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Research article

Multiple nontrivial periodic solutions to a second-order partial difference equation

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Abstract: In this article, applying variational technique as well as critical point theory, we establish a series of criteria to ensure the existence and multiplicity of nontrivial periodic solutions to a second-order nonlinear partial difference equation. Our results generalize some known results. Moreover, numerical stimulations are presented to illustrate applications of our major findings.

Keywords: critical point theory; partial difference equation; nontrivial periodic solutions; variational technique

1. Introduction

Denote the real number set and the integer set by $\mathbb{R}$ and $\mathbb{Z}$, respectively. For any $a, b \in \mathbb{Z}$ with $a \leq b$, define $\mathbb{Z}(a, b)$: $=\{a, a+1, \cdots, b\}$. In this paper, we focus on multiple nontrivial periodic solutions of the following second-order partial difference equation:

$$\begin{aligned} &\triangle_1 \left[p(n-1,m)\left(\triangle_1 u(n-1,m)\right)^{\eta}\right] + \triangle_2 \left[r(n,m-1)\left(\triangle_2 u(n,m-1)\right)^{\eta}\right] \\ &+ q(n,m)(u(n,m))^{\eta} + f\left((n,m),u(n,m)\right) = 0, \qquad n,m \in \mathbb{Z}, \end{aligned} \tag{1.1}$$

where $\triangle_1 u(n,m) = u(n+1,m) - u(n,m)$ and $\triangle_2 u(n,m) = u(n,m+1) - u(n,m)$. $f((n,m),u) : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to u. Given integers $T_1, T_2 > 0$, for any $n, m \in \mathbb{Z}$, let nonzero sequences $\{p(n,m)\}$, $\{r(n,m)\}$ and $\{q(n,m)\}$ satisfy

$$p(n+T_1,m) = p(n,m) = p(n,m+T_2) > 0, \qquad r(n+T_1,m) = r(n,m) = r(n,m+T_2) \geq 0,$$

$$q(n+T_1,m) = q(n,m) = q(n,m+T_2) \leq 0,$$

and

$$f((n+T_1,m),u) = f((n,m),u) = f((n,m+T_2),u), \qquad \forall((n,m),u) \in \mathbb{Z}^2 \times \mathbb{R}.$$

Let η be the ratio of odd positive integers such that $(-1)^\eta = -1$. If a solution $u = \{u(n,m)\}$ satisfies $u(n+T_1,m) = u(n,m) = u(n,m+T_2)$ for any $n,m \in \mathbb{Z}$, we call u a (T_1,T_2)-periodic solution. To help with understanding if a solution u is (T_1,T_2)-periodic, we give an example as a remark.

Remark 1.1. *Consider* (1.1) *with* $T_1 = T_2 = 2$. *Suppose (1.1) possesses four solutions, denoted by*

$$\begin{aligned} u_1 &= (u_{11},u_{12},u_{13},u_{14}), \qquad & u_2 &= (u_{21},u_{22},u_{23},u_{24}), \\ u_3 &= (u_{31},u_{32},u_{33},u_{34}), \qquad & u_4 &= (u_{41},u_{42},u_{43},u_{44}). \end{aligned}$$

If they are $(2,2)$*-periodic, that is,* $u(n+2,m) = u(n,m) = u(n,m+2)$, $n,m = 1,2$, *then*

$$\begin{aligned} u_{31} &= u_{11} = u_{13} = u_{33}, \qquad & u_{32} &= u_{12} = u_{14} = u_{34}; \\ u_{41} &= u_{21} = u_{23} = u_{43}, \qquad & u_{42} &= u_{22} = u_{24} = u_{44}. \end{aligned}$$

Actually, $u_1 = u_3$ *and* $u_2 = u_4$. *Therefore,* $u_{11} = u_{13}$ *and* $u_{12} = u_{14}$ *ensure that the solution* $u_1 = (u_{11},u_{12},u_{13},u_{14})$ *is* $(2,2)$*-periodic.*

In socio-economic activities and natural science research, we often encounter variables similar to time t. Meanwhile, one can often only observe or record values of these variables in discrete cases. Solving this problem is inseparable from difference equations. During past decades, difference equations have been used extensively [1,2], and scholars have studied difference equations in many ways, including period solutions, boundary value problems, homoclinic solutions, heteroclinic solutions [3–8] and so on. It is worth mentioning that Guo and Yu [3] made critical point theory an effective tool to discuss periodic solutions by constructing a new variational structure for the first time. In [9], by critical point theory, Cai and Yu studied the existence of solutions to the following equation:

$$\triangle\left(p_n(\triangle x_{n-1})^\eta\right) + q_n x_n^\eta = f(n,x_n), \qquad n \in \mathbb{Z}. \tag{1.2}$$

Obviously, Eq (1.2), involving only one independent variable, is a special case of (1.1). It has been studied by many authors, and certain conclusions [10–12] have been yielded.

