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ERA, 31(5): 2974–2993.
DOI: 10.3934/era.2023150
Received: 11 January 2023
Revised: 01 March 2023
Accepted: 07 March 2023
Published: 20 March 2023

Research article

Delayed wave equation with logarithmic variable-exponent nonlinearity

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Abstract: A delayed nonlinear wave equation with variable exponents of logarithmic type is discussed in this paper. In the presence of the logarithmic nonlinear source, we established a global existence result under sufficient conditions on the initial data only without imposing the Sobolev Logarithmic Inequality. After that, we established global results of exponential and polynomial types according to the range values of the exponents. At the end, we give a numerical study that supports our theoretical results.

Keywords: decay; nonlinearly damped; delay time; variable exponent

1. Introduction

In this work, we are concerned with the following delayed nonlinear wave problem

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) & \\ +\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) = u\,|u|^{p(x)-2}\ln|u|^k & \text{in } \Omega\times(0,\infty) \\ u(x,t) = 0 & \text{in } \partial\Omega\times[0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega \\ u_t\,(x,t-\tau) = f_0\,(x,t-\tau) & \text{in } \Omega\times(0,\tau), \end{cases} \tag{1.1}$$

where τ, k, $\mu_1 > 0$ and μ_2 is a real number. The functions u_0, u_1, f_0 are the initial and history data to be determined later. The domain $\Omega\subset\mathbb{R}^n$ is an open bounded domain with a smooth boundary $\partial\Omega$. The variable exponents $m(\cdot)$ and $p(\cdot)$ are given measurable functions on Ω and satisfy

$$2 \le m_1 \le m(x) \le m_2 < p_1 \le p(x) \le p_2 \le 2\frac{n-1}{n-2}, \quad n\ge 3, \tag{1.2}$$

where

$$m_1 \ : \ = \text{ess inf}_{x\in\Omega} m(x), \qquad m_2 := \text{ess sup}_{x\in\Omega} m(x),$$
$$p_1 \ : \ = \text{ess inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_2 := \text{ess sup}_{x\in\Omega} p(x)$$

and the log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \tag{1.3}$$

for a.e., $x, y \in \Omega$, with $|x - y| < \delta$, $A > 0$ and $0 < \delta < 1$.

In the absence of delay ($\mu_2 = 0$), the hyperbolic equation in (1.1) is well studied and many blow-up and decay results have been proved. Relaxation or viscoelastic term also were added. See [1–7].

Like physical, chemical, biological, and thermal processes, time delays are frequent occurrences. It is well established that the delay term, if no extra stabilization techniques are included, can be a source of instability. Nicaise and Pignotti [8] did in fact analyze to the following wave equation

$$u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a u_t(x, t-\tau) = 0 \ \text{ in } \ \Omega \times (0, +\infty),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and a, a_0 are positive real parameters. They proved that the system is exponentially stable under the condition ($0 \leq a < a_0$). In the case $a \geq a_0$, they produced a sequence of delays for which the corresponding solution is instable. After that, various types of delay were considered and similar stability results were established. See, in this regard [9–11].

Recently, equations with variable exponents of nonlinearity have been used to model a variety of physical phenomena, including flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through porous media, and image processing. The references in [12–17] provide additional information on these issues.

For instance, a hyperbolic problems with nonlinearities of variable-exponent type presented in the work of Antontsev [18], where he considered the equation

$$u_{tt} - \text{div}\left(a(x,t)\,|\nabla u|^{p(x,t)-2}\,\nabla u\right) - \alpha\Delta u_t = b(x,t)u\,|u|^{\sigma(x,t)-2}, \ \text{in } \Omega \times (0,\infty) \tag{1.4}$$

and demonstrated numerous blow-up results on the variables a, b, p, and σ for some non-positive initial energy solutions. Similarly, in [19], Antontsev used Galerkin approximations in spaces of the Orlicz-Sobolev type to demonstrate the presence of local and global weak solutions of (1.4). The blow up for weak solutions with nonpositive energy functional was then established. Guo and Gao [20] demonstrated that solutions to quasilinear hyperbolic equations with positive initial energy and $p(x,t)$-Laplacian are blow up. We recommend reading Antontsev and Shmarev [21] and Galaktionov [22] for other problems involving variable-exponent nonlinearities. S. Park [23] thought about issues of a similar nature but with constant exponents.

Most literary works impose the Sobolev Logarithmic Inequality (SLI) when a logarithmic source term is present in order to establish specific decay or blow up results. Observe [24–30], for instance. Authors were constrained by the usage of (SLI) by using terms like $u\ln|u|$ or $u^2\ln|u|$ or by imposing additional weaken requirements if the nonlinearity is more challenging.

The presence of a local weak solution for constant exponents was established in [28]. With variable exponent nonlinearity of the logarithmic kind and a delay term present in problem (1.1), our goal is to examine the stability of any strong solution without the use of (SLI). To put it more precisely, we seek to demonstrate a global existence result under sufficient assumptions using only the initial data, variables μ_1, μ_2, m, and p . After that, we developed decay results of polynomial and exponential types based on certain exponent $m(\cdot)$ values. This paper has an introduction and four more sections to serve that aim. In Section 2, we reviewed the meanings of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$. We demonstrate a global existence result in Section 3. For the decay results, see Section 4. While the numerical analysis is covered in Section 5.

2. Preliminaries

The materials required for the assertion and the demonstration of our results are provided in this part. We now provide definitions and characteristics for Lebesgue and Sobolev spaces with varying exponents. See [21,29].

Let Ω be a domain of $\mathbb{R}^n$ with $n \geq 2$ and $p : \Omega \longrightarrow [1, \infty)$ be a measurable function. The Lebesgue space $L^{p(\cdot)}(\Omega)$ with a variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R};\ \text{measurable in } \Omega \text{ and } \int_\Omega |\lambda u(x)|^{p(x)}\, dx < +\infty \right\},$$

for some $\lambda > 0$.

Definition 2.1. *The Luxembourg-type norm is given by*

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The space $L^{p(\cdot)}(\Omega)$, equipped with this norm, is a Banach space (see [21]). The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined as

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

This space is a Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The definition of the space $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. In contrast to the case with constant exponents, the specification of the space $W_0^{1,p(\cdot)}(\Omega)$ is typically different. Both definitions, however, match up under condition (1.3). In the same way as in the classical Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, the dual space of $W_0^{1,p(\cdot)}(\Omega)$ is $W_0^{-1,p'(\cdot)}(\Omega)$.

Lemma 2.2. *[19] **(Poincaré's inequality).*** Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies (1.3), then

$$\|u\|_{p(\cdot)} \leq C \|\nabla u\|_{p(\cdot)},\ \textit{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where the positive constant C depends on $p(\cdot)$ and Ω. In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}.$$

Lemma 2.3. If $p : \overline{\Omega} \longrightarrow [1, \infty)$ is continuous and

$$2 \leq p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}, \quad n \geq 3,$$

then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 2.4. If $p : \Omega \longrightarrow [1, \infty)$ is a measurable function and $p_2 < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Lemma 2.5. *[19] **(Hölder's inequality).*** Let $p, q, s \geq 1$ be measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \textit{for a.e. } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

Lemma 2.6. ***(Unit Ball Property).*** Let p be a measurable function on Ω. Then

$$\|f\|_{p(\cdot)} \leq 1 \ \textit{ if and only if } \varrho_{p(\cdot)}(f) \leq 1,$$

where

$$\varrho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx.$$

Lemma 2.7. If p is a measurable function on Ω satisfying (1.1), then

$$\min\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\} \leq \varrho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\},$$

for a.e., $x \in \Omega$ and for any $u \in L^{p(\cdot)}(\Omega)$.

