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Research article

Existence and uniqueness of solution for a class of non-Newtonian fluids with non-Newtonian potential and damping

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Abstract: This paper discusses the existence and uniqueness of local strong solution for a class of 1D non-Newtonian fluids with non-Newtonian potential and damping term. Here we allow the initial vacuum and viscosity term to be fully nonlinear.

Keywords: strong solution; non-Newtonian fluid; vacuum; damping; non-Newtonian potential

1. Introduction and main result

We consider the following class of 1D non-Newtonian fluids

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ \rho\Phi_x + \left(\rho u^2\right)_x + (\rho u)_t - \left(|u_x|^{p-2}u_x\right)_x + P_x = -\alpha\rho u,\\ \left(\left|\Phi_x^{q-2}\right|\Phi_x\right)_x = 4\pi g\left(\rho - \frac{1}{|\Omega|}\int_\Omega \rho dx\right),\end{cases} \tag{1.1}$$

where u, ρ, $M = \rho u$ and $P = A\rho^\gamma (A > 0, \gamma > 1)$ denote velocity, the unknown density, momentum and pressure, respectively, $g > 0$ is the acceleration of gravity and Φ is the gravitational potential. The constant $\alpha > 0$ models friction. Without losing generality, throughout the paper we take $\alpha = 1$. The initial and boundary value conditions of Eq (1.1) are as follows

$$\begin{cases}(\rho, u, \Phi)|_{t=0} = (\rho_0, u_0, \Phi_0), & \text{for all } x \in [0,1],\\ u(0,t) = u(1,t) = 0, & \text{for all } t \in [0,T],\\ \Phi(0,t) = \Phi(1,t) = 0, & \text{for all } t \in [0,T].\end{cases} \tag{1.2}$$

Ω is considered as a one-dimensional bounded interval here. Furthermore, for simplicity, we only assume the $\Omega = I = (0,1)$, $\Omega_T = I \times (0,T)$. The initial density $\rho_0 \geq 0$, p and q are given constants, and they are both studied in the case of less than 2, where since the method of study is similar for $1 < p < \frac{4}{3}$ and $\frac{4}{3} < p < 2$, we next study only the case of $\frac{4}{3} < p < 2$.

According to classical Newtonian fluid mechanics, in parallel fluids, the shear force is proportional to shear velocity, and its proportion is the viscosity coefficient, i.e.,

$$\Gamma = \Gamma(\rho, \nabla u) = \mu \nabla u, \quad \mu > 0.$$

Generally, we call a fluid with the above properties a Newtonian fluid. Accordingly, a fluid does not have this property is called a non-Newtonian fluid. For non-Newton fluids, $\Gamma(\rho, \nabla u)$ has a reasonable choice (see Ladyzhenskaya [1])

$$\Gamma_{ij} = \left(\mu_0 + \mu_1 |E(\nabla u)|^{p-2}\right) E_{ij}(\nabla u),$$

and

$$E_{ij}(\nabla u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In chemistry, biomechanics, glaciology, geology, and blood rheology, there are many problems in non-Newtonian fluids, which lead to an interest in studying non-Newtonian fluids [2–4]. There are many theoretical and experimental studies in non-Newtonian fluid flow.

In this paper, the vacuum condition i.e., the initial density, is zero. Strictly speaking, a vacuum, a gas state below atmospheric pressure in a given space, is a physical phenomenon. In real life, vacuum distillation, vacuum drying, and vacuum concentration are typical vacuum cases. The role of vacuum in vacuum distillation is mainly to reduce the boiling point temperature of substances and reduce the influence of temperature factors on substances. Vacuum drying and concentration both use a vacuum environment to accelerate the volatilization and evaporation of specific substances or enable the whole process to be completed at lower conditions. In particular, non-Newtonian fluids can expand in a vacuum. According to the experiment, as the air in the vacuum bottle decreases, tiny bubbles gradually appear on the surface of the non-Newtonian fluid, and the bubbles expand until they spill the container out. In any case, the theoretical knowledge of non-Newtonian fluids under vacuum must be continuously refined. To this end, this article allows for an initial vacuum.

In 1996, J. Málek, J. Nečas, M. Rokyta, M. Růžička divided non-Newnewton flows with regard to p in the monograph [5] : when $1 < p < 2$, we call such a fluid a shear thinning fluid, when $p > 2$, we call such a fluid a shear thickening fluid. For non-Newtonian fluids, Yuan Hongjun and Xu Xiaojing [6] studied existence of a solution and whether it is unique of a class of non-Newtonian fluid solution with singularity and vacuum. Takashi Suzuki and Takayuki Kobayashi [7] proved the existence of weak solution to the Navier-Stokes-Poisson equation. Meng Qiu and Yuan Hongjun [8] proved the existence and uniqueness of a class of local solution under conditions where compressible non-Newtonian fluids with a non-Newtonian bit potential in a one-dimensional bounded interval. Song Yukun, Yuan Hongjun, and Yang Chen [9] investigated the existence and uniqueness of a class of local solution in the presence of isentropic compressible non-Newtonian fluids in a one-dimensional bounded area. Liu Hongzhi, Yuan Hongjun, Qiao Jiezeng and Li Fanpei [10] constructed the global existence of robust solution of Navier-Stokes equations with non-Newtonian potential. Li Huapeng and Yuan Hongjun [11] demonstrated the local existence and uniqueness of 1D non-Newtonian fluid solution with damping.

The damping item comes from resistance to fluid motion. Model (1.1) describes more natural phenomena. For example, porous media flow. We can refer to [12–15] contents of the damp item.

However, for non-Newtonian fluid, there is no degradation result with non-Newtonian potential and damp item. We construct a system (1.1)–(1.2) with local existence and uniqueness of the strong solution of non-Newtonian fluids with the non-Newtonian potentials and friction damping. The result is the following theorem:

Theorem 1. *Assume that*

$$\frac{4}{3} < p < 2, 1 < q < 2, 0 \le \rho_0 \in H^1(\Omega), u_0 \in H_0^1(I) \cap H^2(I), \Phi_0 \in H_0^2(I) \cap H^3(I)$$

and that there is a function $g \in L^2(I)$, makes the following equation true almost everywhere on I:

$$-\left(|u_{0x}|^{p-2}u_{0x}\right)_x + P_x(\rho_0) = \rho_0^{\frac{1}{2}}g, \tag{1.3}$$

then there exists a small time $T_* \in (0, +\infty)$ and a unique strong solution (ρ, u, Φ) to the initial boundary value problem (1.1)–(1.2) *such that:*

$$\begin{cases} \rho \in C\left([0\,,\,T_*]\,;H^1(I)\right), & \rho_t \in C\left([0\,,\,T_*]\,;L^2(I)\right), \\ \Phi \in L^\infty\left(0, T_*; H^2(I)\right), & \Phi_t \in L^\infty\left(0, T_*; H^1(I)\right), \\ u \in C\left([0\,,\,T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I)\right), & u_t \in L^2\left(0, T_*; H_0^1(I)\right), \\ \sqrt{\rho}u_t \in L^\infty\left(0, T_*; L^2(I)\right), & \left(|u_x|^{p-2}u_x\right)_x \in C\left([0\,,\,T_*]\,;L^2(I)\right). \end{cases} \tag{1.4}$$

For the above theorem, we will be divided it into four parts to prove. In the first part, we use the iterative method to get the approximate solution system of problems (1.1)–(1.2) and then make a consistent estimate of its approximate solution. In the second part, the convergence of the approximate solution is proved by the weak convergence method. In the third and fourth parts, we demonstrated that a locally strong solution to problems (1.1)–(1.2) exists uniquely.

2. Uniform estimates

Lemma 1. *(Embedding inequality). Assume that $f = 0$ on $\partial\Omega$, here $\Omega \in R^1$ is bounded and open, $f \in C^{2+\alpha}(\bar{\Omega})$. Then*

$$|f'|_{L^\infty(\Omega)} \le d^{\frac{1}{2}}(\Omega)\,|f''|_{L^2(\Omega)}$$

where $d(\Omega)$ denotes the length of Ω.

See the literature [4] for proof.

The system of Eqs (1.1) we studied contains more unknowns, and $(1.1)_2$ and $(1.1)_3$ are non-linear, so we cannot get a direct solution (from Abelian theorem, the fifth and higher order algebraic equations have no analytical solution). Therefore, we are inspired to seek approximate solutions to the system of equations.

