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Research article

Well-posedness of generalized Stokes-Brinkman equations modeling moving solid phases

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Abstract: Fluid flow through a free-fluid region and the adjacent porous medium has been studied in various problems, such as water flow in rice fields. For the problem with self-propelled solid phases, we provide a generalized Stokes equation for the free-fluid domain and the Brinkman equation in a macroscopic scale due to the movement of self-propelled solid phases rather than a single solid in the porous medium. The model is derived with the assumption that the porosity is not a constant. The porosity in the mathematical model varies depending on the propagation of the solid phases. These two models can be matched at the free-fluid/porous-medium interface and are developed for real world problems. We show the proof of the well-posedness of the discretized form of the weak formulation obtained from applying a mixed finite element scheme to the generalized Stokes-Brinkman equations. The proofs of the continuity and coercive property of the linear and bilinear functionals in the discretized equation are illustrated. We present the existence and uniqueness of the generalized Stokes-Brinkman equations for the numerical problem in two dimensions. The system of equations can be applied to fluid flow propelled by moving solid phases, such as mucus flow in the trachea.

Keywords: generalized Stokes-Brinkman equations; finite element approach; well-posedness; varied porosity; permeability tensor

1. Introduction

Fluid flow problems have considerable attention from researchers and appear in many applications such as engineering, industry, biomedical sciences and other areas. The fluid flow problems are investigated in both theoretical and applied research, among which is the study of flow through a free-fluid domain and an adjacent porous medium. For example, Basirata et al. [1] studied the CO_2 gas flow in a porous medium and free air above it. Oangwatcharaparkan and Wuttanachamsri [2] studied the fluid flow in a periciliary layer (PCL) in the human respiratory system where the mucus layer was on the top of a porous layer. In this research, we focus on fluid flow problems in which the fluid is moved by self-propelled solid phases rather than a pressure gradient. That is, the movement of solids affects the fluid flow whether the fluid is in the same layer as the solid phases or above the solid phases. An example domain that illustrates the regions of interest is represented in Figure 1.

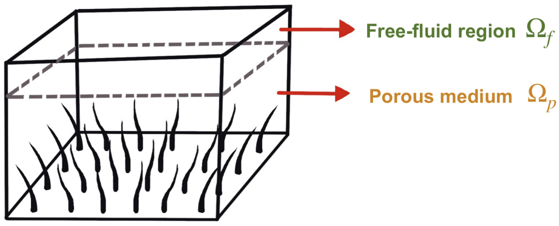

Figure 1. Sample layers of a free-fluid region residing above an adjacent porous medium.

Figure 1 shows a sampling domain consisting of two regions: the layer composed of fluid and self-propelled solid phases, which is considered as a porous medium, Ω_p, and a free-fluid region, Ω_f, residing on the porous medium. In this study, we consider a macroscopic flow where a bundle of solid phases is considered instead of a single self-propelled solid. The locomotion of solids affects the movement of fluids in the nearby areas. If fluid flows through different domains, then mathematical models are also distinct. That is, the equations for the flow above the porous medium and flow in the porous domain are different.

There are several mathematical models used to describe problems of this type [3–11]. Khanafer et al. [3] used Darcy's Law and Brinkman-extended Darcy to investigate the fluid flow inside the hollow fiber bundle of an artificial lung and applied the Navier-Stokes equation for the fluid flow outside the fiber bundle. Ly et al. [4] investigated the problem of fluid flow in coupling a free fluid domain and a porous medium using the Stokes and Darcy equations, respectively. Wuttanachamsri and Schreyer [5] used the Stokes-Brinkman equations to compute the fluid velocities due to the self-propelled solid phases in a three-dimensional domain. Poopra and Wuttanachamsri [6] considered the fluid flow in the periciliary layer (PCL) in human lungs by using the Stokes-Brinkman equations with the Beavers and Joseph boundary condition. Wuttanachamsri [7] used Stokes-Brinkman equations in one dimension with the Stefan problem to estimate the free interface between a porous medium and an adjacent free-fluid region. The well-posedness of the system of the Stokes-Brinkman equations is also provided for both moving or static solid phase [12–16]. For the study of a static solid phase, Ingram [12] applied a finite element discretization to the Brinkman equation and demonstrated that the discretized problem was well-posed. Angot [13] studied the well-posedness of the Stokes-Brinkman and Stokes-Darcy with new jump interface conditions. Chamsri [14] showed the well-posedness of the Stokes-Brinkman model for the case of moving solid phase, while the porosity was assumed to be a constant.

Unlike the usual problems, in this research, we use generalized Stokes-Brinkman equations, where the Brinkman model is developed from the Hybrid Mixture Theory (HMT) [17]. The HMT is a technique for upscaling a multiphase flow by applying an averaging theorem to a microscale equation to obtain a macroscale equation [18, 19]. The macroscopic Brinkman equation is distinct from the models in the above literature because it is derived from the conservation of momentum, where the porosity in the equation is considered as a function of space. Therefore, the porosity is subject to a derivative operator, while the porosity in the Brinkman model in the available research is outside the derivative, although the porosity has been used as a function of space. Our model, the macroscopic Brinkman model, can be used for a bundle of self-propelled solid phases instead of a single solid for fluid flow in a porous medium. In addition, we derive the generalized Stokes equation to apply to the incompressible slow flow in the domain of the free-fluid region next to the porous medium. Extra terms appearing in the generalized Stokes and macroscale Brinkman equations aid to match shear stress at the free-fluid/porous-medium interface. Since our model differs from typical Stokes-Brinkman equations in available research, the well-posedness of the generalized Stokes-Brinkman equations in a macroscopic scale when the fluid is moved by self-propelled solid phases is provided, and the permeability in the model is considered as a second-order tensor, not just a constant.

In Section 2, we derive the generalized Stokes-Brinkman equations. In order to present the well-posedness of the discretized form of the mathematical model using a mixed finite element technique, in Section 3, we present the weak formulation of the governing equations as well as the discretized form of the generalized Stokes-Brinkman equations. In Section 4, the continuity and coercivity of linear and bilinear functionals in the discretized system of equations are presented. The well-posedness of the generalized Stokes-Brinkman equations is illustrated in Section 5. The conclusion is drawn in Section 6. The fundamental definitions, theorems and lemmas proved in available literature and books, which are used in the proof of the well-posedness of the discretized equations, are provided in the Appendix.

2. Generalized Stokes-Brinkman equations

In this section, we present the derivation of our governing equations. For the generalized Stokes equation, we start with the generalized Navier-Stokes equation and then use a nondimensionalization method to obtain the generalized Stokes equation. This is shown in Section 2.1. To derive the Brinkman equation, we begin with a momentum equation obtained from Hybrid Mixture Theory (HMT) [19], an upscaling technique, and then use a nondimensionalization approach to have a macroscopic model in a porous medium, which is illustrated in Section 2.2. We rewrite our governing equations in Section 2.3 in order to summarize and use them in the next sections.

2.1. Derivation of generalized Stokes equation

To obtain the generalized Stokes equation, we start with the generalized Navier-Stokes equation, which is attained from substituting a stress tensor, developed from entropy inequality holding near equilibrium for a viscous fluid, into a momentum equation. The generalized Navier-Stokes equation is [20]

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\cdot\nabla\mathbf{u} - \rho\mathbf{g} = \mathbf{0}, \tag{2.1}$$

where ρ is density, t is time, $\mathbf{u}$ is the velocity, p is pressure, λ is a constant, μ is the dynamic viscosity, and $\mathbf{g}$ is gravity. The generalized Navier-Stokes equation, Eq (2.1), is normalized with dimensionless variables, and we get

$$\rho u_0 f\frac{\partial\widehat{\mathbf{u}}}{\partial\widehat{t}} + \frac{\rho u_0^2}{L}(\widehat{\mathbf{u}}\cdot\widehat{\nabla}\widehat{\mathbf{u}}) + \frac{p_0}{L}\widehat{\nabla}\widehat{p} - \frac{(\lambda+\mu)u_0}{L^2}\widehat{\nabla}(\widehat{\nabla}\cdot\widehat{\mathbf{u}}) - \frac{\mu u_0}{L^2}\widehat{\nabla}\cdot\widehat{\nabla}\widehat{\mathbf{u}} - \rho g_0\widehat{\mathbf{g}} = \mathbf{0}, \tag{2.2}$$

where the characteristic parameter L is the characteristic length; f is the characteristic frequency; u_0 is the characteristic speed; p_0 is the reference pressure; g_0 is the gravitational acceleration. Multiplying Eq (2.2) by $\dfrac{L^2}{\mu u_0}$ on both sides, we have

$$\frac{\rho f L^2}{\mu}\frac{\partial \widehat{\mathbf{u}}}{\partial \widehat{t}} + \frac{\rho u_0 L}{\mu}(\widehat{\mathbf{u}} \cdot \widehat{\nabla}\widehat{\mathbf{u}}) + \frac{p_0 L}{\mu u_0}\widehat{\nabla}\widehat{p} - \frac{(\lambda+\mu)}{\mu}\widehat{\nabla}(\widehat{\nabla} \cdot \widehat{\mathbf{u}}) - \widehat{\nabla} \cdot \widehat{\nabla}\widehat{\mathbf{u}} - \frac{\rho g_0 L^2}{\mu u_0}\widehat{\mathbf{g}} = \mathbf{0}. \tag{2.3}$$

Next, we calculate the coefficients in Eq (2.3), where the values of the characteristic variables and other variables in International System (SI) units are shown in Table 1. The characteristic length is the highest length of cilia in the respiratory system [21], the reference f is the frequency of cilia beat in the human respiratory tract [22], the characteristic velocity u_0 is the maximum speed of cilia for the effective stroke at temperature $37°C$, the reference pressure p_0 is the pressure in the human respiratory tract, which is about one [23], g_0 is the Earth's gravity, and ρ and μ are the density of water and dynamic viscosity at $37°C$, respectively. The constant λ is set equal to zero. The values of the coefficients are shown in Table 2.

