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Research article

Global boundedness of classical solutions to a Keller-Segel-Navier-Stokes system involving saturated sensitivity and indirect signal production in two dimensions

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Abstract: This paper is concerned with the following Keller–Segel–Navier–Stokes system with indirect signal production and tensor-valued sensitivity:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x, n, v, w)\nabla v), & x \in \Omega, t > 0, \\ v_t + u \cdot \nabla v = \Delta v - v + w, & x \in \Omega, t > 0, \\ w_t + u \cdot \nabla w = \Delta w - w + n, & x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases} \qquad (\heartsuit)$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where $\kappa \in \mathbb{R}$, $\phi \in W^{2,\infty}(\Omega)$, and S is a given function with values in $\mathbb{R}^{2\times2}$ which satisfies $|S(x, v, w, u)| \le C_S(n+1)^{-\alpha}$ with $C_S > 0$. If $\alpha > 0$, then for any sufficiently smooth initial data, there exists a globally classical solution which is bounded for the corresponding initial-boundary value problem of system ($\heartsuit$).

Keywords: Keller-Segel-Navier-Stokes system; tensor-valued sensitivity; indirect signal production; classical solution; global boundedness

1. Introduction and main results

Chemotaxis is a biological phenomenon which describes the oriented movement of cells (or organisms) in response to chemical gradients [1, 2]. As early as 1970, Keller and Segel [3] originally introduced a chemotaxis model through a system of parabolic equations. This model reads

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(x, n, c)\nabla c), & x \in \Omega, t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega, t > 0, \end{cases} \qquad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, the unknown functions n and c respectively represent the cell density and the signal concentration, and S denotes the chemotactic sensitivity. This model is primarily used to describe the aggregation phenomenon of Dictyostelium discoideum, where the effects of the chemical signal secreted by themselves are taken into consideration. During the past half a century, the Keller-Segel model has been attracting many scholars' attention. The known results are concentrated on whether the solutions for Neumann boundary problem of (1.1) globally exist or blow up in finite time. Concretely, if $S := S(n)$ is a scalar function fulfilling $S(n) \leq C_S(n+1)^{-\alpha}$ with some $C_S > 0$ and $\alpha > 0$, then for all $\alpha > 1 - \frac{2}{N}$, the corresponding problem has a global solution which is uniformly bounded [4]. However, if S satisfies $S(n) > c_S n^{-\alpha}$ with some $c_S > 0$ and $\alpha < 1 - \frac{2}{N}$ for $N \geq 2$, and Ω is a ball, then the solution of (1.1) will blow up in finite time. So,

$$\alpha = \frac{N-2}{N}$$

is called the critical exponent of the blow-up phenomenon. Recently, some results relating to the well-posedness of the hyperbolic Keller-Segel equation in the Besov framework were obtained in [5]. Afterwards, Zhang et al. [6] improved these results and established two kinds of blow-up criteria of strong solutions in Besov spaces by means of Littlewood-Paley theory. For more results about (1.1) and its variations, we refer interested readers to [7–16]

If we consider the framework where the chemical is produced by the cells indirectly, the corresponding chemotaxis model turns to the following Keller-Segel system with indirect signal production:

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(x,n,v,w)\nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + w, & x \in \Omega, t > 0, \\ w_t = \Delta w - w + n, & x \in \Omega, t > 0, \end{cases} \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary, where the functions n, v and w represent the cells, density, the signal and the chemical concentration, respectively. If $S(x,n,v,w) = \chi$ with $\chi > 0$ and $N \leq 3$, Fujie and Senba [17] showed that the homogeneous Neumann (or mixed) boundary problem of system (1.2) possesses a unique and globally bounded classical solution.

However, in many cases, the migration of cells (or bacteria) is largely affected by their surrounding environment [18, 19]. If the cells consume the chemical signal, Tuval et al. [19] introduced the following chemotaxis-fluid system:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where $f(c)$ measures the rate that cells consume the oxygen, and $S(x,n,c)$ is a tensor-valued (or scalar) chemotactic sensitivity. Most remarkably, by means of the chemical consumption setting and the maximum principle of the parabolic equations, we may directly deduce that c is uniformly bounded from the second equation of (1.3). This significant feature leads to the chemotaxis-fluid model with the framework of signal consumption being more intensively studied than the signal production mechanism. For instance, Winkler [20] proved that the global weak solution of system (1.3) which has enough regularity properties and thereby fulfills the condition of so-called eventual energy solution (this concept is newly proposed in his paper) becomes eventually smooth after some waiting time. For more studies about this system, one can refer to Zheng [21], Winkler [22–25] and other results on the global solvability and asymptotic behavior, such as [26–28], for details.

Considering the framework where the chemical signal is produced by the cells instead of consuming it, the corresponding chemotaxis-fluid model becomes the following Keller-Segel(-Navier)-Stokes system:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases} \tag{1.4}$$

where S is a tensor-valued (or scalar) function, and $\Omega \subset \mathbb{R}^N$ with smooth boundary. Let us just list a few representative results. For the Navier-Stokes fluid (i.e. $\kappa \neq 0$), if $|S(x,n,c)| \leq C_S(n+1)^{-\alpha}$ with $C_S > 0$ and $\alpha > 0$, in 2D case Wang et al. [29] showed that the initial-boundary value problem of (1.4) admits at least one classical solution. In the 3D Stokes case (i.e. $\kappa = 0$) of (1.4), Wang and Xiang [30] got the same results for $\alpha > \frac{1}{2}$. For the 3D Navier-Stokes version of system (1.4), Liu and Wang [31] verified that there exists at least one global weak solution for the corresponding initial-boundary value problem of (1.4) if $|S(x,n,c)| \leq C_S(n+1)^{-\alpha}$ with some $C_S > 0$ and $\alpha > \frac{3}{7}$. Recently, Ke and Zheng [32] improved the restriction admitting a global weak solution from $\alpha > \frac{3}{7}$ to $\alpha > \frac{1}{3}$, which compared with the known result of the fluid-free system is an optimal restriction on α. As for the further results, under assumption $\alpha \geq 1$ and an explicit condition on the size of C_S, Zheng [33] confirmed that the weak solution of system (1.4) would be eventually smooth, and that it is close to a unique spatially homogeneous steady state. Additionally, one can see [34–36] and the references therein to find more conclusions about this system.

Motivated by the above works, in this paper we consider the following initial-boundary value problem of the Keller-Segel-Navier-Stokes system with indirect signal production:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,v,w)\nabla v), & x \in \Omega, t > 0, \\ v_t + u \cdot \nabla v = \Delta v - v + w, & x \in \Omega, t > 0, \\ w_t + u \cdot \nabla w = \Delta w - w + n, & x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \\ (\nabla n - nS(x,n,v,w)\nabla v) \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, \quad u = 0, & x \in \partial\Omega, t > 0, \\ n(x,0) = n_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and $S(x,n,v,w)$ satisfies

$$S \in C^2(\bar{\Omega} \times [0,\infty)^3; \mathbb{R}^{2\times 2}), \tag{1.6}$$

and

$$|S(x,n,v,w)| \leq C_S(n+1)^{-\alpha}, \quad (x,n,v,w) \in \Omega \times [0,\infty)^3 \tag{1.7}$$

with some $C_S > 0$ and $\alpha \geq 0$. To state our main results of this paper, we make the following assumptions that

$$\phi \in W^{2,\infty}(\Omega), \tag{1.8}$$

and the initial data (n_0, v_0, w_0, u_0) satisfies

$$\begin{cases} n_0 \in C^{\iota}(\bar{\Omega}) \text{ with } n_0 \geq 0 \text{ in } \Omega \text{ for certain } \iota > 0, \\ v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0 \text{ in } \bar{\Omega}, \\ w_0 \in W^{1,\infty}(\Omega) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega}, \\ u_0 \in D\left(A_r^{\gamma}\right) \text{ for some } \gamma \in \left(\frac{1}{2}, 1\right) \text{ and any } r \in (1, \infty), \end{cases} \tag{1.9}$$

where A_r denotes the Stokes operator with domain $D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L^r_{\sigma}(\Omega)$ and $L^r_{\sigma}(\Omega) := \{\varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0\}$ [37].

With these assumptions at hand, we can state the following main results.

Theorem 1.1. *If (1.6), (1.7), (1.8) and (1.9) hold, then for any*

$$\alpha > 0, \tag{1.10}$$

there exists a global classical solution (n, v, w, u, P) of problem (1.5) which fulfills

$$\begin{cases} n \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ v \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \cap L^{\infty}\left([0, \infty); W^{1,p}(\Omega)\right), \\ w \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \cap L^{\infty}\left([0, \infty); W^{1,q}(\Omega)\right), \\ u \in C^0\left(\bar{\Omega} \times [0, \infty); \mathbb{R}^2\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, \infty); \mathbb{R}^2\right) \cap L^{\infty}\left([0, \infty); D\left(A^{\gamma}\right)\right), \\ P \in C^{1,0}(\bar{\Omega} \times (0, \infty)) \end{cases} \tag{1.11}$$

with $p, q > 1$ and $\gamma \in (0, 1)$, where n, v and w are nonnegative in $\Omega \times (0, \infty)$. Moreover, the solution is bounded, and there exists $C(\gamma) > 0$ such that

$$\|n(\cdot, t)\|_{L^{\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^{\gamma} u(\cdot, t)\|_{L^2(\Omega)} \leq C(\gamma) \tag{1.12}$$

for all $t > 0$, where γ is given by (1.9).

Remark 1.1. Theorem 1.1 improves the result of Yu [38], which showed that if $\alpha > 0$, then the Stokes version of problem (1.5) possesses a global classical solution.

This paper is organized as follows. In Section 2, we claim that the regularized problem possesses at least one local classical solution which is nonnegative. Relying on a series of ε-independent a priori estimates obtained in Section 3, in Section 4 we verify the local existence of a classical solution for regularized problem can be extended to the global. In Section 5, we will construct a global weak solution which has enough regularity to become a classical solution to prove our main results.

2. Preliminaries

Compared with the classical Keller-Segel model, the convection term presenting in the Navier-Stokes equations engenders more mathematical difficulties. We define

$$S_{\varepsilon}(x, n, v, w) := \rho_{\varepsilon}(x)\chi_{\varepsilon}(n)S(x, n, v, w) \text{ for all } (x, n, v, w) \in \bar{\Omega} \times [0, \infty)^3, \tag{2.1}$$

where $\{\rho_{\varepsilon}\}_{\varepsilon \in (0,1)} \in C_0^{\infty}(\Omega)$ is a family of standard cut-off functions fulfilling $0 \leq \rho_{\varepsilon} \leq 1$ in Ω and $\rho_{\varepsilon} \nearrow 1$ in Ω as $\varepsilon \searrow 0$, and $\chi_{\varepsilon} \in C_0^{\infty}([0, \infty))$ satisfies $0 \leq \chi_{\varepsilon} \leq 1$ in $[0, \infty)$ and $\chi_{\varepsilon} \nearrow 1$ in $[0, \infty)$ as $\varepsilon \searrow 0$.

