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ERA, 31(3): 1613–1624.
DOI: 10.3934/era.2023083
Received: 02 August 2022
Revised: 04 January 2023
Accepted: 08 January 2023
Published: 31 January 2023

Research article

Relative cluster tilting subcategories in an extriangulated category

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Abstract: Let $\mathscr{B}$ be an extriangulated category which admits a cluster tilting subcategory $\mathcal{T}$. We firstly introduce notions of $\mathcal{T}$-cluster tilting subcategories and related subcategories. Then we prove there is a correspondence between $\mathcal{T}$-cluster tilting subcategories of $\mathscr{B}$ and support τ-tilting pairs of $mod\underline{\Omega(\mathcal{T})}$, which recovers several main results from the literature. Note that the generalization is nontrivial and we give a new proof technique.

Keywords: extriangulated category; relative rigid subcategory; relative cluster tilting subcategory; support τ-tilting pair

1. Introduction

In [1] (see [2] for type A), the authors introduced cluster categories which were associated to finite dimensional hereditary algebras. It is well known that cluster-tilting theory gives a way to construct abelian categories from some triangulated and exact categories.

Recently, Nakaoka and Palu introduced extriangulated categories in [3], which are a simultaneous generalization of exact categories and triangulated categories, see also [4–6]. Subcategories of an extriangulated category which are closed under extension are also extriangulated categories. However, there exist some other examples of extriangulated categories which are neither exact nor triangulated, see [6–8].

When $\mathcal{T}$ is a cluster tilting subcategory, the authors Yang, Zhou and Zhu [9, Definition 3.1] introduced the notions of $\mathcal{T}[1]$-cluster tilting subcategories (also called ghost cluster tilting subcategories) and weak $\mathcal{T}[1]$-cluster tilting subcategories in a triangulated category $\mathscr{C}$, which are generalizations of cluster tilting subcategories. In these works, the authors investigated the relationship between $\mathscr{C}$ and $mod\mathcal{T}$ via the restricted Yoneda functor $\mathbb{G}$ more closely. More precisely, they gave a bijection between the class of $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ and the class of support τ-tilting pairs of $mod\mathcal{T}$, see [9, Theorems 4.3 and 4.4].

Inspired by Yang, Zhou and Zhu [9] and Liu and Zhou [10], we introduce the notion of relative cluster tilting subcategories in an extriangulated category $\mathscr{B}$. More importantly, we want to investigate the relationship between relative cluster tilting subcategories and some important subcategories of $mod\underline{\Omega(\mathcal{T})}$ (see Theorem 3.9 and Corollary 3.10), which generalizes and improves the work by Yang, Zhou and Zhu [9] and Liu and Zhou [10].

It is worth noting that the proof idea of our main results in this manuscript is similar to that in [9, Theorems 4.3 and 4.4], however, the generalization is nontrivial and we give a new proof technique.

2. Preliminaries

Throughout the paper, let $\mathscr{B}$ denote an additive category. The subcategories considered are full additive subcategories which are closed under isomorphisms. Let $[\mathscr{X}](A, B)$ denote the subgroup of $Hom_{\mathscr{B}}(A, B)$ consisting of morphisms which factor through objects in a subcategory $\mathscr{X}$. The quotient category $\mathscr{B}/[\mathscr{X}]$ of $\mathscr{B}$ by a subcategory $\mathscr{X}$ is the category with the same objects as $\mathscr{B}$ and the space of morphisms from A to B is the quotient of group of morphisms from A to B in $\mathscr{B}$ by the subgroup consisting of morphisms factor through objects in $\mathscr{X}$. We use Ab to denote the category of abelian groups.

In the following, we recall the definition and some properties of extriangulated categories from [4], [11] and [3].

Suppose there exists a biadditive functor $\mathbb{E} : \mathscr{B}^{op} \times \mathscr{B} \to Ab$. Let $A, C \in \mathscr{B}$ be two objects, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Zero element in $\mathbb{E}(C, A)$ is called the split $\mathbb{E}$-extension.

Let $\mathfrak{s}$ be a correspondence, which associates any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ to an equivalence class $\mathfrak{s}(\delta) = [A \overset{x}{\to} B \overset{y}{\to} C]$. Moreover, if $\mathfrak{s}$ satisfies the conditions in [3, Definition 2.9], we call it a *realization* of $\mathbb{E}$.

Definition 2.1. *[3, Definition 2.12]* A triplet $(\mathscr{B}, \mathbb{E}, \mathfrak{s})$ is called an externally triangulated category, or for short, extriangulated category if

(ET1) $\mathbb{E} : \mathscr{B}^{op} \times \mathscr{B} \to Ab$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) For a pair of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, realized as $\mathfrak{s}(\delta) = [A \overset{x}{\to} B \overset{y}{\to} C]$ and $\mathfrak{s}(\delta') = [A' \overset{x'}{\to} B' \overset{y'}{\to} C']$. If there exists a commutative square,

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

then there exists a morphism $c : C \to C'$ which makes the above diagram commutative.

(ET3)^{op} Dual of $(ET3)$.

(ET4) Let δ and δ' be two $\mathbb{E}$-extensions realized by $A \overset{f}{\to} B \overset{f'}{\to} D$ and $B \overset{g}{\to} C \overset{g'}{\to} F$, respectively. Then there exist an object $E \in \mathscr{B}$, and a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

and an $\mathbb{E}$-extension δ'' realized by $A \overset{h}{\to} C \overset{h'}{\to} E$, which satisfy the following compatibilities:
(i). $D \overset{d}{\to} E \overset{e}{\to} F$ realizes $\mathbb{E}(F, f')(\delta')$,
(ii). $\mathbb{E}(d, A)(\delta'') = \delta$,
(iii). $\mathbb{E}(E, f)(\delta'') = \mathbb{E}(e, B)(\delta')$.

$(ET4^{op})$ Dual of $(ET4)$.

Let $\mathscr{B}$ be an extriangulated category, we recall some notations from [3, 6].

