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Research article

Least energy sign-changing solutions for Kirchhoff-Schrödinger-Poisson system on bounded domains

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Abstract: We investigate the following nonlinear system

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2 dx)\Delta u + \phi u = \lambda u + \mu |u|^2 u, & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases}$$

with $a, b > 0$, $\lambda, \mu \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^3$ is bounded with smooth boundary. Let $\lambda_1 > 0$ be the first eigenvalue of $(-\Delta u, H_0^1(\Omega))$. We get that for certain $\widetilde{\mu} > 0$ there exists at least one least energy sign-changing solution for the above system if $\lambda < a\lambda_1$ and $\mu > \widetilde{\mu}$. In addition, we remark that the nonlinearity $\lambda u + \mu|u|^2 u$ does not satisfy the growth conditions.

Keywords: Kirchhoff type equation; Schrödinger-Poisson problem; sign-changing solutions; Nehari manifold; the growth conditions

1. Introduction and main results

We consider the following nonlinear problem in $\mathbb{R}^3$

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2 dx)\Delta u + \phi u = \lambda u + \mu |u|^2 u, & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

with $a, b > 0$, $\lambda, \mu \in \mathbb{R}$, $\Omega \subset \mathbb{R}^3$ is a bounded and $\partial\Omega$ is the smooth boundary of Ω.

In paper [1] Kirchhoff firstly introduced two Kirchhoff type equations with onnlocal term. There is an extensive literature concerning the sign-changing solutions for the Kirchhoff type equations, see e.g., [2–19]. System (1.1) is in connection with these equations.

System (1.1) also stems from the Schrödinger-Poisson problem

$$\begin{cases} -\Delta u + V(x) + \lambda\phi(x)u = f(u), \\ -\Delta\phi = u^2, \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^3$. For physical reason the appearance of nonlocal term in these equations makes them important and interesting. In the past several decades, the above problems or similar problems have captured the attention of many mathematicians. Especially, many authors (for example, [20–29]) used variational methods to prove the existence of sign-changing solutions for these problems. However, as far as we know, the nonlinearity always satisfies the growth conditions of super-linear near zero or super-three-linear near infinity except [3, 19, 22]. We remark that, although Cheng and Tang [3] obtained such solution when the nonlinearity satisfies asymptotically linear growth at infinity about u, their results still depend on the fact that nonlinearity is super-linear near zero about u. In [19], Zhong and Tang investigated the similar system

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda u + |u|^2 u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.3}$$

with $a, b > 0$ and Ω be smooth bounded in $\mathbb{R}^N, N = 1, 2, 3$. By Nehari manifold argument, they obtained that there exists $\wedge > 0$ such that the above Kirchhoff-type system has at least one such solution for all $0 < b < \wedge$ and $\lambda < a\lambda_1$, where λ_1 is the principal eigenvalue of $(-\Delta, H_0^1(\Omega))$. Obviously, the nonlinearity $\lambda u + |u|^2 u$ does not satisfy such super-linear growth conditions. However, since their results strongly depend on the condition $0 < b < \wedge$, the methods used in [19] seem not valid for all $b > 0$. Recently, Khoutir studied the Schrödinger-Poisson type system in $\mathbb{R}^3$ where the nonlinear term is a combination of linear terms and cubic terms. He also established the existence of such solutions.

Motivated by these works, we study the least energy sign-changing solution for (1.1). Our method is in collection with the works in [30], where authors dealt with p-Laplacian equation.

Let $L^p(\Omega)$ with $1 \le p < \infty$ be the Lebesgue space equipped with p-norm, and $H_0^1(\Omega)$ be the usual Sobolev space equipped with the inner product and the norm

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v dx, \ \|u\| = (u, u)^{\frac{1}{2}}.$$

Since Ω is bounded, the embedding from $H_0^1(\Omega)$ to $L^p(\Omega)$ is continuous and compact for $p \in [1, 6)$, and the embedding is not compact but continuous for $p = 6$.

The Lax-Milgram theorem shows that, given $u \in H_0^1(\Omega)$, there exists a unique $\phi_u \in H_0^1(\Omega)$ satisfying

$$-\Delta\phi_u = u^2,$$

where

$$\phi_u(x) = \frac{1}{4\pi}\int_\Omega \frac{u^2(y)}{|x - y|}dy.$$

It is easy to get that $\phi_u \ge 0$ and $\phi_{tu} = t^2\phi_u$, for any $t > 0$ and $u \in H_0^1(\Omega)$. Moreover ϕ_u has some properties.