3. Global existence

We prove a global existence result, referring to the method used in [10]. We first introduce the new variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \qquad x \in \Omega,\ \rho \in (0, 1),\ t > 0.$$

Thus, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad x \in \Omega,\ \rho \in (0, 1),\ t > 0.$$

Then, problem (1.1) takes the form

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t(x,t) |u_t(x,t)|^{m(x)-2} & \\ +\mu_2 z(x,1,t) |z(x,1,t)|^{m(x)-2} = u |u|^{p(x)-2} \ln |u|^k, & \text{in} \quad \Omega \times (0, \infty) \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in} \quad \Omega \times (0,1) \times (0,\infty) \\ z(x,\rho,0) = f_0(x, -\rho\tau), & \text{in} \quad \Omega \times (0,1) \\ u(x,t) = 0, & \text{on} \quad \partial\Omega \times [0, \infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

Definition 3.1. For $T > 0$ fixed, we call (u, z) a strong solution if

$$\begin{aligned} u &\in C^2([0,T); L^2(\Omega)) \cap C^1([0,T); H_0^1(\Omega)) \cap C([0,T); H^2(\Omega) \cap H_0^1(\Omega)), \\ u_t &\in L^{m(\cdot)}(\Omega \times (0,T)), \\ z &\in C^1([0,1] \times [0,T); L^2(\Omega)) \cap L^\infty((0,T); L^{m(\cdot)}((0,1) \times \Omega)) \end{aligned}$$

and satisfies the equations of (3.1) in $H^{-1}(\Omega)$ and $L^2(\Omega)$ respectively and the initial data.

The energy functional associated to (3.1) is given by

$$\begin{aligned} \mathrm{E}(t) \ : \ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\ &+ \int_0^1 \int_\Omega \frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho + k \int_\Omega \frac{|u|^{p(x)}}{p^2(x)} dx - \int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx, \end{aligned} \tag{3.2}$$

for $t \geq 0$ and ζ is a continuous function satisfying

$$\tau\,|\mu_2|\,(m(x) - 1) < \zeta(x) < \tau\,(\mu_1 m(x) - |\mu_2|)\,, \quad x \in \overline{\Omega}. \tag{3.3}$$

One can take, for instance,

$$\begin{aligned} \zeta(x) &= \frac{\tau}{2}\,[|\mu_2|\,(m(x) - 1) + (\mu_1 m(x) - |\mu_2|)] \\ &= \frac{\tau}{2}\,[(\mu_1 + |\mu_2|)\,m(x) - 2\,|\mu_2|] > 0 \quad \text{on} \quad \overline{\Omega}. \end{aligned}$$

The following lemma shows that the associated energy of the problem is nonincreasing under the condition $\mu_1 > |\mu_2|$.

Lemma 3.2. Let (u, z) be the solution of (3.1) . Then, for some $C_0 > 0$,

$$\mathrm{E}'(t) \leq -C_0 \left[\int_\Omega \left(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)} \right) dx \right] \leq 0. \tag{3.4}$$

Proof. Multiplying Eq $(3.1)_1$ by u_t and integrating over Ω and multiplying $(3.1)_2$ by $\frac{1}{\tau}\zeta(x)\,|z|^{m(x)-2}\,z$ and integrating over $\Omega \times (0, 1)$, then summing up, we get

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \int_0^1 \int_\Omega \frac{\zeta\,(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho \right. \\ &\left. +k \int_\Omega \frac{|u|^{p(x)}}{p^2(x)} dx - \int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx \right] \\ = \ &-\mu_1 \int_\Omega |u_t|^{m(x)}\,dx - \frac{1}{\tau} \int_\Omega \int_0^1 \zeta\,(x)\,|z(x,\rho,t)|^{m(x)-2}\,z z_\rho(x,\rho,t) d\rho dx \\ &-\mu_2 \int_\Omega u_t z(x,1,t)\,|z(x,1,t)|^{m(x)-2}\,dx. \end{aligned} \tag{3.5}$$

We, now, estimate the last two terms of the right-hand side of (3.5) as follows,

$$-\frac{1}{\tau} \int_\Omega \int_0^1 \zeta\,(x)\,|z(x,\rho,t)|^{m(x)-2}\,z z_\rho(x,\rho,t) d\rho dx$$

$$
\begin{aligned}
&= -\frac{1}{\tau}\int_{\Omega}\int_0^1 \frac{\partial}{\partial\rho}\left(\frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}\right)d\rho dx \\
&= \frac{1}{\tau}\int_{\Omega}\frac{\zeta(x)}{m(x)}\left(|z(x,0,t)|^{m(x)}-|z(x,1,t)|^{m(x)}\right)dx \\
&= \int_{\Omega}\frac{\zeta(x)}{\tau m(x)}\,|u_t|^{m(x)}\,dx-\int_{\Omega}\frac{\zeta(x)}{\tau m(x)}\,|z(x,1,t)|^{m(x)}\,dx.
\end{aligned}
$$

For the last term, we use Young's inequality with $q=\frac{m(x)}{m(x)-1}$ and $q'=m(x)$ to get

$$
|u_t|\,|z(x,1,t)|^{m(x)-1}\le\frac{1}{m(x)}\,|u_t|^{m(x)}+\frac{m(x)-1}{m(x)}\,|z(x,1,t)|^{m(x)}.
$$

Consequently, we arrive at

$$
\begin{aligned}
&-\mu_2\int_{\Omega}u_t z\,|z(x,1,t)|^{m(x)-2}\,dx \\
\le\ &|\mu_2|\left(\int_{\Omega}\frac{1}{m(x)}\,|u_t(t)|^{m(x)}\,dx+\int_{\Omega}\frac{m(x)-1}{m(x)}\,|z(x,1,t)|^{m(x)}\,dx\right).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\frac{d\mathrm{E}(t)}{dt} \le\ &-\int_{\Omega}\left[\mu_1-\left(\frac{\zeta(x)}{\tau m(x)}+\frac{|\mu_2|}{m(x)}\right)\right]|u_t(t)|^{m(x)}\,dx \\
&-\int_{\Omega}\left(\frac{\zeta(x)}{\tau m(x)}-\frac{|\mu_2|\,(m(x)-1)}{m(x)}\right)|z(x,1,t)|^{m(x)}\,dx.
\end{aligned}
$$

Finally, the relation (3.3) yields, $\forall\ x\in\overline{\Omega}$,

$$
f_1(x)=\mu_1-\left(\frac{\zeta(x)}{\tau m(x)}+\frac{|\mu_2|}{m(x)}\right)>0 \text{ and } f_2(x)=\frac{\zeta(x)}{\tau m(x)}-\frac{|\mu_2|\,(m(x)-1)}{m(x)}>0.
$$

Since $m(x)$ is bounded, hence $\zeta(x)$, we deduce that $f_1(x)$ and $f_2(x)$ are bounded. Therefore, if we define

$$
C_0(x)=\min\{f_1(x),\ f_2(x)\}>0,\quad \text{for any } x\in\overline{\Omega}
$$

and take $C_0=\inf_{\overline{\Omega}}C_0(x)$, then $C_0(x)\ge C_0>0$.
Hence,

$$
\mathrm{E}'(t)\le -C_0\left[\int_{\Omega}|u_t(t)|^{m(x)}\,dx+\int_{\Omega}|z(x,1,t)|^{m(x)}\,dx\right]\le 0.
$$

Now, we show that the solution of (3.1) is uniformly bounded and global in time.
For this purpose, we set

$$
\begin{aligned}
I(t) &= \|\nabla u\|_2^2-\int_{\Omega}|u|^{p(x)}\ln|u|^k\,dx, \\
J(t) &= \frac{1}{2}\|\nabla u\|_2^2+k\int_{\Omega}\frac{|u|^{p(x)}}{p^2(x)}dx+\int_0^1\int_{\Omega}\frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho-\int_{\Omega}\frac{|u|^{p(x)}\ln|u|^k}{p(x)}dx.
\end{aligned}
$$

Hence,

$$
\mathrm{E}(t)=J(t)+\frac{1}{2}\|u_t\|_2^2.
$$

Lemma 3.3. Suppose that the initial data $u_0, u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying $I(0) > 0$ and

$$\beta = C_{p_2+k}\left(\frac{2p_1\mathrm{E}(0)}{p_1 - 2}\right)^{\frac{p_2+k-2}{2}} < 1.$$

Then $I(t) > 0$, for any $t \in [0, T]$ and $\gamma > 0$ to be specified later.