- We call a sequence $X \overset{x}{\to} Y \overset{y}{\to} Z$ a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(Z, X)$, where the morphism x is called an *inflation*, the morphism y is called an *deflation* and $X \overset{x}{\to} Y \overset{y}{\to} Z \overset{\delta}{\dashrightarrow}$ is called an $\mathbb{E}$-*triangle*.
- When $X \overset{x}{\to} Y \overset{y}{\to} Z \overset{\delta}{\dashrightarrow}$ is an $\mathbb{E}$-triangle, X is called the *CoCone* of the deflation y, and denote it by $\mathrm{CoCone}(y)$; C is called the *Cone* of the inflation x, and denote it by $\mathrm{Cone}(x)$.

Remark 2.2. 1) Both inflations and deflations are closed under composition.
2) We call a subcategory $\mathscr{T}$ extension-closed if for any $\mathbb{E}$-triangle $X \overset{x}{\to} Y \overset{y}{\to} Z \overset{\delta}{\dashrightarrow}$ with $X,\ Z \in \mathscr{T}$, then $Y \in \mathscr{T}$.

Denote $\mathcal{I}$ by the subcategory of all injective objects of $\mathscr{B}$ and $\mathcal{P}$ by the subcategory of all projective objects.

In an extriangulated category having enough projectives and injectives, Liu and Nakaoka [4] defined the higher extension groups as

$$\mathbb{E}^{i+1}(X, Y) = \mathbb{E}(\Omega^i(X), Y) = \mathbb{E}(X, \Sigma^i(Y)) \text{ for } i \geq 0.$$

By [3, Corollary 3.5], there exists a useful lemma.

Lemma 2.3. *For a pair of $\mathbb{E}$-triangles $L \overset{l}{\to} M \overset{m}{\to} N \dashrightarrow$ and $D \overset{d}{\to} E \overset{e}{\to} F \dashrightarrow$. If there is a commutative diagram*

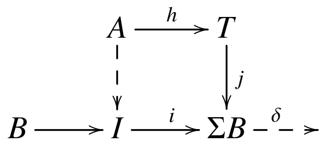
$$\begin{array}{ccccccc} L & \xrightarrow{l} & M & \xrightarrow{m} & N & \dashrightarrow & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ D & \xrightarrow{d} & E & \xrightarrow{e} & F & \dashrightarrow & \end{array}$$

f factors through l if and only if h factors through e.

3. Results

In this section, $\mathscr{B}$ is always an extriangulated category and $\mathcal{T}$ is always a cluster tilting subcategory [6, Definition 2.10].

Let $A,\ B \in \mathscr{B}$ be two objects, denote by $[\overline{\mathcal{T}}](A, \Sigma B)$ the subset of $\mathscr{B}(A, \Sigma B)$ such that $f \in [\overline{\mathcal{T}}](A, \Sigma B)$ if we have $f:\ A \to T \to \Sigma B$ where $T \in \mathcal{T}$ and the following commutative diagram



where I is an injective object of $\mathscr{B}$ [10, Definition 3.2].

Let $\mathscr{M}$ and $\mathscr{N}$ be two subcategories of $\mathscr{B}$. The notation $[\overline{\mathcal{T}}](\mathscr{M}, \Sigma(\mathscr{N})) = [\mathcal{T}](\mathscr{M}, \Sigma(\mathscr{N}))$ will mean that $[\overline{\mathcal{T}}](M, \Sigma N) = [\mathcal{T}](M, \Sigma N)$ for every object $M \in \mathscr{M}$ and $N \in \mathscr{N}$.

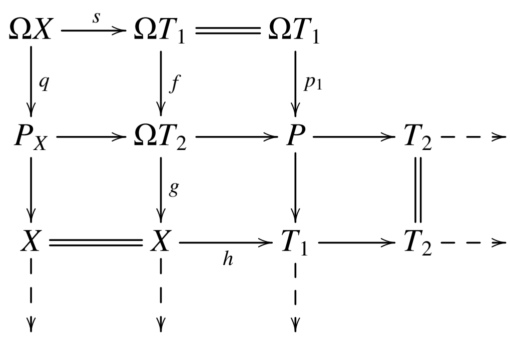
Now, we give the definition of $\mathcal{T}$-cluster tilting subcategories.

Definition 3.1. Let $\mathcal{X}$ be a subcategory of $\mathscr{B}$.

1) [11, Definition 2.14] $\mathcal{X}$ is called $\mathcal{T}$-rigid if $[\overline{\mathcal{T}}](\mathcal{X}, \Sigma\mathcal{X}) = [\mathcal{T}](\mathcal{X}, \Sigma\mathcal{X})$;
2) $\mathcal{X}$ is called $\mathcal{T}$-cluster tilting if $\mathcal{X}$ is strongly functorially finite in $\mathscr{B}$ and $\mathcal{X} = \{M \in \mathscr{C} \mid [\overline{\mathcal{T}}](\mathcal{X}, \Sigma M) = [\mathcal{T}](\mathcal{X}, \Sigma M) \text{ and } [\overline{\mathcal{T}}](M, \Sigma\mathcal{X}) = [\mathcal{T}](M, \Sigma\mathcal{X})\}$.

Remark 3.2. 1) Rigid subcategories are always $\mathcal{T}$-rigid by [6, Definition 2.10];
2) $\mathcal{T}$-cluster tilting subcategories are always $\mathcal{T}$-rigid;
3) $\mathcal{T}$-cluster tilting subcategories always contain the class of projective objects $\mathcal{P}$ and injective objects $\mathcal{I}$.

Remark 3.3. Since $\mathcal{T}$ is a cluster tilting subcategory, $\forall X \in \mathscr{B}$, there exists a commutative diagram by [6, Remark 2.11] and Definition 2.1($(ET4)^{op}$), where $T_1,\ T_2 \in \mathcal{T}$ and h is a left $\mathcal{T}$-approximation of X:



Hence $\forall X \in \mathscr{B}$, there always exists an $\mathbb{E}$-triangle

$$\Omega(T_1) \overset{f_X}{\to} \Omega(T_2) \to X \dashrightarrow \text{ with } T_i \in \mathcal{T}.$$

By Remark 3.2(3), $\mathcal{P} \subseteq \mathcal{T}$ and $\mathscr{B} = CoCone(\mathcal{T}, \mathcal{T})$ by [6, Remark 2.11(1),(2)]. Following from [4, Theorem 3.2], $\underline{\mathscr{B}} = \mathscr{B}/\mathcal{T}$ is an abelian category. $\forall f \in \mathscr{B}(A, C)$, denote by $\underline{f}$ the image of f under the natural quotient functor $\mathscr{B} \to \underline{\mathscr{B}}$.