Therefore, we apply an iterative approach to problems (1.1)–(1.2), which yields its approximate solution system.

$$\rho_t^k + u^{k-1}\rho_x^k + u_x^{k-1}\rho^k = 0, \tag{2.1}$$

$$\rho^k u_t^k + \rho^k u^{k-1} u_x^k + \rho^k \Phi_x^k + Ł_p^\varepsilon u^k + P_x^k = -\rho^k u^k, \tag{2.2}$$

$$\left(\left|\Phi_x^k\right|^{q-2}\Phi_x^k\right)_x = 4\pi g\left(\rho^k - m_0\right), \tag{2.3}$$

$$\begin{cases}\left(\rho^k, u^k, \Phi^k\right)\Big|_{t=0} = \left(\rho_0^\delta, u_0^\varepsilon, \Phi_0^\varepsilon\right), & x \in [0,1],\\ u^k(0,t) = u^k(1,t) = 0, & t \in [0,T],\end{cases} \tag{2.4}$$

we take the initial mass $m_0 = \int_0^1 \rho_0(x)\mathrm{d}x > 0$, $P^k = \mathrm{P}\left(\rho^k\right) = A\left(\rho^k\right)^\gamma$, $A > 0$, $\gamma > 1$, $\rho_0^\delta = \delta + \rho_0 * J_\delta$, $\delta > 0$,

$$Ł_p^\varepsilon u^k = -\left[\left(\frac{\varepsilon\left(u_x^k\right)^2 + 1}{\left(u_x^k\right)^2 + \varepsilon}\right)^{\frac{2-p}{2}} u_x^k\right]_x,$$

For problem (2.5), $u_0^\varepsilon \in H^2(I) \cap H_0^1(I)$ is a smooth solution to it

$$\begin{cases}-\left[\left(\frac{\varepsilon\left(u_{0x}^\varepsilon\right)^2+1}{\left(u_{0x}^\varepsilon\right)^2+\varepsilon}\right)^{\frac{2-p}{2}} u_{0x}\right]_x + P_x(\rho_0^\delta) = (\rho_0^\delta)^{\frac{1}{2}} g,\\ u_0^\varepsilon(0) = u_0^\varepsilon(1) = 0.\end{cases} \tag{2.5}$$

Then, we will conduct a consistent estimation of the approximate solution and prove that the limit of the approximate solution is just the solution of the Eqs (1.1)–(1.2).

In order to do this, we will first get the uniform estimate on u_0^ε. The u_0^ε is known from the smooth solution of the boundary value problem

$$u_{0xx}^\varepsilon = \left(\frac{\varepsilon\left(u_{0x}^\varepsilon\right)^2 + 1}{\left(u_{0x}^\varepsilon\right)^2 + \varepsilon}\right)^{\frac{p}{2}} \frac{\left(\left(u_{0x}^\varepsilon\right)^2 + \varepsilon\right)^2\left(P_x\left(\rho_0^\delta\right) - \left(\rho_0^\delta\right)^{\frac{1}{2}} g\right)}{\left(\varepsilon\left(u_{0x}^\varepsilon\right)^2 + 1\right)\left(\left(u_{0x}^\varepsilon\right)^2 + \varepsilon\right) - (2-p)\left(1-\varepsilon^2\right)\left(u_{0x}^\varepsilon\right)^2}, \tag{2.6}$$

then

$$\begin{aligned}\left|u_{0xx}^\varepsilon\right|_{L^2(I)} &\le \left|\left(\frac{\left(u_{0x}^\varepsilon\right)^2 + \varepsilon}{\varepsilon\left(u_{0x}^\varepsilon\right)^2 + 1}\right)^{1-\frac{p}{2}}\right|_{L^\infty(I)} \left|P_x\left(\rho_0^\delta\right) - \left(\rho_0^\delta\right)^{\frac{1}{2}} g\right|_{L^2(I)}\\ &\le \left(\left|u_{0x}^\varepsilon\right|_{L^\infty(I)}^2 + 1\right)^{1-\frac{p}{2}}\left(\left|(\rho_0^\delta)^{\frac{1}{2}} g\right|_{L^2(I)} + \left|P_x\left(\rho_0^\delta\right)\right|_{L^2(I)}\right)\\ &\le \left(\left|u_{0xx}^\varepsilon\right|_{L^2(I)}^2 + 1\right)^{1-\frac{p}{2}}\left(\left|\left(\rho_0^\delta\right)^{\frac{1}{2}} g\right|_{L^2(I)} + \left|P_x\left(\rho_0^\delta\right)\right|_{L^2(I)}\right).\end{aligned} \tag{2.7}$$

Using Young's inequality, we have

$$\left|u_{0xx}^\varepsilon\right|_{L^2(I)} \le C, \tag{2.8}$$

with the help of the Lemma 1, we get

$$\left|u_0^\varepsilon\right|_{L^\infty(I)} + \left|u_{0x}^\varepsilon\right|_{L^\infty(I)} + \left|u_{0xx}^\varepsilon\right|_{L^2(I)} \le C, \tag{2.9}$$

where $C > 0$ is a constant that depends only on M_0, which may not necessarily be fixed. Next, we denote

$$M_0 = 1 + |\rho_0|_{H^1(I)} + |\ u_0\ |_{H_0^1(I)\bigcap H_0^2(I)} + |g|_{L^2(I)}.$$

For any fixed integer K, define

$$J_K(t) = \max_{1\le k\le K}\sup_{0\le s\le t}\left(1 + \left|\rho^k(s)\right|_{H^1(I)} + \left|u^k(s)\right|_{W_0^{1,p}(I)} + \left|\ \sqrt{\rho^k}u_t^k(s)\right|_{L^2(I)}\right), \tag{2.10}$$

then we will prove that $J_K(t)$ is locally bounded for $\frac{4}{3} < p < 2$. We estimate each term in $J_K(t)$ in the following sections.

2.1. Estimate for $\left|u^k(t)\right|_{w_0^{1,p}(I)}$

Multiplying (2.2) by u_t^k, Integrating over $(0,1)$ concerning x and integrating over $(0,t)$ to s gives, we can get

$$\begin{aligned}&\int_0^t\int_0^1 \rho^k\left|u_t^k\right|^2 \mathrm{d}x\mathrm{d}s + \int_0^t\int_0^1\left[\left(\frac{\varepsilon\left(u_x^k\right)^2+1}{(u_x^k)^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x^k\right]u_{xt}^k\ \mathrm{d}x\mathrm{d}s\\ &= -\int_0^1 P^k u_x^k(0)\mathrm{d}x + \int_0^1 P^k u_x^k(t)\mathrm{d}x - \int_0^t\int_0^1\left(P_t^k u_x^k + \rho^k\Phi_x^k u_t^k + \rho^k u^k u_t^k + \rho^k u^{k-1}u_x^k u_t^k\right)\mathrm{d}x\mathrm{d}s. \end{aligned}\tag{2.11}$$

We firstly compute the second term of (2.11), we obtain

$$\int_0^1\left[\left(\frac{\varepsilon\left(u_x^k\right)^2+1}{(u_x^k)^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x^k\right]u_{xt}^k\ \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\int_0^{(u_x^k)^2}\left(\frac{\varepsilon s+1}{s+\varepsilon}\right)^{\frac{2-p}{2}}\mathrm{d}s\right)\mathrm{d}x, \tag{2.12}$$

and

$$\int_0^{(u_x^k)^2}\left(\frac{\varepsilon s+1}{s+\varepsilon}\right)^{\frac{2-p}{2}}\ \mathrm{d}s \ge \int_0^{(u_x^k)^2}(s+1)^{\frac{p-2}{2}}\ \mathrm{d}s = \frac{2}{p}\left(\left(\left(u_x^k\right)^2+1\right)^{\frac{p}{2}}-1\right). \tag{2.13}$$

Substituting (2.12), (2.13) into (2.11), by (2.9) and Young's inequality, we have

$$\begin{aligned}&\int_0^t\left|\ \sqrt{\rho^k}u_t^k(s)\right|_{L^2(I)}^2\mathrm{d}s + \frac{1}{p}\int_0^1\left|u_x^k(t)\right|^p\ \mathrm{d}x\\ &\le C + \int_0^1\left|P^k u_x^k(\mathrm{t})\right|\mathrm{d}x - \int_0^t\int_0^1\left|P_t^k u_x^k\right|\mathrm{d}x\mathrm{d}s - \int_0^t\int_0^1\rho^k\left|u^k\right|\left|u_t^k\right|\mathrm{d}x\mathrm{d}s - \int_0^t\int_0^1\rho^k\left|u^{k-1}\right|\left|u_x^k\right|\left|u_t^k\right|\mathrm{d}x\mathrm{d}s\\ &-\int_0^t\int_0^1\rho^k\left|\Phi_x^k\right|\left|u_t^k\right|\mathrm{d}x\mathrm{d}s. \end{aligned}\tag{2.14}$$

By (2.1), we get

$$P_t^k = -\gamma P^k u^{k-1} - P_x^k u^{k-1}.$$

Then the above inequality can be expressed as

$$
\begin{aligned}
&\int_0^t \left|\sqrt{\rho^k}u_t^k(s)\right|_{L^2(I)}^2 \mathrm{d}s + \left|u_x^k(t)\right|_{L^p(I)}^p \\
\leq &\int_0^t \int_0^1 \left(\left|\rho^k u^k u_t^k\right| + \left|\rho^k u^{k-1} u_x^k u_t^k\right| + \left|\rho^k \Phi_x^k u_t^k\right| \mathrm{d}x\mathrm{d}s + \int_0^1 \left|P^k u_x^k\right| \mathrm{d}x \right.\\
&\left. + \int_0^t \int_0^1 \left|P_x^k u^{k-1} u_x^k\right| + \gamma \left|P u_x^{k-1} u_x^k\right|\right) \mathrm{d}x\mathrm{d}s + C \\
\leq &C_\eta \int_0^t \left|\sqrt{\rho^k}u^k\right|_{L^2(I)}^2 \mathrm{d}s + C_\eta \int_0^t \left|\rho^k(s)\right|_{L^\infty(I)} \left|u_x^{k-1}(s)\right|_{L^p(I)}^2 \left|u_{xx}^k(s)\right|_{L^2(I)}^2 \mathrm{d}s + C\left|P^k(t)\right|_{L^{\frac{p}{p-1}}(I)}^{\frac{p}{p-1}} \\
&+ \int_0^t \left(A\gamma \left|\rho^k\right|_{L^\infty(I)}^{\gamma-1} \left|\rho_x^k\right|_{L^2(I)} \left|u^{k-1}\right|_{L^\infty(I)} \left|u_x^k(s)\right|_{L^\infty(I)} + \gamma |P(s)|_{L^\infty(I)} \left|u_x^{k-1}(s)\right|_{L^p(I)} \left|u_x^k(s)\right|_{L^\infty(I)}\right) \mathrm{d}s \\
&+ C_\eta \int_0^t \left|\rho^k\right|_{H^1(I)} \left|\Phi_{xx}^k\right|_{L^2(I)}^2 \mathrm{d}s + \frac{1}{2}\int_0^t \left|\sqrt{\rho^k}u_t^k\right|_{L^2(I)}^2 (s)\mathrm{d}s + \frac{1}{2}\left|u_x^k(t)\right|_{L^p(I)}^p + C, \qquad (2.15)
\end{aligned}
$$

where $0 < \eta \ll 1$. To estimate the right part of the (2.14), we have the following estimates

$$
\left|\rho^k(t)\right|_{L^\infty(I)} + \left|P^k(t)\right|_{H^1(I)} \leq C J_K^\gamma(t). \qquad (2.16)
$$

