n\to\infty} f_n(s) = f(s)$, $(f_n)_0 = f_0$ and $(f_n)_\infty = n$. Lemma 2.1 implies that there exists a sequence of unbounded continua C_n of solutions to problem (2.3) emanating from $(\frac{\mu_1}{f_0}, 0)$ and joining to $(\frac{\mu_1}{n}, \infty)$.

By Lemma 2.5, there exists an unbounded component C of $\limsup_{n\to\infty} C_n$ such that $(\frac{\mu_1}{f_0}, 0) \in C$ and $(0, \infty) \in C$. This completes the proof. □

3. Direction turn of bifurcation

In this section, we show that there are at least two direction turns of the continuum under conditions (H3) and (H5), and accordingly we finish the proof of Theorem 1.1.

Lemma 3.1. Assume that (H1)–(H3) hold. Let $\{(\lambda_n, u_n)\}$ be a sequence of positive solutions of (1.1) which satisfies $\|u_n\| \to 0$ and $\lambda_n \to \frac{\mu_1}{f_0}$. Let $\phi_1(x)$ be the eigenfunction of (1.4) which satisfies $\|\phi_1\| = 1$. Then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|}$ converges uniformly to ϕ_1 on $[0, 1]$.

Proof. As the proof is very similar to that in [11, Lemma 2.3], we omit it.

Lemma 3.2. Assume that (H2) holds. Let $\alpha > 0$ and let $\phi_1 > 0$ be a first eigenfunction of (1.4). Then

$$\int_0^1 m(x)[\phi_1(x)]^{p+\alpha} dx > 0.$$

Proof. Multiplying the equation of (1.4) by $\phi_1^{\alpha+1}$ and integrating it over $[0, 1]$, we obtain

$$\begin{aligned}\mu_1 \int_0^1 m(x)[\phi_1(x)]^{p+\alpha} dx &= -\int_0^1 (\varphi_p(\phi_1'(x)))'[\phi_1(x)]^{\alpha+1} dx = (\alpha+1)\int_0^1 \varphi_p(\phi_1'(x))[\phi_1(x)]^{\alpha}\phi_1'(x) dx\\ &= (\alpha+1)\int_0^1 |\phi_1'(x)|^p [\phi_1(x)]^{\alpha} dx > 0.\end{aligned}$$

□

Lemma 3.3. Let the hypotheses of Lemma 2.1 hold. Then there exists $\delta > 0$ such that $(\lambda, u) \in C$ and $|\lambda - \mu_1/f_0| + \|u\| \le \delta$ imply $\lambda < \mu_1/f_0$.

Proof. For contradiction we assume that there exists a sequence $\{(\beta_n, u_n)\}$ such that $(\beta_n, u_n) \in C$ satisfying

$$\beta_n \to \mu_1/f_0,\ \|u_n\| \to 0 \text{ and } \beta_n \ge \mu_1/f_0.$$

By Lemma 3.1, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\|$ converges uniformly to ϕ_1 on $[0, 1]$, where ϕ_1 is the eigenfunction of (1.4) which satisfies $\|\phi_1\| = 1$. Multiplying the equation of (1.1) with $(\lambda, u) = (\beta_n, u_n)$ by u_n and integrating it over $[0, 1]$, we obtain

$$\int_0^1 \beta_n m(r) f(u_n(r)) u_n(r) dr = \int_0^1 |u_n'(r)|^p dr,$$

that is,

$$\beta_n \int_0^1 m(r)\frac{f(u_n(r))}{\|u_n\|^{p-1}}\frac{u_n(r)}{\|u_n\|}dr = \int_0^1 \frac{|u_n'(r)|^p}{\|u_n\|^p}dr.$$

From Lemma 3.1, after taking a subsequence and relabeling if necessary, $u_n/\|u_n\|$ converges to ϕ_1 in E.

$$\int_0^1 |\phi_1'(r)|^p dr = \mu_1 \int_0^1 m(r)|\phi_1(r)|^p dr,$$

it follows that

$$\beta_n \int_0^1 m(r)\frac{f(u_n(r))}{\|u_n\|^{p-1}}\frac{u_n(r)}{\|u_n\|}dr = \mu_1 \int_0^1 m(r)\frac{|u_n(r)|^p}{\|u_n\|^p}dr - \hat{\zeta}(n),$$

and accordingly,

$$\beta_n \int_0^1 m(r)f(u_n(r))u_n(r)dr = \mu_1 \int_0^1 m(r)|u_n(r)|^p dr - \hat{\zeta}(n)\|u_n\|^p$$

with $\hat{\zeta} : \mathbb{N} \to \mathbb{N}$ satisfying $\lim_{n\to\infty} \hat{\zeta}(n) = 0$.