Let $\Omega(\mathcal{T})$ =CoCone$(\mathcal{P},\mathcal{T})$, then $\underline{\Omega(\mathcal{T})}$ is the subcategory consisting of projective objects of $\underline{\mathscr{B}}$ by [4, Theorem 4.10]. Moreover, $mod\underline{\Omega(\mathcal{T})}$ denotes the category of coherent functors over the category of $\underline{\Omega(\mathcal{T})}$ by [4, Fact 4.13].

Let $\mathbb{G}: \mathscr{B} \to mod\underline{\Omega(\mathcal{T})}$, $M \mapsto Hom_{\underline{\mathscr{B}}}(-, M) \mid_{\underline{\Omega(\mathcal{T})}}$ be the restricted Yoneda functor. Then $\mathbb{G}$ is homological, i.e., any $\mathbb{E}$-triangle $X \to Y \to Z \dashrightarrow$ in $\mathscr{B}$ yields an exact sequence $\mathbb{G}(X) \to \mathbb{G}(Y) \to \mathbb{G}(Z)$ in $mod\underline{\Omega(\mathcal{T})}$. Similar to [9, Theorem 2.8], we obtain a lemma:

Lemma 3.4. *Denote $proj(mod\underline{\Omega(\mathcal{T})})$ the subcategory of projective objects in $mod\underline{\Omega(\mathcal{T})}$. Then*

1) *$\mathbb{G}$ induces an equivalence $\underline{\Omega(\mathcal{T})} \xrightarrow{\sim} proj(mod\underline{\Omega(\mathcal{T})})$.*
2) *For $N \in mod\underline{\Omega(\mathcal{T})}$, there exists a natural isomorphism*
$$Hom_{mod\underline{\Omega(\mathcal{T})}}(\mathbb{G}(\Omega(\mathcal{T})), N) \simeq N(\Omega(\mathcal{T})).$$

In the following, we investigate the relationship between $\mathscr{B}$ and $mod\underline{\Omega(\mathcal{T})}$ via $\mathbb{G}$ more closely.

Lemma 3.5. *Let $\mathscr{X}$ be any subcategory of $\mathscr{B}$. Then*

1) *any object $X \in \mathscr{X}$, there is a projective presentation in mod $\underline{\Omega(\mathcal{T})}$*
$$P_1^{\mathbb{G}(X)} \xrightarrow{\pi^{\mathbb{G}(X)}} P_0^{\mathbb{G}(X)} \to \mathbb{G}(X) \to 0.$$
2) *$\mathscr{X}$ is a $\mathcal{T}$-rigid subcategory if and only if the class $\{\pi^{\mathbb{G}(X)} \mid X \in \mathscr{X}\}$ has property $((S)$ [9, Definition 2.7(1)]).*

Proof. 1). By Remark 3.3, there exists an $\mathbb{E}$-triangle:
$$\Omega(T_1) \xrightarrow{f_X} \Omega(T_0) \to X \dashrightarrow$$
When we apply the functor $\mathbb{G}$ to it ,there exists an exact sequence $\mathbb{G}(\Omega(T_1)) \to \mathbb{G}(\Omega(T_0)) \to \mathbb{G}(X) \to 0$. By Lemma 3.4(1), $\mathbb{G}(\Omega(T_i))$ is projective in mod $\underline{\Omega(\mathcal{T})}$. So the above exact sequence is the desired projective presentation.

2). For any $X_0 \in \mathscr{X}$, using the similar proof to [9, Lemma 4.1], we get the following commutative diagram

$$\begin{array}{ccc} Hom_{mod\underline{\Omega(\mathcal{T})}}(\mathbb{G}(\Omega(T_0)), \mathbb{G}(X_0)) & \xrightarrow{\alpha} & Hom_{mod\underline{\Omega(\mathcal{T})}}(\mathbb{G}(\Omega(T_1)), \mathbb{G}(X_0)) \\ \downarrow & & \downarrow \\ Hom_{\underline{\mathscr{B}}}(\Omega(T_0), X_0) & \xrightarrow{Hom_{\underline{\mathscr{B}}}(f_X, X_0)} & Hom_{\underline{\mathscr{B}}}(\Omega(T_1), X_0) \end{array}$$

where $\alpha = Hom_{mod\underline{\Omega\mathcal{T}}}(\pi^{\mathbb{G}(X)}, \mathbb{G}(X_0))$. By Lemma 3.4(2), both the left and right vertical maps are isomorphisms. Hence the set $\{\pi^{\mathbb{G}(X)} \mid X \in \mathscr{X}\}$ has property $((S)$ iff α is epic iff $Hom_{\underline{\mathscr{B}}}(f_X, X_0)$ is epic iff $\mathscr{X}$ is a $\mathcal{T}$-rigid subcategory by [10, Lemma 3.6].

Lemma 3.6. *Let $\mathscr{X}$ be a $\mathcal{T}$-rigid subcategory and $\mathcal{T}_1$ a subcategory of $\mathcal{T}$. Then $\mathscr{X} \vee \mathcal{T}_1$ is a $\mathcal{T}$-rigid subcategory iff $\mathbb{E}(\mathcal{T}_1, \mathscr{X}) = 0$.*

Proof. For any $M \in \mathscr{X} \vee \mathcal{T}_1$, then $M = X \oplus T_1$ for $X \in \mathscr{X}$ and $T_1 \in \mathcal{T}_1$. Let $h: X \to T$ be a left $\mathcal{T}$-approximation of X and $y: T_1 \to \Sigma(X')$ for $X' \in \mathscr{X}$ any morphism. Then there exists the following commutative diagram

$$\begin{array}{ccccc}
 & & X\oplus T_1 & \xrightarrow{\ f\ } & T\oplus T_1 \\
 & & & & \Big\downarrow{\scriptstyle (0\ 1)} \\
\Omega(T_1) & \longrightarrow & P_1 & \longrightarrow & T_1 \dashrightarrow \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle y} \\
X' & \longrightarrow & I & \xrightarrow{\ i_0\ } & \Sigma(X') \dashrightarrow \\
\Big\downarrow{\scriptstyle \binom{1}{0}} & & \Big\downarrow{\scriptstyle \binom{1}{0}} & & \Big\downarrow{\scriptstyle \binom{1}{0}} \\
X'\oplus T_1' & \longrightarrow & I\oplus I_T & \xrightarrow{\ \beta\ } & \Sigma(X')\oplus\Sigma(T_1') \dashrightarrow
\end{array}$$

with $P_1 \in \mathcal{P}$, $f = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} i_0 & 0 \\ 0 & i_1 \end{pmatrix}$.