Proof. If

$$\int_\Omega |u|^{p(x)} \ln |u|^k \, dx \le 0,$$

then the result is straightforward. So, we will assume

$$\int_\Omega |u|^{p(x)} \ln |u|^k \, dx > 0.$$

Since $I(0) > 0$ we deduce by continuity that there exists $T^* \le T$ such that $I(t) \ge 0$ for all $t \in [0, T^*]$. This implies that, for all $t \in [0, T^*]$,

$$\begin{aligned} J(t) &\ge \frac{1}{2}\|\nabla u\|_2^2 + \frac{k}{p_2}\int_\Omega |u|^{p(x)}\, dx + \int_0^1 \int_\Omega \frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho \\ &\quad -\frac{1}{p_1}\int_\Omega |u|^{p(x)} \ln |u|^k \, dx \\ &\ge \frac{p_1 - 2}{2p_1}\|\nabla u\|_2^2 + \frac{k}{p_2}\int_\Omega |u|^{p(x)}\, dx + \int_0^1 \int_\Omega \frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho + \frac{1}{p_1} I(t) \\ &\ge \frac{p_1 - 2}{2p_1}\|\nabla u\|_2^2. \end{aligned}$$

Thus,

$$\|\nabla u\|_2^2 \le \frac{2p_1}{p_1 - 2} J(t) \le \frac{2p_1}{p_1 - 2}\mathrm{E}(t) \le \frac{2p_1}{p_1 - 2}\mathrm{E}(0).$$

On the other hand, using the facts that $\ln |u| < |u|$ and $|u| > 1$, we get

$$\int_\Omega |u|^{p(x)} \ln |u|^k \, dx < \int_\Omega |u|^{p(x)+k}\, dx \le \int_\Omega |u|^{p_2+k}\, dx.$$

If we choose $0 < k < \frac{2}{n-2}$, then the embedding $H_0^1(\Omega) \hookrightarrow L^{p_2+k}(\Omega)$ yields

$$\begin{aligned} \int_\Omega |u|^{p_2+k}\, dx &\le C_{p_2+k}\|\nabla u\|_2^{p_2+k} = C_{p_2+k}\|\nabla u\|_2^2 \|\nabla u\|_2^{p_2+k-2} \\ &= C_{p_2+k}\|\nabla u\|_2^2 \left(\|\nabla u\|_2^2\right)^{\frac{p_2+k-2}{2}} \\ &\le C_{p_2+k}\left(\frac{2p_1\mathrm{E}(0)}{p_1 - 2}\right)^{\frac{p_2+k-2}{2}} \|\nabla u\|_2^2, \end{aligned} \tag{3.6}$$

where C_{p_2+k} is the embedding constant. So,

$$\int_\Omega |u|^{p(x)} \ln |u|^k \, dx \le \beta \|\nabla u\|_2^2, \tag{3.7}$$

Consequently, from (3.6) and (3.7) we deduce that

$$I(t) > (1-\beta)\,\|\nabla u\|_2^2 > 0, \quad \forall t \in [0, T^*].$$

By repeating this procedure, T^* can be extended to T.

Theorem 3.4. If the initial data u_0, u_1 satisfy the conditions of Lemma 3.3, then the solution of (3.1) is uniformly bounded and global in time.

Proof. It suffices to show that $\|\nabla u\|_2^2 + \|u_t\|_2^2$ is bounded independently of t. Clearly,

$$\begin{aligned} \mathrm{E}(0) &\geq \mathrm{E}(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{p_1-2}{2p_1}\|\nabla u\|_2^2 + \frac{k}{p_2}\int_\Omega |u|^{p(x)}\,dx + \int_0^1\int_\Omega \frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho + \frac{1}{p_1}I(t) \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p_1}\,(1-\beta)\,\|\nabla u\|_2^2. \end{aligned}$$

Therefore,

$$\|\nabla u\|_2^2 + \|u_t\|_2^2 \leq C\mathrm{E}(0),$$

where C is a positive constant depending only on k, p_1 and p_2.

4. Decay

Lemma 4.1. *(Komornik [31] p. 103 and 124).* Let $\mathrm{E} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a nonincreasing function. Assume that there exist $\sigma > 0, \omega > 0$ such that

$$\int_s^\infty \mathrm{E}^{1+\sigma}(t)dt \leq \frac{1}{\omega}\mathrm{E}^\sigma(0)\mathrm{E}(s) = c\mathrm{E}(s), \quad \forall s > 0.$$

Then, $\forall t \geq 0$,

$$\begin{aligned} \mathrm{E}(t) &\leq c\mathrm{E}(0)/(1+t)^{1/\sigma}, \quad \textit{if } \sigma > 0, \\ \mathrm{E}(t) &\leq c\mathrm{E}(0)e^{-\omega t}, \qquad \textit{if } \sigma = 0. \end{aligned}$$

Before we state the main theorem, we need the following technical lemma.

Lemma 4.2. The functional

$$F(t) = \tau\int_0^1\int_\Omega e^{-\rho\tau}\zeta\,(x)\,|z(x,\rho,t)|^{m(x)}dxd\rho,$$

satisfies, along the solution of (3.1),

$$F'(t) \leq \int_\Omega \zeta\,(x)\,|u_t|^{m(x)}dx - \tau e^{-\tau}\int_0^1\int_\Omega \zeta\,(x)\,|z(x,\rho,t)|^{m(x)}dxd\rho.$$

Proof. A direct differentiation of $F(t)$, using $(3.1)_2$, leads to

$$
\begin{aligned}
F'(t) &= -\int_0^1 \int_\Omega e^{-\rho\tau} m(x)\zeta(x) |z|^{m(x)-1} z_\rho dx d\rho \\
&= -\int_0^1 \int_\Omega \frac{d}{d\rho}\left(e^{-\rho\tau}\zeta(x)|z|^{m(x)}\right) dx d\rho - \tau \int_0^1 \int_\Omega e^{-\rho\tau}\zeta(x)|z|^{m(x)} dx d\rho \\
&\leq \int_\Omega e^{-\tau}\zeta(x)|z(x,0,t)|^{m(x)} dx - \tau \int_0^1 \int_\Omega e^{-\rho\tau}\zeta(x)|z|^{m(x)} dx d\rho \\
&\leq \int_\Omega \zeta(x)|u_t|^{m(x)} dx - \tau e^{-\tau} \int_0^1 \int_\Omega \zeta(x)|z|^{m(x)} dx d\rho.
\end{aligned}
$$

Our main result reads as follows.