Using (2.1), we have

$$
\begin{aligned}
\int_0^1 \left|P^k(t)\right|^{\frac{p}{p-1}} \mathrm{d}x = &\int_0^1 \left|P^k(0)\right|^{\frac{p}{p-1}} \mathrm{d}x + \int_0^t \frac{\partial}{\partial s}\left(\int_0^1 \left(P^k(s)\right)^{\frac{p}{p-1}} \mathrm{d}x\right)\mathrm{d}s \\
\leq &C\left(1 + \int_0^t J_K^{\frac{2\gamma+1}{p-1}}(s)\mathrm{d}s\right). \qquad (2.17)
\end{aligned}
$$

By virtue of (2.2), we have

$$
\left[\left(\frac{\varepsilon\left(u_x^k\right)^2 + 1}{(u_x^k)^2 + \varepsilon}\right)^{\frac{2-p}{2}} u_x^k\right]_x = \rho^k u_t^k + \rho^k \Phi_x^k + \rho^k u^{k-1} u_x^k + P_x^k + \rho^k u^k,
$$

then we have

$$
\begin{aligned}
\left|u_{xx}^k\right| = &\left(\frac{\varepsilon\left(u_x^k\right)^2 + 1}{(u_x^k)^2 + \varepsilon}\right)^{\frac{p}{2}} \frac{\left[\left(u_x^k\right)^2 + \varepsilon\right]^2}{\left(\varepsilon(u_x^k)^2 + 1\right)\left((u_x^k)^2 + \varepsilon\right) - (2-p)(1-\varepsilon^2)(u_x^k)^2} \\
&\cdot \left|\rho^k u_t^k + \rho^k u^{k-1} u_x^k + \rho^k \Phi_x^k + P_x^k + \rho^k u^k\right| \\
\leq &\frac{1}{p-1}\left(\left|u_x^k\right|^{2-p} + 1\right)\left|\rho^k u_t^k + \rho^k u^{k-1} u_x^k + \rho^k \Phi_x^k + P_x^k + \rho^k u^k\right|, \qquad (2.18)
\end{aligned}
$$

taking the above inequality by L^2 norm, using Young's inequality, we obtain

$$
\left|u_{xx}^k\right|_{L^2(I)}^{p-1}
$$

$$\le C\left[1+\left|\rho^k u_t^k\right|_{L^2(I)}+\left|\rho^k u^{k-1}u_x^k\right|_{L^2(I)}+\left|\rho^k\Phi_x^k\right|_{L^2(I)}+\left|P_x^k\right|_{L^2(I)}+\left|\rho^k u^k\right|_{L^2(I)}\right]$$

$$\le C\left[1+|\rho_k|_{L^\infty(I)}^{\frac{1}{2}}\left|\sqrt{\rho^k}u_t^k\right|_{L^2(I)}+\left(\left|\rho^k\right|_{L^\infty(I)}\left|u_x^{k-1}\right|_{L^p(I)}\left|u_x^k\right|_{L^p(I)}^{\frac{p}{2}}\right)^{\frac{2(p-1)}{3p-4}}+\left|\rho^k\right|_{H^1(I)}\left|\Phi_{xx}^k\right|_{L^2(I)}+\left|P_x^k\right|_{L^2(I)}\right.$$
$$\left.+\left|\rho^k u^k\right|_{L^2(I)}\right]+\frac{1}{2}\left|u_{xx}^k\right|_{L^2(I)}^{p-1}.$$

We deal with $\left|\Phi_{xx}^k\right|_{L^2(I)}$, by (2.3) we have

$$\left|\Phi_{xx}^k\right|\le\frac{1}{q-1}\left|\Phi_x^k\right|^{2-q}\left|4\pi g\left(\rho^k-m_0\right)\right|,$$

taking it by L^2-norm, using Young's inequality and Lemma 1, we get

$$\left|\Phi_{xx}^k\right|_{L^2(I)}\le CJ_K^{\frac{1}{q-1}}(t),\tag{2.19}$$

then

$$\left|u_{xx}^k(t)\right|_{L^2(I)}\le CJ_K^{\frac{(4+p)\gamma}{3p-4}}(t)\le CJ_K^{\frac{6\gamma}{3p-4}}(t).\tag{2.20}$$

Using (2.14) and the above inequality, we get

$$\int_0^t\left|\sqrt{\rho^k}u_t^k(s)\right|_{L^2(I)}^2\mathrm{d}s+\left|u_x^k(t)\right|_{L^p(I)}^p\le C\left(1+\int_0^t J_K^{\frac{24\gamma}{3p-4}}(s)\mathrm{d}s\right),\tag{2.21}$$

for all $k, 1\le k\le K$.

2.2. Estimate for $\left|\sqrt{\rho^k}u_t^k(t)\right|_{L^2(I)}$

We differentiate (2.2) with respect to t, and multiply it by u_t^k, and integrating it over $(0,1)$ with respect to x, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\rho^k\left|u_t^k\right|^2(t)\mathrm{d}x+\int_0^1\left[\left(\frac{\varepsilon\left(u_x^k\right)^2+1}{(u_x^k)^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x^k\right]_t u_{xt}^k(t)\mathrm{d}x$$
$$=\int_0^1\left[\left(-u^k-u_t^k-u^{k-1}u_x^k-\Phi_x^k\right)\rho_t^k-\rho^k u_t^{k-1}u_x^k-\rho^k\Phi_{xt}^k-\rho^k u_t^k\right]u_t^k\mathrm{d}x+\int_0^1 P_t^k u_{xt}^k\mathrm{d}x.\tag{2.22}$$

Since

$$\left[\left(\frac{\varepsilon\left(u_x^k\right)^2+1}{(u_x^k)^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x^k\right]_t u_{xt}^k$$
$$=\left(\frac{\varepsilon\left(u_x^k\right)^2+1}{(u_x^k)^2+\varepsilon}\right)^{-\frac{p}{2}}\frac{\left(\varepsilon\left(u_x^k\right)^2+1\right)\left(\left(u_x^k\right)^2+\varepsilon\right)-(2-p)\left(1-\varepsilon^2\right)\left(u_x^k\right)^2}{\left((u_x^k)^2+\varepsilon\right)^2}\left(u_{xt}^k\right)^2$$

$$\geq (p-1)\left(\left(u_x^k\right)^2+1\right)^{\frac{p-2}{2}}\left(u_{xt}^k\right)^2, \tag{2.23}$$

let

$$\beta_k=\left(\left(u_x^k\right)^2+1\right)^{\frac{p-2}{4}}.$$

by (2.20), we have

$$\left|\beta_k^{-1}\right|_{L^\infty(I)}=\left|\left(\left(u_x^k\right)^2+1\right)^{\frac{2-p}{4}}\right|_{L^\infty(I)}\leq\left(\left|u_x^k\right|^2_{L^\infty(I)}+1\right)^{\frac{2-p}{4}}\leq\left|u_x^k\right|^{\frac{2-p}{2}}_{L^\infty(I)}+1\leq CJ_K^{\frac{3\gamma}{3p-4}}(t).$$

