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Research article

Periodic solutions of a class of indefinite singular differential equations

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Abstract: In this article, some sufficient conditions for the existence of positive periodic solutions of a more general indefinite singular differential equation are established. The results are applicable to strong singularities as well as weak singularities. Some results in literature are generalized.

Keywords: indefinite singular equations; periodic solutions; fixed point theorem

1. Introduction and main results

In this article, we consider the following indefinite singular equation

$$u'' + a(t)u = f(t,u) - h(t)g(u), \tag{1.1}$$

where $a \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$, $h \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ may be allowed to change sign, $T > 0$, $f \in C((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^+, \mathbb{R})$, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ may be singular at the origin and

$$\lim_{u \to 0^+} g(u) = +\infty.$$

Since h is a sign changing function, the singular Eq (1.1) is called indefinite singular equation. Hess and Kato in [1] first mentioned the terminology indefinite. In the past few decades, scholars have mainly focused on the periodic problem of the singular equations with repulsive or attractive singularities, and many excellent results have been obtained. We just refer the reader to the literature [2] and the monographs [3, 4] for the surveys on this problem. But, when nonlinearities with indefinite singularities are taken into account, the periodic problem becomes slightly more tricky and still remains insufficiently investigated. The periodic problem of indefinite singular equations has been considered by researchers in the past few years. For example, Godoy and Zamora in [5–7] obtained some existence results for the periodic problem of the equation with indefinite weak singularities. Boscaggin et al. in [8] studied the uniqueness of positive solutions of the ϕ-Laplacian equations with indefinite singularities. Relying on the continuation theorem, Lu et al. [9–12] studied the periodic problem of indefinite singular Liénard equations. Chen in [13] considered the indefinite singular equation

$$u'' + u - \frac{h(t)}{u^{1-p}} = 0, \tag{1.2}$$

where $-2 \le p \le 0$ and $h \in C(\mathbb{R}/\pi\mathbb{Z}, \mathbb{R})$ may be allowed to change sign. Relying on a variational method, Chen proved that if one of the following hypotheses holds:

(A_1) $-1 < p \le 0$ and $\int_0^\pi h(s)\mathrm{d}s > 0$,

(A_2) $p = -2$, $\int_0^\pi h(s)\mathrm{d}s > 0$ and h is a $\frac{\pi}{k}$-periodic function for integer $k > 1$,

(A_3) $-2 < p \le -1$ and h is positive somewhere,

then Eq (1.2) has a π-periodic solution. The Eq (1.2) has also attracted attentions of Dou and Zhu in [14] for $p \le -2$, Jiang [15], Torres and Zamora [16] for $p \le 0$.

Cheng et al. in [17, 18] got some existence results for the following indefinite singular equation

$$u'' + a(t)u = \frac{b(t)}{u^q} + e(t), \tag{1.3}$$

where $b \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ may be allowed to change sign, $q > 0$, $a, e \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$. After that, Cheng and his coauthors in [19] generalized the similar results to the differential equation with two indefinite singularities. When $a(t) = e(t) = 0$, Bravo and Torres in [20] studied the periodic problem of the Eq (1.3) with b as a piecewise function that changes sign, and Ureña in [21] obtained some nonexistent results for the Eq (1.3). Moreover, Hakl and Zamora in [22–24] considered the following second-order indefinite singular equation

$$u'' - h(t)g(u) = 0,$$

where $h \in L(\mathbb{R}/T\mathbb{Z})$ may be allowed to change sign and $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ may be singular at the origin. Based on the Leray-Schauder degree theory, they established some sufficient conditions for the existence of positive T-periodic solutions.