Table 1. The values of characteristic and constant variables in Eq (2.3) in International System units.

Variables	L	f	u_0	p_0	g_0	ρ	μ
SI Units	m	1/s	m/s	kg/(m·s^2)	m/s^2	kg/m^3	kg/(m·s)
Values	7×10^{-6}	10	2.5×10^{-4}	1	9.807	993.3	0.6913×10^{-3}

Table 2. The values of the coefficients in Eq (2.3).

Coefficients	$\dfrac{\rho f L^2}{\mu}$	$\dfrac{\rho u_0 L}{\mu}$	$\dfrac{p_0 L}{\mu u_0}$	$\dfrac{\lambda+\mu}{\mu}$	$\dfrac{\rho g_0 L^2}{\mu u_0}$
Values	7.0406×10^{-4}	2.5×10^{-3}	40.5034	1	2.7619

From Table 2, the coefficients of the first two terms in Eq (2.3) are comparatively small compared with the others. Therefore, the unsteady and nonlinear terms in the equation are neglected, and then Eq (2.3) becomes

$$\nabla p \; - \; (\lambda+\mu)\nabla(\nabla \cdot \mathbf{u}) \; - \; \mu\nabla \cdot \nabla\mathbf{u} \; - \; \rho\mathbf{g} \; = \; \mathbf{0}. \tag{2.4}$$

In this work, we assume that the velocity $\mathbf{u}$ in two dimensions is smooth enough that the order of the derivative can be interchanged, that is, $\dfrac{\partial}{\partial x}\dfrac{\partial \mathbf{u}}{\partial y} = \dfrac{\partial}{\partial y}\dfrac{\partial \mathbf{u}}{\partial x}$. Then, we obtain

$$\nabla(\nabla \cdot \mathbf{u}) \; = \; \nabla \cdot (\nabla\mathbf{u})^T. \tag{2.5}$$

Substituting Eq (2.5) into the second term of Eq (2.4), we have

$$\nabla p \; - \; \lambda\nabla(\nabla \cdot \mathbf{u}) \; - \; \mu\nabla \cdot (\nabla\mathbf{u})^T \; - \; \mu\nabla \cdot \nabla\mathbf{u} \; - \; \rho\mathbf{g} \; = \; \mathbf{0}, \tag{2.6}$$

or

$$\nabla p \; - \; \lambda\nabla(\nabla \cdot \mathbf{u}) \; - \; \mu\nabla \cdot (\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \; - \; \rho\mathbf{g} = \mathbf{0}. \tag{2.7}$$

Since the rate of deformation for the liquid phase $\mathbf{d} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, we rewrite Eq (2.7) as

$$\nabla p - \lambda\nabla(\nabla\cdot\mathbf{u}) - \nabla\cdot(2\mu\mathbf{d}) - \rho\mathbf{g} = \mathbf{0}. \tag{2.8}$$

Notice that if $\lambda = 0$, then Eq (2.8) becomes

$$\nabla p - \nabla\cdot(2\mu\mathbf{d}) - \rho\mathbf{g} = \mathbf{0}, \tag{2.9}$$

which is the generalized Stokes equation. If the matrix $\nabla\mathbf{u}$ is symmetric, then Eq (2.9) is the following Stokes equation:

$$\nabla p - 2\mu\Delta\mathbf{u} - \rho\mathbf{g} = \mathbf{0}. \tag{2.10}$$

2.2. Derivation of the Brinkman equation

In this section, we show the derivation of the Brinkman equation in a macroscopic scale derived using Hybrid Mixture Theory (HMT). HMT is an upscaling technique used to derive multiphase equations such as the combination of solid and liquid phases. This method uses the averaging theorem to upscale equations from a microscale equation to a macroscale equation [19]. In this study, we focus on developing a model for fluid flow due to the movement of self-propelled solid phases. Here, we follow the procedure provided in [18]. We begin with the multiphase equation upscaled from the conservation of momentum [18] when the porosity is a function, not a constant:

$$\varepsilon^l\rho^l\frac{D^l\mathbf{u}^l}{Dt} + \varepsilon^l\nabla p + p\nabla\varepsilon^l - \nabla\cdot(\varepsilon^l 2\mu\mathbf{d}^l) - \varepsilon^l\rho^l\mathbf{g}^l = p\nabla\varepsilon^l - \varepsilon^l\mathbf{R}\cdot(\mathbf{u}^l - \mathbf{u}^s), \tag{2.11}$$

where l and s mean the liquid and solid phases, respectively. The function ε^l is the porosity, which is a variable in space; $\mathbf{d}^l = 0.5\left(\nabla\mathbf{u}^l + (\nabla\mathbf{u}^l)^T\right)$ is the rate of deformation tensor; $\mathbf{R}$ is a second-order tensor; $\mathbf{u}^l$ and $\mathbf{u}^s$ are the velocities of liquid and solid phases, respectively. Substituting $\mathbf{R} = \mu\varepsilon^l\mathbf{k}^{-1}$, where $\mathbf{k}^{-1}$ is the inverse of the permeability tensor, and taking $\frac{D^l\mathbf{u}^l}{Dt} = \frac{\partial\mathbf{u}^l}{\partial t} + \mathbf{u}^l\cdot\nabla\mathbf{u}^l$ into Eq (2.11), subtracting from both sides by $p\nabla\varepsilon^l$ and dividing by ε^l on both sides, we have

$$\rho\left(\frac{\partial\mathbf{u}^l}{\partial t} + \mathbf{u}^l\cdot\nabla\mathbf{u}^l\right) + \mu\mathbf{k}^{-1}\cdot(\varepsilon^l\mathbf{u}^l - \varepsilon^l\mathbf{u}^s) + \nabla p - \frac{\mu}{\varepsilon^l}\nabla\cdot(2\varepsilon^l\mathbf{d}^l) = \rho\mathbf{g}. \tag{2.12}$$

Then, we normalize Eq (2.12). We use the same characteristic parameters as in the previous subsection. The dimensionless form of Eq (2.12) is

$$\frac{\mathbf{k}\rho f}{\mu}\frac{\partial\widehat{\mathbf{u}^l}}{\partial\widehat{t}} + \frac{\mathbf{k}\rho u_0}{\mu L}(\widehat{\mathbf{u}^l}\cdot\widehat{\nabla}\widehat{\mathbf{u}^l}) + (\varepsilon^l\widehat{\mathbf{u}^l} - \varepsilon^l\widehat{\mathbf{u}^s}) + \frac{\mathbf{k}p_0}{\mu u_0 L}\widehat{\nabla}\widehat{p} - \frac{\mathbf{k}}{\varepsilon^l L^2}\widehat{\nabla}\cdot(2\varepsilon^l\widehat{\mathbf{d}^l}) = \frac{\mathbf{k}\rho g_0}{\mu u_0}\widehat{\mathbf{g}}. \tag{2.13}$$

Using the values in Table 1 to calculate the coefficients in Eq (2.13) with the permeability $k = 10^{-14}$ m^2 and porosity $\varepsilon^l = 1$, which are the maximum values employed from [24], we obtain the values of the coefficients as illustrated in Table 3.

Table 3. The values of the coefficients in Eq (2.13).