Then (2.22) can be rewritten as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\rho^k\left|u_t^k\right|^2\,\mathrm{d}x+(p-1)\int_0^1\left(\left(u_x^k\right)^2+1\right)^{\frac{p-2}{2}}\left(u_{xt}^k\right)^2\,\mathrm{d}x\\
\leq&\int_0^1 2\rho^k\left|u^{k-1}\right|\left|u_t^k\right|\left|u_{xt}^k\right|\mathrm{d}x+\int_0^1\left|\rho_x^k\right|\left|u^{k-1}\right|^2\left|u_x^k\right|\left|u_t^k\right|\mathrm{d}x+\int_0^1\left|\rho_x^k\left|u^{k-1}\right|u^k\right|\left|u_t^k\right|\mathrm{d}x\\
&+\int_0^1\rho^k|u^{k-1}||u_x^{k-1}||u_x^k||u_t^k|\,\mathrm{d}x+\int_0^1\rho^k|u_x^{k-1}||u^k||u_t^k|\,\mathrm{d}x+\int_0^1|P_x^k||u^{k-1}||u_{xt}^k|\mathrm{d}x\\
&+\int_0^1\gamma P^k|u_x^{k-1}||u_{xt}^k(t)|\,\mathrm{d}x+\int_0^1|\rho_x^k||u^{k-1}||\Phi_x^k||u_t^k|\,\mathrm{d}x+\int_0^1\rho^k|u_x^{k-1}||\Phi_x^k||u_t^k|\mathrm{d}x\\
&+\int_0^1\rho^k\left|u_t^{k-1}\right|\left|u_x^k\right|\left|u_t^k\right|\mathrm{d}x+\int_0^1\rho^k\left|u_t^k\right|\left|u_t^k\right|\mathrm{d}x+\int_0^1\rho^k\left|\Phi_{xt}^k\|u_t^k\right|\mathrm{d}x=\sum_{j=1}^{12}I_j.
\end{aligned}\tag{2.24}$$

Using Sobolev embedding theorem and Young's inequality, we obtain

$$\begin{aligned}
I_1=&\int_0^1 2\rho^k\left|u^{k-1}\right|\left|u_t^k\right|\left|u_{xt}^k\right|\mathrm{d}x\leq CJ_K^{\frac{16\gamma}{3p-4}}(t)+\frac{p-1}{8}\left|\beta_k u_{xt}^k(t)\right|^2_{L^2(I)},\\
I_2=&\int_0^1|\rho_x^k|\left|u^{k-1}\right|^2\left|u_x^k\right|\left|u_t^k\right|\mathrm{d}x\leq CJ_K^{\frac{30\gamma}{3p-4}}(t)+\frac{p-1}{8}\left|\beta_k u_{xt}^k(t)\right|^2_{L^2(I)},\\
I_3=&\int_0^1\left|\rho_x^k\right|\left|u^{k-1}\right|\left|u_t^k\right|\left|u^k\right|\mathrm{d}x\leq CJ_K^{\frac{24\gamma}{3p-4}}(t)+\frac{p-1}{8}\left|\beta_k u_{xt}^k(t)\right|^2_{L^2(I)},\\
I_4=&\int_0^1\rho^k\left|u^{k-1}\right|\left|u_x^{k-1}\right|\left|u_t^k\right|\left|u_x^k\right|\mathrm{d}x\leq CJ_K^{\frac{34\gamma}{3p-4}}(t),\\
I_5=&\int_0^1\rho^k\left|u_x^{k-1}\right||u^k|\left|u_t^k\right|\mathrm{d}x\leq\left|\rho^k\right|^{\frac{1}{2}}_{L^\infty(I)}\left|u_x^{k-1}\right|_{L^\infty(I)}|u^k|_{L^2(I)}\left|\sqrt{\rho^k}u_t^k\right|_{L^2(I)}\leq CJ_K^{\frac{24\gamma}{3p-4}}(t),\\
I_6=&\int_0^1\left|P_x^k\right|\left|u^{k-1}\right|\left|u_{xt}^k\right|\mathrm{d}x\leq CJ_K^{\frac{14\gamma}{3p-4}}(t)+\frac{p-1}{8}\left|\beta_k u_{xt}^k(t)\right|^2_{L^2(I)},\\
I_7=&\int_0^1\gamma P^k\left|u_x^{k-1}\right|\left|u_{xt}^k\right|\mathrm{d}x\leq CJ_K^{\frac{22\gamma}{3p-4}}(t)+\frac{p-1}{8}\left|\beta_k u_{xt}^k\right|^2_{L^2(I)},\\
I_8=&\int_0^1\left|\rho_x^k\right|\left|u^{k-1}\|\Phi_x^k\right|\left|u_t^k\right|\mathrm{d}x\leq CJ_K^{\frac{34\gamma}{3p-4}}(t)+\frac{p-1}{8}\left|\beta_k u_{xt}^k\right|^2_{L^2(I)},
\end{aligned}$$

$$\begin{aligned}I_9 &= \int_0^1 \rho^k \left|u_x^{k-1}\right| \left|\Phi_x^k\right| \left|u_t^k\right| \mathrm{d}x \le \left|\rho^k\right|_{L^{\frac{p}{p-1}}(I)} \left|u_x^{k-1}\right|_{L^p(I)} \left|\Phi_x^k\right|_{L^\infty(I)} \left|u_t^k\right|_{L^\infty(I)} \\ &\le C J_K^{\frac{34\gamma}{3p-4}}(t) + \frac{p-1}{8} \left|\beta_k u_{xt}^k\right|_{L^2(I)}^2, \\ I_{10} &= \int_0^1 \rho^k \left|u_t^{k-1}\right| \left|u_x^k\right| \left|u_t^k\right| \mathrm{d}x \le \left|\rho^k\right|_{L^\infty(I)}^{\frac12} \left|u_t^{k-1}\right|_{L^\infty(I)} \left|u_x^k\right|_{L^\infty(I)} \left|\sqrt{\rho^k} u_t^k\right|_{L^2(I)} \\ &\le C J_K^{\frac{24\gamma}{3p-4}}(t) + \frac{p-1}{2} \left|\beta_{k-1} u_{xt}^{k-1}\right|_{L^2(I)}^2 + \frac{p-1}{8} \left|\beta_k u_{xt}^k\right|_{L^2(I)}^2, \\ I_{11} &= \int_0^1 \rho^k \left|u_t^k\right| \left|u_t^k\right| \mathrm{d}x \le \left|\rho^k\right|_{L^\infty(I)}^{\frac12} \left|\sqrt{\rho^k} u_t^k\right|_{L^2(I)} \left|u_t^k\right|_{L^2(I)} \le C J_K^{\frac32}(t).\end{aligned}$$

In order to estimate I_{12}, we need to deal with Φ_{xt}^k. Differentiating (2.3) with respect to t, multiplying it by Φ_t^k and integrating over $(0, 1)$, we have

$$\int_0^1 \left[\left(\frac{\varepsilon\left(\Phi_x^k\right)^2 + 1}{(\Phi_x^k)^2 + \varepsilon}\right)^{\frac{2-q}{2}} \Phi_x^k\right]_t \Phi_{xt}^k \,\mathrm{d}x = -4\pi g \int_0^1 \left(\rho^k u^{k-1}\right)_x \Phi_t^k \,\mathrm{d}x.$$

By (2.23), we have

$$\int_0^1 \left[\left(\frac{\varepsilon\left(\Phi_x^k\right)^2 + 1}{(\Phi_x^k)^2 + \varepsilon}\right)^{\frac{2-q}{2}} \Phi_x^k\right]_t \Phi_{xt}^k \,\mathrm{d}x \ge (q-1) \int_0^1 \left[\left(\Phi_x^k\right)^2 + 1\right]^{\frac{q-2}{2}} \left|\Phi_{xt}^k\right|^2 \mathrm{d}x.$$

Let

$$\beta_k^q = \left[\left(\Phi_x^k\right)^2 + 1\right]^{\frac{q-2}{4}},$$

then

$$\left|\left(\beta_k^q\right)^{-1}\right|_{L^\infty(I)} = \left|\left[\left(\Phi_x^k\right)^2 + 1\right]^{\frac{2-q}{4}}\right|_{L^\infty(I)} \le C\left(\left|\Phi_{xx}^k\right|_{L^2(I)}^{\frac{2-q}{2}} + 1\right) \le C J_K^{\frac{2-q}{2(q-1)}}(t),$$

we have

$$\int_0^1 \left|\beta_k^q \Phi_{xt}^k\right|^2 \,\mathrm{d}x = C \int_0^1 \left(\rho^k u^{k-1}\right) \Phi_{xt}^k \,\mathrm{d}x \le C \left|\rho^k\right|_{L^2(I)} \left|u^{k-1}\right|_{L^\infty(I)} \left|\beta_k^q \Phi_{xt}^k\right|_{L^2(I)} \left|\left(\beta_k^q\right)^{-1}\right|_{L^\infty(I)}.$$

Using Young's inequality, combining the above estimate we obtain

$$I_{12} = \int_0^1 \rho^k \left|\Phi_{xt}^k\right| \left|u_t^k\right| \mathrm{d}x \le \left|\rho^k\right|_{L^\infty(I)} \left|\sqrt{\rho^k} u_t^k\right|_{L^2(I)} \left|\beta_k^q \Phi_{xt}^k\right|_{L^2(I)} \left|\left(\beta_k^q\right)^{-1}\right|_{L^\infty(I)} \le C J_K^6(t).$$

Substituting $I_j (j = 1, 2, \ldots, 12)$ into (2.24), integrating over (τ, t) on time variable, we have

$$\begin{aligned}\left|\sqrt{\rho^k} u_t^k(t)\right|_{L^2(I)}^2 + (p-1) \int_\tau^t \left|\beta_k u_{xt}^k\right|_{L^2(I)}^2 (s)\mathrm{d}s \le& C \int_\tau^t J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s + \sup_{0\le k\le K}\left(1 + \left|\sqrt{\rho^k} u_t^k(\tau)\right|_{L^2(I)}^2\right) \\ &+ \frac{p-1}{2} \int_\tau^t \left|\beta_{k-1} u_{xt}^{k-1}\right|_{L^2(I)}^2 (s)\mathrm{d}s, \qquad (2.25)\end{aligned}$$

then, from the above recursive relation, for $1 \le k \le K$, we obtain

$$(p-1)\int_\tau^t \left|\beta_k u_{xt}^k\right|_{L^2(I)}^2(s)\mathrm{d}s \le \left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^K}\right)C\left[\int_\tau^t J_K^{\frac{34\gamma}{3p-4}}(s)\mathrm{d}s + \sup_{0\le k\le K}\left(1+\left|\sqrt{\rho^k}u_t^k(\tau)\right|_{L^2(I)}^2\right)\right]$$
$$\le 2C\left[\int_\tau^t J_K^{\frac{34\gamma}{3p-4}}(s)\mathrm{d}s + \sup_{0\le k\le K}\left(1+\left|\sqrt{\rho^k}u_t^k(\tau)\right|_{L^2(I)}^2\right)\right].$$

Thus, we deduce from (2.25) that

$$\left|\sqrt{\rho^k}u_t^k(s)\right|_{L^2(I)}^2 + \int_\tau^t \left|\beta_k u_{xt}^k\right|_{L^2(I)}^2(s)\mathrm{d}s \le C\int_\tau^t J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s + \sup_{0\le k\le K}\left(1+\left|\sqrt{\rho^k}u_t^k(\tau)\right|_{L^2(I)}^2\right), \tag{2.26}$$

where C is a positive constant, depending only on M_0.