On the other side, as modern technology advances rapidly, the use of mathematical modeling to solve problems is not only becoming more and more frequent, but also there are more and more factors needing to be considered. As a result, partial difference equations, containing multiple independent integer variables, have widespread applications in image processing, life sciences, quantum mechanics, and other fields [13] and capture great interest of many scholars. For example, [14–16] obtain multiple results on discrete Kirchhoff problems, and [17–20] concern second order partial difference equations via Morse theory. Very recently, [21] investigated periodic solutions of the equation

$$\triangle_1\left[p(n-1,m)\triangle_1 u(n-1,m)\right] + \triangle_2\left[r(n,m-1)\triangle_2 u(n,m-1)\right] + f((n,m),u(n,m)) = 0, \tag{1.3}$$

via critical point theory. Clearly, letting $\eta \equiv 1$, (1.1) is just (1.3), and (1.1) is more general than (1.3). Moreover, via critical point theory, [22, 23] deal with the existence of multiple solutions for a partial discrete Dirichlet boundary value problem with mean curvature operator and homoclinic solutions for a differential inclusion system involving the $p(t)$-Laplacian, respectively. In view of the abovementioned results, we find that critical point theory serves as a robust method for studying both differential equations and difference equations. Therefore, motivated by the above obtained results, we intend to study periodic solutions to (1.1) by critical point theory. We also provide numerical stimulations to illustrate applications of our theoretical results. Our results generalize some results in [9] and [21]. The resulting problem engages two major difficulties: First, to estimate relations between norms, we need to transfer (1.1) into an equivalent form to compute its eigenvalues. Another difficulty we must overcome is verifying the link geometry and certifying boundedness of the Palais-Smale sequence.

For the rest of this paper, we organize in the following way. In Section 2, we give a variational structure and look for the corresponding functional to (1.1). Moreover, some definitions and lemmas are recalled. Our main results and detailed proofs are provided in Section 3. Finally, Section 4 presents three examples to demonstrate the application of our main results.

2. Preliminaries and notations

In this section, we establish the corresponding variational framework to (1.1) and state some preliminaries and notations to make preparation for our main results.

Write

$$u = (\cdots;\cdots, u(1,1), u(2,1), u(3,1), \cdots;\cdots, u(1,2), u(2,2), u(3,2), \cdots;\cdots),$$

and let

$$S = \{u = \{u(n,m)\} | u(n,m) \in \mathbb{R}, \quad n,m \in \mathbb{Z}\}$$

be a vector space which is composed of all $u = \{u(n,m)\}_{n,m\in\mathbb{Z}}$. Define

$$E = \{u = \{u(n,m)\} \in S \,| u(n+T_1, m) = u(n,m) = u(n, m+T_2), \quad n,m \in \mathbb{Z}\}$$

as a subset of S. Define an inner product $\langle\cdot,\cdot\rangle$ on E as

$$\langle u, v\rangle = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} u(n,m)v(n,m), \qquad \forall u, v \in E.$$

Then, the induced norm $||\cdot||$ is

$$||u|| = \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |u(n,m)|^2\right)^{\frac{1}{2}}, \qquad \forall u \in E.$$

Clearly, the dimension of the Hilbert space E is T_1T_2-dimensional. Thus, E is homeomorphic to $\mathbb{R}^{T_1T_2}$.

For $s > 1$, define another norm $||\cdot||_s$ on E as

$$||u||_s = \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |u(n,m)|^s\right)^{\frac{1}{s}}, \qquad \forall u \in E.$$

Then, $\|u\|_2 = \|u\|$, and for $\beta > \eta + 1$ there exist constants $C_2 \geq C_1 > 0$, $C_4 \geq C_3 > 0$ such that

$$C_1\|u\| \leq \|u\|_{\eta+1} \leq C_2\|u\|, \quad C_3\|u\| \leq \|u\|_\beta \leq C_4\|u\|, \qquad \forall u \in E. \tag{2.1}$$

Moreover, there holds

$$\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |u(n,m)|^2 \leq (T_1T_2)^{\frac{\eta-1}{\eta+1}} \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |u(n,m)|^{\eta+1}\right)^{\frac{2}{\eta+1}}.$$

Then,

$$(T_1T_2)^{-\frac{\eta-1}{2(\eta+1)}} \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |u(n,m)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |u(n,m)|^{\eta+1}\right)^{\frac{1}{\eta+1}},$$

which means that we can choose $C_1 = (T_1T_2)^{-\frac{\eta-1}{2(\eta+1)}}$.