Lemma 1.1. [20,31] *For every $u \in H_0^1(\Omega)$:*

(i) there exists $C > 0$ such that $\|\phi_u\| \leq C\|u\|^2$ and

$$\int_\Omega |\nabla \phi_u|^2 dx = \int_\Omega \phi_u u^2 dx \leq C\|u\|^4;$$

(ii) $\phi_{u_n} \to \phi_u$ in $H_0^1(\Omega)$ when $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, and

$$\lim_{n\to+\infty} \int_\Omega \phi_{u_n} u_n^2 dx = \int_\Omega \phi_u u^2 dx.$$

Let

$$\begin{aligned} J(u) &= \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}(\int_\Omega |\nabla u|^2 dx)^2 + \frac{1}{4}\int_\Omega \phi_u u^2 dx - \frac{\lambda}{2}\int_\Omega |u^2| dx - \frac{\mu}{4}\int_\Omega |u|^4 dx \\ &= \frac{1}{2}A_\lambda(u) + \frac{1}{4}B_\mu(u), \end{aligned}$$

where

$$A_\lambda(u) = a\|u\|^2 - \lambda \int_\Omega |u|^2 dx,$$

$$B_\mu(u) = b\|u\|^4 + \int_\Omega \phi_u u^2 dx - \mu \int_\Omega |u|^4 dx.$$

Obviously $J(u) \in C^1(H_0^1(\Omega), \mathbb{R})$ and

$$\langle J'(u), v\rangle = a\int_\Omega \nabla u \cdot \nabla v dx + b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega \phi_u u v dx - \lambda \int_\Omega u v dx - \mu \int_\Omega |u|^2 u v dx,$$

with $u, v \in H_0^1(\Omega)$.

In this article, we write

$$u^+ = \max\{u(x), 0\},\ u^- = \min\{u(x), 0\},$$

so, $u = u^+ + u^-$ and $|u| = u^+ - u^-$.

u is called a least energy sign-changing solution of system (1.1) if $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$ is a solution of system (1.1) and satisfies

$$J(u) = \inf\{J(v) : v \in H_0^1(\Omega), v^\pm \neq 0, J^{'}(v) = 0\}.$$

Let

$$\mathcal{M} = \{u \in H_0^1(\Omega) : u^\pm \neq 0, \langle J'(u), u^\pm\rangle = 0\},$$

and set

$$m := \inf_{u\in\mathcal{M}} J(u).$$

In order to get that u is a least energy sign-changing solution of the system (1.1), we will show that m is achieved at the critical point of $J(u)$ with $u \in \mathcal{M}$.

Let λ_1 be the first eigenvalue of the equation:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases}$$

and the following characterization holds

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{|u|_2^2}.$$

If $u \in H_0^1(\Omega)$ is a sign-changing solution of (1.1), then

$$\langle J'(u), u^{\pm} \rangle = 0.$$

Consequently, for any $\lambda < a\lambda_1$, we obtain that

$$b\|u\|^2\|u^{\pm}\|^2 + \int_\Omega \phi_u (u^{\pm})^2 dx - \mu \int_\Omega |u^{\pm}|^4 dx = -A_\lambda(u^{\pm}) < 0,$$

and

$$\mu > \max\left\{ \frac{b\|u\|^2\|u^+\|^2 + \int_\Omega \phi_u (u^+)^2 dx}{\int_\Omega |u^+|^4 dx}, \frac{b\|u\|^2\|u^-\|^2 + \int_\Omega \phi_u (u^-)^2 dx}{\int_\Omega |u^-|^4 dx} \right\}, \forall u \in \mathcal{M}.$$

On the contrary, if

$$\mu \leq \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \max\left\{ \frac{b\|u\|^2\|u^+\|^2 + \int_\Omega \phi_u (u^+)^2 dx}{\int_\Omega |u^+|^4 dx}, \frac{b\|u\|^2\|u^-\|^2 + \int_\Omega \phi_u (u^-)^2 dx}{\int_\Omega |u^-|^4 dx} \right\},$$

then for any $u \in H_0^1(\Omega)$ with $u^{\pm} \neq 0$, we obtain

$$b\|u\|^2\|u^+\|^2 + \int_\Omega \phi_u (u^+)^2 dx - \mu \int_\Omega |u^+|^4 dx \geq 0$$

or

$$b\|u\|^2\|u^-\|^2 + \int_\Omega \phi_u (u^-)^2 dx - \mu \int_\Omega |u^-|^4 dx \geq 0.$$

Due to $A_\lambda(u^{\pm}) > 0$, we get

$$\langle J'(u), u^+ \rangle \neq 0 \text{ or } \langle J'(u), u^- \rangle \neq 0.$$

Hence $\mathcal{M} = \emptyset$ and the sign-changing solution for (1.1) does not exist.