To obtain the estimate of $\left|\sqrt{\rho^k}u_t^k(t)\right|_{L^2(I)}^2$, we need to estimate

$$\lim_{\tau\to 0}\sup_{0\le k\le K}\left(1+\left|\sqrt{\rho^k}u_t^k(\tau)\right|_{L^2(I)}^2\right).$$

Using (2.2), we get

$$\int_0^1 \rho^k\left|u_t^k\right|^2(t)\mathrm{d}x \le 2\int_0^1\left(\rho^k\left|u^{k-1}\right|^2\left|u_x^k\right|^2 + \rho^k\Phi_x^k + \rho^k\left|u^k\right|^2 + \left(\rho^k\right)^{-1}\left|Ł_p^\varepsilon u^k + P_x^k\right|^2\right)\mathrm{d}x,$$

$$Ł_p^\varepsilon u^k = -\left[\left(\frac{\varepsilon\left(u_x^k\right)^2+1}{(u_x^k)^2+\varepsilon}\right)^{\frac{2-p}{2}} u_x^k\right]_x.$$

Since $\left(\rho^k, u^k, \Phi^k\right)$ is a smooth solution, we obtain

$$\lim_{t\to 0}\int_0^1\left(\rho^k\left|u^{k-1}\right|^2\left|u_x^k\right|^2 + \rho^k\left|u^k\right|^2 + \rho^k\Phi_x^k + \left(\rho^k\right)^{-1}\left|Ł_p^\varepsilon u^k + P_x^k\right|^2\right)(x,t)\mathrm{d}x$$
$$\le |\rho_0|_{L^\infty(I)}\left|u_0^\varepsilon\right|_{L^\infty(I)}\left|u_{0x}^\varepsilon\right|_{L^2(I)}^2 + |\rho_0|_{L^\infty(I)}|u^k|_{L^2(I)} + |\rho_0|_{L^\infty(I)}|\Phi_x^k|_{L^2(I)} + |g|_{L^2(I)}^2.$$

Thus, using (2.9), we deduce

$$\lim_{\tau\to 0}\sup\int_0^1 \rho^k\left|u_t^k\right|^2(\tau)dx \le C.$$

Taking a limit on τ for inequality (2.26), we obtain, as $\tau\to 0$,

$$\left|\sqrt{\rho^k}u_t^k(t)\right|_{L^2(I)}^2 + \int_0^t\left|\beta_k u_{xt}^k\right|_{L^2(I)}^2(s)\mathrm{d}s \le C\left(1+\int_0^t J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s\right). \tag{2.27}$$

2.3. Estimate $|\rho^k(s)|_{H^1(I)}$

We differentiate (2.1) concerning x, multiply it by ρ_x^k, integrating it over (0, 1) for x, and using Sobolev embedding theorem, we obtain

$$\begin{aligned}\frac{d}{dt}\left|\rho_x^k\right|_{L^2(I)}^2 \, \mathrm{d}x &= -\int_0^1\left(\frac{3}{2}u_x^{k-1}\left(\rho_x^k\right)^2+\rho^k\rho_x^k u_{xx}^{k-1}\right)(t)\mathrm{d}x\\ &\leq \frac{3}{2}\left(\left|u_x^{k-1}\right|_{L^\infty(I)}\left|\rho_x^k\right|_{L^2(I)}^2+\left|\rho^k\right|_{L^\infty(I)}\left|u_{xx}^{k-1}\right|_{L^2(I)}\right)\\ &\leq 3\left|\rho_x^k\right|_{L^2(I)}^2\left|u_{xx}^{k-1}\right|_{L^2(I)},\end{aligned}$$

applying Gronwall's inequality, it follows that

$$\sup_{0\leq t\leq T}\left|\rho^k(t)\right|_{H^1(I)}^2 \leq |\rho_0^k|_{H^1(I)}^2\exp\left(C\int_0^t\left|u_{xx}^{k-1}(\cdot,s)\right|_{L^2(I)}ds\right). \tag{2.28}$$

Substituting (2.20) into the above inequality, we get

$$\left|\rho^k(t)\right|_{H^1(I)}^2 \leq C|\rho_0^k(t)|_{H^1(I)}^2\exp\left(\int_0^t J_K^{\frac{6\gamma}{3p-4}}(s)ds\right). \tag{2.29}$$

Using (2.29) and (2.1), we have

$$\left|\rho_t^k(t)\right|_{L^2(I)} \leq \left|\rho_x^k(t)\right|_{L^2(I)}\left|u^{k-1}(t)\right|_{L^\infty(I)}+\left|\rho^k(t)\right|_{L^\infty(I)}\left|u_{xx}^{k-1}(t)\right|_{L^2(I)} \leq J_K^{\frac{8\gamma}{3p-4}}(t). \tag{2.30}$$

By virtue of (2.20), (2.27), (2.29) and (2.30), we conclude that

$$\begin{aligned}&\left|u_x^k(t)\right|_{L^p(I)}^p+|u_{xx}(t)|_{L^2(I)}+\left|\rho^k(t)\right|_{H^1(I)}+\left|\sqrt{\rho}^k u_t^k(t)\right|_{L^2(I)}+\int_0^t\left(\left|\sqrt{\rho}^k u_t^k(s)\right|_{L^2(I)}^2+\left|u_{xt}^k(s)\right|_{L^2(I)}^2\right)\mathrm{d}s\\ &\leq C_1\exp\left(C_2\int_0^t J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s\right),\end{aligned} \tag{2.31}$$

where C_1, C_2 are two positive constants, depending only on M_0. By the definition of $J_K(t)$, we obtain

$$J_K(t) \leq C_1\exp\left(C_2\int_0^t J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s\right). \tag{2.32}$$

If

$$\int_0^T J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s \leq 1,$$

then we take $T_1 = T$. On the other hand, if

$$\int_0^t J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s > 1,$$

we can find $t_0 \in (0, T)$, such that

$$\int_0^{t_0} J_K^{\frac{48\gamma}{3p-4}}(s)\mathrm{d}s = 1.$$

So we have

$$\sup_{0\le t\le t_0} J_K(t) \le C_1 e^{c_2},$$

and

$$\int_0^{t_0} J_K^{\frac{48\gamma}{3p-4}}(s)\,\mathrm{d}s = 1 \le \int_0^{t_0} C^{\frac{48\gamma}{3p-4}} e^{\frac{48\gamma c}{3p-4}}\,\mathrm{d}s \le C^{\frac{48\gamma}{3p-4}} e^{\frac{48\gamma c}{3p-4}} t_0,$$

so

$$T_1 = c^{\frac{-48\gamma}{3p-4}} e^{\frac{-48\gamma C}{3p-4}},$$

then we have

$$\sup_{0\le t\le T_1} J_K(s) \le Ce^c \le C. \tag{2.33}$$

Given this inequality, we can acquire a short-time $T_1 > 0$ such that:

$$\operatorname{ess}\sup_{0\le t\le T_1}\left(\left|\rho^k(s)\right|_{H^1(I)} + \left|u^k\right|_{w_0^{1,p}(I)\cap H^2(I)} + \left|\sqrt{\rho^k}u_t^k(s)\right|_{L^2(I)} + \left|\rho_t^k\right|_{L^2(I)}\right) + \int_0^{T_1} |u_{xt}|^2_{L^2(I)}\,\mathrm{d}t \le C. \tag{2.34}$$