Coefficients	$\frac{\mathbf{k}\rho f}{\mu}$	$\frac{\mathbf{k}\rho u_0}{\mu L}$	$\frac{\mathbf{k}p_0}{\mu u_0 L}$	$\frac{\mathbf{k}}{\varepsilon^l L^2}$	$\frac{\mathbf{k}\rho g_0}{\mu u_0}$
Values	1.4369×10^{-7}	5.1316×10^{-7}	8.3×10^{-3}	2.0408×10^{-4}	5.6365×10^{-4}

From the calculation demonstrated in Table 3, we neglect the first two terms in Eq (2.13), the time-dependent and nonlinear terms, because these expressions are significantly small in comparison with others. Therefore, Eq (2.12) becomes

$$\mu\mathbf{k}^{-1}\cdot(\varepsilon^l\mathbf{u}^l-\varepsilon^l\mathbf{u}^s)\;+\;\nabla p\;-\;\frac{\mu}{\varepsilon^l}\nabla\cdot(2\varepsilon^l\mathbf{d}^l)\;=\;\rho\mathbf{g}, \tag{2.14}$$

which is called the Brinkman equation in a macroscopic scale. Notice that the macroscale Brinkman equation is distinct from the Brinkman in literature such as in [25],

$$\mu\mathbf{k}^{-1}\cdot\varepsilon^l\mathbf{u}^l\;+\;\nabla p\;-\;\mu\Delta\mathbf{u}^l=\;\rho\mathbf{g}, \tag{2.15}$$

because our model starts with the momentum equation that the porosity is a function and cannot be moved out of the derivative as a constant, as usually used in research. The porosity in the parentheses in the third term in Eq (2.14) cannot be canceled out with the denominator. Moreover, the first-order derivative of the rate of deformation times the porosity in Eq (2.14) cannot be changed to be the second-order derivative of the velocity as shown in Eq (2.15). It may seem that the difference is not much, but finding the numerical results of Eq (2.14) is more complicated than for Eq (2.15), including the proof of the well-posedness of the equation.

Notice that the Brinkman equation in the macroscopic scale, Eq (2.14), has a good engagement with the generalized Stokes equation, Eq (2.9). Because there are no solid phases in the adjacent free-fluid domain, the porosity becomes one, and the permeability tends to infinity in this region. Therefore, the first term in Eq (2.14) disappears, and then the Brinkman equation, Eq (2.14), becomes generalized Stokes equation, Eq (2.9). Thus, the solutions in these two layers can be matched in the transition zone at the free-fluid/porous-medium interface by using the generalized Stokes-Brinkman model. The mathematical model is summarized in the next section.

2.3. Governing equations

The models for a problem of this kind are summarized in this section. The models in both free-fluid layer and porous medium consist of two unknowns, which are the velocity $\mathbf{u}^l$ and the pressure p. Therefore, in each region, we need one more equation, which is a continuity equation obtained from conservation of mass. Since a bundle of self-propelled solid phases effects the fluid flow, we employ the continuity equation for two-phase flow in the porous medium derived by HMT [26, 27], which is

$$\nabla\cdot(\varepsilon^l\mathbf{u}^l)\;=\;f, \tag{2.16}$$

where $f=-\dot{\varepsilon}^l/(1-\varepsilon^l)+\nabla\cdot(\varepsilon^l\mathbf{u}^s)$ and $\dot{\varepsilon}^l=\partial\varepsilon^l/\partial t+\mathbf{u}^s\cdot\nabla\varepsilon^l$. Let $\Omega=\Omega_p\cup\Omega_f$ be our domain, and $\partial\Omega$ is the boundary of the domain. Define the vectors

$$\mathbf{u}=\varepsilon^l\mathbf{u}^l\quad\text{and}\quad\mathbf{f}=\rho\mathbf{g}+\mu\mathbf{k}^{-1}\cdot\varepsilon^l\mathbf{u}^s. \tag{2.17}$$

From Eqs (2.14), (2.16) and (2.17), the system of equations used in domain Ω_p is

$$\mu\mathbf{k}^{-1}\cdot\mathbf{u}-\frac{\mu}{\varepsilon^l}\nabla\cdot(2\varepsilon^l\mathbf{d}^l)+\nabla p\;=\;\mathbf{f}\qquad\text{in }\Omega_p, \tag{2.18}$$

$$\nabla\cdot\mathbf{u}\;=\;f\qquad\text{in }\Omega_p. \tag{2.19}$$

For the free-fluid domain Ω_f, the flow is considered incompressible. Then, the divergence of velocity is zero. Therefore, in domain Ω_f, we have the system of equations

$$-\mu\nabla\cdot(2\varepsilon^l\mathbf{d}^l)+\nabla p = 0 \quad \text{in } \Omega_f, \tag{2.20}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_f. \tag{2.21}$$

Before we prove the well-posedness of the generalized Stokes-Brinkman equations, Eqs (2.18) and (2.19), in the next section we provide the discretized form of the governing equations by using a finite element method.

3. Discretized model

In this section, we first formulate the weak formulation of the generalized Stokes-Brinkman equations, Eqs (2.18) and (2.19), by using a mixed finite element method. To obtain the weak form of Eq (2.18), we multiply a weight function $\mathbf{w} \in H_0^1(\Omega)$ and integrate Eq (2.18) over the domain Ω on both sides, and we have

$$\int_\Omega \mu(\mathbf{k}^{-1}\cdot\mathbf{u})\cdot\mathbf{w} - \int_\Omega \frac{\mu}{\varepsilon^l}\left(\nabla\cdot(2\varepsilon^l\mathbf{d}^l)\right)\cdot\mathbf{w} + \int_\Omega \nabla p\cdot\mathbf{w} = \int_\Omega \mathbf{f}\cdot\mathbf{w}. \tag{3.1}$$

Applying Green's first identity to the second and third terms on the left hand side of Eq (3.1) and using the property that the weight function is zero at the boundary, we have

$$\int_\Omega \mu(\mathbf{k}^{-1}\cdot\mathbf{u})\cdot\mathbf{w} + \int_\Omega 2\mu\varepsilon^l\mathbf{d}^l : \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) - \int_\Omega (\nabla\cdot\mathbf{w})p = \int_\Omega \mathbf{f}\cdot\mathbf{w}. \tag{3.2}$$

Substituting $\mathbf{d}^l = 0.5\left[\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)+\left(\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right)^T\right]$ into Eq (3.2), we obtain the weak formulation of Eq (2.18), which is

$$\int_\Omega \mu(\mathbf{k}^{-1}\cdot\mathbf{u})\cdot\mathbf{w} + \int_\Omega \mu\varepsilon^l\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right) : \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) + \int_\Omega \mu\varepsilon^l\left(\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right)^T : \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) - \int_\Omega (\nabla\cdot\mathbf{w})p = \int_\Omega \mathbf{f}\cdot\mathbf{w}. \tag{3.3}$$

Similarly, multiplying both sides of Eq (2.19) by another weight function $q \in L_0^2(\Omega)$ and integrating both sides, we obtain the weak formulation of Eq (2.19):

$$\int_\Omega (\nabla\cdot\mathbf{u})\,q = \int_\Omega fq. \tag{3.4}$$

The weak formulation of the generalized Stokes-Brinkman equations can be written in linear and bilinear functionals as follows.

Problem 1. *The weak form of the generalized Stokes-Brinkman equations is to find* $\mathbf{u} \in H_s^1(\Omega)$ *and* $p \in L_0^2(\Omega)$ *such that*

$$a(\mathbf{u},\mathbf{w}) + b(\mathbf{w},p) = c_1(\mathbf{w}), \quad \forall\mathbf{w} \in H_0^1(\Omega), \tag{3.5}$$

$$b(\mathbf{u},q) = c_2(q), \quad \forall q \in L_0^2(\Omega), \tag{3.6}$$

where the linear and bilinear functionals are defined as

$$a(\mathbf{u},\mathbf{w}) = \int_\Omega \mu(\mathbf{k}^{-1}\cdot\mathbf{u})\cdot\mathbf{w} + \int_\Omega \mu\varepsilon^l \nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right) : \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) + \int_\Omega \mu\varepsilon^l \left(\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right)^T : \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right), \tag{3.7}$$

$$b(\mathbf{u},q) = -\int_\Omega (\nabla\cdot\mathbf{u})\, q, \tag{3.8}$$

$$c_1(\mathbf{w}) = \langle \mathbf{f},\mathbf{w}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \tag{3.9}$$

$$c_2(q) = -\int_\Omega fq, \tag{3.10}$$

where the space $H_s^1(\Omega) = \{\mathbf{w}\in H^1(\Omega) : \mathbf{w}|_{\partial\Omega} = \mathbf{s}\}$*, and* $\langle\cdot,\cdot\rangle$ *is the duality pairing.*

Notice that the space $L_0^2(\Omega)$ is used instead of $L^2(\Omega)$ because the system of Eqs (2.18) and (2.19) demonstrates pressure up to an additive constant; see [28] on page 157 for details. The linear and bilinear functionals in Problem 1 can be written in the form of a linear operator as in Problem 2 (the definition of the linear operator is in the Appendix).