Therefore, we denote

$$\widetilde{\mu} := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \left\{ \max\left\{ \frac{b\|u\|^2\|u^+\|^2 + \int_\Omega \phi_u (u^+)^2 dx}{|u^+|_4^4}, \frac{b\|u\|^2\|u^-\|^2 + \int_\Omega \phi_u (u^-)^2 dx}{|u^-|_4^4} \right\} \right\}.$$

We have the main theorem.

Theorem 1.1. *If $\lambda < a\lambda_1$ and $\mu > \widetilde{\mu}$, there is at least one nontrivial least energy sign-changing solution for system* (1.1).

2. Technical lemmas

When $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, we have

$$\phi_u(x) = \phi_{u^-}(x) + \phi_{u^+}(x), \tag{2.1}$$

$$A_\lambda(u) = A_\lambda(u^-) + A_\lambda(u^+), \tag{2.2}$$

$$B_\mu(u) = B_\mu(u^+) + B_\mu(u^-) + 2b\|u^-\|^2\|u^+\|^2 + \int_\Omega \phi_{u^+}(u^-)^2dx + \int_\Omega \phi_{u^-}(u^+)^2dx, \tag{2.3}$$

$$J(u) = J(u^-) + J(u^+) + \frac{b}{2}\|u^-\|^2\|u^+\|^2 + \frac{1}{4}\int_\Omega \phi_{u^+}(u^-)^2dx + \frac{1}{4}\int_\Omega \phi_{u^-}(u^+)^2dx, \tag{2.4}$$

$$\langle J'(u), u^\pm\rangle = \langle J'(u^\pm), u^\pm\rangle + b\|u^-\|^2\|u^+\|^2 + \int_\Omega \phi_{u^\mp}(u^\pm)^2dx. \tag{2.5}$$

Defining

$$\begin{aligned}\mathcal{H}_\mu &:= \left\{u \in H_0^1(\Omega), u^\pm \neq 0 : B_\mu(u^\pm) + b\|u^\mp\|^2\|u^\pm\|^2 + \int_\Omega \phi_{u^\mp}(u^\pm)^2dx < 0\right\}\\ &= \left\{u \in H_0^1(\Omega), u^\pm \neq 0 : b\|u\|^2\|u^\pm\|^2 + \int_\Omega \phi_u(u^\pm)^2dx - \mu\int_\Omega |u^\pm|^4dx < 0\right\}.\end{aligned} \tag{2.6}$$

Lemma 2.1. *If $\lambda < a\lambda_1$ and $\mu > \widetilde{\mu}$, then $\mathcal{H}_\mu \neq \emptyset$ and $\mathcal{M} \subset \mathcal{H}_\mu$.*

Proof. Assume that $\mu > \widetilde{\mu}$, then there exists $v \in H_0^1(\Omega)$ with $v^\pm \neq 0$ satisfying

$$\mu > \max\left\{\frac{b\|v\|^2\|v^+\|^2 + \int_\Omega \phi_v(v^+)^2dx}{|v^+|_4^4}, \frac{b\|v\|^2\|v^-\|^2 + \int_\Omega \phi_v(v^-)^2dx}{|v^-|_4^4}\right\} \geq \widetilde{\mu}.$$

So,

$$b\|v\|^2\|v^\pm\|^2 + \int_\Omega \phi_v(v^\pm)^2dx - \mu|v^\pm|_4^4 < 0.$$

This means that $v \in \mathcal{H}_\mu$. Thus $\mathcal{H}_\mu \neq \emptyset$.

For any $u \in \mathcal{M}$, we have

$$b\|u\|^2\|u^\pm\|^2 + \int_\Omega \phi_u(u^\pm)^2dx - \mu\int_\Omega |u^\pm|^4dx = -A_\lambda(u^\pm). \tag{2.7}$$

If $0 < \lambda < a\lambda_1$, we get

$$A_\lambda(u^\pm) = a\|u^\pm\|^2 - \lambda\int_\Omega |u^\pm|^2dx \geq (a - \frac{\lambda}{\lambda_1})\|u^\pm\|^2 > 0. \tag{2.8}$$

If $\lambda \leq 0$, it is obviously that $A_\lambda(u^\pm) > 0$. Then (2.7) and (2.8) implies that

$$b\|u\|^2\|u^\pm\|^2 + \int_\Omega \phi_u(u^\pm)^2dx - \mu\int_\Omega |u^\pm|^4dx < 0.$$

This means $u \in \mathcal{H}_\mu$. So we get that $\mathcal{M} \subset \mathcal{H}_\mu$.