When $\mathscr{X} \vee \mathcal{T}_1$ is a $\mathcal{T}$-rigid subcategory, we can get a morphism $g : X \oplus T_1 \to \Sigma(X') \oplus \Sigma(T_1')$ such that $\beta g = \binom{1}{0} y (0\ 1) f$. i.e., $\exists b : T_1 \to I$ such that $y = i_0 b$. So $\mathbb{E}(T_1, X') = 0$ and then $\mathbb{E}(\mathcal{T}_1, \mathscr{X}) = 0$.

Let $\gamma = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$: $T \oplus T_1 \to \Sigma(X') \oplus \Sigma(T_1')$ be a morphism. As $\mathscr{X}$ is $\mathcal{T}$-rigid, $r_{11}h : X \to \Sigma(X')$ factors through i_0. Since $\mathbb{E}(\mathcal{T}, \mathscr{X}) = 0$, $r_{12} : T_1 \to \Sigma(X')$ factors through i_0. As $\mathcal{T}$ is rigid, the morphism $r_{21}h : X \to T \to \Sigma(T_1')$ factors through i_1, and the morphism $r_{22} : T_1 \to \Sigma(T_1')$ factors through i_1. So the morphism γf can factor through $\beta = \begin{pmatrix} i_0 & 0 \\ 0 & i_1 \end{pmatrix}$. Therefore $\mathscr{X} \vee \mathcal{T}_1$ is an $\mathcal{T}$-rigid subcategory.

For the definition of τ-rigid pair in an additive category, we refer the readers to see [9, Definition 2.7].

Lemma 3.7. *Let $\mathscr{U}$ be a class of $\mathcal{T}$-rigid subcategories and $\mathscr{V}$ a class of τ-rigid pairs of mod$\underline{\Omega(\mathcal{T})}$. Then there exists a bijection $\varphi : \mathscr{U} \to \mathscr{V}$, given by : $\mathscr{X} \mapsto (\mathbb{G}(\mathscr{X}), \Omega(\mathcal{T}) \cap \Omega(\mathscr{X}))$.*

Proof. Let $\mathscr{X}$ be $\mathcal{T}$-rigid. By Lemma 3.5, $\mathbb{G}(\mathscr{X})$ is a τ-rigid subcategory of mod $\underline{\Omega(\mathcal{T})}$.

Let $Y \in \Omega(\mathcal{T}) \cap \Omega(\mathscr{X})$, then there exists $X_0 \in \mathscr{X}$ such that $Y = \Omega(X_0)$. Consider the $\mathbb{E}$-triangle $\Omega(X_0) \to P \to X_0 \dashrightarrow$ with $P \in \mathcal{P}$. $\forall X \in \mathscr{X}$, applying $Hom_{\mathscr{B}}(-, X)$ yields an exact sequence $Hom_{\mathscr{B}}(P, X) \to Hom_{\mathscr{B}}(\Omega(X_0), X) \to \mathbb{E}(X_0, X) \to 0$. Hence in $\underline{\mathscr{B}} = \mathscr{B}/\mathcal{T}$, $Hom_{\underline{\mathscr{B}}}(\Omega(X_0), X) \cong \mathbb{E}(X_0, X)$.

By Remark 3.3, for X_0, there is an $\mathbb{E}$-triangle $\Omega(T_1) \to \Omega(T_2) \to X_0 \dashrightarrow$ with $T_1,\ T_2 \in \mathcal{T}$. Applying $Hom_{\underline{\mathscr{B}}}(-, X)$, we obtain an exact sequence $Hom_{\underline{\mathscr{B}}}(\Omega(T_2), X) \to Hom_{\underline{\mathscr{B}}}(\Omega(T_1), X) \to \mathbb{E}(X_0, X) \to \mathbb{E}(\Omega(T_2), X)$. By [10, Lemma 3.6], $Hom_{\underline{\mathscr{B}}}(\Omega(T_2), X) \to Hom_{\underline{\mathscr{B}}}(\Omega(T_1), X)$ is epic. Moreover, $\underline{\Omega(T_2)}$ is projective in $\underline{\mathscr{B}}$ by [4, Proposition 4.8]. So $\mathbb{E}(\Omega(T_2), X) = 0$. Thus $\mathbb{E}(X_0, X) = 0$. Hence $\forall X \in \mathscr{X}$,

$$\mathbb{G}(X)(Y) = Hom_{\underline{\mathscr{B}}}(\Omega(X_0), X) = 0.$$

So $(\mathbb{G}(\mathscr{X}), \Omega(\mathcal{T}) \cap \Omega(\mathscr{X}))$ is a τ-rigid pairs of *mod*$\underline{\Omega(\mathcal{T})}$.

We will show φ is a surjective map.

Let $(\mathscr{N}, \sigma)$ be a τ-rigid pair of mod$\underline{\Omega(\mathcal{T})}$. $\forall N \in \mathscr{N}$, consider the projective presentation

$$P_1 \xrightarrow{\pi^N} P_0 \to N \to 0$$

such that the class $\{\pi^N | N \in \mathcal{N}\}$ has Property (S). By Lemma 3.4, there exists a unique morphism $f_N : \Omega(T_1) \to \Omega(T_0)$ in $\underline{\Omega(\mathcal{T})}$ satisfying $\mathbb{G}(f_N) = \pi^N$ and $\mathbb{G}(Cone(f_N)) \cong N$. Following from Lemma 3.5, $\mathscr{X}_1 := \{cone(f_N) \mid N \in \mathscr{N}\}$ is a $\mathcal{T}$-rigid subcategory.