Recently, in [25], the first three authors of this paper considered the following indefinite singular radially symmetric system

$$\ddot{u} + l^2 u = \left(\frac{b(t)}{|u|^q} + e(t)\right)\frac{u}{|u|}, \quad u \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \tag{1.4}$$

where $0 < l < \frac{\pi}{T}$, $T > 0$, $b \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ may be allowed to change sign, $q > 0$, $e \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$. We introduce the following polar coordinates

$$u(t) = \big(r(t)\cos\theta(t), r(t)\sin\theta(t)\big), \quad \forall t \in \mathbb{R}.$$

Here $\theta(t) \in \mathbb{R}$, $r(t) > 0$, and then Eq (1.4) is transformed into

$$r'' + l^2 r = \frac{\mu^2}{r^3} + \frac{b(t)}{r^q} + e(t), \tag{1.5}$$

where $\mu = r^2\theta'$ is the angular momentum of $u(t)$. First, we got some existence results of the periodic problem of Eq (1.5) with $\mu = 0$ (see [25, Lemma 3.2]). Then, by [26, Theorem 14.C], we proved that there is a positive constant $\mathfrak{A}$ such that, for every $\mu \in [0, \mathfrak{A}]$, Eq (1.5) has a T-periodic solution. Finally, we proved that Eq (1.4) has a mT-periodic solution and mT is the minimal period.

Since the indefinite singular radially symmetric systems like Eq (1.4) are quite complex, we can not generalize the main results in [25] to a more general indefinite singular radially symmetric system. But some results in [25] can be generalized. Motivated by [25, Lemma 3.2], we study the periodic problem of the general indefinite singular Eq (1.1).

Suppose that $G(t, s)$ is the Green's function of

$$\begin{cases} u'' + a(t)u = 0, \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases}$$

According to [27, Corollary 2.3 and Remark 1], we know that if

$$\|a\|_1 \le \frac{4}{T} \ \text{ or } \ \|a\|_\infty \le (\frac{\pi}{T})^2, \tag{1.6}$$

then

$$G(t, s) > 0, \quad \forall (t, s) \in [0, T]^2.$$

In this case, the periodic solutions of Eq (1.1) can be written as

$$u(t) = \int_0^T G(t, s)[f(s, u(s)) - h(s)g(u(s))]\mathrm{d}s.$$

For simplicity, for a given $h \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, we denote

$$\bar{h} = \frac{1}{T}\int_0^T h(s)\mathrm{d}s, \quad h_* = \min_{t\in[0,T]} h(t), \quad h^* = \max_{t\in[0,T]} h(t),$$

$$h^-(t) = \max\{-h(t), 0\}, \quad h^+(t) = \max\{h(t), 0\}.$$

We state our main result as follows:

Theorem 1.1. *Assume that Eq* (1.6) *holds and there exists* $p > 0$*, such that*

$$g(u) \ge \frac{1}{u^p}, \quad \forall u \in \mathbb{R}^+. \tag{1.7}$$

Suppose further that there exist

$$0 < \rho < \sigma(G_* T\overline{h^-})^{\frac{1}{p+1}}$$

and a continuous function φ*, such that*

$$h^* g(u) \le f(t, u) \le \varphi(u), \quad \forall (t, u) \in \mathbb{R} \times [\rho, +\infty), \tag{1.8}$$

where $\varphi(u) > 0$ *is non-increasing in* $u \in [\rho, +\infty)$ *and*

$$G^* = \max_{(t,s)\in[0,T]\times[0,T]} G(t, s), \quad G_* = \min_{(t,s)\in[0,T]\times[0,T]} G(t, s), \quad \sigma = \frac{G_*}{G^*}.$$

Then, Eq (1.1) *has a positive* T*-periodic solution.*

Notice that the Eq (1.5) with $\mu = 0$ is a special case of Eq (1.1), where

$$g(u) = u^{-q}, \quad f(t,u) = e(t), \quad h(t) = -b(t).$$

Choosing positive constants $p = q, \rho = (\frac{-b_*}{e_*})^{\frac{1}{p}}$ and non-increasing function $\varphi(u) = e^*$, notice that

$$g(u) = \frac{1}{u^p}, \quad \forall u \in \mathbb{R}^+$$

and

$$e^* \geq f(t,u) \geq e_* \geq \frac{-b_*}{u^p}, \quad \forall (t,u) \in \mathbb{R} \times [\rho, +\infty),$$

which imply that Eqs (1.7) and (1.8) hold for the Eq (1.5) with $\mu = 0$. Therefore, Theorem 1.1 can be directly applied to the Eq (1.5) with $\mu = 0$, as the following corollary shows.