Lemma 2.2. *Suppose that $\lambda < a\lambda_1$ and $\mu > \widetilde{\mu}$ hold, then for every $u \in \mathcal{H}_\mu$, there exists exactly one pair of positive numbers (l_u, n_u) such that $l_u u^+ + n_u u^- \in \mathcal{M}$ and $J(l_u u^+ + n_u u^-) = \max_{l,n>0} J(lu^+ + nu^-)$. And if $\langle J'(u), u^\pm\rangle \le 0$, we have $0 < l_u, n_u \le 1$.*

Proof. By the definition of $\mathcal{M}$ and (2.5), $lu^+ + nu^- \in \mathcal{M}$ with $u \in \mathcal{H}_\mu$ if and only if (l, n) with $l, n > 0$ satisfies

$$\begin{cases} \langle J'(lu^+ + nu^-), lu^+\rangle = l^2 A_\lambda(u^+) + l^4 B_\mu(u^+) + bl^2n^2\|u^+\|^2\|u^-\|^2 + l^2n^2 \int_\Omega \phi_{u^-}(u^+)^2 dx = 0, \\ \langle J'(lu^+ + nu^-), nu^-\rangle = n^2 A_\lambda(u^-) + n^4 B_\mu(u^-) + bl^2n^2\|u^+\|^2\|u^-\|^2 + l^2n^2 \int_\Omega \phi_{u^+}(u^-)^2 dx = 0. \end{cases}$$

Thus

$$\begin{cases} l^2 B_\mu(u^+) + n^2[b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx] = -A_\lambda(u^+), \\ n^2 B_\mu(u^-) + l^2[b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx] = -A_\lambda(u^-), \end{cases} \tag{2.9}$$

which can be rewritten as

$$\begin{bmatrix} B_\mu(u^+) & b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx \\ b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx & B_\mu(u^-) \end{bmatrix} \begin{bmatrix} l^2 \\ n^2 \end{bmatrix} = \begin{bmatrix} -A_\lambda(u^+) \\ -A_\lambda(u^-) \end{bmatrix}.$$

Then $u \in \mathcal{H}_\mu$ implies

$$\begin{vmatrix} B_\mu(u^+) & b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx \\ b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx & B_\mu(u^-) \end{vmatrix}$$

$$= B_\mu(u^+)B_\mu(u^-) - [b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx][b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx] > 0.$$

Hence, due to $A_\lambda(u^\pm) > 0$, there exists exactly one solution (l_u, n_u) with $l_u, n_u > 0$ for system (2.9).

For $u \in \mathcal{H}_\mu$, we define $\varphi_u : {\mathbb{R}_+}^2 \to \mathbb{R}$ by

$$\varphi_u(l, n) = J(lu^+ + nu^-).$$

Next, we show that

$$\varphi_u(l_u, n_u) = J(l_u u^+ + n_u u^-) = \max_{l,n>0} J(lu^+ + nu^-),$$

where (l_u, n_u) is the unique solution of system (2.9).

Straightforward computations yield

$$\begin{aligned} \nabla\varphi_u(l, n) &= \left(\frac{\partial\varphi_u}{\partial l}(l, n), \frac{\partial\varphi_u}{\partial n}(l, n)\right) \\ &= (\langle J'(lu^+ + nu^-), u^+\rangle, \langle J'(lu^+ + nu^-), u^-\rangle) \\ &= \left(\frac{1}{l}\langle J'(lu^+ + nu^-), lu^+\rangle, \frac{1}{n}\langle J'(lu^+ + nu^-), nu^-\rangle\right), \end{aligned}$$

which means that a positive pair (l, n) is a critical point of φ_u if and only if $lu^+ + nu^- \in \mathcal{M}$. Then we deduce that (l_u, n_u) with $l_u, n_u > 0$ is the unique critical point of the function φ_u.

Since $u \in \mathcal{H}_\mu$, we get

$$\begin{cases} \frac{\partial^2 \varphi_u}{\partial l^2}(l_u, n_u) = A_\lambda(u^+) + 3l_u^2 B_\mu(u^+) + n_u^2[b\|u^-\|^2\|u^+\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx] = 2l_u^2 B_\mu(u^+) < 0, \\ \frac{\partial^2 \varphi_u}{\partial n^2}(l_u, n_u) = A_\lambda(u^-) + 3n_u^2 B_\mu(u^-) + l_u^2[b\|u^-\|^2\|u^+\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx] = 2n_u^2 B_\mu(u^-) < 0, \\ \frac{\partial^2 \varphi_u}{\partial l \partial n}(l_u, n_u) = 2l_u n_u[b\|u^-\|^2\|u^+\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx], \\ \frac{\partial^2 \varphi_u}{\partial n \partial l}(l_u, n_u) = 2l_u n_u[b\|u^-\|^2\|u^+\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx]. \end{cases} \tag{2.10}$$