Theorem 4.3. Assume that the conditions (1.2) and (1.3) are satisfied. Then there exist two positive constants c and α such that any global solution of (3.1) satisfies

$$
\begin{aligned}
\mathrm{E}(t) &\leq c\mathrm{E}(0)/(1+t)^{2/(m_2-2)}, \quad \textit{if } m_2 > 2 \\
\mathrm{E}(t) &\leq ce^{-\alpha t}, \qquad\qquad\qquad \textit{if } \ m(\cdot) = 2.
\end{aligned}
$$

Proof. Multiply $(3.1)_1$ by $u\mathrm{E}^q(t)$, for $q > 0$ to be specified later, and integrate over $\Omega \times (s, T)$, $s < T$, to obtain

$$
\begin{aligned}
&\int_s^T \mathrm{E}^q(t) \int_\Omega \Big(uu_{tt} - u\Delta u + \mu_1 uu_t |u_t|^{m(x)-2} \\
&+\mu_2 uz(x,1,t)|z(x,1,t)|^{m(x)-2} - u|u|^{p(x)-2}\ln|u|^k\Big) dx dt = 0,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_s^T \mathrm{E}^q(t) \int_\Omega \left(\frac{d}{dt}(uu_t) - u_t^2 + |\nabla u|^2 + \mu_1 uu_t(x,t)|u_t(x,t)|^{m(x)-2}\right. \\
&\left.+\mu_2 uz(x,1,t)|z(x,1,t)|^{m(x)-2} - u|u|^{p(x)-2}\ln|u|^k\right) dx dt = 0.
\end{aligned} \tag{4.1}
$$

Recalling the definition of $\mathrm{E}(t)$ given in (3.2), adding and subtracting some terms and using the relation

$$
\frac{d}{dt}\left(\mathrm{E}^q(t)\int_\Omega uu_t dx\right) = q\mathrm{E}^{q-1}(t)\mathrm{E}'(t)\int_\Omega uu_t dx + \mathrm{E}^q(t)\frac{d}{dt}\int_\Omega uu_t dx,
$$

Eq (4.1) becomes

$$
\begin{aligned}
2\int_s^T \mathrm{E}^{q+1}(t)dt &= -\int_s^T \frac{d}{dt}\left(\mathrm{E}^q(t)\int_\Omega uu_t dx\right) + q\int_s^T \mathrm{E}^{q-1}(t)\mathrm{E}'(t)\int_\Omega uu_t dx dt \\
&\quad +2\int_s^T \mathrm{E}^q(t)\int_\Omega u_t^2 dx - \mu_1 \int_s^T \mathrm{E}^q(t)\int_\Omega uu_t|u_t|^{m(x)-2} dx dt
\end{aligned}
$$

$$
\begin{aligned}
&-\mu_2 \int_s^T \mathrm{E}^q(t) \int_\Omega uz(x,1,t)\,|z(x,1,t)|^{m(x)-2}\, dxdt \\
&+\int_s^T \mathrm{E}^q(t) \int_\Omega u\,|u|^{p(x)-2} \ln |u|^k \, dxdt \\
&+2\int_s^T \mathrm{E}^q(t) \int_0^1 \int_\Omega \frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)} dxd\rho.
\end{aligned} \tag{4.2}
$$

The first term in the right hand side of (4.2) is estimated as follows.

$$
\begin{aligned}
\left| -\int_s^T \frac{d}{dt}\left(\mathrm{E}^q(t)\int_\Omega uu_t dx\right)\right| &= \left|\mathrm{E}^q(s)\int_\Omega uu_t(x,s)dx - \mathrm{E}^q(T)\int_\Omega uu_t(x,T)dx\right| \\
&\le \frac{1}{2}\mathrm{E}^q(s)\left[\int_\Omega u^2(x,s)dx + \int_\Omega u_t^2(x,s)dx\right] \\
&\quad +\frac{1}{2}\mathrm{E}^q(T)\left[\int_\Omega u^2(x,T)dx + \int_\Omega u_t^2(x,T)dx\right] \\
&\le \frac{1}{2}\mathrm{E}^q(s)\left[C_p\,\|\nabla u(s)\|_2^2 + 2\mathrm{E}(s)\right] \\
&\quad +\frac{1}{2}\mathrm{E}^q(T)\left[C_p\,\|\nabla u(T)\|_2^2 + 2\mathrm{E}(T)\right] \\
&\le \mathrm{E}^q(s)\left[C_P\mathrm{E}(s)+\mathrm{E}(s)\right] + \mathrm{E}^q(T)\left[C_P\mathrm{E}(T)+\mathrm{E}(T)\right],
\end{aligned}
$$

where C_P is the Poincaré constant. Using the fact that $\mathrm{E}(t)$ is decreasing, we deduce that

$$
\left| -\int_s^T \frac{d}{dt}\left(\mathrm{E}^q(t)\int_\Omega uu_t dx\right)\right| \le c\mathrm{E}^{q+1}(s) \le c\mathrm{E}^q(0)\mathrm{E}(s) \le c\mathrm{E}(s). \tag{4.3}
$$

Similarly, we treat the term:

$$
\begin{aligned}
\left| q\int_s^T \mathrm{E}^{q-1}(t)\mathrm{E}'(t)\int_\Omega uu_t dxdt\right| &\le -q\int_s^T \mathrm{E}^{q-1}(t)\mathrm{E}'(t)\left[C_p\mathrm{E}(t)+2\mathrm{E}(t)\right] \\
&\le -c\int_s^T \mathrm{E}^q(t)\mathrm{E}'(t)dt \le c\mathrm{E}^{q+1}(s) \le c\mathrm{E}(s).
\end{aligned} \tag{4.4}
$$

To handle the next term, we set

$$
\Omega_+ = \{x\in\Omega \mid\ |u_t(x,t)| \ge 1\} \ \text{ and } \ \Omega_- = \{x\in\Omega \mid\ |u_t(x,t)| < 1\}
$$

and use Hölder's and Young's inequalities, to get

$$
\begin{aligned}
\left|\int_s^T \mathrm{E}^q(t)\int_\Omega u_t^2 dx\right| &= \left|\int_s^T \mathrm{E}^q(t)\left[\int_{\Omega_+} u_t^2 dx + \int_{\Omega_-} u_t^2 dx\right]\right| \\
&\le c\int_s^T \mathrm{E}^q(t)\left[\left(\int_{\Omega_+} |u_t|^{m_1}dx\right)^{2/m_1} + \left(\int_{\Omega_-} |u_t|^{m_2}dx\right)^{2/m_2}\right] \\
&\le c\int_s^T \mathrm{E}^q(t)\left[\left(\int_\Omega |u_t|^{m(x)}dx\right)^{2/m_1} + \left(\int_\Omega |u_t|^{m(x)}dx\right)^{2/m_2}\right]
\end{aligned}
$$

$$\leq \ c\int_s^T \mathrm{E}^q(t)\left[(-\mathrm{E}'(t))^{2/m_1} + (-\mathrm{E}'(t))^{2/m_2}\right]$$
$$\leq \ c\varepsilon\int_s^T [\mathrm{E}(t)]^{qm_1/(m_1-2)}\,dt + c(\varepsilon)\int_s^T (-\mathrm{E}'(t))\,dt$$
$$+c\varepsilon\int_s^T \mathrm{E}^{q+1}(t)dt + c(\varepsilon)\int_s^T (-\mathrm{E}'(t))^{2(q+1)/m_2}\,dt.$$

For $m_1 > 2$, the choice of $q = \frac{m_2}{2} - 1$ will make $\frac{qm_1}{m_1-2} = q + 1 + \frac{m_2-m_1}{m_1-2}$. Hence,

$$\left|\int_s^T \mathrm{E}^q(t)\int_\Omega u_t^2 dx\right| \tag{4.5}$$
$$\leq \ c\varepsilon\int_s^T \mathrm{E}^{q+1}(t)dt + c\varepsilon\,[\mathrm{E}(0)]^{\frac{m_2-m_1}{m_1-2}}\int_s^T [\mathrm{E}(t)]^{q+1}\,dt + c(\varepsilon)\mathrm{E}(s)$$
$$\leq \ c\varepsilon\int_s^T \mathrm{E}^{q+1}(t)dt + c(\varepsilon)\mathrm{E}(s).$$

For the case $m_1 = 2$, the choice of $q = \frac{m_2}{2} - 1$, will give a similar result.