Let $\mathscr{X} = \mathscr{X}_1 \vee \mathscr{Y}$, where $\mathscr{Y} = \{T \in \mathcal{T} \mid \Omega(T) \in \sigma\}$. For any $T_0 \in \mathscr{Y}$, there is an $\mathbb{E}$-triangle $\Omega(T_0) \to P \to T_0 \dashrightarrow$ with $P \in \mathcal{P}$. For any $Cone(f_N) \in \mathscr{X}_1$, applying $Hom_{\underline{\mathscr{B}}}(-, Cone(f_N))$, yields an exact sequence $Hom_{\underline{\mathscr{B}}}(\Omega(T_0), Cone(f_N)) \to \mathbb{E}(T_0, Cone(f_N)) \to \mathbb{E}(P, Cone(f_N)) = 0$. Since $(\mathscr{N}, \sigma)$ is a τ-rigid pair, $Hom_{\underline{\mathscr{B}}}(\Omega(T_0), Cone(f_N)) = \mathbb{G}(Cone(f_M))(\Omega(T_0)) = 0$. So $\mathbb{E}(T_0, Cone(f_N)) = 0$. Due to Lemma 3.6, $\mathscr{X} = \mathscr{X}_1 \vee \mathscr{Y}$ is $\mathcal{T}$-rigid. Since $\mathscr{Y} \subseteq \mathcal{T}$, we get $\mathbb{G}(\mathscr{Y}) = Hom_{\underline{\mathscr{B}}}(-, \mathcal{T}) \mid_{\Omega(T)} = 0$ by [4, Lemma 4.7]. So $\mathbb{G}(\mathscr{X}) = \mathbb{G}(\mathscr{X}_1) = \mathscr{N}$.

It is straightforward to check that $\Omega(\mathcal{T}) \cap \Omega(\mathscr{X}_1) = 0$. Let $X \in \Omega(\mathcal{T}) \cap \Omega(\mathscr{X})$, then $X \in \Omega(\mathcal{T})$ and $X \in \Omega(\mathscr{X}) = \Omega(\mathscr{X}_1) \vee \sigma$. So we can assume that $X = \Omega(X_1) \oplus E$, where $E \in \sigma$. Then $\Omega(X_1) \oplus E \in \Omega(\mathcal{T})$. Since $E \in \Omega(\mathcal{T})$, we get $\Omega(X_1) \in \Omega(\mathcal{T}) \cap \Omega(\mathscr{X}_1) = 0$. So $\Omega(\mathcal{T}) \cap \Omega(\mathscr{X}) \subseteq \sigma$. Clearly, $\sigma \subseteq \Omega(\mathcal{T})$. Moreover, $\sigma \subseteq \Omega(\mathscr{X})$. So $\sigma \subseteq \Omega(\mathcal{T}) \cap \Omega(\mathscr{X})$. Hence $\Omega(\mathcal{T}) \cap \Omega(\mathscr{X}) = \sigma$. Therefore φ is surjective.

Lastly, φ is injective by the similar proof method to [9, Proposition 4.2].

Therefore φ is bijective.

Lemma 3.8. *Let $\mathcal{T}$ be a rigid subcategory and $A \xrightarrow{a} B \to C \overset{\delta}{\dashrightarrow}$ an $\mathbb{E}$-triangle satisfying $[\overline{\mathcal{T}}](C, \Sigma(A)) = [\mathcal{T}](C, \Sigma(A))$. If there exist an $\mathbb{E}$-extension $\gamma \in \mathbb{E}(T, A)$ and a morphism $t : C \to T$ with $T \in \mathcal{T}$ such that $t^*\gamma = \delta$, then the $\mathbb{E}$-triangle $A \xrightarrow{a} B \to C \overset{\delta}{\dashrightarrow}$ splits.*

Proof. Applying $Hom_{\mathscr{B}}(T, -)$ to the $\mathbb{E}$-triangle $A \to I \xrightarrow{i} \Sigma(A) \overset{\alpha}{\dashrightarrow}$ with $I \in \mathcal{I}$, yields an exact sequence $Hom_{\mathscr{B}}(T, A) \to \mathbb{E}(T, X) \to \mathbb{E}(T, I) = 0$. So there is a morphism $d \in Hom_{\mathscr{B}}(T, \Sigma(A))$ such that $\gamma = d^*\alpha$. So $\delta = t^*\gamma = t^*d^*\alpha = (dt)^*\alpha$. So we have a diagram which is commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{a} & B & \longrightarrow & C & \overset{\delta}{\dashrightarrow} & \\ \| & & \downarrow & & \downarrow{\scriptstyle dt} & & \\ A & \longrightarrow & I & \xrightarrow{i} & \Sigma(A) & \overset{\alpha}{\dashrightarrow} & \end{array}$$

Since $[\overline{\mathcal{T}}](C, \Sigma(A)) = [\mathcal{T}](C, \Sigma(A))$ and $dt \in [\mathcal{T}](C, \Sigma(A))$, dt can factor through i. So 1_A can factor through a and the result follows.

Now, we will show our main theorem, which explains the relation between $\mathcal{T}$-cluster tilting subcategories and support τ-tilting pairs of $mod\underline{\Omega(\mathcal{T})}$.

The subcategory $\mathscr{X}$ is called a preimage of $\mathscr{Y}$ by $\mathbb{G}$ if $\mathbb{G}(\mathscr{X}) = \mathscr{Y}$.

Theorem 3.9. *There is a correspondence between the class of $\mathcal{T}$-cluster tilting subcategories of $\mathscr{B}$ and the class of support τ-tilting pairs of $mod\underline{\Omega(\mathcal{T})}$ such that the class of preimages of support τ-tilting subcategories is contravariantly finite in $\mathscr{B}$.*

Proof. Let φ be the bijective map, such that $\mathscr{X} \mapsto (\mathbb{G}(\mathscr{X}), \Omega(\mathcal{T} \cap \Omega(\mathscr{X})))$, where $\mathbb{G}$ is the restricted Yoneda functor defined in the argument above Lemma 3.4.

1). The map φ is well-defined.