Setting $Q := \frac{\partial^2 \varphi_u}{\partial l^2}(l_u, n_u)\frac{\partial^2 \varphi_u}{\partial n^2}(l_u, n_u) - \frac{\partial^2 \varphi_u}{\partial l \partial n}(l_u, n_u)\frac{\partial^2 \varphi_u}{\partial n \partial l}(l_u, n_u)$, it follows that

$$\begin{aligned} Q =& 4l_u^2 n_u^2 B_\mu(u^+)B_\mu(u^-) - 4l_u^2 n_u^2[b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx] \\ & [b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx] \\ =& 4l_u^2 n_u^2\{B_\mu(u^+)B_\mu(u^-) - [b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx] \\ & [b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx]\} \\ >& 0, \end{aligned}$$

which implies that the Hessian matrix of φ_u is negative definite at (l_u, n_u). Then we have $\varphi_u(l_u, n_u) = \max_{l,n>0} \varphi_u(lu^+ + nu^-)$, that is

$$J(l_u u^+ + n_u u^-) = \max_{l,n>0} J(lu^+ + nu^-).$$

When $l_u \geq n_u > 0$. For $l_u u^+ + n_u u^- \in \mathcal{M}$, we have

$$\begin{cases} l_u^2 B_\mu(u^+) + n_u^2[b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx] = -A_\lambda(u^+), \\ n_u^2 B_\mu(u^-) + l_u^2[b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^+}(u^-)^2 dx] = -A_\lambda(u^-). \end{cases} \tag{2.11}$$

And $\langle J'(u), u^\pm \rangle \leq 0$ implies

$$b\|u\|^2\|u^\pm\|^2 + \int_\Omega \phi_u(u^\pm)^2 dx - \mu \int_\Omega |u^\pm|^4 dx \leq -A_\lambda(u^\pm) < 0. \tag{2.12}$$

By (2.11) and (2.12) we have

$$\begin{aligned} -l_u^2 A_\lambda(u^+) &\geq l_u^2[b\|u\|^2\|u^+\|^2 + \int_\Omega \phi_u(u^+)^2 dx - \mu \int_\Omega |u^+|^4 dx] \\ &= l_u^2[B_\mu(u^+) + b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx] \\ &\geq l_u^2 B_\mu(u^+) + n_u^2[b\|u^+\|^2\|u^-\|^2 + \int_\Omega \phi_{u^-}(u^+)^2 dx] \\ &= -A_\lambda(u^+). \end{aligned}$$

Due to $A_\lambda(u^+) > 0$, we get that $0 < n_u \leq l_u \leq 1$.

When $n_u \geq l_u > 0$, with similar discussion we get $0 < l_u \leq n_u \leq 1$. In conclusion, we obtain that $0 < l_u, n_u \leq 1$.

Lemma 2.3. *Assume that $\lambda < a\lambda_1$, $\mu > \widetilde{\mu}$, then for all $u \in \mathcal{M}$ there exists $\sigma > 0$ satisfying $\|u^{\pm}\| \geq \sigma$ and $m > 0$.*

Proof. For any $u \in \mathcal{M}$, we have

$$a\|u^{\pm}\|^2 + b\|u\|^2\|u^{\pm}\|^2 + \int_{\Omega} \phi_u (u^{\pm})^2 dx - \lambda \int_{\Omega} |u^{\pm}|^2 dx = \mu \int_{\Omega} |u^{\pm}|^4 dx.$$

When $0 < \lambda < a\lambda_1$, by Sobolev inequalities, we obtain that

$$\begin{aligned} 0 < (a - \frac{\lambda}{\lambda_1})\|u^{\pm}\|^2 &\leq a\|u^{\pm}\|^2 - \lambda \int_{\Omega} |u^{\pm}|^2 dx \\ &\leq a\|u^{\pm}\|^2 + b\|u\|^2\|u^{\pm}\|^2 + \int_{\Omega} \phi_u (u^{\pm})^2 dx - \lambda \int_{\Omega} |u^{\pm}|^2 dx \\ &= \mu \int_{\Omega} |u^{\pm}|^4 dx \\ &\leq \mu C \|u^{\pm}\|^4, \end{aligned}$$

thus

$$\|u^{\pm}\| \geq \sqrt{\frac{a\lambda_1 - \lambda}{\mu C \lambda_1}} := \sigma_1 > 0.$$