Then, we can introduce the following approximate system of (1.5):

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon S_\varepsilon(x, n_\varepsilon, v_\varepsilon, w_\varepsilon)\nabla v_\varepsilon), & x \in \Omega, t > 0, \\ v_{\varepsilon t} + u_\varepsilon \cdot \nabla v_\varepsilon = \Delta v_\varepsilon - v_\varepsilon + w_\varepsilon, & x \in \Omega, t > 0, \\ w_{\varepsilon t} + u_\varepsilon \cdot \nabla w_\varepsilon = \Delta w_\varepsilon - w_\varepsilon + n_\varepsilon, & x \in \Omega, t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon - \kappa (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega, t > 0, \\ \nabla n_\varepsilon \cdot \nu = \nabla v_\varepsilon \cdot \nu = \nabla w_\varepsilon \cdot \nu = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega, t > 0, \\ n_\varepsilon(x,0) = n_0(x), v_\varepsilon(x,0) = v_0(x), w_\varepsilon(x,0) = w_0(x), u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.2}$$

where

$$Y_\varepsilon \xi := (1 + \varepsilon A)^{-1}\xi \quad \text{ for all } \xi \in L^2_\sigma(\Omega)$$

is the standard Yosida approximation, and $A := -\mathcal{P}\Delta$ is the realization of the Stokes operator with $\mathcal{P}$ denoting the Helmholtz projection of $L^2(\Omega)$ onto solenoidal subspace $L^2_\sigma(\Omega)$ [37].

Our main idea is to construct a weak solution which is globally bounded (the concept of weak solution can be found in Definition 5.1), and we claim it possesses adequate regularity to be a classical solution. The biggest obstacle we must deal with is the bad regularity of n caused by the small exponent α. Our main tool is based upon an energy estimate concerning the functional

$$\int_\Omega n_\varepsilon^{1+\alpha}(\cdot, t) + \int_\Omega |\nabla v_\varepsilon(\cdot, t)|^2,$$

which successfully overcomes this difficulty. The appropriately regularized problem (2.2) possesses local-in-time classical solution, which can be stated as follows.

Lemma 2.1. *Suppose $\phi \in W^{2,\infty}(\Omega)$ and $\varepsilon \in (0,1)$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then, there exist $T_{max,\varepsilon} \in (0,\infty]$ and a classical solution $(n_\varepsilon, v_\varepsilon, w_\varepsilon, u_\varepsilon, P_\varepsilon)$ of (2.2) in $\Omega \times (0, T_{max,\varepsilon})$ such that*

$$\begin{cases} n_\varepsilon \in C^0\left(\bar\Omega \times [0, T_{max,\varepsilon})\right) \cap C^{2,1}\left(\bar\Omega \times (0, T_{max,\varepsilon})\right), \\ v_\varepsilon \in C^0\left(\bar\Omega \times [0, T_{max,\varepsilon})\right) \cap C^{2,1}\left(\bar\Omega \times (0, T_{max,\varepsilon})\right), \\ w_\varepsilon \in C^0\left(\bar\Omega \times [0, T_{max,\varepsilon})\right) \cap C^{2,1}\left(\bar\Omega \times (0, T_{max,\varepsilon})\right), \\ u_\varepsilon \in C^0\left(\bar\Omega \times [0, T_{max,\varepsilon})\right) \cap C^{2,1}\left(\bar\Omega \times (0, T_{max,\varepsilon})\right), \\ P_\varepsilon \in C^{1,0}\left(\bar\Omega \times (0, T_{max,\varepsilon})\right) \end{cases} \tag{2.3}$$

solves (2.2) in the classical sence in $\Omega \times [0, T_{max,\varepsilon})$, and that n_ε, v_ε and w_ε are nonnegative in $\Omega \times (0, T_{max,\varepsilon})$. Moreover, if $T_{max,\varepsilon} < \infty$, then we have

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty$$

as $t \to T_{max,\varepsilon}$, where γ is similar to that in (1.9).

Proof. The fixed point argument which is established in [39,40] enables us to immediately substantiate the local existence for a classical solution which complies with (2.3). That n_ε, v_ε and w_ε are nonnegative is a clear conclusion of the maximum principle [41]. □

3. A priori estimates

In this section, we will derive a series ε-independent a priori estimates of the classical solution $(n_\varepsilon, v_\varepsilon, w_\varepsilon, u_\varepsilon)$ of regularized problem (2.2) from Lemma 2.1. By the way, we take $\tau = \min\left\{1, \frac{1}{4}T_{max,\varepsilon}\right\}$. The positive constants C_i $(i \in \mathbb{N}^*)$ appearing in the proof of every lemma are independent of $\varepsilon \in (0,1)$, which only depend on $\Omega, \alpha, C_S, \phi, n_0, v_0, w_0$ and u_0 if there is no especial explanation. Firstly, by simple integration and ODE comparison arguments, we obtain the following boundedness of L^1-norms, which is common in many chemotaxis models.

Lemma 3.1. *For any $\varepsilon \in (0,1)$, the solution of (2.2) satisfies*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \quad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.1}$$

as well as

$$\int_\Omega w_\varepsilon(\cdot,t) \le \max\left\{\int_\Omega n_0, \int_\Omega w_0\right\} \quad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.2}$$

and

$$\int_\Omega v_\varepsilon(\cdot,t) \le \max\left\{\int_\Omega n_0, \int_\Omega v_0, \int_\Omega w_0\right\} \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.3}$$

Lemma 3.2. *If $\alpha > 0$, then for any $\mu > 0$, there exists a constant $C > 0$ independent of $\varepsilon \in (0,1)$ such that*

$$\|n_\varepsilon(\cdot,t)\|^2_{L^{1+\alpha}(\Omega)} \le \mu \left\|\nabla n_\varepsilon^\alpha(\cdot,t)\right\|^2_{L^2(\Omega)} + C \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.4}$$

Proof. For any $\mu > 0$, the Gagliardo-Nirenberg inequality and the Young inequality provide positive constants C_1 and C_2 such that

$$\begin{aligned}
&\|n_\varepsilon\|^2_{L^{1+\alpha}(\Omega)}\\
=&\left\|n_\varepsilon^\alpha\right\|^{\frac{2}{\alpha}}_{L^{\frac{\alpha+1}{\alpha}}}\\
\le& C_1\left\|\nabla n_\varepsilon^\alpha\right\|^{\frac{2}{1+\alpha}}_{L^2(\Omega)}\left\|n_\varepsilon^\alpha\right\|^{\frac{2}{\alpha(1+\alpha)}}_{L^{\frac{1}{\alpha}}(\Omega)} + C_1\left\|n_\varepsilon^\alpha\right\|^{\frac{2}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)}\\
\le& \mu\left\|\nabla n_\varepsilon^\alpha\right\|^2_{L^2(\Omega)} + C_2 \text{ for all } t \in (0, T_{max,\varepsilon}),
\end{aligned} \tag{3.5}$$

where the boundedness of $\|n_\varepsilon\|_{L^1(\Omega)}$ from (3.1) and the fact that $\frac{2}{1+\alpha} < 2$ by $\alpha > 0$ are used. □

Lemma 3.3. *If $\alpha > 0$ and $p \in [2, 2 + \frac{2\alpha}{1+\alpha})$, then there exists some $C > 0$ such that for all $\varepsilon \in (0,1)$ the solution of (2.2) satisfies*

$$\begin{cases}
\int_\Omega n_\varepsilon^{2\alpha}(\cdot,t) + \int_\Omega v_\varepsilon^2(\cdot,t) + \int_\Omega w_\varepsilon^p(\cdot,t) \le C \quad \text{for all } t \in (0, T_{max,\varepsilon}) & \text{if } \alpha \ne \dfrac{1}{2}\\
\int_\Omega n_\varepsilon(\cdot,t)\ln n_\varepsilon(\cdot,t) + \int_\Omega v_\varepsilon^2(\cdot,t) + \int_\Omega w_\varepsilon^p(\cdot,t) \le C \quad \text{for all } t \in (0, T_{max,\varepsilon}) & \text{if } \alpha = \dfrac{1}{2}
\end{cases} \tag{3.6}$$

and

$$\int_t^{t+\tau}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 + \int_t^{t+\tau}\int_\Omega |\nabla v_\varepsilon|^2 + \int_t^{t+\tau}\int_\Omega \left|\nabla w_\varepsilon^{\frac{p}{2}}\right|^2 \le C \quad \text{for all } t \in (0, T_{max,\varepsilon} - \tau). \tag{3.7}$$

Proof. This proof may be divided into two cases:

Case 1, $\alpha \neq \frac{1}{2}$**.**

First, multiplying the first equation of (2.2) by $n_\varepsilon^{2\alpha-1}$, employing the fact that $\nabla \cdot u_\varepsilon = 0$, integrating by parts, we derive

$$\begin{aligned}&\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\alpha} + \frac{2\alpha-1}{\alpha^2}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 \\ =&\int_\Omega n_\varepsilon^{2\alpha-1}\nabla\cdot(n_\varepsilon S_\varepsilon(x,n_\varepsilon,v_\varepsilon,w_\varepsilon)\nabla v_\varepsilon)\,.\end{aligned} \tag{3.8}$$

Applying the Young inequality and the trivial fact $\frac{n_\varepsilon}{n_\varepsilon+1} \leq 1$, by (1.7), we obtain

$$\begin{aligned}&\frac{sgn(2\alpha-1)}{2\alpha}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\alpha} + \frac{|2\alpha-1|}{\alpha^2}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 \\ =&sgn(2\alpha-1)\int_\Omega n_\varepsilon^{2\alpha-1}\nabla\cdot(n_\varepsilon S_\varepsilon(x,n_\varepsilon,v_\varepsilon,w_\varepsilon)\nabla v_\varepsilon) \\ \leq&|2\alpha-1|\int_\Omega n_\varepsilon^{2\alpha-1}|S_\varepsilon(x,n_\varepsilon,v_\varepsilon,w_\varepsilon)||\nabla v_\varepsilon||\nabla n_\varepsilon| \\ \leq&|2\alpha-1|C_S\int_\Omega n_\varepsilon^{2\alpha-1}(n_\varepsilon+1)^{-\alpha}|\nabla v_\varepsilon||\nabla n_\varepsilon| \\ \leq&\frac{|2\alpha-1|}{2\alpha^2}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 + \frac{|2\alpha-1|}{2}C_S^2\int_\Omega |\nabla v_\varepsilon|^2.\end{aligned} \tag{3.9}$$

Namely,

$$\begin{aligned}&\frac{sgn(2\alpha-1)}{\alpha}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\alpha} + \frac{|2\alpha-1|}{\alpha^2}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 \\ \leq&|2\alpha-1|C_S^2\int_\Omega |\nabla v_\varepsilon|^2.\end{aligned} \tag{3.10}$$