For the next term, we use Young's inequality. So, for a.e., $x \in \Omega$, we have

$$\left|-\mu_1\int_s^T \mathrm{E}^q(t)\int_\Omega u\,|u_t|^{m(x)-1}\,dxdt\right|$$
$$\leq \ \varepsilon\int_s^T \mathrm{E}^q(t)\int_\Omega |u(t)|^{m(x)}\,dxdt + c\int_s^T \mathrm{E}^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt$$
$$\leq \ \varepsilon\int_s^T \mathrm{E}^q(t)\left[\int_{\Omega_+} |u(t)|^{m_1}\,dxdt + \int_{\Omega_-} |u(t)|^{m_2}\,dxdt\right]$$
$$+c\int_s^T \mathrm{E}^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt,$$

where we used Young's inequality with

$$p(x) = \frac{m(x)}{m(x)-1} \quad \text{and} \quad p'(x) = m(x)$$

and, hence,

$$c_\varepsilon(x) = \varepsilon^{1-m(x)}\left(m(x)^{-m(x)}\,(m(x)-1)\right)^{m(x)-1}.$$

Therefore, using the embedding of $H_0^1(\Omega) \hookrightarrow L^{m_1}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{m_2}(\Omega)$, we arrive at

$$\left|-\mu_1\int_s^T \mathrm{E}^q(t)\int_\Omega u\,|u_t|^{m(x)-1}\,dxdt\right| \tag{4.6}$$
$$\leq \ \varepsilon\int_s^T \mathrm{E}^q(t)\left[c\,\|\nabla u(s)\|_2^{m_1} + c\,\|\nabla u(s)\|_2^{m_2}\right]$$
$$+c\int_s^T \mathrm{E}^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt$$

$$\begin{aligned}
&\le \varepsilon \int_s^T \mathrm{E}^q(t)\left[c\mathrm{E}^{\frac{m_1-2}{2}}(0)\mathrm{E}(t) + c\mathrm{E}^{\frac{m_2-2}{2}}(0)\mathrm{E}(t)\right] \\
&\quad + c\int_s^T \mathrm{E}^q(t) \int_\Omega c_\varepsilon(x)\, |u_t(t)|^{m(x)}\, dxdt \\
&\le c\varepsilon \int_s^T \mathrm{E}^{q+1}(t) + \int_s^T \mathrm{E}^q(t) \int_\Omega c_\varepsilon(x)\, |u_t(t)|^{m(x)}\, dxdt \\
&\le c\varepsilon \int_s^T \mathrm{E}^{q+1}(t) + \int_s^T \mathrm{E}^q(t) \int_\Omega c_\varepsilon(x)\, |u_t(t)|^{m(x)}\, dxdt \\
&\le \varepsilon \int_s^T \mathrm{E}^{q+1}(t) + c(\varepsilon)\mathrm{E}(s).
\end{aligned}$$

where $c(\varepsilon)$ is a finite constant depend on ε whence it is fixed because $m(x)$ is bounded.

The next term of (4.2) can be estimated in a similar manner to reach

$$\begin{aligned}
&\left| -\mu_2 \int_s^T \mathrm{E}^q(t) \int_\Omega u\, |z(x,1,t)|^{m(x)-1}\, dxdt \right| \qquad (4.7)\\
&\le \varepsilon \int_s^T \mathrm{E}^q(t)\left[c\, \|\nabla u(s)\|_2^{m_1} + c\, \|\nabla u(s)\|_2^{m_2}\right] \\
&\quad + c\int_s^T \mathrm{E}^q(t) \int_\Omega c_\varepsilon(x)\, |z(x,1,t)|^{m(x)}\, dxdt \\
&\le \varepsilon \int_s^T \mathrm{E}^{q+1}(t)dt + c(\varepsilon)\mathrm{E}(s).
\end{aligned}$$

For the logarithmic term, we use the same idea of (3.6) .We have

$$p_2 - 1 + k < \frac{n}{n-2} + k < \frac{2n}{n-2}.$$

Thus the embedding $H_0^1(\Omega) \hookrightarrow L^{p_2-1+k}(\Omega)$, yields, for some $0 < \widetilde{\beta} < 1$, to

$$\int_\Omega |u|^{p(x)-1} \ln |u|^k\, dx \le \widetilde{\beta} \|\nabla u\|_2^2.$$

Thus, we have

$$\begin{aligned}
&\left| \int_s^T \mathrm{E}^q(t) \int_\Omega u\, |u|^{p(x)-2} \ln |u|^k\, dxdt \right| \qquad (4.8)\\
&= \left| \int_s^T \mathrm{E}^q(t) \int_\Omega |u|^{p(x)-1} \ln |u|^k\, dxdt \right| \\
&\le \widetilde{\beta} \int_s^T \mathrm{E}^q(t) \|\nabla u\|_2^2 dt \le \widetilde{\beta} \int_s^T \mathrm{E}^{q+1}(t)dt.
\end{aligned}$$

The last term of (4.2) can be estimated, using Lemma 4.2, as follows,

$$2\int_s^T \mathrm{E}^q(t) \int_0^1 \int_\Omega \frac{\zeta(x)\, |z(x,\rho,t)|^{m(x)}}{m(x)} dxd\rho$$

$$\leq \frac{2}{m_1}\int_s^T \mathrm{E}^q(t)\int_0^1\int_\Omega \zeta(x)\,|z(x,\rho,t)|^{m(x)}\,dxd\rho$$
$$\leq -\frac{2\tau}{m_1}\left[\mathrm{E}^q(t)\int_0^1\int_\Omega e^{-\rho\tau}\zeta(x)\,|z|^{m(x)}dxd\rho\right]_{t=s}^{t=T} + \frac{2}{m_1}\int_s^T \mathrm{E}^q(t)\int_\Omega \zeta(x)\,|u_t|^{m(x)}dx.$$

As $\zeta(x)$ is bounded, we obtain, for $c > 0$,

$$\begin{aligned} 2\int_s^T \mathrm{E}^q(t)\int_0^1\int_\Omega \frac{\zeta(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho &\leq \frac{2\tau e^{-\tau}}{m_1}\mathrm{E}^q(s)\mathrm{E}(s) + \frac{2c}{m_1}\mathrm{E}^{q+1}(T)\\ &\leq \frac{2\tau e^{-\tau}}{m_1}\mathrm{E}^q(0)\mathrm{E}(s) + \frac{2c}{m_1}\mathrm{E}^q(T)\mathrm{E}(s)\\ &\leq c\mathrm{E}(s). \end{aligned} \tag{4.9}$$

Combining (4.2) – (4.9), we arrive at

$$\int_s^T \mathrm{E}^{q+1}(t)dt \leq \left(\varepsilon + \widetilde{\beta}\right)\int_s^T \mathrm{E}^{q+1}(t) + c\mathrm{E}(s).$$

Recalling that $\widetilde{\beta} < 1$, then the choice of ε small enough will make $\varepsilon + \widetilde{\beta} < 1$. Therefore,

$$\int_s^T \mathrm{E}^{q+1}(t)dt \leq c\mathrm{E}(s).$$

As $T \longrightarrow \infty$, we get

$$\int_s^\infty \mathrm{E}^{q+1}(t)dt \leq c\mathrm{E}(s).$$

Therefore, Komornik's lemma is satisfied with $\sigma = q = \frac{m_2}{2} - 1$ which implies the desired result.

5. Numerical experiments

In this section, we devote some numerical experiments to illustrate the theoretical results in theorems (3.4) and (4.3) on a one-dimensional test problem of the form (3.1) with space variable x. For this purpose, we discretize the system (3.1) using a finite difference method (FDM) in both time and space with second-order accuracy in time and space over the time-space domain $(0, 1] \times [0, 1]$. The spatial space $\Omega = (0, 1)$ is divided into $M = 20$ subintervals in Test 1 and $M = 100$ in Test 2 with a step $\Delta x = \frac{1}{M}$, where the time interval $(0, 1)$ is divided into $N = 1000$ subintervals with a time step $\Delta t = \frac{1}{N}$. We take the initial conditions of the problem as $u_0 = sin(\pi x)$, $u_1 = 0$, $\mu_1 = 10$, $\mu_2 = -5$, and $\tau = 0.4$.