When $\lambda \leq 0$, with similar discussion we have

$$\begin{aligned} 0 < a\|u^{\pm}\|^2 &\leq a\|u^{\pm}\|^2 - \lambda \int_{\Omega} |u^{\pm}|^2 dx \\ &\leq \mu C \|u^{\pm}\|^4, \end{aligned}$$

which shows that

$$\|u^{\pm}\| \geq \sqrt{\frac{a}{\mu C}} := \sigma_2 > 0.$$

Let $\sigma = \min\{\sigma_1, \sigma_2\}$, then $\|u^{\pm}\| \geq \sigma$ for all $u \in \mathcal{M}$. Furthermore

$$\begin{aligned} J(u) &= J(u) - \frac{1}{4}\langle J'(u), u\rangle \\ &= \frac{a}{4}\|u\|^2 - \frac{\lambda}{4}\int_{\Omega} |u|^2 dx \\ &= \frac{1}{4} A_{\lambda}(u) \\ &\geq \frac{1}{4}(a - \frac{\lambda}{\lambda_1})\|u\|^2 \\ &\geq \frac{a\lambda_1 - \lambda}{4\lambda_1}\sigma^2 \\ &> 0. \end{aligned}$$

Thus we obtain that $m > 0$.

3. The proof of Theorem 1.1

It is easy for the situation of $\lambda \leq 0$. Thus we only give the proof in the case of $0 < \lambda < a\lambda_1$. The proof of Theorem 1.1 is divided into the following two steps.

Step 1. We testify that $J(u)$ attains its infimum on $\mathcal{M}$.

Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence of m, that is,

$$J(u_n) \to m \ \ as \ \ n \to \infty. \tag{3.1}$$

Without loss of generality, we assume that $J(u_n) \leq 2m$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} 2m \geq J(u_n) &= J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle \\ &= \frac{a}{4}\|u_n\|^2 - \frac{\lambda}{4}\int_\Omega |u_n|^2 dx \\ &= \frac{1}{4}A_\lambda(u_n) \\ &\geq \frac{1}{4}(a - \frac{\lambda}{\lambda_1})\|u_n\|^2, \end{aligned}$$

which indicates that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then there exist a subsequence still denoted by $\{u_n\}$ and a u in $H_0^1(\Omega)$ satisfying

$$\begin{cases} u_n \rightharpoonup u \ \ in \ \ H_0^1(\Omega), \\ u_n^\pm \rightharpoonup u^\pm \ \ in \ \ H_0^1(\Omega), \\ u_n^\pm \to u^\pm \ \ in \ \ L^p(\Omega), p \in [1, 6), \\ u_n^\pm \to u^\pm \ \ a.e. \ \ x \in \Omega. \end{cases} \tag{3.2}$$

When $\{u_n\} \subset \mathcal{M}$, we get

$$\begin{aligned} a\|u_n^\pm\|^2 &\leq a\|u_n^\pm\|^2 + b\|u_n\|^2\|u_n^\pm\|^2 + \int_\Omega \phi_{u_n}(u_n^\pm)^2 dx \\ &= \lambda\int_\Omega |u_n^\pm|^2 dx + \mu\int_\Omega |u_n^\pm|^4 dx \\ &\leq \frac{\lambda}{\lambda_1}\|u_n^\pm\|^2 + \mu\int_\Omega |u_n^\pm|^4 dx. \end{aligned}$$

Then,

$$\mu\int_\Omega |u_n^\pm|^4 dx \geq (a - \frac{\lambda}{\lambda_1})\|u_n^\pm\|^2 > 0.$$

By Lemma 2.3, we get

$$\int_\Omega |u^\pm|^4 dx \geq \frac{a\lambda_1 - \lambda}{\mu\lambda_1}\|u^\pm\|^2 \geq \frac{a\lambda_1 - \lambda}{\mu\lambda_1}\sigma^2 > 0,$$

which yields that $u^\pm \neq 0$.

Since $\{u_n\} \subset \mathcal{M}$, it follows from Lemma 1.1-(iv), (3.2) and the weakly lower semicontinuity of norm that

$$\begin{aligned}
&a\|u^\pm\|^2 + b\|u\|^2\|u^\pm\|^2 + \int_\Omega \phi_u (u^\pm)^2 dx \\
&\le \liminf_{n\to\infty}\{a\|u_n^\pm\|^2 + b\|u_n\|^2\|u_n^\pm\|^2 + \int_\Omega \phi_{u_n}(u_n^\pm)^2 dx\} \\
&= \liminf_{n\to\infty}\{\lambda \int_\Omega |u_n^\pm|^2 dx + \mu \int_\Omega |u_n^\pm|^4 dx\} \\
&= \lambda \int_\Omega |u^\pm|^2 dx + \mu \int_\Omega |u^\pm|^4 dx,
\end{aligned}$$

thus

$$\langle J'(u), u^\pm \rangle \le 0. \tag{3.3}$$

By (2.6) and (2.12), we obtain that $u \in \mathcal{H}_\mu$.