Next, testing the second equation of (2.2) by v_ε, utilizing the fact that u_ε is divergence-free and the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{1+\alpha}{\alpha}}(\Omega)$, by virtue of the Hölder inequality and the Cauchy-Schwarz inequality, we deduce that there exists a constant $C_1 > 0$ satisfying

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega v_\varepsilon^2 \\ =&\int_\Omega v_\varepsilon w_\varepsilon \\ \leq&\|v_\varepsilon\|_{L^{\frac{1+\alpha}{\alpha}}(\Omega)}\|w_\varepsilon\|_{L^{1+\alpha}(\Omega)} \\ \leq&C_1\|v_\varepsilon\|_{W^{1,2}(\Omega)}\|w_\varepsilon\|_{L^{1+\alpha}(\Omega)} \\ \leq&\frac{1}{2}\int_\Omega v_\varepsilon^2 + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2}C_1^2\|w_\varepsilon\|_{L^{1+\alpha}(\Omega)}^2\,.\end{aligned} \tag{3.11}$$

Hence,

$$\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega v_\varepsilon^2 \leq C_1^2\|w_\varepsilon\|_{L^{1+\alpha}(\Omega)}^2\,. \tag{3.12}$$

This along with a multiple of (3.10) yields that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{sgn(2\alpha-1)}{2\alpha|2\alpha-1|C_S^2}\int_\Omega n_\varepsilon^{2\alpha}+\int_\Omega v_\varepsilon^2\right)+\frac{1}{2\alpha^2C_S^2}\int_\Omega\left|\nabla n_\varepsilon^\alpha\right|^2\\&+\frac{1}{2}\int_\Omega|\nabla v_\varepsilon|^2+\int_\Omega v_\varepsilon^2\le C_1^2\,\|w_\varepsilon\|^2_{L^{1+\alpha}(\Omega)}.\end{aligned}\tag{3.13}$$

Then, multiplying the third equation of (2.2) by w_ε^{p-1} with $p\in[2,2+\frac{2\alpha}{1+\alpha})$, integrating by parts, applying the Hölder and the Cauchy-Schwarz inequalities as well as the fact $\nabla\cdot u_\varepsilon=0$, we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega w_\varepsilon^p+\frac{4(p-1)}{p^2}\int_\Omega\left|\nabla w_\varepsilon^{\frac{p}{2}}\right|^2+\int_\Omega w_\varepsilon^p\\=&\int_\Omega n_\varepsilon w_\varepsilon^{p-1}\\\le&\|n_\varepsilon\|_{L^{1+\alpha}(\Omega)}\left\|w_\varepsilon^{p-1}\right\|_{L^{\frac{1+\alpha}{\alpha}}(\Omega)}\\\le&\frac{1}{2}\|n_\varepsilon\|^2_{L^{1+\alpha}(\Omega)}+\frac{1}{2}\left\|w_\varepsilon^{p-1}\right\|^2_{L^{\frac{1+\alpha}{\alpha}}(\Omega)}.\end{aligned}\tag{3.14}$$

Combining (3.13) and (3.14), we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{sgn(2\alpha-1)}{2\alpha|2\alpha-1|C_S^2}\int_\Omega n_\varepsilon^{2\alpha}+\int_\Omega v_\varepsilon^2+\frac{1}{p}\int_\Omega w_\varepsilon^p\right)+\frac{1}{2\alpha^2C_S^2}\int_\Omega\left|\nabla n_\varepsilon^\alpha\right|^2+\frac{4(p-1)}{p^2}\int_\Omega\left|\nabla w_\varepsilon^{\frac{p}{2}}\right|^2\\&+\frac{1}{2}\int_\Omega|\nabla v_\varepsilon|^2+\int_\Omega v_\varepsilon^2\le C_1^2\,\|w_\varepsilon\|^2_{L^{1+\alpha}(\Omega)}+\frac{1}{2}\|n_\varepsilon\|^2_{L^{1+\alpha}(\Omega)}+\frac{1}{2}\left\|w_\varepsilon^{p-1}\right\|^2_{L^{\frac{1+\alpha}{\alpha}}(\Omega)}\ \text{for all } t\in(0,T_{max,\varepsilon}).\end{aligned}\tag{3.15}$$

To handle these three terms on the right side of (3.15), one can employ the Gagliardo-Nirenberg and the Young inequalities as well as Lemma 3.2 to estimate

$$\begin{aligned}&C_1^2\,\|w_\varepsilon\|^2_{L^{1+\alpha}(\Omega)}\\=&C_1^2\left\|w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4}{p}}_{L^{\frac{2(1+\alpha)}{p}}(\Omega)}\\\le&C_2\left\|\nabla w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4\alpha}{(1+\alpha)p}}_{L^2(\Omega)}\left\|w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4}{(1+\alpha)p}}_{L^{\frac{2}{p}}(\Omega)}+C_2\left\|w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4}{p}}_{L^{\frac{2}{p}}(\Omega)}\\\le&\frac{p-1}{p^2}\left\|\nabla w_\varepsilon^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}+C_3\end{aligned}\tag{3.16}$$

as well as

$$\begin{aligned}&\frac{1}{2}\left\|w_\varepsilon^{p-1}\right\|^2_{L^{\frac{1+\alpha}{\alpha}}}\\=&\frac{1}{2}\left\|w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4(p-1)}{p}}_{L^{\frac{2(p-1)(1+\alpha)}{p\alpha}}(\Omega)}\\\le&C_4\left\|\nabla w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4}{p}\left(p-1-\frac{\alpha}{1+\alpha}\right)}_{L^2(\Omega)}\left\|w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4\alpha}{p(1+\alpha)}}_{L^{\frac{2}{p}}(\Omega)}+C_4\left\|w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{4(p-1)}{p}}_{L^{\frac{2}{p}}(\Omega)}\\\le&\frac{p-1}{p^2}\left\|\nabla w_\varepsilon^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}+C_5\end{aligned}\tag{3.17}$$

and

$$\frac{1}{2}\|n_\varepsilon\|^2_{L^{1+\alpha}(\Omega)} \le \frac{1}{4\alpha^2 C_S^2}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 + C_6 \tag{3.18}$$

with positive constants C_2, C_3, C_4, C_5 and C_6, where we have $\frac{4\alpha}{(1+\alpha)p} < 2$ and $\frac{4}{p}\left(p-1-\frac{\alpha}{1+\alpha}\right) < 2$ by $p \in [2, 2+\frac{2\alpha}{1+\alpha})$. Substituting (3.16), (3.17) and (3.18) into (3.15), one has

$$\begin{aligned}&\frac{d}{dt}\left(\frac{sgn(2\alpha-1)}{2\alpha|2\alpha-1|C_S^2}\int_\Omega n_\varepsilon^{2\alpha} + \int_\Omega v_\varepsilon^2 + \frac{1}{p}\int_\Omega w_\varepsilon^p\right) + \frac{1}{4\alpha^2 C_S^2}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 \\ &+ \frac{2(p-1)}{p^2}\int_\Omega \left|\nabla w_\varepsilon^{\frac{p}{2}}\right|^2 + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega v_\varepsilon^2 \le C_7 \text{ for all } t \in (0, T_{max,\varepsilon})\end{aligned} \tag{3.19}$$

with $C_7 := C_3 + C_5 + C_6$. If $sgn(2\alpha-1) = 1$ (i.e. $\alpha > \frac{1}{2}$), (3.19) in conjunction with some standard arguments implies that (3.6) and (3.7) hold. On the other hand, if $sgn(2\alpha-1) = -1$ (i.e. $0 < \alpha < \frac{1}{2}$), we set

$$f_\varepsilon(t) := -\frac{1}{2\alpha|2\alpha-1|C_S^2}\int_\Omega n_\varepsilon^{2\alpha}(\cdot,t) + \int_\Omega v_\varepsilon^2(\cdot,t) + \frac{1}{p}\int_\Omega w_\varepsilon^p(\cdot,t) \tag{3.20}$$

and

$$g_\varepsilon(t) := \frac{1}{4\alpha^2 C_S^2}\int_\Omega \left|\nabla n_\varepsilon^\alpha(\cdot,t)\right|^2 + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \frac{p-1}{p^2}\int_\Omega \left|\nabla w_\varepsilon^{\frac{p}{2}}(\cdot,t)\right|^2. \tag{3.21}$$

By the Gagliardo-Nirenberg estimate

$$\begin{aligned}&\frac{1}{p}\int_\Omega w_\varepsilon^p \\ =&\frac{1}{p}\left\|w_\varepsilon^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} \\ \le& C_8\left\|\nabla w_\varepsilon^{\frac{p}{2}}\right\|^{2-\frac{2}{p}}_{L^2(\Omega)}\left\|w_\varepsilon^{\frac{p}{2}}\right\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} + C_8\left\|w_\varepsilon^{\frac{p}{2}}\right\|^2_{L^{\frac{2}{p}}(\Omega)} \\ \le&\frac{p-1}{p^2}\int_\Omega \left|\nabla w_\varepsilon^{\frac{p}{2}}\right|^2 + C_9\end{aligned}$$

and (3.19), we deduce

$$\frac{d}{dt}f_\varepsilon(t) + f_\varepsilon(t) + g_\varepsilon(t) \le C_{10} \text{ for all } t \in (0, T_{max,\varepsilon}) \tag{3.22}$$

with $C_{10} := C_7 + C_9$, where we observe the fact that $f_\varepsilon(t) \le \int_\Omega v_\varepsilon^2(\cdot,t) + \frac{1}{p}\int_\Omega w_\varepsilon^p(\cdot,t)$. In view of an ODE comparison argument, from (3.22), we obtain a constant $C_{11} > 0$ such that

$$-\frac{1}{2\alpha|2\alpha-1|C_S^2}\int_\Omega n_\varepsilon^{2\alpha} + \int_\Omega v_\varepsilon^2 + \frac{1}{p}\int_\Omega w_\varepsilon^p \le C_{11}. \tag{3.23}$$

Since $0 < \alpha < \frac{1}{2}$, the boundedness of $\int_\Omega n_\varepsilon^{2\alpha}$ is an immediate consequence by (3.1). Thus, (3.23) guarantees the existence of some constant $C_{12} > 0$ satisfying

$$\int_\Omega v_\varepsilon^2 + \frac{1}{p}\int_\Omega w_\varepsilon^p \le \frac{1}{2\alpha|2\alpha-1|C_S^2}\int_\Omega n_\varepsilon^{2\alpha} + C_{11} \le C_{12} \text{ for all } t \in (0, T_{max,\varepsilon}). \tag{3.24}$$

Integrating (3.19) in time, there exists a $C_{13} > 0$ fulfilling

$$\int_t^{t+\tau}\int_\Omega \left|\nabla n_\varepsilon^\alpha\right|^2 + \int_t^{t+\tau}\int_\Omega |\nabla v_\varepsilon|^2 + \int_t^{t+\tau}\int_\Omega \left|\nabla w_\varepsilon^{\frac{p}{2}}\right|^2 \le C_{13} \text{ for all } t \in (0, T_{max,\varepsilon} - \tau). \tag{3.25}$$

Consequently, (3.6) and (3.7) hold for $\alpha \neq \frac{1}{2}$.