3. Convergence of approximate solution

It is demonstrated that the approximate solution $\left(\rho^k, u^k, \Phi^k\right)$ strongly converge to the solution of the Eqs (1.1)–(1.2) with positive density. We give the following definition

$$\bar\rho^{k+1} = \rho^{k+1} - \rho^k, \quad \bar u^{k+1} = u^{k+1} - u^k, \quad \bar\Phi^{k+1} = \Phi^{k+1} - \Phi^k,$$

then we verify that $\left(\bar\rho^{k+1}, \bar u^{k+1}, \bar\Phi^{k+1}\right)$ satisfy the system of equations

$$\bar\rho_t^{k+1} + \left(\bar\rho^{k+1}u^k\right)_x + \left(\rho^k\bar u^k\right)_x = 0, \tag{3.1}$$

$$\begin{aligned}&\rho^{k+1}\bar u_t^{k+1} + \rho^{k+1}u^k\bar u_x^{k+1} + \left(Ł_p^\varepsilon u^{k+1} - Ł_p^\varepsilon u^k\right) + \left(P_x^{k+1} - P_x^k\right)\\ =&\bar\rho^{k+1}\left(-u^k - \bar u^{k+1} - u_t^k - u^k u_x^k - \Phi_x^k\right) - \rho^k\bar u^{k+1} - \rho^{k+1}\left(\bar u^k u_x^k + \bar\Phi_x^{k+1}\right),\end{aligned} \tag{3.2}$$

$$Ł_q^\varepsilon\Phi^{k+1} - Ł_q^\varepsilon\Phi^k = 4\pi g\bar\rho^{k+1}, \tag{3.3}$$

the initial boundary value conditions are given as follows

$$\begin{aligned}&\bar u^{k+1} = 0, \quad \bar\Phi^{k+1} = 0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ &\rho^{k+1}(x,0) = 0, \quad \bar u^{k+1}(x,0) = 0, \quad x\in\Omega.\end{aligned}$$

Multiplying (3.1) by $\bar\rho^{k+1}$, integrating over I with respect to x, we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|\bar\rho^{k+1}(t)\right|^2_{L^2(I)} &\le C\left|u_x^k(t)\right|_{L^\infty(I)}\left|\bar\rho^{k+1}(t)\right|^2_{L^2(I)} + \left|\rho^k(t)\right|_{H^1(I)}\left|\bar u_x^k(t)\right|_{L^2(I)}\left|\bar\rho^{k+1}(t)\right|_{L^2(I)}\\ &\le B_\eta^k\left|\bar\rho^{k+1}(t)\right|^2_{L^2(I)} + \eta\left|\bar u_x^k(t)\right|^2_{L^2(I)},\end{aligned} \tag{3.4}$$

where $B_\eta^k = C\left|u_{xx}^k\right|_{L^2(I)} + C_\eta|\rho^k|^2_{H^1(I)}$, for all $t\le T_1$ and $k\ge 1$.

Multiplying (3.2) by $\bar{u}^{k+1}$, integrating over I with respect to x, using (3.1), Hölder inequality and Lemma 1, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho^{k+1}\left|\bar{u}^{k+1}\right|^2\,\mathrm{d}x+\int_0^1\left(Ł_p^{\varepsilon}u^{k+1}-Ł_p^{\varepsilon}u^k\right)\bar{u}^{k+1}\,\mathrm{d}x\\
&=\int_0^1\left(\bar{\rho}^{k+1}\left(-u^k-u_t^k-u^{k-1}u_x^k-\Phi_x^k\right)\bar{u}^{k+1}-\rho^{k+1}\left(\bar{\Phi}_x^{k+1}+\bar{u}^k u_x^k\right)\bar{u}^{k+1}-\left(P_x^{k+1}-P_x^k\right)\bar{u}^{k+1}\right)\mathrm{d}x\\
&\le\left|\bar{\rho}^{k+1}\right|_{L^2(I)}\left|u^k\right|_{L^2(I)}\left|\bar{u}_x^{k+1}\right|_{L^2(I)}+\left|\bar{\rho}^{k+1}\right|_{L^2(I)}\left|u_{xt}^k\right|_{L^2(I)}\left|\bar{u}_x^{k+1}\right|_{L^2(I)}+\left|\bar{\rho}^{k+1}\right|_{L^2(I)}\left|\bar{u}_x^{k-1}\right|_{L^2(I)}\left|u_{xx}^k\right|_{L^2(I)}\left|\bar{u}_x^{k+1}\right|_{L^2(I)}\\
&\quad+\left|\bar{\rho}^{k+1}\right|_{L^2(I)}\left|\Phi_{xx}^k\right|_{L^2(I)}\left|\bar{u}_x^{k+1}\right|_{L^2(I)}+\left|\rho^{k+1}\right|_{H^1(I)}\left|\bar{\Phi}_x^{k+1}\right|_{L^2(I)}\left|\bar{u}_x^{k+1}\right|_{L^2(I)}\\
&\quad+\left|\rho^{k+1}\right|_{H^1(I)}^{\frac{1}{2}}\left|\bar{u}_x^k\right|_{L^2(I)}\left|u_{xx}^k\right|_{L^2(I)}\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}\right|_{L^2(I)}+\left|P^{k+1}-P^k\right|_{L^2(I)}\left|\bar{u}_x^{k+1}\right|_{L^2(I)}. \qquad (3.5)
\end{aligned}$$

Let

$$\omega'(s)=\left[\left(\frac{\varepsilon s^2+1}{s^2+\varepsilon}\right)^{\frac{2-p}{2}}s\right]'\ge\frac{p-1}{\left(s^2+\varepsilon\right)^{\frac{2-p}{2}}},$$

so

$$\int_0^1\left[\int_0^1\omega'\left(\theta u_x^{k+1}+(1-\theta)u_x^k\right)\right]\mathrm{d}\theta\left(\bar{u}_x^{k+1}\right)^2\mathrm{d}x\ge C\int_0^1\left(\bar{u}_x^{k+1}\right)^2\mathrm{d}x, \qquad (3.6)$$

using (3.4) and (3.5), we have

$$\int_0^1\left(Ł_p^{\varepsilon}u^{k+1}-Ł_p^{\varepsilon}u^k\right)\bar{u}^{k+1}\mathrm{d}x\ge C\int_0^1\left(\bar{u}_x^{k+1}\right)^2\mathrm{d}x, \qquad (3.7)$$

Using (2.34), (3.7) and Young's inequality, (3.5) could be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\rho^{k+1}\left|\bar{u}^{k+1}\right|^2\mathrm{d}x+\int_0^1\left|\bar{u}_x^{k+1}\right|^2\mathrm{d}x\le E_\eta^k(t)\left|\bar{\rho}^{k+1}(t)\right|_{L^2(I)}^2+\eta|\bar{u}_x^k|_{L^2(I)}^2, \qquad (3.8)$$

where $E_\eta^k(t)=C\left(1+\left|u_{xt}^k(t)\right|_{L^2(I)}^2\right)$, for all $t<T_1$ and $k\ge 1$. Using (2.31), we derive

$$\int_0^t E_\eta^k(s)\mathrm{d}s\le C+Ct,\ \int_0^t B_\eta^k(s)\mathrm{d}s\le C+Ct.$$

According to Eq (2.3)

$$\left[\left|\Phi_x^{k+1}\right|^{q-2}\Phi_x^{k+1}\right]_x-\left[\left|\Phi_x^k\right|^{q-2}\Phi_x^k\right]_x=4\pi g\bar{\rho}^{k+1},$$

let's multiply both sides of this equation by $\bar{\Phi}^{k+1}$, about x in $(0,1)$ integral, available

$$\int_0^1\left(\left[\left|\Phi_x^{k+1}\right|^{q-2}\Phi_x^{k+1}\right]_x-\left[\left|\Phi_x^k\right|^{q-2}\Phi_x^k\right]_x\right)\bar{\Phi}^{k+1}\mathrm{d}x=4\pi g\int_0^1\bar{\Phi}^{k+1}\bar{\rho}^{k+1}\mathrm{d}x,$$

$$z(s)=\left(\frac{\varepsilon s^2+1}{s^2+\varepsilon}\right)^{\frac{2-p}{2}}s$$

then

$$z'(s)=\left[\left(\frac{\varepsilon s^2+1}{s^2+\varepsilon}\right)^{\frac{2-p}{2}}s\right]'\geq\frac{p-1}{(s^2+\varepsilon)^{\frac{2-p}{2}}},$$

$$\int_0^1\left(\left[\left(\Phi_x^{k+1}\right)^{q-2}\Phi_x^{k+1}\right]_x-\left[\left(\Phi_x^{k}\right)^{q-2}\Phi_x^{k}\right]_x\right)\bar{\Phi}^{k+1}\mathrm{d}x$$
$$=\int_0^1\left[\int_0^1 z'\left(\theta\Phi_x^{k+1}+(1-\theta)\Phi_x^k\right)\right]\mathrm{d}\theta\left(\bar{\Phi}_x^{k+1}\right)^2\mathrm{d}x\geq C\int_0^1\left(\bar{\Phi}_x^{k+1}\right)^2\mathrm{d}x.$$

By combining the above formula, Hölder inequality and Lemma 1 are obtained

$$|\bar{\Phi}_x^{k+1}|^2_{L^2(I)}\leq C|\bar{\rho}^{k+1}|^2_{L^2(I)}. \tag{3.9}$$

Collecting (3.4), (3.8) and (3.9), we deduce that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\left|\bar{\rho}^{k+1}(t)\right|^2_{L^2(I)}+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|^2_{L^2(I)}\right)+\left|\bar{u}_x^{k+1}(t)\right|^2_{L^2(I)}+|\bar{\Phi}_x^{k+1}|^2_{L^2(I)}$$
$$\leq C\left(\left|\bar{\rho}^{k+1}(t)\right|^2_{L^2(I)}+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|^2_{L^2(I)}\right)+\eta\left|\bar{u}_x^{k}(t)\right|^2_{L^2(I)}. \tag{3.10}$$