By Lemma 2.2, there exists $0 < l_u, n_u \le 1$ such that $u := l_u u^+ + n_u u^- \in \mathcal{M}$. Since the norm is weakly lower semicontinuous, by (3.2) we have

$$\begin{aligned}
m &\le J(l_u u^+ + n_u u^-) \\
&= J(l_u u^+ + n_u u^-) - \frac{1}{4}\langle J'(l_u u^+ + n_u u^-), l_u u^+ + n_u u^- \rangle \\
&= \frac{l_u^2}{4} A_\lambda(u^+) + \frac{n_u^2}{4} A_\lambda(u^-) \\
&\le \frac{1}{4} A_\lambda(u^+) + \frac{1}{4} A_\lambda(u^-) \\
&= \frac{1}{4} A_\lambda(u) \\
&= \frac{1}{4}(a\|u\|^2 - \lambda |u|_2^2) \\
&\le \liminf_{n\to\infty}\{\frac{1}{4}(a\|u_n\|^2 - \lambda |u_n|_2^2)\} \\
&= \liminf_{n\to\infty}\{J(u_n) - \frac{1}{4}\langle J'(u_n), u_n \rangle\} = m.
\end{aligned}$$

Thus we get $l_u = n_u = 1$ and $J(u) = m$.

Step 2. We testify that $J'(u) = 0$.

Assume that $J'(u) \neq 0$. Since $J'(u)$ is continuous, there exist $\delta, \xi > 0$ satisfying

$$\|J'(v)\| \ge \xi, \forall v \in H_0^1(\Omega) \ \ with \ \ \|v - u\| \le 3\delta. \tag{3.4}$$

For $u \in \mathcal{M}$, we have $\langle J'(u), u^\pm \rangle = 0$. From Step 1 and Lemma 2.2, we obtain that $(1, 1)$ is the global maximum point of $J(lu^+ + nu^-)$ on $\mathbb{R}_+ \times \mathbb{R}_+$. Thus

$$J(lu^+ + nu^-) < J(u^+ + u^-) = J(u) = m, \quad for \ (l, n) \in (\mathbb{R}_+, \mathbb{R}_+)\backslash(1, 1). \tag{3.5}$$

Due to $A_\lambda(u^\pm) > 0$, we can choose a θ_1 small enough in $(0, 1)$ such that

$$\min_{t\in[1-\theta_1,1+\theta_1]} A_\lambda(tu^\pm) > 0. \tag{3.6}$$

Let $\theta_2 \in (0, \min\{\frac{1}{2}, \frac{\delta}{\sqrt{2}\|u\|}\}), \theta = \min\{\theta_1, \theta_2\}$. Define $\Gamma := (1-\theta, 1+\theta) \times (1-\theta, 1+\theta)$ and the function $\mathcal{G} : \Gamma \to H_0^1(\Omega)$ by $\mathcal{G}(l, n) = lu^+ + nu^-$, $(l, n) \in \Gamma$. It follows from (3.5) that

$$m^* := \max_{(l,n)\in\partial\Gamma} J(\mathcal{G}(l, n)) < J(\mathcal{G}(1, 1)) = m. \tag{3.7}$$

Let $\varepsilon := \min\{\frac{m-m^*}{3}, \frac{\xi\delta}{8}\}$ and $S_{n\delta} := \{v \in H_0^1(\Omega)\ \textit{with}\ v^\pm \neq 0 : \|v - u\| \leq n\delta\}, n = 1, 2$. By the deformation lemma in [32], we get that there exists a deformation $\eta \in C([0, 1] \times H_0^1(\Omega), H_0^1(\Omega))$ satisfying

(i) $\eta(\alpha, v) = v$ if $\alpha = 0$ or if $v \notin J^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $J(\eta(\alpha, v)) < m$ for all $v \in S_\delta$ with $J(v) \leq m$ and $\alpha \in (0, 1]$;

(iii) $J(\eta(\alpha, v)) \leq J(v)$ for all $v \in H_0^1(\Omega)$ and $\alpha \in [0, 1]$.

Firstly we will show that

$$\max_{\{(l,n)\in\Gamma\}} J(\eta(\alpha, \mathcal{G}(l, n))) < m. \tag{3.8}$$

By Lemma 2.2, Step 1, (3.5) and the expression of $\mathcal{G}(l, n)$, we obtain that $J(\mathcal{G}(l, n)) \leq m$. Also,

$$\begin{aligned}
\|\mathcal{G}(l, n) - u\|^2 &= \|lu^+ + nu^- - (u^+ + u^-)\|^2 \\
&\leq 2((l - 1)^2\|u^+\|^2 + (n - 1)^2\|u^-\|^2) \\
&\leq 2\theta^2\|u\|^2 \\
&\leq \delta^2,
\end{aligned}$$

this means $\mathcal{G}(l, n) \in S_\delta$. Accordingly (ii) implies

$$\max_{\{(l,n)\in\Gamma,\mathcal{G}(l,n)\in S_\delta\}} J(\eta(\alpha, \mathcal{G}(l, n))) < m, \forall\alpha \in (0, 1].$$