Case 2, $\alpha = \frac{1}{2}$.

By the first equation of (2.2), one may exploit the Young inequality to estimate

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon \\
=&\int_\Omega n_{\varepsilon t} \ln n_\varepsilon + \int_\Omega n_{\varepsilon t} \\
=&\int_\Omega \Delta n_\varepsilon \ln n_\varepsilon - \int_\Omega \ln n_\varepsilon \nabla\cdot(n_\varepsilon S_\varepsilon(x, n_\varepsilon, v_\varepsilon, w_\varepsilon)\nabla v_\varepsilon) \\
\le& -\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_S\int_\Omega (n_\varepsilon+1)^{-\frac{1}{2}}|\nabla n_\varepsilon||\nabla v_\varepsilon| \\
\le& -\frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{1}{2}C_S^2\int_\Omega |\nabla v_\varepsilon|^2.
\end{aligned} \tag{3.26}$$

That is,

$$\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \le \frac{1}{2}C_S^2\int_\Omega |\nabla v_\varepsilon|^2. \tag{3.27}$$

Using the same arguments as proving case $0 < \alpha < \frac{1}{2}$, it is deduced that

$$\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega v_\varepsilon^2 + \int_\Omega w_\varepsilon^p \le C_{14} \text{ for all } t \in (0, T_{max,\varepsilon}), \tag{3.28}$$

and

$$\int_t^{t+\tau}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_t^{t+\tau}\int_\Omega |\nabla v_\varepsilon|^2 + \int_t^{t+\tau}\int_\Omega \left|\nabla w_\varepsilon^{\frac{p}{2}}\right|^2 \le C_{15} \text{ for all } t \in (0, T_{max,\varepsilon} - \tau) \tag{3.29}$$

with positive constants C_{14} and C_{15}. Thus, (3.6) and (3.7) hold for $\alpha = \frac{1}{2}$.

Therefore, we may merge these two cases to conclude that (3.6) and (3.7) hold for any $\alpha > 0$. We complete this proof. □

Lemma 3.4. *There exists a constant $C > 0$ independent of $\varepsilon \in (0, 1)$ such that the solution of (2.2) satisfies*

$$\int_\Omega |u_\varepsilon|^2(\cdot, t) \le C \quad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.30}$$

and

$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2 \le C \quad \text{for all } t \in (0, T_{max,\varepsilon} - \tau)\,. \tag{3.31}$$

Proof. Multiplying the fourth equation of (2.2) by u_ε, recalling the fact that u_ε is divergence-free, integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi \text{ for all } t \in (0, T_{max,\varepsilon}). \tag{3.32}$$

Then, in light of the Hölder inequality as well as the Young inequality and the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{1+\alpha}{\alpha}}(\Omega)$, we apply Lemma 3.2 to estimate

$$\begin{aligned}\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi &\le \|n_\varepsilon\|_{L^{1+\alpha}(\Omega)} \|u_\varepsilon\|_{L^{\frac{1+\alpha}{\alpha}}(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} \\ &\le C_1 \|n_\varepsilon\|_{L^{1+\alpha}(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} \\ &\le \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_2 \|n_\varepsilon\|^2_{L^{1+\alpha}(\Omega)} \\ &\le \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \left\|\nabla n_\varepsilon^\alpha\right\|^2_{L^2(\Omega)} + C_3\end{aligned} \tag{3.33}$$

with positive constants C_1, C_2 and C_3. Now, inserting (3.33) into (3.32) and considering the estimate obtained by (3.7), we obtain

$$\int_\Omega |u_\varepsilon|^2 \le C_4 \text{ for all } t \in (0, T_{max,\varepsilon}) \tag{3.34}$$

and

$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2 \le C_5 \text{ for all } t \in (0, T_{max,\varepsilon} - \tau) \tag{3.35}$$

with positive constants C_4 and C_5. The proof is completed. □

By almost exactly analogous argument with Lemma 6.1 in [29], one can directly derive the higher norm estimate of w_ε.

Lemma 3.5. *For any $q \ge 2$, one can find a constant $C > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\|w_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \le C \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.36}$$

Lemma 3.6. *For any $\varepsilon \in (0, 1)$, there exists a constant $C > 0$ that satisfies*

$$\int_\Omega |\nabla v_\varepsilon(\cdot, t)|^2 \le C \quad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.37}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta v_\varepsilon|^2 \le C \quad \text{for all } t \in (0, T_{max,\varepsilon} - \tau). \tag{3.38}$$

Proof. Testing the second equation in (2.2) by $-\Delta v_\varepsilon$, by applying the Young inequality and integrating by parts, we derive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 \\ =& -\int_\Omega w_\varepsilon \Delta v_\varepsilon + \int_\Omega (u_\varepsilon \cdot \nabla v_\varepsilon)\Delta v_\varepsilon \\ \le& \frac{1}{4}\int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega w_\varepsilon^2 - \int_\Omega \nabla v_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla v_\varepsilon) \\ \le& \frac{1}{4}\int_\Omega |\Delta v_\varepsilon|^2 - \int_\Omega \nabla v_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla v_\varepsilon) + C_1,\end{aligned} \tag{3.39}$$

where the positive constant C_1 satisfies $\int_\Omega w_\varepsilon^2 \le C_1$, and we have the fact that

$$\int_\Omega \nabla v_\varepsilon \cdot \left(D^2 v_\varepsilon \cdot u_\varepsilon\right) = \frac{1}{2}\int_\Omega u_\varepsilon \cdot \nabla |\nabla v_\varepsilon|^2 = 0. \tag{3.40}$$

In view of the standard elliptic regularity theory, the Gagliardo-Nirenberg inequality and the Young inequality provide a constant $C_2 > 0$ fulfilling

$$\begin{aligned} &-\int_\Omega \nabla v_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla v_\varepsilon) \\ \le& \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla v_\varepsilon\|_{L^4(\Omega)}^2 \\ \le& C_2 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla v_\varepsilon\|_{L^2(\Omega)} \|\Delta v_\varepsilon\|_{L^2(\Omega)} \\ \le& C_2^2 \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{4}\|\Delta v_\varepsilon\|_{L^2(\Omega)}^2 . \end{aligned} \tag{3.41}$$

This in conjunction with (3.39) indicates that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 \\ \le& C_2^2 \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 + C_1 \text{ for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{3.42}$$

If we put

$$y_\varepsilon(t) := \frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot, t)|^2$$

and

$$\rho_\varepsilon(t) := 2C_2^2 \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2,$$

then (3.42) yields that

$$\frac{d}{dt} y_\varepsilon(t) + z_\varepsilon(t) \le \rho_\varepsilon(t) y_\varepsilon(t) + C_1 \text{ for all } t \in (0, T_{max,\varepsilon}), \tag{3.43}$$

where

$$z_\varepsilon(t) = \frac{1}{2}\int_\Omega |\Delta v_\varepsilon(\cdot, t)|^2.$$

Recalling the estimates inferred from (3.7) and (3.31), there are two positive constants C_3 and C_4 satisfying

$$\int_t^{t+\tau} y_\varepsilon(s) ds \le C_3 \text{ for all } t \in (0, T_{max,\varepsilon} - \tau)$$

and

$$\int_t^{t+\tau} \rho_\varepsilon(s) ds \le C_4 \text{ for all } t \in (0, T_{max,\varepsilon} - \tau).$$

Furthermore, for any $t \in (0, T_{max,\varepsilon})$, one can pick a $t_0 \in [(t-\tau)_+, t)$ such that $y_\varepsilon(\cdot, t_0) \le C_5$ with some $C_5 > 0$. Invoking the Gronwall inequality, we obtain

$$\begin{aligned} y_\varepsilon(t) &\le y_\varepsilon(t_0) e^{\int_{t_0}^t \rho_\varepsilon(s) ds} + \int_{t_0}^t e^{\int_s^t \rho_\varepsilon(\tau) d\tau} C_1 ds \\ &\le C_5 e^{C_4} + \int_{t_0}^t e^{C_4} C_1 ds \\ &\le C_5 e^{C_4} + C_1 e^{C_4} \text{ for all } t \in (0, T_{max,\varepsilon}), \end{aligned} \tag{3.44}$$

which implies (3.37). Finally, integrating (3.42) in time and exploiting the estimates obtained in (3.37) and (3.31), we can verify (3.38) is valid. □

Lemma 3.7. *If $\alpha > 0$, then there exists a constant $C > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\int_\Omega n_\varepsilon^{1+\alpha}(\cdot, t) \le C \quad \textit{for all } t \in (0, T_{max,\varepsilon}) \tag{3.45}$$

and

$$\int_t^{t+\tau} \int_\Omega n_\varepsilon^{\alpha-1} |\nabla n_\varepsilon|^2 \le C \quad \textit{for all } t \in (0, T_{max,\varepsilon} - \tau)\,. \tag{3.46}$$

Particularly, one has

$$\int_t^{t+\tau} \int_\Omega n_\varepsilon^2 \le C \quad \textit{for all } t \in (0, T_{max,\varepsilon} - \tau)\,. \tag{3.47}$$