That is

$$\int_0^1 m(r)\frac{f(u_n(r)) - f_0|u_n(r)|^{p-2}u_n(r)}{u_n^{p-1+\alpha}(r)}\Big|\frac{u_n(r)}{\|u_n\|}\Big|^{p+\alpha}dr$$
$$=\frac{1}{\|u_n\|^\alpha}\Bigg(\frac{\mu_1 - f_0\beta_n}{\beta_n}\int_0^1 m(r)\Big|\frac{u_n(r)}{\|u_n\|}\Big|^p dr - \hat{\zeta}(n)\Bigg).$$

Lebesgue's dominated convergence theorem, condition (H3) imply that

$$\int_0^1 m(r)\frac{f(u_n(r)) - f_0|u_n(r)|^{p-2}u_n}{|u_n|^{p-2+\alpha}(r)u_n(r)}\Big|\frac{u_n(r)}{\|u_n\|}\Big|^{p+\alpha}dr \to f_1\int_0^1 m(r)|\phi_1|^{p+\alpha}dr > 0$$

and

$$\int_0^1 m(r)\Big|\frac{u_n(r)}{\|u_n\|}\Big|^p dr \to \int_0^1 m(r)|\phi_1|^p dr > 0.$$

This contradicts $\beta_n \geq \mu_1/f_0$. □

Lemma 3.4. Assume that (H1), (H2) and (H5) hold. Let C be as in Lemma 2.6. If $(\lambda, u) \in C$ such that $\|u\|_\infty = s_0$, we have $\lambda > \frac{\lambda_1}{f_0}$.

Proof. Let (λ, u) be a solution of (1.1) with $\|u\|_\infty = u(\tau) = s_0$. Let $f_* = \frac{f_0}{\mu_1\int_0^1 |m(x)|dx}$ be from Lemma 2.2. By condition (H5) and Lemma 2.2, we have

$$\|u\|_\infty = \int_0^\tau u'(x)dx < \int_0^1 |u'(x)|dx \leq \lambda^{\frac{1}{p-1}}\Big(\frac{f_0}{\mu_1\int_0^1 |m(x)|dx}\Big)^{\frac{1}{p-1}}\|u\|_\infty\Big(\int_0^1 |m(x)|dx\Big)^{\frac{1}{p-1}},$$

that is

$$\lambda > \frac{\mu_1}{f_0}.$$

□

Proof of Theorem 1.1. Let C be as in Lemma 2.6. By Lemma 2.6, C is bifurcating from $(\frac{\mu_1}{f_0}, 0)$ and joins $(\mu_1/f_0, 0)$ to $(0, \infty)$. Since C is unbounded, there exists $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, u_n) \in C$ and

$\lambda_n + \|u_n\|_\infty \to \infty$. By Lemma 2.6 , we have that $\|u_n\|_\infty \to \infty$ and $\lambda_n \to 0$, then there exists $(\lambda_0, u_0) \in C$ such that $\|u_0\|_\infty = s_0$ and Lemma 3.4 shows that $\lambda_0 > \frac{\mu_1}{f_0}$.

By Lemmas 2.6, 3.3, 3.4, C passes through some points $(\frac{\mu_1}{f_0}, v_1)$ and $(\frac{\mu_1}{f_0}, v_2)$ with $\|v_1\|_\infty < s_0 < \|v_2\|_\infty$, and there exist $\underline{\lambda}$ and $\overline{\lambda}$ which satisfy $0 < \underline{\lambda} < \frac{\mu_1}{f_0} < \overline{\lambda}$ and both (i) and (ii):

(i) if $\lambda \in (\frac{\mu_1}{f_0}, \overline{\lambda}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in C$ and $\|u\|_\infty < s_0 < \|v\|_\infty$;

(ii) if $\lambda \in (\underline{\lambda}, \frac{\mu_1}{f_0}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in C$ and $\|u\|_\infty < \|v\|_\infty < s_0$.

Define $\lambda^* = \sup\{\overline{\lambda} : \overline{\lambda}$ satisfies (i)$\}$ and $\lambda_* = \inf\{\underline{\lambda} : \underline{\lambda}$ satisfies (ii)$\}$. Then by the standard arguments, (1.1) has a positive solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively. Clearly, C is bifurcating from $(\frac{\mu_1}{f_0}, 0)$ and goes leftward. Moreover, C turns to the right at $(\lambda_*, u_{\lambda_*})$ and to the left at $(\lambda^*, u_{\lambda^*})$, finally to the left near $\lambda = 0$ (see Figure 1). That is, C is a *reversed S-shaped continuum.* By figuring the shape of unbounded continuum C of positive solutions, the statements (i)–(vi) hold. This complete the proof of Theorem 1.1. □

4. Conclusions

The p-Laplacian operator in one and multi-dimensions is a current vigorous area of research. In this paper, we extend the seminal work by Sim and Tanaka [11] on "Three positive solutions for the one-dimensional p-Laplacian problem", where the authors studied $f(s)s^{1-p} = f_\infty$ with $f_\infty \in (0, \infty)$. The current work considers the case where $f_\infty = \infty$ and aims to prove the existence of a *reversed S-shaped continuum*. As a by-product, we assert further that (1.1) has one, or two or three positive solutions under the suitable conditions on the weight function and nonlinearity. More interesting and complex behavior of such problem will further be explored.

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Conflict of interest

The authors declare that they have no competing interests.

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