From Lemma 2.2 and (iii), we can deduce that

$$J(\eta(\alpha, \mathcal{G}(l, n))) \leq J(\mathcal{G}(l, n)),\ \textit{for all}\ \mathcal{G}(l, n) \in H_0^1(\Omega)\ \textit{and}\ \alpha \in [0, 1],$$

furthermore,

$$\begin{aligned}
&\max_{\{(l,n)\in\Gamma,\mathcal{G}(l,n)\notin S_\delta\}} J(\eta(\alpha, \mathcal{G}(l, n))) \\
\leq &\max_{\{(l,n)\in\Gamma,\mathcal{G}(l,n)\notin S_\delta\}} J(\mathcal{G}(l, n)) < m, \forall\alpha \in [0, 1].
\end{aligned}$$

To sum up, (3.8) can be concluded.

Since η and $A_\lambda(u)$ are continuous, by (3.6), there exists $\alpha_0 \in (0, 1)$ satisfying

$$A_\lambda(\eta^\pm(\alpha_0, \mathcal{G}(l, n))) > 0, \forall(l, n) \in \Gamma. \tag{3.9}$$

Secondly, we prove that

$$\eta(\alpha_0, \mathcal{G}(\Gamma)) \cap \mathcal{M} \neq \emptyset. \tag{3.10}$$

Set $g(l,n) = \eta(\alpha_0, \mathcal{G}(l,n))$, and consider the maps $\Phi_1, \Phi_2 : \Gamma \to \mathbb{R} \times \mathbb{R}$ defined as

$$\Phi_1(l,n) = \left(\langle J'(lu^+ + nu^-), lu^+\rangle, \langle J'(lu^+ + nu^-), nu^-\rangle\right),$$
$$\Phi_2(l,n) = \left(\langle J'(g(l,n)), g^+(l,n)\rangle, \langle J'(g(l,n)), g^-(l,n)\rangle\right).$$

Since $u \in \mathcal{M}$, we have

$$\begin{aligned}\Phi_1(1,1) &= (\langle J'(u^+ + u^-), u^+\rangle, \langle J'(u^+ + u^-), u^-\rangle)\\ &= (\langle J'(u), u^+\rangle, \langle J'(u), u^-\rangle) = (0,0),\end{aligned}$$

which implies that $(1,1)$ is the only isolated zero of $\Phi_1(l,n)$. By the degree theory, we obtain that $deg(\Phi_1, \Gamma, 0) = 1$.

For $\varepsilon \le \frac{m-m^*}{3}$, we have

$$m^* := \max_{(l,n)\in\partial\Gamma} J(\mathcal{G}(l,n)) = \max_{(l,n)\in\partial\Gamma} J(lu^+ + nu^-) \le m - 3\varepsilon < m - 2\varepsilon.$$

Therefore, $\mathcal{G}(l,n) \notin J^{-1}([m-2\varepsilon, m+2\varepsilon])$ with $(l,n) \in \partial\Gamma$, and $\mathcal{G}(l,n) \notin J^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}$. It follows from (i) that

$$\eta(\alpha, \mathcal{G}(l,n)) = \mathcal{G}(l,n), \forall (l,n) \in \partial\Gamma \textit{ and } \alpha \in [0,1].$$

Hence, we conclude that

$$\Phi_1(l,n) = \Phi_2(l,n), \forall (l,n) \in \partial\Gamma.$$

The boundary value property of the degree suggests that

$$deg(\Phi_2, \Gamma, 0) = deg(\Phi_1, \Gamma, 0) = 1,$$

which indicates that there exists $(l_u, n_u) \in \Gamma$ such that $\Phi_2(l_u, n_u) = 0$. Namely,

$$\Phi_2(l_u, n_u) = (\langle J'(g(l_u,n_u)), g^+(l_u,n_u)\rangle, \langle J'(g(l_u,n_u)), g^-(l_u,n_u)\rangle) = 0,$$

which means $g(l_u, n_u)) = \eta(\alpha_0, \mathcal{G}(l,n)) \in \mathcal{M}$. Thus $\eta(\alpha_0, \mathcal{G}(\Gamma)) \cap \mathcal{M} \neq \emptyset$, which contradicts with (3.8).

Thus we obtain that $J'(u) = 0$ and $J(u) = m$. We complete the proof of Theorem 1.1.

Conflict of interest

The authors declare there is no conflicts of interest.

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