Proof. Testing the first equation of (2.2) by n_ε^α, noticing the fact that $\nabla \cdot u_\varepsilon = 0$ and integrating by parts, by using the Young inequality and (1.7), we arrive at

$$\begin{aligned}
&\frac{1}{1+\alpha}\frac{d}{dt}\|n_\varepsilon\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} + \alpha\int_\Omega n_\varepsilon^{\alpha-1}|\nabla n_\varepsilon|^2 \\
=&-\int_\Omega n_\varepsilon^\alpha \nabla\cdot(n_\varepsilon S_\varepsilon(x, n_\varepsilon, v_\varepsilon, w_\varepsilon)\nabla v_\varepsilon) \\
\le&\alpha C_S \int_\Omega n_\varepsilon^\alpha (n_\varepsilon+1)^{-\alpha}|\nabla n_\varepsilon||\nabla v_\varepsilon| \\
\le&\frac{\alpha}{4}\int_\Omega n_\varepsilon^{\alpha-1}|\nabla n_\varepsilon|^2 + \alpha C_S^2 \int_\Omega n_\varepsilon^{1+\alpha}(n_\varepsilon+1)^{-2\alpha}|\nabla v_\varepsilon|^2 \\
\le&\frac{\alpha}{4}\int_\Omega n_\varepsilon^{\alpha-1}|\nabla n_\varepsilon|^2 + \frac{\alpha C_S^2}{2}\int_\Omega n_\varepsilon^2 + \frac{\alpha C_S^2}{2}\int_\Omega n_\varepsilon^{2\alpha}(n_\varepsilon+1)^{-4\alpha}|\nabla v_\varepsilon|^4 \\
\le&\frac{\alpha}{4}\int_\Omega n_\varepsilon^{\alpha-1}|\nabla n_\varepsilon|^2 + \frac{\alpha C_S^2}{2}\int_\Omega n_\varepsilon^2 + \frac{\alpha C_S^2}{2}\int_\Omega |\nabla v_\varepsilon|^4\,,
\end{aligned} \tag{3.48}$$

where one can readily see that $n_\varepsilon^{2\alpha}(n_\varepsilon+1)^{-4\alpha} = \left(\frac{n_\varepsilon}{n_\varepsilon+1}\right)^{2\alpha}(n_\varepsilon+1)^{-2\alpha} < 1$ by $\alpha > 0$. By means of the Gagliardo-Nirenberg inequality and the Young inequality, we conclude that

$$\begin{aligned}
&\frac{\alpha C_S^2}{2}\int_\Omega n_\varepsilon^2 \\
=&\frac{\alpha C_S^2}{2}\left\|n_\varepsilon^{\frac{1+\alpha}{2}}\right\|_{L^{\frac{4}{1+\alpha}}(\Omega)}^{\frac{4}{1+\alpha}} \\
\le&C_1\left\|\nabla n_\varepsilon^{\frac{1+\alpha}{2}}\right\|_{L^2(\Omega)}^{\frac{2}{1+\alpha}}\left\|n_\varepsilon^{\frac{1+\alpha}{2}}\right\|_{L^{\frac{2}{1+\alpha}}(\Omega)}^{\frac{2}{1+\alpha}} + C_1\left\|n_\varepsilon^{\frac{1+\alpha}{2}}\right\|_{L^{\frac{2}{1+\alpha}}(\Omega)}^{\frac{4}{1+\alpha}} \\
\le&\frac{\alpha}{4}\int_\Omega n_\varepsilon^{\alpha-1}|\nabla n_\varepsilon|^2 + C_2
\end{aligned} \tag{3.49}$$

with positive constants C_1 and C_2, where we observe the truth that $\frac{2}{1+\alpha} < 2$ by $\alpha > 0$. Moreover, looking back on the estimate in (3.37), we utilize the Gagliardo-Nirenberg inequality and the elliptic regularity to ensure the existence of constants $C_3 > 0$ and $C_4 > 0$ such that

$$\begin{aligned} &\frac{\alpha C_S^2}{2}\int_\Omega |\nabla v_\varepsilon|^4 \\ =&\frac{\alpha C_S^2}{2}\,\|\nabla v_\varepsilon\|_{L^4(\Omega)}^4 \\ \le& C_3\,\|\Delta v_\varepsilon\|_{L^2(\Omega)}^2\,\|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 \\ \le& C_4\,\|\Delta v_\varepsilon\|_{L^2(\Omega)}^2 . \end{aligned} \tag{3.50}$$

Accordingly, (3.48) in combination with (3.49) and (3.50) leads to

$$\begin{aligned} &\frac{1}{1+\alpha}\frac{d}{dt}\,\|n_\varepsilon\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} + \frac{\alpha}{2}\int_\Omega n_\varepsilon^{\alpha-1}\,|\nabla n_\varepsilon|^2 \\ \le& C_4\int_\Omega |\Delta v_\varepsilon|^2 + C_2 \text{ for all } t\in(0,T_{max,\varepsilon}). \end{aligned} \tag{3.51}$$

Recalling the spatio-temporal boundedness of $\int_t^{t+\tau}\int_\Omega |\Delta v_\varepsilon|^2$ inferred from (3.38), (3.51) implies (3.45) and (3.46). Finally, integrating (3.49) in time, (3.46) yields (3.47). □

Relying on the spatio-temporal estimates of $\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2$ (see Lemma 3.4) and $\int_t^{t+\tau}\int_\Omega n_\varepsilon^2$ (see Lemma 3.7), one can improve the regularity features of the corresponding fluid field. Since the proof may be found in many papers [42], the details are omitted in order to avoid duplication.

Lemma 3.8. *There exists some $C > 0$ such that for all $\varepsilon \in (0, 1)$ the solution of (2.2) satisfies*

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le C \quad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{3.52}$$

4. The global solvability of regularized problem (2.2)

In this section, we will prove the local-in-time solutions of regularized problem (2.2) are actually global. Without loss of generality, in this section we presume $0 < \alpha < \frac{1}{2}$. If $\alpha \ge \frac{1}{2}$, at least the boundedness of $\|n_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)}$ can be deduced from (3.45). With the higher regularity of n_ε, it becomes easier than case $0 < \alpha < \frac{1}{2}$ to get our desired conclusion. Thanks to the well-known smoothing properties of the Stokes semigroup and the Neumann heat semigroup, one can derive the following uniform L^∞ estimates for n_ε, ∇v_ε, ∇w_ε and u_ε.

Lemma 4.1. *If $\alpha > 0$ and $\gamma \in \left(\frac{1}{2}, 1\right)$, then there exists a constant $C > 0$ such that for all $\varepsilon \in (0, 1)$, the classical solution of (2.2) satisfies*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}+\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}+\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \text{ for all } t\in(0,T_{max,\varepsilon}). \tag{4.1}$$

Moreover, for $p > 1$, we can find a positive constant $C(p)$ such that

$$\|\nabla u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p) \text{ for all } t\in(0,T_{max,\varepsilon}). \tag{4.2}$$

Proof. For the sake of clarity, this proof is divided into several steps. It is worth mentioning that the following constants C_i ($i \in \mathbb{N}^*$) are independent of $\varepsilon \in (0,1)$.

Step 1. The boundedness of $\|\nabla w_\varepsilon(\cdot,t)\|_{L^{\frac{2}{1-\alpha}}(\Omega)}$ **and** $\|\nabla v_\varepsilon(\cdot,t)\|_{L^{\tilde{p}}(\Omega)}$ **with** $\tilde{p} > 2$ **for all** $t \in (0, T_{max,\varepsilon})$**.**

Since $0 < \alpha < \frac{1}{2}$, we have $\frac{2}{1-\alpha} > 2$. First, utilizing the variation-of-constants formula for w_ε, we obtain

$$\begin{aligned}
&\|\nabla w_\varepsilon(\cdot,t)\|_{L^{\frac{2}{1-\alpha}}(\Omega)}\\
\le&\left\|\nabla e^{-t(-\Delta+1)}w_0\right\|_{L^{\frac{2}{1-\alpha}}(\Omega)} + \int_0^t \left\|\nabla e^{-(t-s)(-\Delta+1)}n_\varepsilon(\cdot,s)\right\|_{L^{\frac{2}{1-\alpha}}(\Omega)} ds\\
&+\int_0^t \left\|\nabla e^{-(t-s)(-\Delta+1)}\nabla\cdot(u_\varepsilon(\cdot,s)w_\varepsilon(\cdot,s))\right\|_{L^{\frac{2}{1-\alpha}}(\Omega)} ds.
\end{aligned} \tag{4.3}$$

With the boundedness of $\|n_\varepsilon(\cdot,s)\|_{L^{1+\alpha}(\Omega)}$ obtained by Lemma 3.7, in view of the $L^p - L^q$ estimates associated heat semigroup, we deduce

$$\left\|\nabla e^{-t(-\Delta+1)}w_0\right\|_{L^{\frac{2}{1-\alpha}}(\Omega)} \le C_{11} \text{ for all } t \in (0, T_{max,\varepsilon}) \tag{4.4}$$

and

$$\begin{aligned}
&\int_0^t \left\|\nabla e^{-(t-s)(-\Delta+1)}n_\varepsilon(\cdot,s)\right\|_{L^{\frac{2}{1-\alpha}}(\Omega)} ds\\
\le& C_1 \int_0^t \left[(t-s)^{-\frac{1}{2}-\left(\frac{1}{1+\alpha}-\frac{1-\alpha}{2}\right)}+1\right] e^{-\lambda(t-s)} \|n_\varepsilon(\cdot,s)\|_{L^{1+\alpha}(\Omega)} ds\\
\le& C_2 \text{ for all } t \in (0, T_{max,\varepsilon})
\end{aligned} \tag{4.5}$$

with $\lambda > 0$, where we have the fact that $-\frac{1}{2} - \left(\frac{1}{1+\alpha} - \frac{1-\alpha}{2}\right) > -1$ by $0 < \alpha < \frac{1}{2}$. Furthermore, taking $\varsigma = \frac{19}{40}$ and $\delta = \frac{1}{80}$ so that $\frac{1}{2} + \left(\frac{1}{5} - \frac{1-\alpha}{2}\right) < \varsigma$ and $-\varsigma - \frac{1}{2} - \delta > -1$, we can infer that

$$\begin{aligned}
&\int_0^t \left\|\nabla e^{-(t-s)(-\Delta+1)}\nabla\cdot(w_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s))\right\|_{L^{\frac{2}{1-\alpha}}(\Omega)} ds\\
\le& C_3 \int_0^t \left\|(-\Delta+1)^{\varsigma} e^{-(t-s)(-\Delta+1)}\nabla\cdot(w_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s))\right\|_{L^5(\Omega)} ds\\
\le& C_4 \int_0^t (t-s)^{-\varsigma-\frac{1}{2}-\delta} e^{-\mu(t-s)} \|w_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\|_{L^5(\Omega)} ds\\
\le& C_5 \int_0^t (t-s)^{-\varsigma-\frac{1}{2}-\delta} e^{-\mu(t-s)} \|w_\varepsilon(\cdot,s)\|_{L^{10}(\Omega)} \|u_\varepsilon(\cdot,s)\|_{L^{10}(\Omega)} ds\\
\le& C_6 \text{ for all } t \in (0, T_{max,\varepsilon}),
\end{aligned} \tag{4.6}$$

where the boundedness of $\|u_\varepsilon(\cdot,s)\|_{L^{10}(\Omega)}$ is derived from Lemma 3.8 along with the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^{10}(\Omega)$, and $\|w_\varepsilon(\cdot,s)\|_{L^{10}(\Omega)}$ is ensured by Lemma 3.5. Therefore, by accumulating (4.3)-(4.6), the boundedness of $\|\nabla w_\varepsilon(\cdot,t)\|_{L^{\frac{2}{1-\alpha}}(\Omega)}$ is obtained. With some very similar arguments, one can derive the boundedness of $\|\nabla v_\varepsilon(\cdot,t)\|_{L^{\tilde{p}}(\Omega)}$ with some $\tilde{p} > 2$.