If $\mathscr{X}$ is $\mathcal{T}$-cluster tilting, then $\mathscr{X}$ is $\mathcal{T}$-rigid. So $\varphi(\mathscr{X})$ is a τ-rigid pair of $mod\underline{\Omega(\mathcal{T})}$ by Lemma 3.7. Therefore $\Omega(\mathcal{T}) \cap \Omega(\mathscr{X}) \subseteq Ker\mathbb{G}(\mathscr{X})$. Assume $\Omega(T_0) \in \Omega(\mathcal{T})$ is an object of $\overline{Ker}\mathbb{G}(\mathscr{X})$. Then

$Hom_{\underline{\mathscr{B}}}(\Omega(T_0), \mathscr{X}) = 0$. Applying $Hom_{\underline{\mathscr{B}}}(-, X)$ with $X \in \mathcal{X}$ to $\Omega(T_0) \to P \to T_0 \dashrightarrow$ with $P \in \mathcal{P}$, yields an exact sequence

$$Hom_{\underline{\mathscr{B}}}(P, X) \to Hom_{\underline{\mathscr{B}}}(\Omega(T), X) \to \mathbb{E}(T_0, X) \to 0.$$

Hence we get $\mathbb{E}(T_0, X) \cong Hom_{\underline{\mathscr{B}}}(\Omega(T_0), X) = 0$.
Applying $Hom_{\mathscr{B}}(T_0, -)$ to $X \to I \to \Sigma(X) \dashrightarrow$, we obtain

$$(3.1) \qquad [\overline{\mathcal{T}}](T_0, \Sigma(\mathscr{X})) = [\mathcal{T}](T_0, \Sigma(\mathscr{X})).$$

For any $ba : X \xrightarrow{a} R \xrightarrow{b} \Sigma(T_0)$ with $R \in \mathcal{T}$, as $\mathcal{T}$ is rigid, we get a commutative diagram:

$$\begin{array}{ccccc}
 & & & & X \\
 & & & & \downarrow{\scriptstyle a} \\
 & & & & T \\
 & & & {\scriptstyle \exists t}\swarrow & \downarrow{\scriptstyle b} \\
T_0 & \longrightarrow & I_R & \longrightarrow & \Sigma(T_0) \dashrightarrow
\end{array}$$

Hence we get $\quad (3.2) \quad [\overline{\mathcal{T}}](\mathscr{X}, \Sigma(T_0)) = [\mathcal{T}](\mathscr{X}, \Sigma(T_0))$.

By the equalities (3.1) and (3.2) and $\mathscr{X}$ being a $\mathcal{T}$-rigid subcategory, we obtain

$$[\overline{\mathcal{T}}](\mathscr{X}, \Sigma(X \oplus T_0)) = [\mathcal{T}](\mathscr{X}, \Sigma(X \oplus T_0)) \text{ and } [\overline{\mathcal{T}}](X \oplus T_0, \Sigma(\mathscr{X})) = [\mathcal{T}](X \oplus T_0, \Sigma(\mathscr{X})).$$

As $\mathscr{X}$ is $\mathcal{T}$-cluster tilting, we get $X \oplus T_0 \in \mathscr{X}$. So $T_0 \in \mathscr{X}$. And thus $\Omega(T_0) \in \Omega(\mathcal{T}) \cap \Omega(\mathscr{X})$. Hence $Ker\mathbb{G}(\mathscr{X}) = \Omega(\mathcal{T}) \cap \Omega(\mathscr{X})$.

Since $\mathscr{X}$ is functorially finte, similar to [6, Lemma 4.1(2)], $\forall \Omega(T) \in \Omega(\mathcal{T})$, we can find an $\mathbb{E}$-triangle $\Omega(T) \xrightarrow{f} X_1 \to X_2 \dashrightarrow$, where $X_1, X_2 \in \mathscr{X}$ and f is a left $\mathscr{X}$-approximation. Applying $\mathbb{G}$, yields an exact sequence

$$\mathbb{G}(\Omega(R)) \xrightarrow{\mathbb{G}(f)} \mathbb{G}(X_1) \to \mathbb{G}(X_2) \to 0.$$

Thus we get a diagram which is commutative, where $Hom_{\underline{\mathscr{B}}}(f, X)$ is surjective.

$$\begin{array}{ccccc}
Hom_{\underline{\mathscr{B}}}(X_1, X) & \xrightarrow{Hom_{\underline{\mathscr{B}}}(f,X)} & Hom_{\underline{\mathscr{B}}}(\Omega(R), X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
Hom_{mod\underline{\Omega(\mathcal{T})}}(\mathbb{G}(X_1), \mathbb{G}(X)) & \xrightarrow{\circ\mathbb{G}(f)} & Hom_{mod\underline{\Omega(\mathcal{T})}}(\mathbb{G}(\Omega(R)), \mathbb{G}(X)) & &
\end{array}$$

By Lemma 3.4, the morphism $\circ\mathbb{G}(f)$ is surjective. So $\mathbb{G}(f)$ is a left $\mathbb{G}(\mathscr{X})$-approximation and $(\mathbb{G}(\mathscr{X}), \Omega(\mathcal{T}) \cap \Omega(\mathscr{X}))$ is a support τ-tilting pair of $mod\underline{\Omega(\mathcal{T})}$ by [3, Definition 2.12].

2). φ is epic.

Assume $(\mathcal{N}, \sigma)$ is a support τ-tilting pair of mod$\underline{\Omega(\mathcal{T})}$. By Lemma 3.7, there is a $\mathcal{T}$-rigid subcategory $\mathscr{X}$ satisfies $\mathbb{G}(\mathscr{X}) = \mathcal{N}$. So $\forall \Omega(T) \in \Omega((T))$, there is an exact sequence $\mathbb{G}(\Omega(T)) \xrightarrow{\alpha} \mathbb{G}(X_3) \to \mathbb{G}(X_4) \to 0$, such that $X_3, X_4 \in \mathscr{X}$ and α is a left $\mathbb{G}(\mathscr{X})$-approximation. By Yoneda's lemma, we have a unique morphism in $mod\underline{\Omega((T))}$:

$$\beta : \ \Omega(\overline{T}) \to X_3 \text{ such that } \alpha = \mathbb{G}(\beta) \text{ and } \mathbb{G}(cone(\beta)) \cong \mathbb{G}(X_4).$$

Moreover, $\forall X \in \mathscr{X}$, consider the following commutative diagram

$$\begin{array}{ccc} Hom_{\underline{\mathscr{B}}}(X_3, X) & \xrightarrow{Hom_{\underline{\mathscr{B}}}(\beta,X)} & Hom_{\underline{\mathscr{B}}}(\Omega(T), X) \\ \downarrow{\scriptstyle \mathbb{G}(-)} & & \downarrow{\scriptstyle \cong} \\ Hom_{mod\underline{\Omega(\mathcal{T})}}(\mathbb{G}(X_3), \mathbb{G}(X)) & \xrightarrow{\circ\alpha} & Hom_{mod\underline{\Omega(\mathcal{T})}}(\mathbb{G}(\Omega(T)), \mathbb{G}(X)) \end{array}$$

By Lemma 3.4, $\mathbb{G}(-)$ is surjective. So the map $Hom_{\underline{\mathscr{B}}}(\beta, X)$ is surjective.