Problem 2. *Let* $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ *and* $B : H_0^1(\Omega) \to L_0^2(\Omega)$ *be linear operators. Find* $\mathbf{u} \in H_s^1(\Omega),\ p \in L_0^2(\Omega)$ *such that*

$$A\mathbf{u} + B'p = \mathbf{f} \quad in\ H^{-1}(\Omega), \tag{3.11}$$

$$B\mathbf{u} = f \quad in\ L_0^2(\Omega), \tag{3.12}$$

where

$$\|\mathbf{f}\|_{H^{-1}(\Omega)} = \sup_{\mathbf{w}\in H_0^1(\Omega),\mathbf{w}\neq 0} \frac{\langle\mathbf{f},\mathbf{w}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}}{\|\mathbf{w}\|_{H^1(\Omega)}}, \tag{3.13}$$

the norm $\|\cdot\|_{H^1(\Omega)}$ *denotes the standard norm on the space* $H^1(\Omega)$*, and the function* $\mathbf{u}^s$ *in* $\mathbf{f}$ *is a bounded continuous function.*

Next, we show that the linear and bilinear functionals in Problem 1 are continuous and coercive, and that will be used to prove the existence and uniqueness of the generalized Stokes-Brinkman equations in Section 5.

4. Continuity and coercivity of linear and bilinear functionals

In this section, we show that the linear and bilinear functionals in Problem 1 are continuous and coercive. These properties are necessary to prove the existence and uniqueness of the governing equations. We first proof the continuity as shown in Theorem 4.1.

Theorem 4.1. *The linear functionals* $c_1(\mathbf{w}), c_2(q)$ *and bilinear functionals* $a(\cdot,\cdot),\ b(\cdot,\cdot)$ *are continuous. In particular,*

$$c_1(\mathbf{w}) \leq \|\mathbf{f}\|_{H^{-1}(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)}, \qquad \forall\mathbf{w}\in H^1(\Omega), \tag{4.1}$$

$$c_2(q) \leq \|f\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}, \qquad \forall q\in L^2(\Omega), \tag{4.2}$$

$$b(\mathbf{u},q) \leq \sqrt{n}|\mathbf{u}|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}, \qquad \forall\mathbf{u}\in H^1(\Omega), \forall q\in L^2(\Omega), \tag{4.3}$$

$$a(\mathbf{u},\mathbf{w}) \quad \le \quad Q_a\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)}, \qquad \forall \mathbf{u},\mathbf{w}\in H^1(\Omega), \tag{4.4}$$

where n is the dimensional number, and

$$Q_a = \max\{\sqrt{6}\mu \max_{1\le i,j\le 2}|k_{ij}^{-1}|,\ 2\mu/\|\varepsilon^l\|_{H^1(\Omega)}\}.$$

Proof. It is obvious that $c_1(\mathbf{w})$ and $c_2(q)$ are linear functionals, and $a(\mathbf{u},\mathbf{w})$ and $b(\mathbf{u},q)$ are bilinear functionals. Next, we show the continuity of $c_1(\mathbf{w})$. Let $\mathbf{w}\in H^1(\Omega)$. Then,

$$\begin{aligned}
|c_1(\mathbf{w})| &= \left|\langle \mathbf{f},\mathbf{w}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\right| \\
&= \left|\frac{\langle \mathbf{f},\mathbf{w}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}}{\|\mathbf{w}\|_{H^1(\Omega)}}\|\mathbf{w}\|_{H^1(\Omega)}\right| \\
&\le \|\mathbf{f}\|_{H^{-1}(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)},
\end{aligned}$$

where we apply the definition of norm on the space $H^{-1}(\Omega)$, Eq (3.13), at the inequality. The proof of the continuities of $c_2(q)$ and $b(\mathbf{u},q)$ has been shown in [16]. Next, we show the continuity of $a(\mathbf{u},\mathbf{w})$ in a two-dimensional domain. Define $\mathbf{u}=(u_1,u_2)$ and $\mathbf{w}=(w_1,w_2)$. Then, from Eq (3.7), we have

$$\begin{aligned}
|a(\mathbf{u},\mathbf{w})| &= \left|\int_\Omega \mu(\mathbf{k}^{-1}\cdot\mathbf{u})\cdot\mathbf{w} + \int_\Omega \mu\varepsilon^l\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right):\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) + \int_\Omega \mu\varepsilon^l\left(\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right)^T:\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right| \\
&\le \left|\int_\Omega \mu(\mathbf{k}^{-1}\cdot\mathbf{u})\cdot\mathbf{w}\right| + \left|\int_\Omega \mu\varepsilon^l\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right):\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right| + \left|\int_\Omega \mu\varepsilon^l\left(\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right)^T:\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right| \\
&\le \mu\left\|\mathbf{k}^{-1}\cdot\mathbf{u}\right\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^2(\Omega)} + \mu\left\|\varepsilon^l\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)} \\
&\quad + \mu\left\|\varepsilon^l\left(\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right)^T\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)} \\
&\le \sqrt{6}\mu\max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right|\|\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^2(\Omega)} + \mu\left\|\varepsilon^l\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)} \\
&\quad + \mu\left\|\varepsilon^l\right\|_{L^2(\Omega)}\left\|\left(\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right)^T\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)} \\
&= \sqrt{6}\mu\max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right|\|\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^2(\Omega)} + 2\mu\left\|\varepsilon^l\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\mathbf{u}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right\|_{L^2(\Omega)} \\
&\le \sqrt{6}\mu\max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right|\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)} + 2\mu\left\|\varepsilon^l\right\|_{L^2(\Omega)}\left\|\frac{\mathbf{u}}{\varepsilon^l}\right\|_{H^1(\Omega)}\left\|\frac{\mathbf{w}}{\varepsilon^l}\right\|_{H^1(\Omega)} \\
&= \sqrt{6}\mu\max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right|\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)} + 2\mu\left\|\varepsilon^l\right\|_{L^2(\Omega)}\frac{\|\mathbf{u}\|_{H^1(\Omega)}}{\left\|\varepsilon^l\right\|_{H^1(\Omega)}}\frac{\|\mathbf{w}\|_{H^1(\Omega)}}{\left\|\varepsilon^l\right\|_{H^1(\Omega)}} \\
&\le \sqrt{6}\mu\max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right|\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)} + \frac{2\mu}{\left\|\varepsilon^l\right\|_{H^1(\Omega)}}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)} \\
&\le \max\left\{\sqrt{6}\mu\max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right|,\ \frac{2\mu}{\left\|\varepsilon^l\right\|_{H^1(\Omega)}}\right\}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)} \\
&= Q_a\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)},
\end{aligned}$$

where $Q_a = \max\left\{\sqrt{6}\mu \max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right|,\ \dfrac{2\mu}{\left\|\varepsilon^l\right\|_{H^1(\Omega)}}\right\}$ and the inequality [14]

$$\left\|\mathbf{k}^{-1}\cdot\mathbf{u}\right\|_{L^2(\Omega)} \le \sqrt{6}\max_{1\le i,j\le 2}\left|k_{ij}^{-1}\right| \|\mathbf{u}\|_{L^2(\Omega)} \tag{4.5}$$

is applied to the third inequality. For the fifth inequality, we use the fact that $\|\cdot\|_{L^2(\Omega)} \le \|\cdot\|_{H^1(\Omega)}$, so $\dfrac{\|\cdot\|_{L^2(\Omega)}}{\|\cdot\|_{H^1(\Omega)}} \le 1$. Therefore, $a(\mathbf{u},\mathbf{w})$ is continuous.

To show that the bilinear form $a(\cdot,\cdot)$ is coercive, we first proof Lemma 4.2, which will be used in the proof of coercivity presented in Theorem 4.3.

Lemma 4.2. *Let* $\mathbf{w}\in H^1(\Omega)$. *Then,*

$$\int_\Omega \mu\varepsilon^l\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right):\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) + \int_\Omega \mu\varepsilon^l\left(\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right)^T:\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) \ge \frac{\mu\sqrt{|V|}}{2n_r^2}\left\|\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)+\left(\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right)^T\right\|_{L^2(\Omega)}^2, \tag{4.6}$$

for some natural number n_r.