Step 2. The boundedness of $\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ **for all** $t \in (0, T_{max,\varepsilon})$**.**

Letting

$$M(T) := \sup_{t\in(0,T)} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$$

and

$$\tilde{h}_\varepsilon := S_\varepsilon(x, n_\varepsilon, v_\varepsilon, v_\varepsilon)\nabla v_\varepsilon + u_\varepsilon,$$

then by the $L^{\tilde{p}}$ estimate of ∇v_ε, we obtain

$$\left\|\tilde{h}_\varepsilon(\cdot, t)\right\|_{L^{\tilde{p}}(\Omega)} \leq C_7 \text{ for all } t \in (0, T_{max,\varepsilon}). \tag{4.7}$$

Exploiting the associate variation-of-constants formula for n_ε, in light of the fact that $\nabla \cdot u_\varepsilon = 0$, we obtain

$$n_\varepsilon(\cdot, t) = e^{(t-t_0)\Delta} n_\varepsilon(\cdot, t_0) - \int_{t_0}^{t} e^{(t-s)\Delta}\nabla \cdot \left(n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s)\right) ds \quad \text{for } t \in (t_0, T) \text{ with } t_0 := (t-1)_+. \tag{4.8}$$

If $0 < t \leq 1$, then in view of the maximum principle, we have

$$\left\|e^{(t-t_0)\Delta} n_\varepsilon(\cdot, t_0)\right\|_{L^\infty(\Omega)} \leq \|n_0\|_{L^\infty(\Omega)}. \tag{4.9}$$

If $t > 1$, then by the $L^p - L^q$ estimates of the Neumann heat semigroup, we deduce

$$\left\|e^{(t-t_0)\Delta} n_\varepsilon(\cdot, t_0)\right\|_{L^\infty(\Omega)} \leq C_8 (t-t_0)^{-\frac{2}{2}} \|n_\varepsilon(\cdot, t_0)\|_{L^1(\Omega)} \leq C_9. \tag{4.10}$$

Next, fixing $q \in (2, \tilde{p})$, we may utilize the well-known smoothing properties of the Neumann heat semigroup and the Hölder inequality to conclude

$$\begin{aligned}
&\int_{t_0}^{t} \|e^{(t-s)\Delta}\nabla \cdot (n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s))\|_{L^\infty(\Omega)} ds \\
\leq &C_{10} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{2}{2q}} \|n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s)\|_{L^q(\Omega)} ds \\
\leq &C_{10} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{2}{2q}} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{q\tilde{p}}{\tilde{p}-q}}(\Omega)} \|\tilde{h}_\varepsilon(\cdot, s)\|_{L^{\tilde{p}}(\Omega)} ds \\
\leq &C_{10} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{2}{2q}} \|n_\varepsilon(\cdot, s)\|^{\sigma}_{L^\infty(\Omega)} \|n_\varepsilon(\cdot, s)\|^{1-\sigma}_{L^1(\Omega)} \|\tilde{h}_\varepsilon(\cdot, s)\|_{L^{\tilde{p}}(\Omega)} ds \\
\leq &C_{11} M^{\sigma}(T) \text{ for all } t \in (0, T),
\end{aligned} \tag{4.11}$$

where $\sigma := \frac{q\tilde{p}-\tilde{p}+q}{q\tilde{p}} \in (0, 1)$, and $-\frac{1}{2} - \frac{2}{2q} > -1$ by $q > 2$. Collecting (4.7)–(4.11) and utilizing the definition of $M(T)$, there is a $C_{12} > 0$ such that

$$M(T) \leq C_{12} + C_{12} M^{\sigma}(T) \text{ for all } T \in (0, T_{max,\varepsilon}).$$

Since $\sigma < 1$, by some basic calculation we have

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{13} \text{ for all } t \in (0, T_{max,\varepsilon}).$$

Step 3. The boundedness of $\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$ and $\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}$ for all $t \in (0, T_{max,\varepsilon})$.

Employing the Helmholtz projection $\mathcal{P}$ to the fourth equation in (2.2), we get the variation-of-constants formula of u_ε

$$u_\varepsilon(\cdot, t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} h_\varepsilon(\cdot, s) ds \text{ for all } t \in (0, T_{max,\varepsilon}),$$

where $h_\varepsilon(\cdot,s)=\mathcal{P}\left[n_\varepsilon(\cdot,s)\nabla\phi-\kappa\left(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla\right)u_\varepsilon(\cdot,s)\right]$. With the standard smoothing properties of the Stokes semigroup, we derive that for all $t\in(0,T_{max,\varepsilon})$ and any $\gamma\in\left(\frac{1}{2},1\right)$, there exist $C_{14}>0$ and $C_{15}>0$ fulfilling

$$\begin{aligned}
&\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\\
\le&\|A^\gamma u_0\|_{L^2(\Omega)}+\int_0^t\left\|A^\gamma e^{-(t-s)A}h_\varepsilon(\cdot,s)\right\|_{L^2(\Omega)}ds\\
\le&C_{14}+C_{15}\int_0^t(t-s)^{-\gamma-\left(\frac{1}{p_0}-\frac{1}{2}\right)}e^{-\lambda(t-s)}\|h_\varepsilon(\cdot,s)\|_{L^{p_0}(\Omega)}ds.
\end{aligned}\tag{4.12}$$

Choosing $p_0\in\left(\frac{2}{3-2\gamma},2\right)$ such that

$$-\gamma-\left(\frac{1}{p_0}-\frac{1}{2}\right)>-1,\tag{4.13}$$

the L^∞-estimate of n_ε provides a $C_{16}>0$ fulfilling

$$\|n_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)}\le C_{16}\text{ for all }t\in(0,T_{max,\varepsilon}).$$

Next, considering the embedding $W^{1,2}(\Omega)\hookrightarrow L^{\frac{2p_0}{2-p_0}}(\Omega)$ and the boundedness of $\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ (see Lemma 3.8), we employ the Hölder inequality and the fact that $\mathcal{P}$ is continuous in $L^p\left(\Omega;\mathbb{R}^2\right)$ to achieve that

$$\begin{aligned}
&\|h_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)}\\
\le&C_{17}\left\|\left(Y_\varepsilon u_\varepsilon(\cdot,t)\cdot\nabla\right)u_\varepsilon(\cdot,t)\right\|_{L^{p_0}(\Omega)}+C_{17}\|n_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)}\\
\le&C_{17}\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^{\frac{2p_0}{2-p_0}}(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}+C_{18}\\
\le&C_{19}\|\nabla Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}+C_{18}\\
\le&C_{20}\text{ for all }t\in(0,T_{max,\varepsilon}),
\end{aligned}\tag{4.14}$$

where we notice the fact that

$$\|\nabla Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)}=\|A^{\frac{1}{2}}Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)}=\|Y_\varepsilon A^{\frac{1}{2}}u_\varepsilon\|_{L^2(\Omega)}\le\|A^{\frac{1}{2}}u_\varepsilon\|_{L^2(\Omega)}=\|\nabla u_\varepsilon\|_{L^2(\Omega)}.$$

Assembling (4.12), (4.13) and (4.14), we conclude that

$$\begin{aligned}
&\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\\
\le&C_{21}+C_{21}\int_0^t(t-s)^{-\gamma-\left(\frac{1}{p_0}-\frac{1}{2}\right)}e^{-\lambda(t-s)}\|h_\varepsilon(\cdot,s)\|_{L^{p_0}(\Omega)}ds\\
\le&C_{22}\text{ for all }t\in(0,T_{max,\varepsilon}),
\end{aligned}\tag{4.15}$$

which in combination with the continuous embedding $D\left(A^\gamma\right)\hookrightarrow L^\infty(\Omega)$ by $\gamma\in\left(\frac{1}{2},1\right)$ yields that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C_{23}\text{ for all }t\in(0,T_{max,\varepsilon}).\tag{4.16}$$

Step 4. The boundedness of $\|\nabla u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$ with $p>1$ for all $t\in(0,T_{max,\varepsilon})$.

For any $p>1$, we can pick suitable $\gamma\in\left(\frac{1}{2},1\right)$ satisfying $\gamma>1-\frac{1}{p}$. By means of the embedding $D(A^\gamma)\hookrightarrow W^{1,p}(\Omega;\mathbb{R}^2)$ (see [37]), (4.2) holds.

Step 5. The boundedness of $\|w_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ and $\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ for all $t\in(0,T_{max,\varepsilon})$.

Fixing $\theta \in \left(\frac{1}{2} + \frac{1-\alpha}{2}, 1\right)$, the domain of the fractional power $D\left((-\Delta + 1)^{\theta}\right)$ can be embedded into $W^{1,\infty}(\Omega)$ [4]. Accordingly, exploiting the $L^p - L^q$ estimates associated heat semigroup, one has

$$\begin{aligned}
&\|w_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\\
\leq & C_{24}\left\|(-\Delta+1)^{\theta} w_\varepsilon(\cdot,t)\right\|_{L^{\frac{2}{1-\alpha}}(\Omega)}\\
\leq & C_{25} t^{-\theta} e^{-\mu t}\|w_0\|_{L^{\frac{2}{1-\alpha}}(\Omega)} + C_{25}\int_0^t (t-s)^{-\theta} e^{-\mu(t-s)}\|(n_\varepsilon - u_\varepsilon\cdot\nabla w_\varepsilon)(\cdot,s)\|_{L^{\frac{2}{1-\alpha}}(\Omega)}\,ds\\
\leq & C_{26} + C_{26}\int_0^t (t-s)^{-\theta} e^{-\mu(t-s)}\left[\|n_\varepsilon(\cdot,s)\|_{L^{\frac{2}{1-\alpha}}(\Omega)} + \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla w_\varepsilon(\cdot,s)\|_{L^{\frac{2}{1-\alpha}}(\Omega)}\right]ds\\
\leq & C_{27} \text{ for all } t\in(\tau_0, T_{max,\varepsilon})
\end{aligned}$$

with $\tau_0 \in (0, T_{max,\varepsilon})$. An application of the local solvability of (2.2) indicates that for some $C_{28} > 0$,

$$\|w_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_{28} \text{ for all } t\in(0, T_{max,\varepsilon}). \tag{4.17}$$

Meanwhile, a similar argument yields a $C_{29} > 0$ satisfying

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_{29} \text{ for all } t\in(0, T_{max,\varepsilon}). \tag{4.18}$$

The proof is completed. □

With the uniform L^∞ bounds of $n_\varepsilon, \nabla v_\varepsilon, \nabla w_\varepsilon$ and u_ε at hand, we claim that the local classical solution of regularized problem (2.2) which is constructed in Lemma 2.1 can be extended to the global.