We compare the following numerical two tests based on the the value of $m(x)$:

- **Test 1:** Exponential decaying. We take $m(x) = 2$ with $p(x) = 6 + x^2$ to verify the second case of theorem 4.3,
$$\mathrm{E}(t) \sim c\,e^{-\alpha t}, \qquad \text{for all } t \geq 0,$$
where the notation $\sim$ means the two quantities have the same order.
- **Test 2:** Polynomial decaying. We take $m(x) = 2 + x^2$ with $p(x) = 6 + x^2$ to verify the first case,
$$\mathrm{E}(t) \sim \frac{c\,\mathrm{E}(0)}{(1+t)^{\frac{2}{m_2-2}}}, \qquad \text{for all } t \geq 0.$$

5.1. Numerical method

Now, we introduce our numerical scheme of the problem by using finite difference method for time and space discretization. For this purpose, we divide the special domain into M subintervals and the time interval into N subintervals:

In finite difference method, we find an approximate solution at the points (nodes): $x_1, x_2, \ldots, x_M$ for each time level $t_1, t_2, t_3, \ldots, t_N$, such that $U(i, n) \approx u(x_i, t_n)$, where $u(x_i, t_n)$ is the exact solution at (x_i, t_n).

The approximation of the problem (3.1) is accomplished by replacing the derivatives with appropriate difference quotients as the following:

$$
\begin{aligned}
u_t(x_i, t_n) &= \frac{U(i,n) - U(i,n-1)}{\Delta t}, \\
u_{tt}(x_i, t_n) &= \frac{U(i,n+1) - 2U(i,n) + U(i,n-1)}{(\Delta t)^2}, \\
\Delta u(x_i, t_n) &= \frac{U(i+1,n) - 2U(i,n) + U(i-1,n)}{(\Delta x)^2}, \\
u_t(x_i, t_n - \tau) &= \begin{cases} f_0(x_i, t_n - \tau), & t_n \leq \tau; \\ \frac{U(i, n - \frac{\tau}{\Delta t}) - U(i, n - \frac{\tau}{\Delta t} - 1)}{\Delta t}, & t_n > \tau. \end{cases}
\end{aligned}
$$

Then we substitute the above difference quotients in the first equation of the system problem (3.1) and we solve the resultant equation for $U(i, n+1)$ to get the numerical scheme.

By iteration method, we find $U(i, n+1)$ for $n = 1, 2, 3, \ldots, N$ and $i = 1, 2, 3, \ldots, M$ with $U(i, 0) = u_0(x_i)$, $U(i, 1) = u_0(x_i) + \Delta t u_1(x_i)$ and $U(0, n) = U(M, n) = 0$.

5.2. Numerical results

We use the MATLAB tool to perform the numerical scheme. In this section, we discuss the numerical results and compare them with the results of theorem (4.3):

Exponential decaying ($m(x) = 2$): Figure 1 shows that the solution $u(t)$ is a function of t at some fixed values of $x = 0.25, 0.5, 0.7, 0.9$ which are oscillating and decaying at each cross section cut of x.

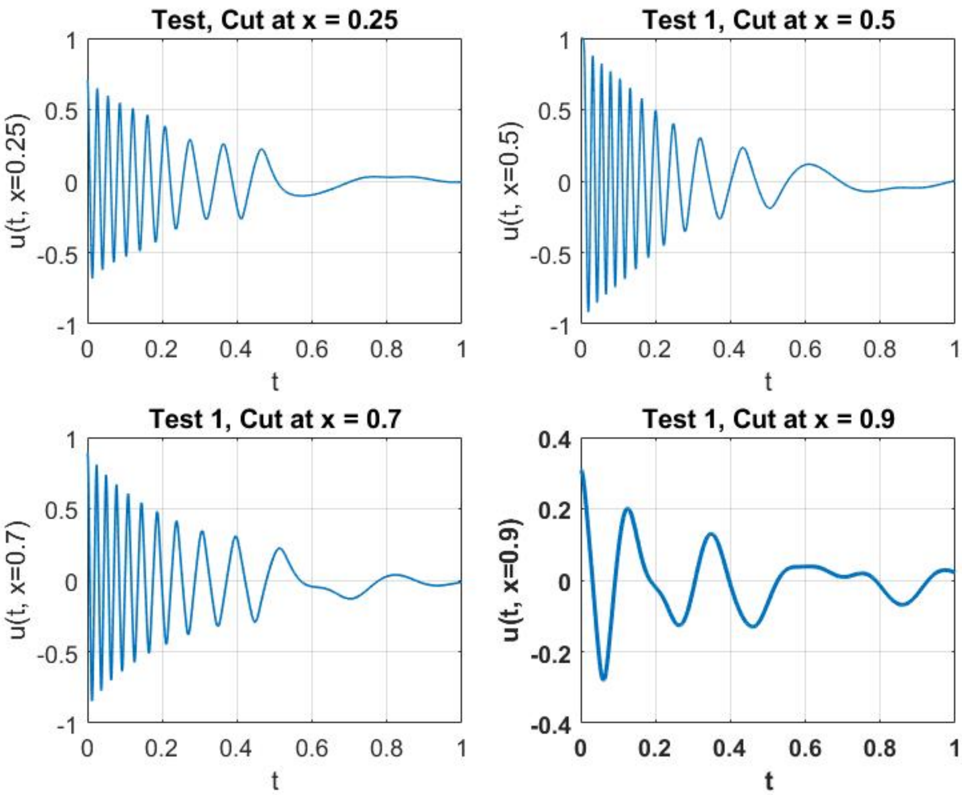

Figure 1. The exponential decay of the solution $u(t)$ at fixed values of x.

Figure 2 shows that the energy function E(t) is decaying exponentially. The left graph is for E(t) vs t with linear scale axes, while the right one is for E(t) vs t with log scale on the $y-$axis and linear scale on the $x-$axis. As we see, the graph is line (linear relation between log[E(t)] and t with negative slope). This means that the energy is decaying exponentially of the form:

$$\mathrm{E}(t) \sim c\, e^{-\alpha t}$$

where by taking log to both sides, we get:

$$log[\mathrm{E}(t)] \sim log(c) + log(e^{-\alpha t}),$$

$$log[\mathrm{E}(t)] \sim -\alpha t + log(c).$$

This result agrees with the second case ($m(x) = 2$, exponential decay) of theorem (4.3).

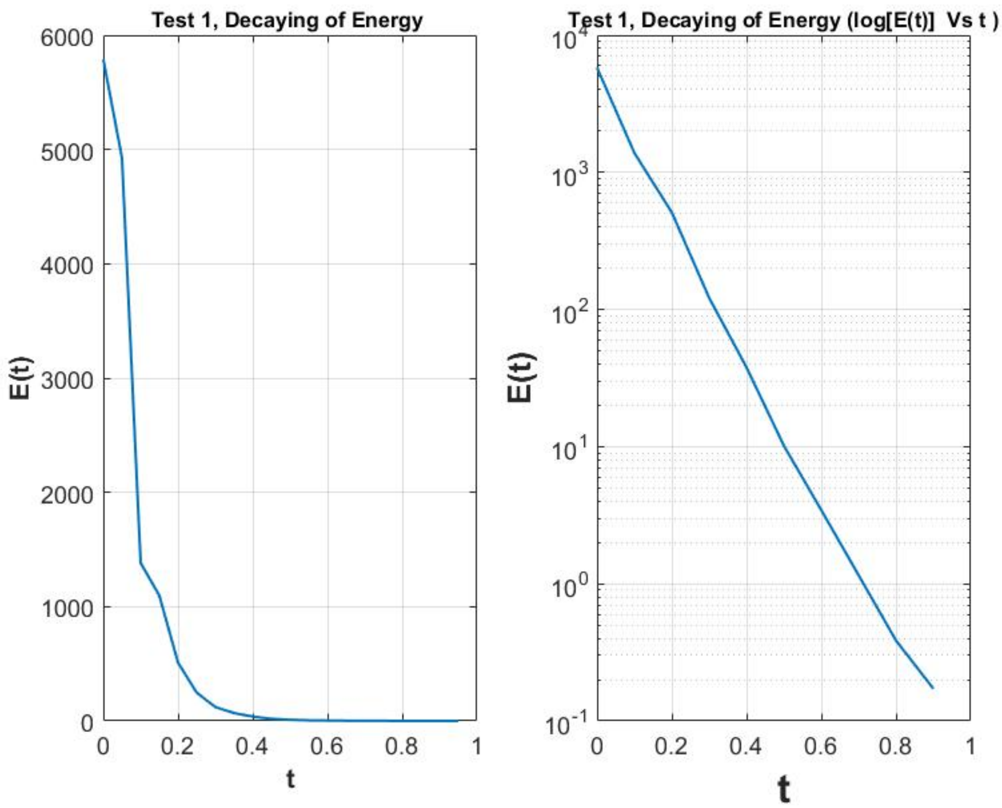

Figure 2. The exponential decay of the energy functional.