Proof. Let $\mathbf{w}\in H^1(\Omega)$. Since the porosity $\varepsilon^l > 0$ is a real number, by the Archimedean property, there exists $n_r\in\mathbb{N}$ such that $\varepsilon^l \ge \dfrac{1}{n_r}$. Given $\mathbf{W} = \dfrac{\mathbf{w}}{\varepsilon^l}$, then,

$$\begin{aligned}
\int_\Omega \mu\varepsilon^l\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right):\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) + \int_\Omega \mu\varepsilon^l\left(\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right)^T:\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) &= \int_\Omega \mu\varepsilon^l\nabla\mathbf{W}:\nabla\mathbf{W} + \int_\Omega \mu\varepsilon^l(\nabla\mathbf{W})^T:\nabla\mathbf{W}\\
&= \int_\Omega \mu\varepsilon^l\left[\nabla\mathbf{W}:\nabla\mathbf{W} + (\nabla\mathbf{W})^T:\nabla\mathbf{W}\right]\\
&= \int_\Omega \mu\varepsilon^l\left[\nabla\mathbf{W} + (\nabla\mathbf{W})^T\right]:\nabla\mathbf{W}\\
&= \int_\Omega 2\mu\varepsilon^l\frac{1}{2}\left[\nabla\mathbf{W} + (\nabla\mathbf{W})^T\right]:\nabla\mathbf{W}\\
&= \int_\Omega 2\mu\varepsilon^l\mathbf{D}:\nabla\mathbf{W}\\
&= \int_\Omega 2\mu\varepsilon^l D_{ij}\frac{\partial W_i}{\partial x_j}\\
&= \int_\Omega 2\mu\varepsilon^l\left(D_{11}\frac{\partial W_1}{\partial x_1} + D_{12}\frac{\partial W_1}{\partial x_2} + D_{21}\frac{\partial W_2}{\partial x_1} + D_{22}\frac{\partial W_2}{\partial x_2}\right)\\
&= \int_\Omega 2\mu\varepsilon^l\left(D_{11}^2 + D_{12}\left(\frac{\partial W_1}{\partial x_2} + \frac{\partial W_2}{\partial x_1}\right) + D_{22}^2\right)\\
&= \int_\Omega 2\mu\varepsilon^l\left(D_{11}^2 + 2D_{12}\frac{1}{2}\left(\frac{\partial W_1}{\partial x_2} + \frac{\partial W_2}{\partial x_1}\right) + D_{22}^2\right)\\
&= \int_\Omega 2\mu\varepsilon^l\left(D_{11}^2 + 2D_{12}^2 + D_{22}^2\right)\\
&= \int_\Omega 2\mu\varepsilon^l\left(D_{11}^2 + D_{12}^2 + D_{21}^2 + D_{22}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq 2\mu \int_\Omega \left(\varepsilon^l\right)^2 \left(D_{11}^2 + D_{12}^2 + D_{21}^2 + D_{22}^2\right) \\
&= 2\mu \left(\int_\Omega (\varepsilon^l D_{11})^2 + \int_\Omega (\varepsilon^l D_{12})^2 + \int_\Omega (\varepsilon^l D_{21})^2 + \int_\Omega (\varepsilon^l D_{22})^2 \right) \\
&= 2\mu \left(\left\|\varepsilon^l D_{11}\right\|_{L^2(\Omega)}^2 + \left\|\varepsilon^l D_{12}\right\|_{L^2(\Omega)}^2 + \left\|\varepsilon^l D_{21}\right\|_{L^2(\Omega)}^2 + \left\|\varepsilon^l D_{22}\right\|_{L^2(\Omega)}^2 \right) \\
&\geq 2\mu \left(\left\|(\varepsilon^l D_{11})^2\right\|_{L^2(\Omega)} + \left\|(\varepsilon^l D_{12})^2\right\|_{L^2(\Omega)} + \left\|(\varepsilon^l D_{21})^2\right\|_{L^2(\Omega)} + \left\|(\varepsilon^l D_{22})^2\right\|_{L^2(\Omega)} \right) \\
&\geq 2\mu \left\|(\varepsilon^l D_{11})^2 + (\varepsilon^l D_{12})^2 + (\varepsilon^l D_{21})^2 + (\varepsilon^l D_{22})^2\right\|_{L^2(\Omega)} \\
&= 2\mu \left\| \sum_{i=1}^{2} \sum_{j=1}^{2} \left|\varepsilon^l D_{ij}\right|^2 \right\|_{L^2(\Omega)} \\
&\geq 2\mu \left\| \left(\sigma_{\max}\left(\varepsilon^l \mathbf{D}\right)\right)^2 \right\|_{L^2(\Omega)} \\
&= 2\mu \left(\int_\Omega \left(\sigma_{\max}\left(\varepsilon^l \mathbf{D}\right)\right)^4 d\Omega \right)^{\frac{1}{2}} \\
&= 2\mu \left(\left(\sigma_{\max}\left(\varepsilon^l \mathbf{D}\right)\right)^4 \int_\Omega 1 \, d\Omega \right)^{\frac{1}{2}} \\
&= 2\mu \left(\left(\sigma_{\max}\left(\varepsilon^l \mathbf{D}\right)\right)^4 |V| \right)^{\frac{1}{2}} \\
&= 2\mu \sqrt{|V|} \left(\sigma_{\max}\left(\varepsilon^l \mathbf{D}\right)\right)^2 \\
&= 2\mu \sqrt{|V|} \left\|\varepsilon^l \mathbf{D}\right\|_{L^2(\Omega)}^2 \\
&= 2\mu \sqrt{|V|} \left\| \frac{\varepsilon^l}{2} \left(\nabla \mathbf{W} + (\nabla \mathbf{W})^T\right) \right\|_{L^2(\Omega)}^2 \\
&\geq 2\mu \sqrt{|V|} \left\| \frac{1}{2n_r} \left(\nabla \mathbf{W} + (\nabla \mathbf{W})^T\right) \right\|_{L^2(\Omega)}^2 \\
&= \frac{\mu \sqrt{|V|}}{2n_r^2} \left\|\nabla \mathbf{W} + (\nabla \mathbf{W})^T\right\|_{L^2(\Omega)}^2 \\
&= \frac{\mu \sqrt{|V|}}{2n_r^2} \left\| \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) + \left(\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right)^T \right\|_{L^2(\Omega)}^2 ,
\end{aligned}
$$

where $\mathbf{D} = \dfrac{1}{2}(\nabla \mathbf{W} + (\nabla \mathbf{W})^T)$ is symmetric with the indicial notation $D_{ij} = \dfrac{1}{2}\left(\dfrac{\partial W_i}{\partial x_j} + \dfrac{\partial W_j}{\partial x_i}\right)$. The variable D_{ij} is considered to be a real number for all $i, j = 1, 2$, and $|V|$ is the volume of the domain Ω. For the first inequality, we use the fact that the porosity $\varepsilon^l \leq 1$. For the second and third inequalities, we apply Hölder's inequality and the triangle inequality, respectively. For the fourth inequality, we apply the spectral norm of a matrix $\mathbf{E}$ [29], i.e.,

$$
\|\mathbf{E}\|_{L^2(\Omega)} = \sigma_{\max}(\mathbf{E}) \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{s} \left|a_{ij}\right|^2 \right)^{\frac{1}{2}}, \tag{4.7}
$$

where $\sigma_{\max}(\mathbf{E})$ represents the largest singular value of a matrix $\mathbf{E}$, and s and m are the dimensional numbers. For the last inequality, we use the Archimedean property [30]. Therefore,

$$\int_\Omega \mu\varepsilon^l \nabla\mathbf{W} : \nabla\mathbf{W} \;+\; \int_\Omega \mu\varepsilon^l (\nabla\mathbf{W})^T : \nabla\mathbf{W} \;\geq\; \frac{\mu\sqrt{|V|}}{2n_r^2}\left\|\nabla\mathbf{W} + (\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)}. \tag{4.8}$$

Hence, the proof is complete.

Next, we show the coercivity of the bilinear functional $a(\cdot,\cdot)$.

Theorem 4.3. *The bilinear functional* $a(\cdot,\cdot)$ *is coercive, such that*

$$a(\mathbf{w},\mathbf{w}) \;\geq\; Q_c\|\mathbf{w}\|^2_{H^1(\Omega)}, \quad \forall\mathbf{w}\in H^1(\Omega), \tag{4.9}$$

where $Q_c = \min\left\{\mu Q_k,\ \frac{\mu\sqrt{|V|}}{n_r^2 K}(N-1)^2\right\}$ *and* Q_k *is a positive number.*