Using Gronwall's inequality, we have

$$\left|\bar{\rho}^{k+1}(t)\right|^2_{L^2(I)}+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|^2_{L^2(I)}+\int_0^t\left(\left|\bar{u}_x^{k+1}(s)\right|^2_{L^2(I)}+|\bar{\Phi}_x^{k+1}|^2_{L^2(I)}\right)\mathrm{d}s$$
$$\leq C\exp\left(C_\eta t\right)\int_0^t\left(\left|\sqrt{\rho^k}\bar{u}^k(s)\right|^2_{L^2(I)}+\left|\bar{u}_x^k(s)\right|^2_{L^2(I)}\right)\mathrm{d}s.$$

Then, we choose $\eta>0$ and then $T_*>0$ so small that $T_*<T_1$ and $C\exp\left(C_\eta T_*\right)<1/2$, we get

$$\left|\bar{\rho}^1(t)\right|^2_{L^2(I)}+\left|\sqrt{\rho^1}\bar{u}^1(t)\right|^2_{L^2(I)}+\int_0^t\left(\left|\bar{u}_x^1(s)\right|^2_{L^2(I)}+|\bar{\Phi}_x^1|^2_{L^2(I)}\right)\mathrm{d}s\leq\frac{1}{2}\int_0^t\left(\left|\sqrt{\rho^0}\bar{u}^0(s)\right|^2_{L^2(I)}+\left|\bar{u}_x^0(s)\right|^2_{L^2(I)}\right)\mathrm{d}s,$$
$$\left|\bar{\rho}^2(t)\right|^2_{L^2(I)}+\left|\sqrt{\rho^2}\bar{u}^2(t)\right|^2_{L^2(I)}+\int_0^t\left(\left|\bar{u}_x^2(s)\right|^2_{L^2(I)}+|\bar{\Phi}_x^2|^2_{L^2(I)}\right)\mathrm{d}s\leq\frac{1}{2}\int_0^t\left(\left|\sqrt{\rho^1}\bar{u}^1(s)\right|^2_{L^2(I)}+\left|\bar{u}_x^1(s)\right|^2_{L^2(I)}\right)\mathrm{d}s,$$
$$\cdots\cdots$$
$$\left|\bar{\rho}^{k+1}(t)\right|^2_{L^2(I)}+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|^2_{L^2(I)}+\int_0^t\left(\left|\bar{u}_x^{k+1}(s)\right|^2_{L^2(I)}+|\bar{\Phi}_x^{k+1}|^2_{L^2(I)}\right)\mathrm{d}s$$
$$\leq\frac{1}{2}\int_0^t\left(\left|\sqrt{\rho^k}\bar{u}^k(s)\right|^2_{L^2(I)}+\left|\bar{u}_x^k(s)\right|^2_{L^2(I)}\right)\mathrm{d}s.$$

Hence, we combine the above inequalities, in view of Gronwall's inequality, we deduce that

$$\sum_{k=1}^{K}\left[\sup_{0\leq t\leq T_*}\left(\left|\bar{\rho}^{k+1}(t)\right|^2_{L^2(I)}+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|^2_{L^2(I)}\right)+\int_0^{T_*}\left(\left|\bar{u}_x^{k+1}(s)\right|^2_{L^2(I)}+|\bar{\Phi}_x^{k+1}|^2_{L^2(I)}\right)\mathrm{d}t\right]<C. \tag{3.11}$$

Therefore, we conclude that the full sequence $\left(\rho^k,u^k,\Phi^k\right)$ converges to a limit (ρ,u,Φ) in the following strong sense:

$$\rho^k\to\rho\quad\text{in }L^\infty\left(0,T_*;L^2(I)\right), \tag{3.12}$$

$$u^k \to u \quad \text{in } L^\infty\left(0, T_*; L^2(I)\right) \cap L^2\left(0, T_*; H_0^1(I)\right). \tag{3.13}$$

Combining (3.3) and the convergence of (3.12), we can get

$$\Phi^k \to \Phi \quad \text{in } L^\infty\left(0, T_*; H^2(I)\right). \tag{3.14}$$

From the lower semi-continuity of the norm, we get:

$$\text{ess} \sup_{0\le t\le T_1} \left(|\rho(t)|_{H^1(I)} + |u(t)|_{W_0^{1,p}\cap H^2(I)} + \left|\sqrt{\rho}u_t(t)\right|_{L^2(I)} + |\rho_t(t)|_{L^2(I)}\right) + \int_0^{T_*} |u_{xt}(t)|^2_{L^2(I)}\,\mathrm{d}t \le C. \tag{3.15}$$

4. Existence

The proof of existence should be completed in three steps, namely, taking limits on $k \to \infty$, $\varepsilon \to 0^+$ and $\delta \to 0^+$. Since the method is similar, we will only describe the process of taking limits on $\delta \to 0$ below. The first two steps can be found in the literature [4].

We take δ to be a very small positive number, let $\rho_0^\delta = J_\delta * \rho_0 + \delta$, J_δ is a mollifier on I, $u_0^\delta \in H_0^1(I) \cap H^2(I)$ is the unique smooth solution of the boundary value problem:

$$\begin{cases} -\left(\left|u_{0x}^\delta\right|^{p-2} u_{0x}^\delta\right)_x = -P_x\left(\rho_0^\delta\right) + \left(\rho_0^\delta\right)^{\frac{1}{2}} g^\delta, \\ u_0^\delta(0) = u_0^\delta(1) = 0, \end{cases}$$

there exists $g^\delta \in C_0^\infty(I)$ satisfies

$$\left|g^\delta\right|_{L^2(I)} \le |g|_{L^2(I)}, \quad \lim_{\delta\to 0^+} \left|g^\delta - g\right|_{L^2(I)} = 0.$$

For $\rho_0^\delta = J_\delta * \rho_0 + \delta$, there is a subsequence $\left\{\left(\rho_0^{\delta_j}, u_0^{\delta_j}\right)\right\} of \left\{\left(\rho_0^\delta, u_0^\delta\right)\right\}$, as $\delta_j \to 0^+$ satisfing

$$-P_x\left(\rho_0^{\delta_j}\right) + \left(\rho_0^{\delta_j}\right)^{\frac{1}{2}} g^{\delta_j} \to -P_x(\rho_0) + \rho_0^{\frac{1}{2}} g \quad \text{in } L^2(I),$$

$$-\left(\left|u_{0x}^{\delta_j}\right|^{p-2} u_{0x}^{\delta_j}\right)_x \to -\left(|u_{0x}|^{p-2} u_{0x}\right)_x \quad \text{in } L^2(I).$$

Therefore, (ρ_0, u_0) satisfies the following problem

$$-\left(|u_{0x}|^{p-2} u_{0x}\right)_x = -P_x(\rho_0) + \rho_0^{\frac{1}{2}} g \quad \text{a.e } x \in I.$$

There exists a $T_* \in (0, +\infty)$, the initial-boundary value problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, & (x,t) \in \Omega_{T_*} \\ (\rho u)_t + \left(\rho u^2\right)_x + \rho\Phi_x - \left(|u_{0x}|^{p-2} u_{0x}\right)_x + P_x = -\rho u, & (x,t) \in \Omega_{T_*} \\ \left[\left|\Phi_x^k\right|^{q-2} \Phi_x^k\right]_x = 4\pi g\left(\rho^k - \frac{1}{|\Omega|}\int_\Omega \rho^k \mathrm{d}x\right), & \\ P \equiv P(\rho) = A\rho\gamma, \quad A > 0, \gamma > 1, & \\ (\rho, u, \Phi)|_{t=0} = \left(\rho_0^\delta, u_0^\delta, \Phi_0^\delta\right), & x \in [0,1] \\ u|_{x=0} = u|_{x=1} = 0, & t \in [0, T_*] \end{cases}$$

admits a unique solution $\left(\rho^\delta, u^\delta, \Phi^\delta\right)$. Moreover, $\left(\rho^\delta, u^\delta, \Phi^\delta\right)$ satisfies the uniform estimate

$$\operatorname{ess}\sup_{0\le t\le T_1}\left(\left|\rho^\delta(t)\right|_{H^1(I)}+\left|u^\delta(t)\right|_{W_0^{1,p}(I)\cap H^2(I)}+\left|\sqrt{\rho^\delta}u_t^\delta(t)\right|_{L^2(I)}+\left|\rho_t^\delta(t)\right|_{L^2(I)}\right)+\int_0^{T_*}\left|u_{xt}^\delta(t)\right|_{L^2(I)}^2\mathrm{d}t\le C.$$

According to the above uniform estimate, by the lower semi-continuity of norm, as $\delta_j\to 0$, we deduce the following uniform estimate:

$$\operatorname{ess}\sup_{0\le t\le T_1}\left(|\rho(t)|_{H^1(I)}+|u(t)|_{W_0^{1,p}(I)\cap H^2(I)}+\left|\sqrt{\rho}u_t(t)\right|_{L^2(I)}+|\rho_t(t)|_{L^2(I)}\right)+\int_0^{T_*}|u_{xt}(t)|_{L^2(I)}^2\,\mathrm{dt}\le C.$$

5. Uniqueness

Suppose (ρ_1,u_1,Φ_1) is a strong solution to the problem (1.1)–(1.2), (ρ_2,u_2,Φ_2) is also a strong solution to the problem (1.1)–(1.2), then we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1\rho_1(u_1-u_2)^2\,\mathrm{d}x+\int_0^t\int_0^1\left(Ł_pu_1-Ł_pu_2\right)(u_1-u_2)\,\mathrm{d}x\mathrm{d}s\\
&=\int_0^t\int_0^1\Big((\rho_1-\rho_2)(h-\Phi_{2x})(u_1-u_2)+\rho_1(u_1-u_2)^2u_{2x}-\rho_1(u_1-u_2)^2+(P_1-P_2)(u_1-u_2)_x\Big)\mathrm{d}x\mathrm{d}s\\
&\quad+\int_0^t\int_0^1\rho_1|(\Phi_1-\Phi_2)_x||u_1-u_2|\mathrm{d}x\mathrm{d}s\\
&=\sum_{j=1}^5 I_j,
\end{aligned}\tag{5.1}$$

where $h=-u_2-u_{2t}-u_2u_{2x}\in L^2\left(0,T_*;L^2(I)\right)$.