Polynomial decay ($m(x) = 2 + x^2$): Figure 3 shows that the solution $u(t)$ as a function of t at some fixed values of $x = 0.25, 0.5, 0.7, 0.9$ which are oscillating and decaying at each cross section cut of x.

According to the first part of the theorem (4.3),

$$\mathrm{E}(t) \sim \frac{c\mathrm{E}(0)}{(1+t)^{\frac{2}{m_2-2}}},$$

means that

$$log[\mathrm{E}(t)] \sim -\frac{2}{m_2-2} log[1+t] + log[c\mathrm{E}(0)].$$

So, the relation between $log[\mathrm{E}(t)]$ vs $log[1+t]$ is a decreasing linear relation. Hence, the energy is decaying polynomially.

This theoretical result coincides with the numerical result in Figure 4; the right graph of $\mathrm{E}(t)$ vs $(t+1)$ with $log - log$ scale axes is linear with negative slope. Also, if we compare the left graph of Figure 2 (Test 1) and the left graph of Figure 4 (Test 2), we observe that the first one (exponential decay) is decaying faster than the second one (polynomial decaying).

Finally, Figure 5 shows the solution $u(x,t)$ in $3D$ with $m(x) = 2 + x$, $p(x) = 3 + x$ and μ_1, μ_2, τ, u_0, and u_1 are the same.

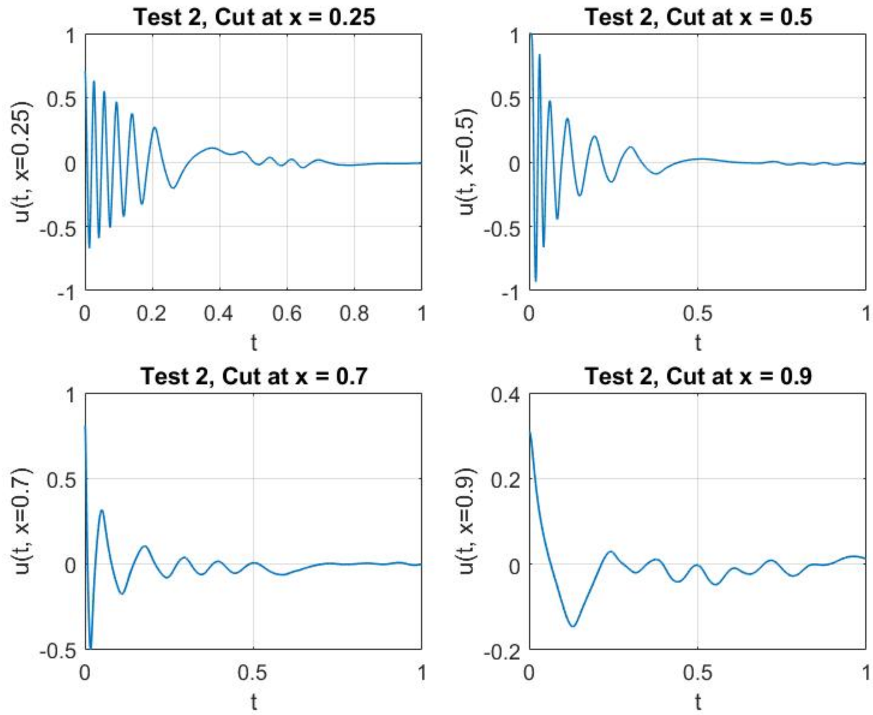

Figure 3. The polynomial decay of the solution $u(t)$ at fixed values of x.

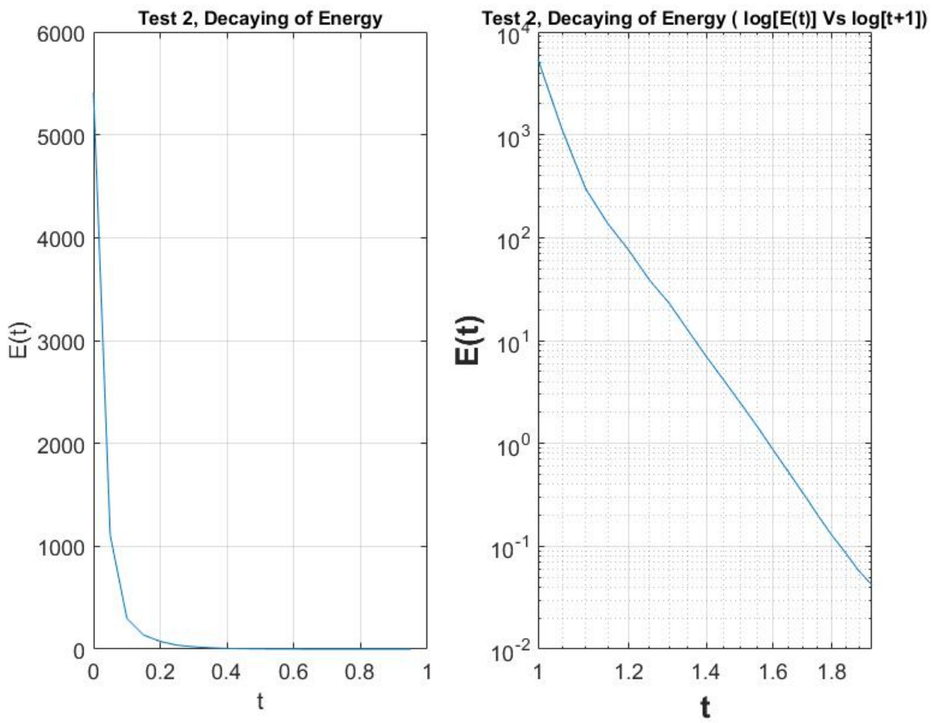

Figure 4. The polynomial decay of the energy functional.