Proof. Let $\mathbf{w}\in H^1(\Omega)$. Then,

$$\begin{aligned}
a(\mathbf{w},\mathbf{w}) &= \int_\Omega \mu\left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} + \int_\Omega \mu\varepsilon^l\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) : \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right) + \int_\Omega \mu\varepsilon^l\left(\nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\right)^T : \nabla\left(\frac{\mathbf{w}}{\varepsilon^l}\right)\\
&= \int_\Omega \mu\left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} + \int_\Omega \mu\varepsilon^l\nabla\mathbf{W} : \nabla\mathbf{W} \;+\; \int_\Omega \mu\varepsilon^l(\nabla\mathbf{W})^T : \nabla\mathbf{W}\\
&\geq \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{2n_r^2}\left\|\nabla\mathbf{W} + (\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)}\\
&= \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{2n_r^2}\Big(\left\|\nabla\mathbf{W} + (\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)} + \left\|\nabla\mathbf{W} - (\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)}\\
&\quad - \left\|\nabla\mathbf{W} - (\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)}\Big)\\
&\geq \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{2n_r^2}\left[2\|\nabla\mathbf{W}\|^2_{L^2(\Omega)} + 2\left\|(\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)} - \left(\|\nabla\mathbf{W}\|^2_{L^2(\Omega)} + \left\|(\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)}\right)\right]\\
&= \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{2n_r^2}\left(\|\nabla\mathbf{W}\|^2_{L^2(\Omega)} + \left\|(\nabla\mathbf{W})^T\right\|^2_{L^2(\Omega)}\right)\\
&\geq \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{n_r^2}\|\nabla\mathbf{W}\|^2_{L^2(\Omega)}\\
&= \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{n_r^2}|\mathbf{W}|^2_{H^1(\Omega)}\\
&= \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{n_r^2}\left|\frac{\mathbf{w}}{\varepsilon^l}\right|^2_{H^1(\Omega)}\\
&\geq \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{n_r^2 K}\left\|\frac{\mathbf{w}}{\varepsilon^l}\right\|^2_{H^1(\Omega)}\\
&\geq \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{n_r^2 K}\|(N-1)\,\mathbf{w}\|^2_{H^1(\Omega)}\\
&= \mu Q_k\|\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\mu\sqrt{|V|}}{n_r^2 K}(N-1)^2\,\|\mathbf{w}\|^2_{H^1(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \min\left\{\mu Q_k,\ \frac{\mu\sqrt{|V|}}{n_r^2 K}(N-1)^2\right\}\left(||\mathbf{w}||^2_{L^2(\Omega)} + ||\mathbf{w}||^2_{H^1(\Omega)}\right)\\
&\geq \min\left\{\mu Q_k,\ \frac{\mu\sqrt{|V|}}{n_r^2 K}(N-1)^2\right\}||\mathbf{w}||^2_{H^1(\Omega)}\\
&= Q_c\,||\mathbf{w}||^2_{H^1(\Omega)},
\end{aligned}
$$

where $Q_c = \min\left\{\mu Q_k,\ \frac{\mu\sqrt{|V|}}{n_r^2 K}(N-1)^2\right\} > 0$. The property [14]

$$
\int_\Omega \left(\mathbf{k}^{-1}\cdot\mathbf{w}\right)\cdot\mathbf{w} \ \geq\ Q_k\,||\mathbf{w}||^2_{L^2(\Omega)} \qquad \text{where } \ Q_k > 0 \tag{4.10}
$$

and inequality (4.8) are used in the first inequality. Parallelogram law [31] $\forall\mathbf{u},\mathbf{w}\in U,\ ||\mathbf{u}+\mathbf{w}||^2 + ||\mathbf{u}-\mathbf{w}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{w}||^2\right)$ and the fact that $||\mathbf{u}-\mathbf{w}||^2 \leq ||\mathbf{u}||^2 + ||\mathbf{w}||^2$ are applied to the second inequality. Poincaré inequality [31] $\exists K > 0$ such that $|\mathbf{w}|_{H^m(\Omega)} \geq \frac{1}{K}||\mathbf{w}||_{H^m(\Omega)},\ \forall\mathbf{w}\in H_0^m(\Omega)$ where $m \geq 0$, is applied to the fourth inequality. For the fifth inequality, we apply Archimedean property [30] $\forall r\in\mathbb{R},\ \exists N\in\mathbb{N}$ such that $N-1 \leq r \leq N$ and use the fact that ε^l is a positive real number. Hence, the proof of the coercivity of the bilinear form $a(\mathbf{w},\mathbf{w})$ is complete.

5. The existence and uniqueness of the generalized Stokes-Brinkman model

In this section, we provide the proof of the existence and uniqueness of the generalized Stokes-Brinkman equations. Before illustrating the proof of the well-posedness, we present the following proposition.

Proposition 5.1. *Let $f\in L^2(\Omega)$ and $\mathbf{s}\in H^{1/2}(\partial\Omega)$. Then, there exist $\mathbf{u_s}\in H^1(\Omega)$ and a unique $\mathbf{u_0}\in V^\perp$ such that*

$$
||\mathbf{u_s}+\mathbf{u_0}||_{H^1(\Omega)} \leq \frac{1}{\beta}||f||_{L^2(\Omega)} + \left(1 + \frac{\sqrt{n}}{\beta}\right)Q_s||\mathbf{s}||_{H^{1/2}(\partial\Omega)}, \tag{5.1}
$$

where n is the dimensional number, and β and Q_s are positive constants.

Proof. Let $f\in L^2(\Omega)$, and $\mathbf{s}\in H^{1/2}(\partial\Omega)$. Therefore, there exist $\mathbf{u_s}\in H^1(\Omega)$, a unique $\mathbf{u_0}\in V^\perp\subset H_0^1(\Omega)$ and $Q_s > 0$ such that $\mathbf{u_s}|_{\partial\Omega} = \mathbf{s}$, $f - \nabla\cdot\mathbf{u_s}\in L_0^2(\Omega)$, $\nabla\cdot\mathbf{u_0} = f - \nabla\cdot\mathbf{u_s}$,

$$
||\mathbf{u_s}||_{H^1(\Omega)} \leq Q_s||\mathbf{s}||_{H^{1/2}(\partial\Omega)}, \tag{5.2}
$$

and there exists $\beta > 0$ such that

$$
||\mathbf{u_0}||_{H^1(\Omega)} \leq \beta^{-1}||f - \nabla\cdot\mathbf{u_s}||_{L^2(\Omega)}. \tag{5.3}
$$

See Lemma A.4 for the details. Next, we show that $\mathbf{u_s}+\mathbf{u_0}$ is bounded, as follows.

$$
\begin{aligned}
||\mathbf{u_s}+\mathbf{u_0}||_{H^1(\Omega)} &\leq ||\mathbf{u_s}||_{H^1(\Omega)} + ||\mathbf{u_0}||_{H^1(\Omega)}\\
&\leq Q_s||\mathbf{s}||_{H^{1/2}(\partial\Omega)} + \frac{1}{\beta}||f - \nabla\cdot\mathbf{u_s}||_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq Q_s\|\mathbf{s}\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\beta}\|f\|_{L^2(\Omega)} + \frac{1}{\beta}\|\nabla\cdot\mathbf{u_s}\|_{L^2(\Omega)} \\
&\leq Q_s\|\mathbf{s}\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\beta}\|f\|_{L^2(\Omega)} + \frac{\sqrt{n}}{\beta}\|\mathbf{u_s}\|_{H^1(\Omega)} \\
&\leq \frac{1}{\beta}\|f\|_{L^2(\Omega)} + \left(1+\frac{\sqrt{n}}{\beta}\right)Q_s\|\mathbf{s}\|_{H^{1/2}(\partial\Omega)},
\end{aligned}
$$

where the fact that [16] $\|\nabla\cdot\phi\|_{L^2(\Omega)} \leq \sqrt{n}|\phi|_{H^1\Omega}$, where n is the dimensional number, is applied to the fourth inequality.

Next, we show the theorem of the well-posedness of the generalized Stokes-Brinkman equations.

Theorem 5.2. *Let $\mathbf{f}\in H^{-1}(\Omega)$, $f\in L^2(\Omega)$ and $\mathbf{s}\in H^{1/2}(\partial\Omega)$. There exist a unique $\mathbf{u}\in H^1_s(\Omega)$ and $p\in L^2_0(\Omega)$ satisfying Problem 1. Moreover,*

$$
\|\mathbf{u}\|_{H^1(\Omega)} \leq \frac{1}{Q_c}\|\mathbf{f}\|_{H^{-1}(\Omega)} + \left(1+\frac{Q_a}{Q_c}\right)\|\hat{\mathbf{u}}\|_{H^1(\Omega)}, \tag{5.4}
$$

$$
\|p\|_{L^2(\Omega)} \leq \frac{1}{\beta}\|\mathbf{f}\|_{H^{-1}(\Omega)} + \frac{1}{\beta}Q_a\|\mathbf{u}\|_{H^1(\Omega)}, \tag{5.5}
$$

where $\hat{\mathbf{u}} = \mathbf{u_s}+\mathbf{u_0}$, $\beta>0$, as presented in Proposition 5.1, and Q_a and Q_c are defined in Theorem 4.1 and Theorem 4.3, respectively.