Corollary 1.2. *Suppose that* $0 < l < \frac{\pi}{T}$,

$$\Big(\frac{-b_*}{e_*}\Big)^{\frac{1}{q}} < \sigma(G_* T\overline{b^+})^{\frac{1}{q+1}}. \tag{1.9}$$

Then, Eq (1.5) *with* $\mu = 0$ *has a positive* T*-periodic solution.*

Moreover, if Eqs (1.6) and (1.9) hold, the same conclusion also holds for the Eq (1.3). Corollary 1.2 is a consequence of [25, Lemma 3.2].

Besides the above applications, Theorem 1.1 can also be applied to some other indefinite singular differential equations. See the examples in Section 3 for details. We will give the proof of Theorem 1.1 in Section 2. Finally, it is worth noting that Theorem 1.1 is applicable to strong singularities as well as weak singularities.

2. Proof of Theorem 1.1

We first recall the following result:

Lemma 2.1 ([28]). *Suppose that* $\Omega_1 \subset X$, $\Omega_2 \subset X$ *are open and bounded,* $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, *where* X *is a Banach space. Let* $K \subset X$ *be a cone and*

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \mapsto K$$

be a continuous and compact operator, such that

(H_1) $\|\Phi u\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$;
(H_2) $\exists \theta \in K \setminus \{0\}$ *such that* $u \neq \Phi u + \lambda\theta$, $\lambda > 0$, $\forall u \in K \cap \partial\Omega_1$.

Then, Φ *has at least one fixed point in* $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Define

$$\Phi : X \to X : \quad (\Phi u)(t) = \int_0^T G(t,s)[f(s,u(s)) - h(s)g(u(s))]\mathrm{d}s,$$

where $X = C(\mathbb{R}/T\mathbb{Z})$ with $\|u\| = \max_{t\in\mathbb{R}} |u(t)|$. Notice that the fixed point u of Φ is a T-periodic solution of Eq (1.1).

Let

$$\begin{aligned}&\Omega_1 = \{u \in X : \|u\| < \alpha\}, \quad \Omega_2 = \{u \in X : \|u\| < \beta\},\\ &K = \{u \in X : \min_{t\in\mathbb{R}} u(t) \geq \sigma\|u\|\},\end{aligned}$$

where $\beta > \alpha > 0$ are constants with

$$\beta > \max\{(G_* T\overline{h^-})^{\frac{1}{p+1}}, G^* T\Big(\frac{\| - h\|}{h^*} + 1\Big)\varphi(\rho)\} \tag{2.1}$$

and

$$\alpha = (G_* T\overline{h^-})^{\frac{1}{p+1}}.$$

Firstly, we show that

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \mapsto K$$

is a continuous and compact operator. Obviously,

$$\beta \geq u(t) \geq \sigma\alpha = \sigma(G_* T\overline{h^-})^{\frac{1}{p+1}} > \rho > 0, \quad \forall u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1), \quad \forall t \in \mathbb{R}.$$