Then using Hölder inequality together with Lemma 1, we have

$$\begin{aligned}
I_1&\le\int_0^t|\rho_1-\rho_2|_{L^2(I)}|h-\Phi_{2x}|_{L^2(I)}|u_1-u_2|_{L^\infty(I)}(I)\mathrm{d}s\\
&\le C_\varepsilon\int_0^t|\rho_1-\rho_2|_{L^2(I)}^2|h-\Phi_{2x}|_{L^2(I)}^2\mathrm{d}s+\varepsilon\int_0^t|u_{1x}-u_{2v}|_{L^2(I)}^2\,\mathrm{d}s,\\
I_2&=\int_0^t\int_0^1\rho_1|u_1-u_2|^2u_{2x}\mathrm{d}x\mathrm{d}s\le\int_0^t\left|\sqrt{\rho_1}(u_1-u_2)\right|_{L^2(I)}^2|u_{2x}|_{L^\infty(I)}\mathrm{d}s,\\
\mathrm{I}_3&=\int_0^t\int_0^1\rho_1|u_1-u_2|^2\,\mathrm{d}x\mathrm{d}s\le\int_0^t|\sqrt{\rho_1}(u_1-u_2)|_{L^2(I)}^2\mathrm{d}s,\\
\mathrm{I}_4&=\int_0^t\int_0^1|P_1-P_2||u_{1x}-u_{2x}|\,\mathrm{d}x\mathrm{d}s\le\int_0^t|P_1-P_2|_{L^2(I)}|u_{1x}-u_{2x}|_{L^2(I)}\,\mathrm{d}s\\
&\le C_\varepsilon\int_0^t|P_1-P_2|_{L^2(I)}^2\,\mathrm{d}s+\varepsilon\int_0^t|u_{1x}-u_{2x}|_{L^2(I)}^2\,\mathrm{d}s,\\
I_5&=\int_0^t\int_0^1\rho_1|(\Phi_1-\Phi_2)_x||u_1-u_2|\mathrm{d}x\mathrm{d}s\le\int_0^t|\rho_1(\rho_1-\rho_2)|_{L^2(I)}|u_1-u_2|_{L^2(I)}\mathrm{d}s.
\end{aligned}$$

By (3.7), we have

$$\int_0^t \int_0^1 \left(Ł_p^\varepsilon u_1 - Ł_p^\varepsilon u_2\right)(u_1 - u_2)\,\mathrm{d}x\mathrm{d}s \geq \mu_0^{\frac{p-2}{2}} \int_0^t |u_{1x} - u_{2x}|_{L^2(I)}^2\,\mathrm{d}s,$$

where $\mu_0 = \frac{1}{\left(|u_x(t)|_{L^\infty(0,t;L^\infty(I))} + |\bar{u}_x(t)|_{L^\infty(0,t;L^\infty(I))}\right)^{2-p}}$.

Then, following from (5.1), by choosing $\varepsilon = (\mu_0)^{\frac{p-2}{2}}/8$, we derive

$$\begin{aligned}
&\frac{1}{2}\left|\sqrt{\rho_1}(u_1 - u_2)\right|_{L^2(I)}^2 + \frac{3}{4}(\mu_0)^{\frac{p-2}{2}} \int_0^t |u_{1x} - u_{2x}|_{L^2(I)}^2\,\mathrm{d}s \\
&\leq \int_0^t \left(C|\rho_1 - \rho_2|_{L^2(I)}^2 |h - \Phi_{2x}|_{L^2(I)}^2 + \left|\sqrt{\rho_1}(u_1 - u_2)\right|_{L^2(I)}^2 \left(1 + |u_{2x}|_{L^\infty(I)}\right) + C|P_1 - P_2|_{L^2(I)}^2\right)\mathrm{d}s, \qquad (5.2)
\end{aligned}$$

where $0 < A(t) = C\left(1 + |h - \Phi_{2x}|_{L^2(I)}^2 + |u_{2x}|_{L^\infty(I)}^2\right) \in L^1(0, T_*)$.

As is known from the definition of a strong solution, we take $\varphi = \rho_1 - \rho_2$, then

$$\begin{aligned}
\frac{1}{2}\int_0^1 |\rho_1 - \rho_2|^2\,\mathrm{d}x &= \int_0^t \int_0^1 (\rho_1 u_1 - \rho_2 u_2)(\rho_1 - \rho_2)_x\,\mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_0^1 (\rho_1(u_1 - u_2) + (\rho_1 - \rho_2)u_2)(\rho_1 - \rho_2)\,\mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_0^1 (\rho_{1x}(u_1 - u_2)(\rho_1 - \rho_2) + \rho_1(u_1 - u_2)_x(\rho_1 - \rho_2) \\
&\leq \int_0^t \Big(|\rho_{1x}|_{L^2(I)} |u_1 - u_2|_{L^\infty(I)} |\rho_1 - \rho_2|_{L^2(I)} + |\rho_1|_{L^\infty(I)} |u_{1x} - u_{2x}| \\
&\quad + \frac{1}{2}|u_{2x}|_{L^2(I)} |\rho_1 - \rho_2|_{L^2(I)}^2\Big)\mathrm{d}s \\
&\leq \int_0^t \left(B(s)|\rho_1 - \rho_2|_{L^2(I)}^2 + \frac{1}{8}(\mu_0)^{\frac{p-2}{2}} |u_{1x} - u_{2x}|_{L^2(I)}^2\right)\mathrm{d}s, \qquad (5.3)
\end{aligned}$$

where $0 < B(t) = C\left(|\rho_1|_{H^1(I)} + |u_{2x}|_{L^\infty(I)}\right) \in L^1(0, T_*)$. Similarly, we have

$$\int_0^t \frac{1}{2}\frac{d}{dt}\int_0^1 |P_1 - P_2|^2\,\mathrm{d}x\mathrm{d}s \leq \int_0^t \left(D(s)|P_1 - P_2|_{L^2(I)}^2 + \frac{1}{8}(\mu_0)^{\frac{p-2}{2}} |u_{1x} - u_{2x}|_{L^2(I)}^2\right)\mathrm{d}s, \qquad (5.4)$$

where $0 < B(t) = C\left(|\rho_1|_{H^1(I)} + |u_{2x}|_{L^\infty(I)}\right) \in L^1(0, T_*)$. Similarly, we have

Combining (5.2), (5.3) and (5.4), we obtain

$$\begin{aligned}
&\frac{1}{2}(\mu_0)^{\frac{p-2}{2}} \int_0^t |u_{1x} - u_{2x}|_{L^2(I)}^2\,\mathrm{d}s + \frac{1}{2}\left[\left|\sqrt{\rho_1}(u_1 - u_2)\right|_{L^2(I)}^2 + |\rho_1 - \rho_2|_{L^2(I)}^2 + |P_1 - P_2|_{L^2(I)}^2\right] \\
&\leq \int_0^t H(t)\left(\left|\sqrt{\rho_1}(u_1 - u_2)\right|_{L^2(I)}^2 + |\rho_1 - \rho_2|_{L^2(I)}^2 + |P_1 - P_2|_{L^2(I)}^2\right)\mathrm{d}s,
\end{aligned}$$

where $H(t) = A(t) + B(t) + D(t) \in L^1(0, T_*)$. Using Gronwall's inequality, we can get

$$\frac{1}{2}(\mu_0)\frac{p-2}{2}|(u_{1x} - u_{2x})(t)|_{L^2(I)}^2$$

$$+\frac{1}{2}\operatorname{ess}\sup_{0\le t\le T_1}\left(\left|\sqrt{\rho_1}\left(u_1-u_2\right)(t)\right|_{L^2(I)}^2+\left|\left(\rho_1-\rho_2\right)(t)\right|_{L^2(I)}^2+\left|\left(P_1-P_2\right)(t)\right|_{L^2(I)}^2\right)\le 0,$$

then

$$\rho_1=\rho_2\,,\qquad \sqrt{\rho_1}\left(u_1-u_2\right)=0,\qquad u_{1x}=u_{2x},$$

we can get

$$\int_0^1\left(\Phi_{1x}-\Phi_{2x}\right)^2\,\mathrm{d}x\le C|\rho_1-\rho_2|_{L^2(I)}^2.$$

Therefore

$$\rho_1=\rho_2\,,\qquad u_1=u_2,\qquad \Phi_1=\Phi_2.$$

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Conflict of interest

The authors declare there is no conflicts of interest.

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