Proof. Let $\mathbf{f}\in H^{-1}(\Omega)$, $f\in L^2(\Omega)$ and $\mathbf{s}\in H^{1/2}(\partial\Omega)$. From Proposition 5.1, we have $\mathbf{u_s}\in H^1(\Omega)$ and $\mathbf{u_0}\in V^{\perp}\subset H^1_0(\Omega)$ such that $\mathbf{u_s}|_{\partial\Omega}=\mathbf{s}$ and $\nabla\cdot\mathbf{u_0} = f-\nabla\cdot\mathbf{u_s}$. Let $\hat{\mathbf{u}} = \mathbf{u_s}+\mathbf{u_0}$ and $L(\mathbf{w}) = c_1(\mathbf{w}) - a(\hat{\mathbf{u}},\mathbf{w})$ for any $\mathbf{w}\in V$. By adopting the linearity and continuity of $c_1(\cdot)$ and bilinearity and continuity of $a(\cdot,\cdot)$ along with the coercivity of $a(\cdot,\cdot)$, we have that $L(\cdot)$ is a linear and continuous function. Then, from the Lax-Milgram theorem, there exists a unique $\tilde{\mathbf{u}}\in V\subset H^1_0(\Omega)$ such that $a(\tilde{\mathbf{u}},\mathbf{w}) = L(\mathbf{w})$. Let $\mathbf{u}=\tilde{\mathbf{u}}+\hat{\mathbf{u}}$. Then,

$$
\begin{aligned}
\mathbf{u}|_{\partial\Omega} &= \tilde{\mathbf{u}}|_{\partial\Omega} + \mathbf{u_s}|_{\partial\Omega} + \mathbf{u_0}|_{\partial\Omega} \\
&= 0+\mathbf{s}+0 \\
&= \mathbf{s}.
\end{aligned}
$$

Therefore, $\mathbf{u}\in H^1_s(\Omega)$. Next, we show the uniqueness of $\mathbf{u}$. Since $a(\tilde{\mathbf{u}},\mathbf{w}) = L(\mathbf{w}) = c_1(\mathbf{w}) - a(\hat{\mathbf{u}},\mathbf{w})$, by using the bilinear property of $a(\cdot,\cdot)$, we have

$$
a(\mathbf{u},\mathbf{w}) = a(\tilde{\mathbf{u}}+\hat{\mathbf{u}},\mathbf{w}) = c_1(\mathbf{w}). \tag{5.6}
$$

Suppose that $\mathbf{u_1}$ and $\mathbf{u_2}$ are two such solutions satisfying

$$
a(\mathbf{u_1},\mathbf{w}) = c_1(\mathbf{w}) \text{ and } a(\mathbf{u_2},\mathbf{w}) = c_1(\mathbf{w}). \tag{5.7}
$$

Then, substituting $\mathbf{w}=\mathbf{u_1}-\mathbf{u_2}$ in Eq (5.7) and subtracting one equation from another equation, we get

$$
0 = a(\mathbf{u_1}-\mathbf{u_2},\mathbf{u_1}-\mathbf{u_2}) \geq Q_c\|\mathbf{u_1}-\mathbf{u_2}\|^2_{H^1(\Omega)} \geq 0,
$$

where we apply the coercivity of $a(\cdot,\cdot)$, Theorem 4.3, at the first inequality. Since the constant $Q_c > 0$, it implies that $||\mathbf{u_1} - \mathbf{u_2}||_{H^1(\Omega)} = 0$. Then, $\mathbf{u_1} = \mathbf{u_2}$. Therefore, $\mathbf{u}$ is unique, satisfying $a(\mathbf{u}, \mathbf{w}) = c_1(\mathbf{w})$, for all $\mathbf{w} \in V$.

Next, we show that $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u_s} + \mathbf{u_0}$ satisfies the continuity equation. Since $\tilde{\mathbf{u}} \in V$, $\nabla \cdot \tilde{\mathbf{u}} = 0$. Then,

$$\begin{aligned}\nabla \cdot \mathbf{u} &= \nabla \cdot (\tilde{\mathbf{u}} + \mathbf{u_s} + \mathbf{u_0}) \\ &= \nabla \cdot \tilde{\mathbf{u}} + \nabla \cdot \mathbf{u_s} + \nabla \cdot \mathbf{u_0} \\ &= 0 + \nabla \cdot \mathbf{u_s} + f - \nabla \cdot \mathbf{u_s} \\ &= f,\end{aligned}$$

where $\nabla \cdot \mathbf{u_0} = f - \nabla \cdot \mathbf{u_s}$ is applied at the third equality. Hence, $\mathbf{u}$ satisfies the continuity equation.

We prove the existence and uniqueness of $p \in L_0^2(\Omega)$. Define L_1 such that $\langle L_1, \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle = c_1(\mathbf{w})$. Then, from Eq (5.6), we have

$$a(\tilde{\mathbf{u}}, \mathbf{w}) + a(\hat{\mathbf{u}}, \mathbf{w}) - \langle L_1, \mathbf{w}\rangle = 0, \ \forall \mathbf{w} \in V. \tag{5.8}$$

By the definition of the linear operator, seen in Appendix A, we can rewrite Eq (5.8) in operator notation as $A\tilde{\mathbf{u}} + A\hat{\mathbf{u}} - L_1 = 0$. Then, $A\tilde{\mathbf{u}} + A\hat{\mathbf{u}} - L_1 \in V^0$, where V^0 is the polar set of V; see Appendix A for the definition. Given that $B' : L_0^2(\Omega) \to V^0$ is an isomorphism **grad** operator, from Theorem A.2 and the property of isomorphism, there exists a unique $p \in L_0^2(\Omega)$ such that $B'p = A\tilde{\mathbf{u}} + A\hat{\mathbf{u}} - L_1 = A\mathbf{u} - L_1$. Then,

$$A\mathbf{u} + B'p = L_1 = \mathbf{f}.$$

Therefore, there exist a unique $\mathbf{u} \in H_s^1(\Omega)$ and $p \in L_0^2(\Omega)$ satisfying Problems 1 and 2.

Next, we show the variables $\mathbf{u}$ and p are bounded. We first illustrate that $\tilde{\mathbf{u}}$ ia bounded. Employing Eq (4.9), we have

$$\begin{aligned}Q_c||\tilde{\mathbf{u}}||^2_{H^1(\Omega)} &\leq a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \\ &= c_1(\tilde{\mathbf{u}}) - a(\hat{\mathbf{u}}, \tilde{\mathbf{u}}) \\ &\leq ||\mathbf{f}||_{H^{-1}(\Omega)}||\tilde{\mathbf{u}}||_{H^1(\Omega)} + Q_a||\hat{\mathbf{u}}||_{H^1(\Omega)}||\tilde{\mathbf{u}}||_{H^1(\Omega)},\end{aligned} \tag{5.9}$$

where the second inequality is obtained from Theorem 4.1. Dividing both sides of Eq (5.9) by $||\tilde{\mathbf{u}}||_{H^1(\Omega)}$ and Q_c, we get

$$||\tilde{\mathbf{u}}||_{H^1(\Omega)} \leq \frac{1}{Q_c}||\mathbf{f}||_{H^{-1}(\Omega)} + \frac{Q_a}{Q_c}||\hat{\mathbf{u}}||_{H^1(\Omega)}. \tag{5.10}$$

Thus,

$$\begin{aligned}||\mathbf{u}||_{H^1(\Omega)} &= ||\tilde{\mathbf{u}} + \hat{\mathbf{u}}||_{H^1(\Omega)} \\ &\leq ||\tilde{\mathbf{u}}||_{H^1(\Omega)} + ||\hat{\mathbf{u}}||_{H^1(\Omega)} \\ &\leq \frac{1}{Q_c}||\mathbf{f}||_{H^{-1}(\Omega)} + \left(1 + \frac{Q_a}{Q_c}\right)||\hat{\mathbf{u}}||_{H^1(\Omega)}.\end{aligned}$$

To show that p is bounded, we first employ Eq (3.5):

$$b(\mathbf{w}, p) = c_1(\mathbf{w}) - a(\mathbf{u}, \mathbf{w})$$

$$\leq \quad \|\mathbf{f}\|_{H^{-1}(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)} + Q_a\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)}, \tag{5.11}$$

where inequalities (4.1) and (4.4) are applied to the inequality. Since

$$\sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{b(\mathbf{w}, p)}{\|\mathbf{w}\|_{H^1(\Omega)}\|p\|_{L^2(\Omega)}}$$

is independent of p, from Eq (A.11),

$$\|p\|_{L^2(\Omega)}^{-1} \sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{b(\mathbf{w}, p)}{\|\mathbf{w}\|_{H^1(\Omega)}} = \sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{b(\mathbf{w}, p)}{\|\mathbf{w}\|_{H^1(\Omega)}\|p\|_{L^2(\Omega)}} \geq \inf_{p\in L_0^2(\Omega)} \sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{b(\mathbf{w}, p)}{\|\mathbf{w}\|_{H^1(\Omega)}\|p\|_{L^2(\Omega)}} \geq \beta > 0. \tag{5.12}$$