Then, it follows from Eq (1.8) that

$$f(t, u) - h(t)g(u) \geq (h^* - h(t))g(u) \geq 0.$$

Note that

$$\begin{aligned}\min_{t\in[0,T]} (\Phi u)(t) &= \min_{t\in[0,T]} \int_0^T G(t, s)(f(s, u(s)) - h(s)g(u(s)))\mathrm{d}s\\ &\geq G_* \int_0^T (f(s, u(s)) - h(s)g(u(s)))\mathrm{d}s\\ &= \sigma G^* \int_0^T (f(s, u(s)) - h(s)g(u(s)))\mathrm{d}s\\ &\geq \sigma \max_{t\in[0,T]} \int_0^T G(t, s)(f(s, u(s)) - h(s)g(u(s)))\mathrm{d}s\\ &= \sigma\|\Phi u\|,\end{aligned}$$

which means that

$$K \supset \Phi(K \cap (\bar{\Omega}_2 \setminus \Omega_1)).$$

The Arzelà-Ascoli theorem guarantees that

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \mapsto K$$

is a continuous and compact operator.

Secondly, we verify that

$$\|u\| \geq \|\Phi u\|, \quad \forall u \in K \cap \partial\Omega_2.$$

Obviously, we have

$$\beta \geq u(t) \geq \sigma\beta > \rho, \quad \forall u \in K \cap \partial\Omega_2, \quad \forall t \in \mathbb{R}.$$

By Eqs (1.8) and (2.1), we get

$$\begin{aligned}
\|\Phi u\| = &\max_{t\in[0,T]} \left| \int_0^T G(t,s)[f(s,u(s)) - h(s)g(u(s))]\mathrm{d}s \right| \\
\leq &G^* \int_0^T \max_{s\in[0,T]} \left| f(s,u(s)) - h(s)g(u(s)) \right| \mathrm{d}s \\
\leq &G^* \int_0^T \left(\max_{s\in[0,T]} |-h(s)| \frac{\varphi(u)}{h^*} + \varphi(u) \right) \mathrm{d}s \\
\leq &TG^* (\frac{\|-h\|}{h^*} + 1)\varphi(\rho) \leq \beta.
\end{aligned}$$

Then, we get that

$$\|u\| \geq \|\Phi u\|, \forall u \in K \cap \partial\Omega_2.$$

Finally, we prove that there is $\theta \in K \setminus \{0\}$, such that

$$u \neq \Phi u + \lambda\theta, \quad \lambda > 0 \ \text{ and } \ \forall u \in K \cap \partial\Omega_1.$$

Let $\theta = 1 \in K \setminus \{0\}$, and we can affirm that

$$u \neq \Phi u + \lambda, \quad \lambda > 0, \quad \forall u \in K \cap \partial\Omega_1. \tag{2.2}$$

Conversely, suppose that there are $u_1 \in K \cap \partial\Omega_1$, $\lambda_1 > 0$, such that $u_1 = \Phi u_1 + \lambda_1$. Then, owing to $u_1 \in K \cap \partial\Omega_1$, note that

$$\rho < \sigma\alpha \leq u_1(t) \leq \alpha = (G_* T\overline{h^-})^{\frac{1}{p+1}}, \quad \forall t \in \mathbb{R}. \tag{2.3}$$

Because h can be allowed to change sign, we have

$$h^* > 0 \ \text{ and } \ h(t) = h^+(t) - h^-(t).$$

Moreover, notice that

$$f(t, u_1) - h^+(t)g(u_1) \geq h^* g(u_1) - h^* g(u_1) = 0, \quad \forall t \in \mathbb{R}.$$

Then, it follows from Eq (1.7) and above facts that

$$\begin{aligned}
u_1(t) =&(\Phi u_1)(t) + \lambda_1 \\
=& \int_0^T G(t,s)\Big[f(s,u_1(s)) - h(s)g(u_1(s))\Big]\mathrm{d}s + \lambda_1 \\
=& \int_0^T G(t,s)\Big[f(s,u_1(s)) + h^-(s)g(u_1(s)) - h^+(s)g(u_1(s))\Big]\mathrm{d}s + \lambda_1
\end{aligned}$$

$$
\begin{aligned}
&>G_* \int_0^T h^-(s) g(u_1(s)) \mathrm{d}s \\
&\geq G_* \int_0^T \frac{h^-(s)}{u_1^p(s)} \mathrm{d}s \\
&\geq TG_* \frac{\overline{h^-}}{\alpha^p} = \alpha,
\end{aligned}
$$

which contradicts Eq (2.3). Hence Eq (2.2) is satisfied.