Denote $Cone(\beta)$ by Y_R and $\mathscr{X} \vee add\{Y_R \mid \Omega(T) \in \Omega(\mathcal{T})\}$ by $\widetilde{\mathscr{X}}$.

We claim $\widetilde{\mathscr{X}}$ is $\mathcal{T}$-rigid.

(I). Assume $a:\ Y_R \xrightarrow{a_1} T_0 \xrightarrow{a_2} \Sigma(X)$ with $T_0 \in \mathcal{T}$ and $X \in \mathscr{X}$. Consider the following diagram:

$$\begin{array}{ccccc} \Omega(T) & \xrightarrow{\beta} & X_3 & \xrightarrow{\gamma} & Y_R \dashrightarrow \\ \downarrow{\scriptstyle \exists g} & & \downarrow{\scriptstyle \exists f} & & \downarrow{\scriptstyle a} \\ X & \longrightarrow & I & \xrightarrow{i} & \Sigma X \dashrightarrow \end{array}$$

Since $\mathscr{X}$ is $\mathcal{T}$-rigid, $\exists f:\ X_3 \to I$ such that $a\gamma = if$. So there is a morphism $g: \Omega(T) \to X$ making the upper diagram commutative. Since $Hom_{\underline{\mathscr{B}}}(\beta, X)$ is surjective, g factors through β. Hence a factors through i, i.e., $[\overline{\mathcal{T}}](Y_R, \Sigma(\mathscr{X})) = [\mathcal{T}](Y_R, \Sigma(\mathscr{X}))$.

(II). For any morphism $b:\ X \xrightarrow{b_1} T_0 \xrightarrow{b_2} \Sigma(Y_R)$ with $T_0 \in \mathcal{T}$ and $X \in \mathscr{X}$. Consider the following diagram:

$$\begin{array}{ccccc} \Omega(T) & \xrightarrow{\beta} & X_3 & \xrightarrow{\gamma} & Y \dashrightarrow \\ \downarrow & & \downarrow{\scriptstyle a\gamma} & & \downarrow{\scriptstyle a} \\ P & \longrightarrow & I & = & I \\ \downarrow & & \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ T & \longrightarrow & \Sigma(X_3) & \xrightarrow{\gamma_1} & \Sigma(Y_T) \\ \vdots & & \vdots & & \vdots \end{array}$$

By [3, Lemma 5.9], $R \to \Sigma(X_3) \to \Sigma(Y_T) \dashrightarrow$ is an $\mathbb{E}$-triangle. Because $\mathcal{T}$ is rigid, b_2 factors through γ_1. By the fact that $\mathscr{X}$ is $\mathcal{T}$-rigid, $b = b_2 b_1$ can factor through i_X. Since $\gamma_1 i_X = i_Y$, we get that b factors through i_Y. So $[\overline{\mathcal{T}}](\mathscr{X}, \Sigma(Y_T)) = [\mathcal{T}](\mathscr{X}, \Sigma(Y_T))$.

By (I) and (II), we also obtain $[\overline{\mathcal{T}}](Y_T, \Sigma(Y_T)) = [\mathcal{T}](Y_T, \Sigma(Y_T))$.

Therefore $\widetilde{\mathscr{X}} = \mathscr{X} \vee add\{Y_T \mid \Omega(T) \in \Omega(\mathcal{T})\}$ is $\mathcal{T}$-rigid.

Let $M \in \mathscr{B}$ satisfying $[\overline{\mathcal{T}}](M, \Sigma(\widetilde{\mathscr{X}})) = [\mathcal{T}](M, \Sigma(\widetilde{\mathscr{X}}))$ and $[\overline{\mathcal{T}}](\widetilde{\mathscr{X}}, \Sigma M) = [\mathcal{T}](\widetilde{\mathscr{X}}, \Sigma M)$. Consider the $\mathbb{E}$-triangle:

$$\Omega(T_5) \xrightarrow{f} \Omega(T_6) \xrightarrow{g} M \dashrightarrow$$

where $T_5,\ T_6 \in \mathcal{T}$. By the above discussion, there exist two $\mathbb{E}$-triangles:

$$\Omega(T_6) \xrightarrow{u} X_6 \xrightarrow{v} Y_6 \dashrightarrow \text{ and } \Omega(T_5) \xrightarrow{u'} X_5 \xrightarrow{v'} Y_5 \dashrightarrow .$$

where $X_5,\ X_6 \in \mathcal{X}$, u and $u^{'}$ are left $\mathscr{X}$-approximations of $\Omega(T_6)$, $\Omega(T_5)$, respectively. So there exists a diagram of $\mathbb{E}$-triangles which is commutative:

$$\begin{array}{cccccc}
\Omega(T_5) & \xrightarrow{f} & \Omega(T_6) & \xrightarrow{g} & M & \dashrightarrow \\
\Vert & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle a} & \\
\Omega(T_5) & \xrightarrow{x=uf} & X_6 & \xrightarrow{y} & N & \dashrightarrow \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle b} & \\
 & & Y & = & Y & \\
 & & \downarrow & & \downarrow &
\end{array}$$