Proposition 4.1. *Let $\alpha > 0$, $\gamma \in \left(\frac{1}{2}, 1\right)$. Let $(n_\varepsilon, v_\varepsilon, w_\varepsilon, u_\varepsilon, P_\varepsilon)_{\varepsilon\in(0,1)}$ be classical solutions of (2.2) constructed in Lemma* 2.1 *on* $[0, T_{max,\varepsilon})$. *Then, we have $T_{max,\varepsilon} = \infty$. Moreover, one can find a $C > 0$ which is independent of $\varepsilon \in (0,1)$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C \textit{ for all } t\in(0,\infty). \tag{4.19}$$

In addition, there is a $C(p) > 0$ fulfilling

$$\|\nabla u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C(p) \textit{ for all } t\in(0,\infty). \tag{4.20}$$

As the straightforward result of Proposition 4.1, in light of the standard parabolic regularity (see e.g. Lemmata 3.18 and 3.19 in [43]), we can get the following Hölder continuity of $v_\varepsilon, \nabla v_\varepsilon$ as well as $w_\varepsilon, \nabla w_\varepsilon$ and u_ε.

Lemma 4.2. *If $\alpha > 0$, then there exist $\mu \in (0,1)$ and some $C > 0$ such that*

$$\|v_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} + \|w_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} + \|u_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \leq C \textit{ for all } t\in(0,\infty). \tag{4.21}$$

Moreover, for any $\tau > 0$, one can find a $C(\tau) > 0$ satisfying

$$\|\nabla v_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} + \|\nabla w_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \leq C(\tau) \textit{ for all } t\in(\tau,\infty). \tag{4.22}$$

5. The proof of main results

With all the results established above, we are adequately prepared for proving Theorem 1.1. First, we state the concept of global weak solution.

Definition 5.1. *Let (n_0, v_0, w_0, u_0) satisfy (1.9) and $T \in (0, \infty]$. Then, a fourfold of functions (n, v, w, u) which fulfills*

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0, T)), \\ v \in L^1_{loc}\left([0, T); W^{1,1}(\Omega)\right), \\ w \in L^1_{loc}\left([0, T); W^{1,1}(\Omega)\right), \\ u \in L^1_{loc}\left([0, T); W^{1,1}(\Omega)\right), \end{cases} \tag{5.1}$$

and n as well as v and w are nonnegative in $\Omega \times (0, T)$ and u is divergence-free in $\Omega \times (0, T)$, and

$$\begin{aligned} u \otimes u \in L^1_{loc}\left(\bar{\Omega} \times [0, \infty); \mathbb{R}^{2\times 2}\right) \text{ and } n \in L^1_{loc}(\bar{\Omega} \times [0, \infty)), \\ vu, wu, nu \text{ and } nS(x, n, v, w)\nabla v \in L^1_{loc}\left(\bar{\Omega} \times [0, \infty); \mathbb{R}^2\right) \end{aligned} \tag{5.2}$$

is called a weak solution of problem (1.5) if the following integral identities are satisfied:

$$\begin{aligned} &-\int_0^T \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot, 0) \\ &= \int_0^T \int_\Omega \nabla n \cdot \nabla\varphi + \int_0^T \int_\Omega nS(x, n, v, w)\nabla v \cdot \nabla\varphi + \int_0^T \int_\Omega nu \cdot \nabla\varphi \end{aligned} \tag{5.3}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0, T)$ and

$$\begin{aligned} &-\int_0^T \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot, 0) \\ &= -\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^T \int_\Omega v\varphi + \int_0^T \int_\Omega w\varphi + \int_0^T \int_\Omega vu \cdot \nabla\varphi \end{aligned} \tag{5.4}$$

as well as

$$\begin{aligned} &-\int_0^T \int_\Omega w\varphi_t - \int_\Omega w_0\varphi(\cdot, 0) \\ &= -\int_0^T \int_\Omega \nabla w \cdot \nabla\varphi - \int_0^T \int_\Omega w\varphi + \int_0^T \int_\Omega n\varphi + \int_0^T \int_\Omega wu \cdot \nabla\varphi \end{aligned} \tag{5.5}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ and

$$\begin{aligned} &-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot, 0) \\ &= \kappa\int_0^T \int_\Omega u \otimes u \cdot \nabla\varphi - \int_0^T \int_\Omega \nabla u \cdot \nabla\varphi - \int_0^T \int_\Omega n\nabla\phi \cdot \varphi \end{aligned} \tag{5.6}$$

for any $\varphi \in C_0^\infty\left(\bar{\Omega} \times [0, T); \mathbb{R}^2\right)$ which is divergence-free in $\Omega \times (0, T)$. If $\Omega \times (0, \infty) \longrightarrow \mathbb{R}^5$ is a weak solution of (1.5) in $\Omega \times (0, T)$ for all $T > 0$, then (n, v, w, u) is called a global weak solution of (1.5).

In the following auxiliary outcome, we will derive the regularity property of time derivative so as to invoke the Aubin-Lions compactness lemma, which plays a prominent role in proving Theorem 1.1.

Lemma 5.1. *If $\alpha > 0$, then for any $T > 0$ and all $\varepsilon \in (0, 1)$, there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega |\nabla n_\varepsilon|^2 \le C(T) \tag{5.7}$$

and

$$\int_0^T \|\partial_t n_\varepsilon(\cdot, t)\|_{\left(W_0^{1,2}(\Omega)\right)^*} dt \le C(T). \tag{5.8}$$

Proof. Firstly, in view of Proposition 4.1, there exists a $C_1 > 0$ such that

$$n_\varepsilon \le C_1, \quad |\nabla v_\varepsilon| \le C_1 \text{ and } |u_\varepsilon| \le C_1 \text{ in } \Omega \times (0, \infty). \tag{5.9}$$

Then, testing the first equation in (2.2) by n_ε, by virtue of (5.9), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|n_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla n_\varepsilon|^2 &= -\int_\Omega n_\varepsilon \nabla \cdot (n_\varepsilon S_\varepsilon(x, n_\varepsilon, v_\varepsilon, w_\varepsilon) \nabla v_\varepsilon) \\ &\le C_S \int_\Omega n_\varepsilon |\nabla n_\varepsilon| \, |\nabla v_\varepsilon| \\ &\le \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + \frac{1}{2} C_S^2 C_1^4 |\Omega|. \end{aligned} \tag{5.10}$$

Integrating (5.10) over $(0, T)$, (5.7) is valid. Testing the first equation in (2.2) by $\varphi \in C_0^\infty(\Omega)$, we conclude there is a $\tilde{C} := C(C_1, \Omega, C_S) > 0$ such that

$$\begin{aligned} &\int_\Omega n_{\varepsilon t}(\cdot, t) \cdot \varphi \\ =& \int_\Omega [\Delta n_\varepsilon - \nabla \cdot (n_\varepsilon S_\varepsilon(x, n_\varepsilon, v_\varepsilon, w_\varepsilon) \nabla v_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon] \cdot \varphi \\ =& -\int_\Omega \nabla n_\varepsilon \cdot \nabla \varphi + \int_\Omega n_\varepsilon S_\varepsilon(x, n_\varepsilon, v_\varepsilon, w_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \varphi \\ \le& \tilde{C} \left(\|\nabla n_\varepsilon\|_{L^2(\Omega)} + \|\nabla v_\varepsilon\|_{L^2(\Omega)} \right) \|\varphi\|_{W_0^{1,2}(\Omega)}. \end{aligned} \tag{5.11}$$

Therefore, by the definition of the operator norm, one has

$$\|n_{\varepsilon t}(\cdot, t)\|^2_{\left(W_0^{1,2}(\Omega)\right)^*} \le \tilde{C} \left(\|\nabla n_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla v_\varepsilon\|^2_{L^2(\Omega)} \right). \tag{5.12}$$

Recalling the estimates obtained in (5.7) and (3.37), integrating (5.12) in time, we finally get (5.8). □

As an application of the parabolic regularity theory, we may further derive the following Hölder continuity of n_ε.

Lemma 5.2. *For any $\varepsilon \in (0, 1)$, there exist a positive constant C and $\theta \in (0, 1)$ such that*

$$\|n_\varepsilon(\cdot, t)\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} \le C \text{ for all } t \in (0, \infty). \tag{5.13}$$

Proof. Firstly, the first equation of (2.2) can be rewritten as the following sub-problem:

$$\begin{cases} n_{\varepsilon t} = \nabla \cdot a(x,t,\nabla n_\varepsilon) + b(x,t,\nabla n_\varepsilon), & x \in \Omega, t > 0, \\ a(x,t,\nabla n_\varepsilon) \cdot \nu = 0, & x \in \partial\Omega, t > 0, \\ n_\varepsilon(x,0) = n_0(x), & x \in \Omega, \end{cases} \tag{5.14}$$

where $a(x,t,\xi) := \xi - n_\varepsilon S_\varepsilon(x,n_\varepsilon,v_\varepsilon,w_\varepsilon)\nabla v_\varepsilon$ and $b(x,t,\xi) := -u_\varepsilon \cdot \xi$ with $(x,t,\xi) \in \Omega \times (0,\infty) \times \mathbb{R}^2$. By means of the Young inequality and basic analysis as well as Proposition 4.1, we obtain

$$\xi \cdot a(x,t,\xi) = |\xi|^2 - n_\varepsilon S_\varepsilon(x,n_\varepsilon,v_\varepsilon,w_\varepsilon)\nabla v_\varepsilon \cdot \xi \ge \frac{1}{2}|\xi|^2 - C_1|\nabla v_\varepsilon|^2, \quad (x,t,\xi) \in \Omega \times (0,\infty) \times \mathbb{R}^2 \tag{5.15}$$

and

$$|a(x,t,\xi)| \le C_2|\nabla v_\varepsilon| + |\xi|, \quad (x,t,\xi) \in \Omega \times (0,\infty) \times \mathbb{R}^2 \tag{5.16}$$

as well as

$$|b(x,t,\xi)| \le \frac{1}{2}|\xi|^2 + C_3, \quad (x,t,\xi) \in \Omega \times (0,\infty) \times \mathbb{R}^2 \tag{5.17}$$

with positive constants C_1, C_2 and C_3. Moreover, Proposition 4.1 points out that $|\nabla v_\varepsilon|$ and $|\nabla v_\varepsilon|^2$ belong to $L^\infty((0,\infty); L^p(\Omega))$ for any $p > 1$. In light of the parabolic regularity theory [44], for any $\tau > 0$, there exist $\theta := \theta(\tau) \in (0,1)$ and some constant $C(\tau) > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega \times [t,t+1])} \le C(\tau) \text{ for all } t \ge \tau, \tag{5.18}$$

which completes the proof. □

According to classical Schauder estimates, we may exploit the same arguments with Lemmata 9.1, 9.2 and 9.3 in [29] to derive the Hölder estimates in $C^{2+\theta,1+\frac{\theta}{2}}$ for v_ε, w_ε and u_ε, so we leave out the details.

Lemma 5.3. *If $\alpha > 0$, then for $\tau > 0$, there exist $\theta \in (0,1)$ and $C(\tau) > 0$ such that the solution of (2.2) satisfies*

$$\|u_\varepsilon(\cdot,t)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega \times [t,t+1])} + \|v_\varepsilon(\cdot,t)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega \times [t,t+1])} + \|w_\varepsilon(\cdot,t)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega \times [t,t+1])} \le C(\tau) \text{ for all } t \ge \tau. \tag{5.19}$$

Based on above preparations, Theorem 1.1 may be proved by utilizing some standard compactness arguments and the parabolic regularity theory.