Up to now, by Lemma 2.1, we know that there exists a $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, such that $\Phi(u) = u$. Then, the proof is finished.

3. Applications

In this section, we present two examples.

Example 1. *Consider*

$$
u'' + a(t)u + \frac{h(t)}{u^p} = \frac{b(t)}{u^q}, \tag{3.1}
$$

where $h \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ may be allowed to change sign, $p > q > 0$, $a, b \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$.

Corollary 3.1. *Suppose that*

$$
\sigma(G_* T\overline{h^-})^{\frac{1}{p+1}} > \Big(\frac{h^*}{b_*}\Big)^{\frac{1}{p-q}} \tag{3.2}
$$

and Eq (1.6) holds. Then, Eq (3.1) has a T-periodic solution.

Proof. Corresponding to Eq (1.1),

$$
f(t,u) = \frac{b(t)}{u^q}, \quad g(u) = \frac{1}{u^p}.
$$

Choose the positive constant ρ and non-increasing function $\varphi(u)$ as follows:

$$
\rho = \Big(\frac{h^*}{b_*}\Big)^{\frac{1}{p-q}}, \quad \varphi(u) = \frac{b^*}{u^q}.
$$

By Eq (3.2), we have

$$
\rho < \sigma(G_* T\overline{h^-})^{\frac{1}{p+1}}.
$$

Moreover, notice that

$$
g(u) = \frac{1}{u^p}, \quad \forall u \in \mathbb{R}^+
$$

and

$$
\frac{b^*}{u^q} \geq \frac{b(t)}{u^q} \geq \frac{h^*}{u^p}, \quad \forall (t,u) \in \mathbb{R} \times [\rho, +\infty),
$$

which imply that Eqs (1.7) and (1.8) hold. Then, by Theorem 1.1, Corollary 3.1 is proved.

Example 2. *Consider*

$$u'' + a(t)u = b(t)(d\sin^2 u + c) - \frac{h(t)}{u^p}, \tag{3.3}$$

where $h \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ *may be allowed to change sign,* $a, b \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$*,* $c, d \in \mathbb{R}^+$*.*

Corollary 3.2. *Suppose that Eq* (1.6) *holds,*

$$\Big(\frac{h^*}{cb_*}\Big)^{\frac{1}{p}} < \sigma(G_*T\overline{h^-})^{\frac{1}{p+1}}. \tag{3.4}$$

Then, Eq (3.3) *has a* T*-periodic solution.*

Proof. Corresponding to Eq (1.1),

$$f(t,u) = b(t)(d\sin^2 u + c) \ \text{ and } \ g(u) = u^{-p}.$$

Let the positive constant ρ and non-increasing function $\varphi(u)$ be as follows:

$$\rho = \Big(\frac{h^*}{cb_*}\Big)^{\frac{1}{p}}, \ \ \varphi(u) = b^*(c+d).$$

By Eq (3.4), we have

$$\rho < \sigma(G_*T\overline{h^-})^{\frac{1}{p+1}}.$$

Moreover, notice that

$$g(u) = \frac{1}{u^p}, \ \ \forall u \in \mathbb{R}^+$$

and

$$b^*(c+d) \geq b(t)(c + d\sin^2 u) \geq cb_* \geq \frac{h^*}{u^p}, \forall (t,u) \in \mathbb{R} \times [\rho, +\infty),$$

which lead to that Eqs (1.7) and (1.8) hold. Then, by Theorem 1.1, Corollary 3.2 is proved.

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Conflict of interest

We declare that there are no conflicts of interest.

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