We claim that the morphism $x = uf$ is a left $\mathscr{X}$-approximation of $\Omega(T_5)$. In fact, let $X \in \mathscr{X}$ and $d: \Omega(T_5) \to X$, we can get a commutative diagram of $\mathbb{E}$-triangles:

$$\begin{array}{ccccc}
\Omega(T_5) & = & \Omega(T_5) & \xrightarrow{d} & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_1} & & \downarrow \\
\Omega(T_6) & \longrightarrow & P & \xrightarrow{d_1} & I_X \\
\downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle i_X} \\
M & \xrightarrow[h]{} & T_5 & \xrightarrow[d_2]{} & \Sigma(X) \\
\downarrow & & \downarrow & & \downarrow
\end{array}$$

where $P \in \mathcal{P}$. By the assumption, $[\overline{\mathcal{T}}](M, \Sigma(X)) = [\mathcal{T}](M, \Sigma(X))$. So $d_2 h$ factors through i_X. By Lemma 2.3, d factors through f. Thus $\exists f_1 : \Omega(T_6) \to X$ such that $d = f_1 f$. Moreover, u is a left $\mathscr{X}$-approximation of $\Omega(T_6)$. So $\exists u_1 : X_6 \to X$ such that $f_1 = u_1 u$. Thus $d = f_1 f = u_1 u f = u_1 x$. So $x = uf$ is a left $\mathscr{X}$-approximation of $\Omega(T_5)$.

Hence there is a commutative diagram:

$$\begin{array}{cccccc}
\Omega(T_5) & \xrightarrow{x} & X_6 & \xrightarrow{y} & N & \dashrightarrow \\
\Vert & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \varphi} & \\
\Omega(T_5) & \xrightarrow{u^{'}} & X_5 & \xrightarrow{v^{'}} & Y_5 & \overset{\delta_5}{\dashrightarrow}
\end{array}$$

By [3, Corollary 3.16], we get an $\mathbb{E}$-triangle $X_6 \overset{\binom{y}{\lambda}}{\to} N \oplus X_5 \to Y_5 \overset{x_*\delta_5}{\dashrightarrow}$

Since $u^{'}$ is a left $\mathscr{X}$-approximation of $\Omega(T_5)$, there is also a commutative diagram with $P \in \mathcal{P}$:

$$\begin{array}{cccccc}
\Omega(T_5) & \xrightarrow{u^{'}} & X_5 & \xrightarrow{v} & Y_5 & \overset{\delta_5}{\dashrightarrow} \\
\Vert & & \downarrow & & \downarrow{\scriptstyle \exists t} & \\
\Omega(T_5) & \longrightarrow & P & \longrightarrow & T_5 & \overset{\mu}{\dashrightarrow}
\end{array}$$

such that $\delta_5 = t^*\mu$. So $x_*\delta_5 = x_*t^*\mu = t^*x_*\mu$. By Lemma 3.8, the $\mathbb{E}$-triangle $x_*\delta_5$ splits. So $N \oplus X_5 \simeq X_6 \oplus Y_5 \in \widetilde{\mathscr{X}}$. hence $N \in \widetilde{\mathscr{X}}$.

Similarly, consider the following commutative diagram with $P \in \mathcal{P}$:

$$\begin{array}{ccccccc}
\Omega(T_6) & \xrightarrow{u} & X_6 & \xrightarrow{v} & Y_6 & \overset{\delta_6}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle t} & & \\
\Omega(T_6) & \xrightarrow{u'} & P & \xrightarrow{v'} & T_6 & \overset{\delta}{\dashrightarrow} &
\end{array}$$

and the $\mathbb{E}$-triangle $M \to N \to Y \overset{g_*\delta_6}{\dashrightarrow}$. Then $\exists t: Y \to T_6$ such that $\delta_6 = t^*\delta$. Then $g_*\delta_6 = g_*t^*\delta = t^*(g_*\delta)$. Since $[\overline{\mathcal{T}}](\widetilde{\mathscr{X}}, \Sigma M) = [\underline{\mathcal{T}}](\widetilde{\mathscr{X}}, \Sigma M)$, the $\mathbb{E}$-triangle $g_*\delta_6$ splits by Lemma 3.5 and M is a direct summands of N. Hence $M \in \widetilde{\mathscr{X}}$.

By the above, we get $\widetilde{\mathscr{X}}$ is a $\mathcal{T}$-cluster tilting subcategory.

By the definition of Y_R, $\mathbb{G}(Y_R) \in \mathbb{G}(\mathscr{X})$. So $\mathbb{G}(\widetilde{\mathscr{X}}) \simeq \mathbb{G}(\mathscr{X}) \simeq \mathcal{N}$. Moreover, $\sigma = \Omega(\mathcal{T}) \cap \Omega(\mathscr{X}) \subseteq \Omega(\mathcal{T}) \cap \Omega(\widetilde{\mathscr{X}})$ and $\Omega(\mathcal{T}) \cap \Omega(\widetilde{\mathscr{X}}) \subseteq ker\mathbb{G}(\mathscr{X}) = \sigma$. So $\Omega(\mathcal{T}) \cap \Omega(\widetilde{\mathscr{X}}) = \sigma$. Hence φ is surjective.

3). φ is injective following from the proof of Lemma 3.7.

By [4, Proposition 4.8 and Fact 4.13], $\underline{\mathscr{B}} \simeq mod\underline{\Omega(\mathcal{T})}$. So it is easy to get the following corollary by Theorem 3.9:

Corollary 3.10. *Let $\mathscr{X}$ be a subcategory of $\mathscr{B}$.*

1) $\mathscr{X}$ is $\mathcal{T}$-rigid iff $\underline{\mathscr{X}}$ is τ-rigid subcategory of $\underline{\mathscr{B}}$.

2) $\mathscr{X}$ is $\mathcal{T}$-cluster tilting iff $\underline{\mathscr{X}}$ is support τ-tilting subcategory of $\underline{\mathscr{B}}$.

If let $\mathcal{H} = CoCone(\mathcal{T}, \mathcal{T})$, then $\mathcal{H}$ can completely replace $\mathscr{B}$ and draw the corresponding conclusion by the proof Lemma 3.7 and Theorem 3.9, which is exactly [12, Theorem 3.8]. If let $\mathscr{B}$ is a triangulated category, then Theorem 3.9 is exactly [9, Theorem 4.3].

Acknowledgments

This research was supported by the National Natural Science Foundation of China (No. 12101344) and Shan Dong Provincial Natural Science Foundation of China (No.ZR2015PA001).

Conflict of interest

The authors declare they have no conflict of interest.

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