Rearranging Eq (5.12), we obtain that

$$\begin{aligned}
\|p\|_{L^2(\Omega)} &\leq \frac{1}{\beta} \sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{b(\mathbf{w}, p)}{\|\mathbf{w}\|_{H^1(\Omega)}} \\
&= \frac{1}{\beta} \sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{c_1(\mathbf{w}) - a(\mathbf{u}, \mathbf{w})}{\|\mathbf{w}\|_{H^1(\Omega)}} \\
&\leq \frac{1}{\beta} \sup_{\mathbf{w}\in H_0^1(\Omega)} \left(\frac{\|\mathbf{f}\|_{H^{-1}(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)} + Q_a\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{w}\|_{H^1(\Omega)}}{\|\mathbf{w}\|_{H^1(\Omega)}} \right) \\
&= \frac{1}{\beta} \sup_{\mathbf{w}\in H_0^1(\Omega)} \left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + Q_a\|\mathbf{u}\|_{H^1(\Omega)} \right).
\end{aligned}$$

Since $\|\mathbf{f}\|_{H^{-1}(\Omega)} + Q_a\|\mathbf{u}\|_{H^1(\Omega)}$ is independent of $\mathbf{w} \in H_0^1(\Omega)$,

$$\sup_{\mathbf{w}\in H_0^1(\Omega)} \left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + Q_a\|\mathbf{u}\|_{H^1(\Omega)} \right) = \|\mathbf{f}\|_{H^{-1}(\Omega)} + Q_a\|\mathbf{u}\|_{H^1(\Omega)}.$$

Therefore,

$$\|p\|_{L^2(\Omega)} \leq \frac{1}{\beta} \left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + Q_a\|\mathbf{u}\|_{H^1(\Omega)} \right).$$

Hence, the proof of existence and uniqueness of the generalized Stokes-Brinkman equations is complete.

6. Conclusions

In this research, we focus on the fluid movement in a porous medium induced by self-propelled solid phases and the adjacent free-fluid region. In the porous medium, we employ a macroscale equation derived from an upscaling technique called Hybrid Mixture Theory (HMT) because we consider a bundle of solid phases. Then, we apply a non-dimensionalization scheme to the macroscale equation to obtain the Brinkman equation. Our model is more general than the Brinkman equation in the literature because the porosity in our equation is considered as a function since the beginning of the derivation. Then, our model in the porous medium conserves the reality of the problem. In the free-fluid region, we start with the generalized Navier-Stokes equation and use the non-dimensionalization method to derive the generalized Stokes equation. Our Brinkman and generalized Stokes equations can be matched at the free-fluid/porous-medium interface. We name them as the generalized Stokes-Brinkman equations. We also show that the discretized form of the mathematical model is a well-posed system with the use of a mixed finite element method. The weak formulation of the generalized Stokes-Brinkman equations is rewritten in linear and bilinear functional structures. We show that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive. Then, we present the well-posedness of the generalized Stokes-Brinkman equations in the last theorem. The system of equations can be useful for more complexity of a real problem in a macroscopic scale than a typical Stokes-Brinkman equation and can be applied to the fluid flow through the regions where fluid is driven by the moving solid phases such as hairlike structures and animal hair. The numerical research will be presented in the next study.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix

A. Fundamental definition and theorems

In this Appendix, we provide notations, definitions, theorems and lemmas used in the proof of the well-posedness of the generalized Stokes-Brinkman equations. They are all presented and proved in [14, 32–34].

Notations and spaces

In this section, we introduce some notations and spaces used in the proof of the well-posedness of the governing equations [14]. Define

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q d\Omega = 0\}, \tag{A.1}$$

$$H_0^1(\Omega) = \{\mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\partial\Omega} = 0\}, \tag{A.2}$$

$$H_s^1(\Omega) = \{\mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\partial\Omega} = \mathbf{s}\}, \tag{A.3}$$

$$H^{-1}(\Omega) = (H_0^1(\Omega))', \text{the dual of } H_0^1(\Omega), \tag{A.4}$$

$$V = \{\mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\partial\Omega} = 0 \text{ and } \nabla \cdot \mathbf{w} = 0\}, \tag{A.5}$$

$$V^\perp = \{\mathbf{w}^\perp \in H_0^1(\Omega) : \int_\Omega \mathbf{w}^\perp \cdot \mathbf{w} = 0 \ \ \forall\, \mathbf{w} \ \in V\}, \tag{A.6}$$

$$V^0 = \{\mathbf{w}' \in H^{-1}(\Omega) : \langle \mathbf{w}', \mathbf{w} \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0 \ \ \forall \mathbf{w} \in V\}, \tag{A.7}$$

where $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Notice that for n dimension, $\mathbf{w} \in H^1(\Omega)^n$ and $\nabla\mathbf{w} \in H^1(\Omega)^{n\times n}$. In any case, for the sake of simplicity, we write $\mathbf{w} \in H^1(\Omega)$, and the implication comes from the context of the surrounding sentences.

Next, we provide the fundamental definition, theorems and lemma used in the proof of the existence and uniqueness of Problems 1 and 2 [12, 14, 31–35].

Definition A.1. *Let $\mathbf{u}, \mathbf{w} \in H^1(\Omega)$ and $q \in L_0^2(\Omega)$. Define linear operators $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $B : H_0^1(\Omega) \to L_0^2(\Omega)$ by*

$$\langle A\mathbf{u}, \mathbf{w} \rangle_{H_0^1(\Omega)\times H^{-1}(\Omega)} := a(\mathbf{u}, \mathbf{w}), \quad \forall \mathbf{u}, \mathbf{w} \in H_0^1(\Omega), \tag{A.8}$$

$$\langle B\mathbf{u}, q \rangle_{H_0^1(\Omega)\times L_0^2(\Omega)} := b(\mathbf{u}, q), \quad \forall \mathbf{u} \in H_0^1(\Omega), \forall q \in L_0^2(\Omega). \tag{A.9}$$

Let $B' \in \mathcal{L}\left(L_0^2(\Omega); H^{-1}(\Omega)\right)$ be the dual operator of B. Then,

$$\langle B'q, \mathbf{u} \rangle = \langle q, B\mathbf{u} \rangle := b(\mathbf{u}, q), \quad \forall \mathbf{u} \in H_0^1(\Omega), \forall q \in L_0^2(\Omega), \tag{A.10}$$

where the dual spaces of $L_0^2(\Omega) = \left(L_0^2(\Omega)\right)'$ and the dual spaces of $H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$.

Theorem A.2. *Let Ω be connected. Then,*

1. *the operator **grad** is an isomorphism of $L_0^2(\Omega)$ onto V^0,*
2. *the operator **div** is an isomorphism of $V^\perp$ onto $L_0^2(\Omega)$.*

Therefore, there exists $\beta > 0$ such that

$$\inf_{q\in L_0^2(\Omega)} \sup_{\mathbf{w}\in H_0^1(\Omega)} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}} \geq \beta > 0 \tag{A.11}$$

and for any $q \in L_0^2(\Omega)$, there exists a unique $\mathbf{u} \in V^\perp \subset H_0^1(\Omega)$ satisfying

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq \beta^{-1}\|q\|_{L^2(\Omega)}. \tag{A.12}$$

Theorem A.3. *There exist positive constants Q_t and Q_s such that, for each $\mathbf{v} \in H^1(\Omega)$, its trace on $\partial\Omega$ belongs to $H^{1/2}(\partial\Omega)$, and*

$$\|\mathbf{v}\|_{H^{1/2}(\partial\Omega)} \leq Q_t\|\mathbf{v}\|_{H^1(\Omega)}. \tag{A.13}$$

Conversely, for each given function $\mathbf{s} \in H^{1/2}(\partial\Omega)$, there exists $\mathbf{u_s} \in H^1(\Omega)$ such that its trace on $\partial\Omega$ coincides with $\mathbf{s}$ and

$$\|\mathbf{u_s}\|_{H^1(\Omega)} \leq Q_s\|\mathbf{s}\|_{H^{1/2}(\partial\Omega)}. \tag{A.14}$$

Lemma A.4. *Suppose that* $\mathbf{f} \in H^{-1}(\Omega)$, $f \in L^2(\Omega)$ *and* $\mathbf{s} \in H^{1/2}(\partial\Omega)$. *Then, there exist* $\mathbf{u_s} \in H^1(\Omega)$, $Q_s > 0$ *and* $\beta > 0$ *such that*

$$\mathbf{u_s}|_{\partial\Omega} = \mathbf{s} \quad \textit{and} \quad \|\mathbf{u_s}\|_{H^1(\Omega)} \leq Q_s \|\mathbf{s}\|_{H^{1/2}(\partial\Omega)}, \tag{A.15}$$

$$\exists!\mathbf{u_0} \in V^\perp \subset H_0^1(\Omega) \quad \textit{satisfying} \quad \nabla \cdot \mathbf{u_0} = f - \nabla \cdot \mathbf{u_s}, \tag{A.16}$$

and

$$\|\mathbf{u_0}\|_{H^1(\Omega)} \leq \beta^{-1} \|f - \nabla \cdot \mathbf{u_s}\|_{L^2(\Omega)}. \tag{A.17}$$