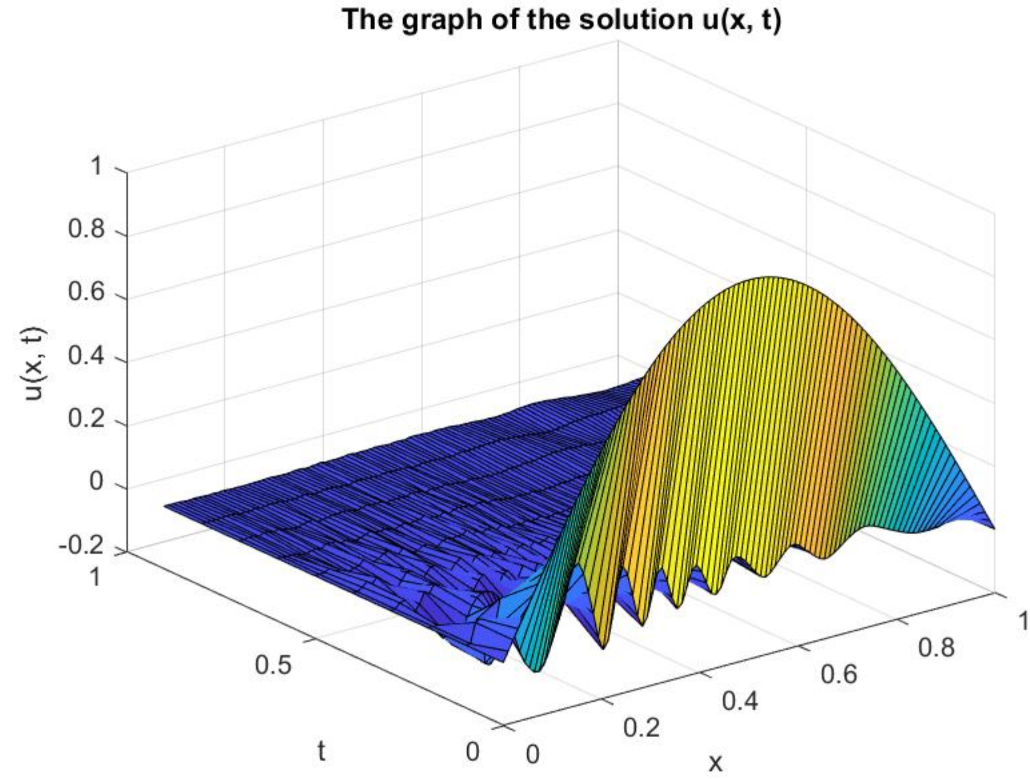

Figure 5. Decay of the solution function $u(x, t)$ in 3D.

Acknowledgment

The authors would like to express their sincere thanks to the interdisciplinary research center in construction and building materials, King Fahd University of Petroleum and Minerals (KFUPM) for their support.

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.*, **5** (1974), 138–146. https://doi.org/10.1137/0505015
2. M. Kopáčková, Remarks on bounded solutions of a semilinear dissipative hyperbolic equation, *Commentat. Math. Univ. Carol.*, **30** (1989), 713–719.
3. E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, *Arch. Ration. Mech. Anal.*, **149** (1999), 155–182. https://doi.org/10.1007/s002050050171

4. H. Levine, J. Serrin, Global nonexistence theorems for quasilinear evolution equations with dissipation, *Arch. Ration. Mech. Anal.*, **137** (1997), 341–361. https://doi.org/10.1007/s002050050032
5. Y. Wang, A global nonexistence theorem for viscoelastic equations with arbitrary positive initial energy, *Appl. Math. Lett.*, **22** (2009), 1394–1400. https://doi.org/10.1016/j.aml.2009.01.052
6. E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, *Commun. Partial Differ. Equations*, **15** (1990), 205–235. https://doi.org/10.1080/03605309908820684
7. Y. Ye, Global existence and blow-up of solutions for higher-order viscoelastic wave equation with a nonlinear source term, *Nonlinear Anal. Theory Methods Appl.*, **112**, (2015), 129–146. https://doi.org/10.1016/j.na.2014.09.001
8. S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, **45** (2006), 1561–1585. https://doi.org/10.1137/060648891
9. S. Nicaise, C. Pignotti, J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *Discrete Contin. Dyn. Syst. - Ser. S*, **4** (2011), 693–722. https://doi.org/10.3934/dcdss.2011.4.693
10. S. Nicaise, C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Differ. Integr. Equations*, **2008** (2008), 935–958. https://doi.org/10.57262/die/1356038593
11. M. Kafini, S. A. Messaoudi, S. Nicaise, A blow-up result in a nonlinear abstract evolution system with delay, *Nonlinear Differ. Equations Appl.*, **23** (2016), 1–14. https://doi.org/10.1007/s00030-016-0354-5
12. R. Aboulaich, D. Meskine, A. Souissi, New diffusion models in image processing, *Comput. Math. Appl.*, **56** (2008), 874–882. https://doi.org/10.1016/j.camwa.2008.01.017
13. S. Lian, W. Gao, C. Cao, H. Yuan, Study of the solutions to a model porous medium equation with variable exponent of nonlinearity, *J. Math. Anal. Appl.*, **342** (2008), 27–38. https://doi.org/10.1016/j.jmaa.2007.11.046
14. Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.*, **66** (2006), 1383–1406. https://doi.org/10.1137/050624522
15. K. Ahmad, K. Bibi, New function solutions of ablowitz-kaup-newell-segur water wave equation via power index method, *J. Funct. Spaces*, **2022** (2022), 9405644.
16. A. M. Alghamdi, S. Gala, M. A. Ragusa, Global regularity for the 3d micropolar fluid flows, *Filomat*, **36** (2022), 1967–1970. https://doi.org/10.2298/FIL2206967A
17. H. Yüksekkaya, E. Piskin, Blow-up and decay of solutions for a delayed timoshenko equation with variable-exponents, *Miskolc Math. Notes*, **23** (2022), 1001–1022. https://doi.org/10.18514/MMN.2022.3890
18. S. Antontsev, Wave equation with p (x, t)-laplacian and damping term: blow-up of solutions, *C.R. Mec.*, **339** (2011), 751–755. https://doi.org/10.1016/j.crme.2011.09.001
19. S. Antontsev, Wave equation with p (x, t)-laplacian and damping term: existence and blow-up, *Differ. Equations Appl.*, **3** (2011), 503–525. https://doi.org/10.7153/dea-03-32

20. B. Guo, W. Gao, Blow-up of solutions to quasilinear hyperbolic equations with p (x, t)-laplacian and positive initial energy, *C.R. Mec.*, **342** (2014), 513–519. https://doi.org/10.1016/j.crme.2014.06.001
21. S. Antontsev, S. Shmarev, *Evolution PDEs with Nonstandard Growth Conditions*, Atlantis Press, Paris, France, 2015.
22. V. Galaktionov, S. Pohozaev, Blow-up and critical exponents for nonlinear hyperbolic equations, *Nonlinear Anal. Theory Methods Appl.*, **53** (2003), 453–466. https://doi.org/10.1016/S0362-546X(02)00311-5
23. S. H. Park, Blowup for nonlinearly damped viscoelastic equations with logarithmic source and delay terms, *Adv. Differ. Equations*, **2021** (2021), 1–14. https://doi.org/10.1186/s13662-020-03162-2
24. T. Yu, H. Yang, Initial boundary value problem for a class of strongly damped nonlinear wave equation, *J. Harbin Eng. Univ.*, **25** (2004), 254–256. https://doi.org/10.1057/palgrave.jphp.3190034
25. T. G. Ha, S. H. Park, Blow-up phenomena for a viscoelastic wave equation with strong damping and logarithmic nonlinearity, *Adv. Differ. Equations*, **2020** (2020), 1–17. https://doi.org/10.1186/s13662-019-2438-0
26. L. Ma, Z. B. Fang, Energy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source, *Math. Methods Appl. Sci.*, **41** (2018), 2639–2653. https://doi.org/10.1002/mma.4766
27. S. H. Park, Global nonexistence for logarithmic wave equations with nonlinear damping and distributed delay terms, *Nonlinear Anal. Real World Appl.*, **68** (2022), 103691. https://doi.org/10.1016/j.nonrwa.2022.103691
28. M. Kafini, S. Messaoudi, Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay, *Appl. Anal.*, **99** (2020), 530–547. https://doi.org/10.1080/00036811.2018.1504029
29. M. Kafini, S. Messaoudi, On the decay and global nonexistence of solutions to a damped wave equation with variable-exponent nonlinearity and delay, in *Annales Polonici Mathematici*, Instytut Matematyczny Polskiej Akademii Nauk, **122** (2019), 49–70.
30. B. Feng, Global well-posedness and stability for a viscoelastic plate equation with a time delay, *Math. Probl. Eng.*, **2015** (2015), 585021.
31. V. Komornik, V. Gattulli, Exact controllability and stabilization. the multiplier method, *SIAM Rev.*, **39** (1997), 351–351.