Lemma 5.4. *If $\alpha > 0$, then there exist $\theta \in (0,1)$, $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1)$ and functions*

$$\begin{cases} n \in C^{\theta,\frac{\theta}{2}}_{loc}(\bar\Omega \times [0,\infty)) \cap C^{2+\theta,1+\frac{\theta}{2}}_{loc}(\bar\Omega \times (0,\infty)), \\ v \in C^{\theta,\frac{\theta}{2}}_{loc}(\bar\Omega \times [0,\infty)) \cap C^{2+\theta,1+\frac{\theta}{2}}_{loc}(\bar\Omega \times (0,\infty)), \\ w \in C^{\theta,\frac{\theta}{2}}_{loc}(\bar\Omega \times [0,\infty)) \cap C^{2+\theta,1+\frac{\theta}{2}}_{loc}(\bar\Omega \times (0,\infty)), \\ u \in C^{\theta,\frac{\theta}{2}}_{loc}\left(\bar\Omega \times [0,\infty); \mathbb{R}^2\right) \cap C^{2+\theta,1+\frac{\theta}{2}}_{loc}\left(\bar\Omega \times (0,\infty); \mathbb{R}^2\right), \\ P \in C^{1,0}(\bar\Omega \times (0,\infty)) \end{cases} \tag{5.20}$$

such that n, v and w are nonnegative in $\Omega \times (0, T)$, and that

$$\begin{cases} n_\varepsilon \to n \in C^0_{loc}\,(\bar{\Omega} \times [0,\infty)), \\ v_\varepsilon \to v \in C^0_{loc}\,(\bar{\Omega} \times [0,\infty)), \\ w_\varepsilon \to w \in C^0_{loc}\,(\bar{\Omega} \times [0,\infty)), \\ u_\varepsilon \to u \in C^0_{loc}\,\left(\bar{\Omega} \times [0,\infty); \mathbb{R}^2\right) \end{cases} \tag{5.21}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and (n, v, w, u, P) solves (1.5) classically in $\Omega \times (0, \infty)$.

Proof. By virtue of Proposition 4.1, Lemmata 4.2 and 5.1 and the Arzelà-Ascoli theorem, we can find a sequence $\varepsilon = \varepsilon_j \searrow 0$ as $j \to \infty$ such that

$$n_\varepsilon \rightharpoonup n \text{ weakly star in } L^\infty(\Omega \times (0,\infty)), \tag{5.22}$$
$$\nabla n_\varepsilon \rightharpoonup \nabla n \text{ weakly in } L^2_{loc}(\bar{\Omega} \times [0,\infty)), \tag{5.23}$$
$$v_\varepsilon \to v \text{ in } C^0_{loc}(\bar{\Omega} \times [0,\infty)), \tag{5.24}$$
$$\nabla v_\varepsilon \to \nabla v \text{ in } C^0_{loc}(\bar{\Omega} \times (0,\infty)), \tag{5.25}$$
$$\nabla v_\varepsilon \rightharpoonup \nabla v \text{ weakly star in } L^\infty(\Omega \times (0,\infty)), \tag{5.26}$$
$$w_\varepsilon \to w \text{ in } C^0_{loc}(\bar{\Omega} \times [0,\infty)), \tag{5.27}$$
$$\nabla w_\varepsilon \to \nabla w \text{ in } C^0_{loc}(\bar{\Omega} \times (0,\infty)), \tag{5.28}$$
$$\nabla w_\varepsilon \rightharpoonup \nabla w \text{ weakly star in } L^\infty(\Omega \times (0,\infty)) \tag{5.29}$$

as well as

$$u_\varepsilon \to u \text{ in } C^0_{loc}(\bar{\Omega} \times [0,\infty)), \tag{5.30}$$

and

$$Du_\varepsilon \rightharpoonup Du \text{ weakly star in } L^\infty(\Omega \times (0,\infty)) \tag{5.31}$$

hold with some limit functions n, v, w and u.

By Lemma 5.1, we assert that n_ε belongs to $L^2((0,T); W^{1,2}(\Omega))$, and $\partial_t n_\varepsilon$ is bounded in $L^1((0,T); (W^{1,2}_0(\Omega))^*)$ for any $T > 0$. Noticing the embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow (W^{1,2}_0(\Omega))^*$, the Aubin-Lions lemma ([45]) along with some standard arguments allows us to derive

$$n_\varepsilon \to n \quad \text{a.e. in } \Omega \times (0,\infty). \tag{5.32}$$

Now, we may verify the limit functions n, v, w and u exactly comply with the properties of a weak solution which are stated by Definition 5.1. The integrability conditions in (5.1) and (5.2) and the nonnegativity of n, v and w are evident by (5.22), (5.23), (5.24), (5.26), (5.27), (5.29), (5.30) and (5.32). Applying the dominated convergence theorem and some standard arguments to the corresponding weak formulations in the regularized problem (2.2) as $\varepsilon = \varepsilon_j \searrow 0$, one can derive the integral identities (5.3)–(5.6) by using (5.22)–(5.32). Moreover, we have

$$n_\varepsilon S_\varepsilon\,(x, n_\varepsilon, v_\varepsilon, w_\varepsilon)\,\nabla v_\varepsilon \to nS\,(x, n, v, w)\,\nabla v \quad \text{a.e. in } \Omega \times (0,\infty). \tag{5.33}$$

Thus, (n, v, w, u) becomes a global weak solution which exactly enjoys the conditions in Definition 5.1.

Lastly, we claim that this weak solution is virtually a solution in the classical sense. Our method is strongly inspired by Lemma 4.3 in [46]. By means of Lemmata 5.2 and 5.3, we obtain

$$\begin{cases} n_\varepsilon \to n \in C_{loc}^{\theta_1,\frac{\theta_1}{2}}(\bar{\Omega}\times[0,\infty)), \\ v_\varepsilon \to v \in C_{loc}^{\theta_1,\frac{\theta_1}{2}}(\bar{\Omega}\times[0,\infty)) \cap C_{loc}^{2+\theta_1,1+\frac{\theta_1}{2}}(\bar{\Omega}\times(0,\infty)), \\ w_\varepsilon \to w \in C_{loc}^{\theta_1,\frac{\theta_1}{2}}(\bar{\Omega}\times[0,\infty)) \cap C_{loc}^{2+\theta_1,1+\frac{\theta_1}{2}}(\bar{\Omega}\times(0,\infty)), \\ u_\varepsilon \to u \in C_{loc}^{\theta_1,\frac{\theta_1}{2}}\left(\bar{\Omega}\times[0,\infty);\mathbb{R}^2\right) \cap C_{loc}^{2+\theta_1,1+\frac{\theta_1}{2}}\left(\bar{\Omega}\times(0,\infty);\mathbb{R}^2\right) \end{cases} \tag{5.34}$$

with some $\theta_1 \in (0,1)$ and subsequence $\varepsilon = \varepsilon_j$. In view of (5.7) and the Hölder regularities provided by (5.34), n possesses the needed regularity properties of a well-established result concerning the gradient Hölder continuity [47], which entails

$$n \in C_{loc}^{1+\theta_2,\frac{1+\theta_2}{2}}(\bar{\Omega}\times(0,\infty)) \quad \text{for some } \theta_2 \in (0,1). \tag{5.35}$$

Now, we consider the sub-problem $n_t - \Delta n = g(x,t)$ with boundary condition $\frac{\partial n}{\partial \nu} = h(x,t)\cdot\nu$, where $g := -\nabla\cdot(nu + nS(x,n,v,w)\nabla v)$ and $h := nS(x,n,v,w)\nabla v$. As the desired Hölder estimates

$$\begin{aligned} &\|g(x,t)\|_{C_{loc}^{\alpha_1,\frac{\alpha_1}{2}}(\bar{\Omega}\times(0,\infty))} \\ \leq &\|u\cdot\nabla n\|_{C_{loc}^{\alpha_1,\frac{\alpha_1}{2}}(\bar{\Omega}\times(0,\infty))} + \|nS(x,n,v,w)\nabla v\|_{C_{loc}^{\alpha_1,\frac{\alpha_1}{2}}(\bar{\Omega}\times(0,\infty))} \\ \leq &C_1 \quad \text{for some } \alpha_1 \in (0,1) \end{aligned} \tag{5.36}$$

and

$$\|h(x,t)\|_{C_{loc}^{1+\alpha_2,\frac{1+\alpha_2}{2}}(\bar{\Omega}\times(0,\infty))} = \|nS(x,n,v,w)\nabla v\|_{C_{loc}^{1+\alpha_2,\frac{1+\alpha_2}{2}}(\bar{\Omega}\times(0,\infty))} \leq C_2 \quad \text{for some } \alpha_2 \in (0,1) \tag{5.37}$$

are warranted by (5.34) and (5.35), invoking the standard parabolic regularity theory [48], we can find a $\theta_3 \in (0,1)$ such that

$$n \in C_{loc}^{2+\theta_3,1+\frac{\theta_3}{2}}(\bar{\Omega}\times(0,\infty)). \tag{5.38}$$

This in combination with (5.34) yields a $\theta_4 \in (0,1)$ such that

$$\begin{cases} n \in C_{loc}^{\theta_4,\frac{\theta_4}{2}}(\bar{\Omega}\times[0,\infty)) \cap C_{loc}^{2+\theta_4,1+\frac{\theta_4}{2}}(\bar{\Omega}\times(0,\infty)), \\ v \in C_{loc}^{\theta_4,\frac{\theta_4}{2}}(\bar{\Omega}\times[0,\infty)) \cap C_{loc}^{2+\theta_4,1+\frac{\theta_4}{2}}(\bar{\Omega}\times(0,\infty)), \\ w \in C_{loc}^{\theta_4,\frac{\theta_4}{2}}(\bar{\Omega}\times[0,\infty)) \cap C_{loc}^{2+\theta_4,1+\frac{\theta_4}{2}}(\bar{\Omega}\times(0,\infty)), \\ u \in C_{loc}^{\theta_4,\frac{\theta_4}{2}}\left(\bar{\Omega}\times[0,\infty);\mathbb{R}^2\right) \cap C_{loc}^{2+\theta_4,1+\frac{\theta_4}{2}}\left(\bar{\Omega}\times(0,\infty);\mathbb{R}^2\right), \end{cases} \tag{5.39}$$

which guarantees the sufficient Hölder regularity of (n,v,w,u) to be a solution in the classical sense and thereby completes the proof. □

Finally, Theorem 1.1 is immediate.

Proof of Theorem 1.1. The statement follows from Lemma 5.4 in conjunction with Proposition 4.1.

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Conflict of interest

The author declares there is no conflict of interest.

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