Consider a functional $I : E \to \mathbb{R}$ in the following form:

$$\begin{aligned} I(u) =& \frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\Big[p(n-1,m)\left(\triangle_1 u(n-1,m)\right)^{\eta+1} + r(n,m-1)\left(\triangle_2 u(n,m-1)\right)^{\eta+1} \\ & -q(n,m)(u(n,m))^{\eta+1}\Big] - \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F\left((n,m),u(n,m)\right), \end{aligned} \tag{2.2}$$

where $F\left((n,m),u\right) = \int_0^u f\left((n,m),s\right)ds$. Since $f((\cdot,\cdot),u)$ is continuous with respect to u, it follows that $I \in C^1(E,\mathbb{R})$. Moreover, for any $u \in E$, using the periodic condition, direct computation yields

$$\begin{aligned} \frac{\partial I}{\partial u(n,m)} =& p(n-1,m)\left(\triangle_1 u(n-1,m)\right)^\eta - p(n,m)\left(\triangle_1 u(n,m)\right)^\eta + r(n-1,m)\left(\triangle_2 u(n,m-1)\right)^\eta \\ & - r(n,m)\left(\triangle_2 u(n,m)\right)^\eta - q(n,m)(u(n,m))^\eta - f\left((n,m),u(n,m)\right) \\ =& -\triangle_1\left[p(n-1,m)(\triangle_1 u(n-1,m))^\eta\right] - \triangle_2\left[r(n,m-1)(\triangle_2 u(n,m-1))^\eta\right] \\ & - q(n,m)(u(n,m))^\eta - f\left((n,m),u(n,m)\right). \end{aligned}$$

Hence, $u \in E$ being a critical point for I is equivalent to

$$\begin{aligned} & \triangle_1\left[p(n-1,m)\left(\triangle_1 u(n-1,m)\right)^\eta\right] + \triangle_2\left[r(n,m-1)\left(\triangle_2 u(n,m-1)\right)^\eta\right] + q(n,m)(u(n,m))^\eta \\ & + f((n,m),u(n,m)) = 0, \qquad \forall u \in E, \end{aligned}$$

which is just (1.1). Therefore, we transform the problem to find (T_1,T_2)-periodic solutions to (1.1) to the problem to seek critical points of I on E.

For convenience, write $u \in E$ as

$$u = (u(1,1),\cdots,u(T_1,1);u(1,2),\cdots,u(T_1,2);\cdots;u(1,T_2),\cdots,u(T_1,T_2))^T,$$

where $\cdot^T$ denotes the transpose of vector $\cdot$. Let matrices A and B be defined by

$$A = \begin{pmatrix} B & & & 0 \\ & B & & \\ & & \ddots & \\ 0 & & & B \end{pmatrix}_{T_1T_2 \times T_1T_2.} \qquad B = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{T_1 \times T_1,}$$

By matrix theory, we find that the matrix B is semi-definite positive, and its eigenvalues are

$$\lambda_k = 2(1 - \cos\frac{2k\pi}{T_1}), \quad k = 0, 1, 2, \cdots, T_1 - 1.$$

Then, $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{T_1}$, and

$$\lambda_2 = 2(1 - \cos\frac{2\pi}{T_1}).$$

Moreover, matrices A and B have the same eigenvalues $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{T_1}$, and the multiplicity of each eigenvalue λ_k of matrix A is T_2. Direct computation gives

$$\begin{aligned} \|\triangle_1 u(n-1,m)\|^2 &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\triangle_1 u(n-1,m)]^2 = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}(u(n,m) - u(n-1,m))^2 \\ &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u^2(n,m) - 2\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u(n,m)u(n-1,m) + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u^2(n-1,m) \\ &= 2\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u^2(n,m) - 2\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u(n,m)u(n-1,m) \\ &= \langle Au, u\rangle, \end{aligned}$$

and

$$\begin{aligned} \|\triangle_1 u(n,m)\|^2 &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\triangle_1 u(n,m)]^2 = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}(u(n+1,m) - u(n,m))^2 \\ &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u^2(n+1,m) - 2\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u(n+1,m)u(n,m) + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u^2(n,m) \\ &= 2\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u^2(n,m) - 2\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}u(n,m)u(n-1,m) \\ &= \langle Au, u\rangle = \|\triangle_1 u(n-1,m)\|^2. \end{aligned}$$

Define an orthogonal matrix

$$P = \begin{array}{c}
\begin{array}{ccccccccccccc} & & & T_1 & & & & 2T_1 & & (T_2-1)T_1+1 & & & \end{array} \\
\left(\begin{array}{ccccccccccccc}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array}\right)
\begin{array}{l}
\\ \\ \\ T_2 \\ \\ \\ \\ 2T_2 \\ \\ (T_1-1)T_2+1 \\ \\ \\ {}_{T_1T_2\times T_1T_2}
\end{array}
\end{array}$$

such that

$$v = Pu = (u(1,1), \cdots, u(1,T_2); u(2,1), \cdots, u(2,T_2); \cdots; u(T_1,1), \cdots, u(T_1,T_2))^T .$$

Then, the matrix P is a rearrangement transformation of u, and $||u||_s = ||v||_s$ for any $s > 0$.

Similarly, given matrices C and D as

$$C = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{T_2\times T_2,} \qquad D = \begin{pmatrix} C & & & 0 \\ & C & & \\ & & \ddots & \\ 0 & & & C \end{pmatrix}_{T_1T_2\times T_1T_2} ,$$

it follows that eigenvalues of matrix C are $\mu_1 = 0 < \mu_2 \le \mu_3 \le \cdots \le \mu_{T_2}$, and

$$\mu_2 = 2(1 - \cos\frac{2\pi}{T_2}).$$

In the same manner, we have that eigenvalues of matrix D are also $\mu_1 = 0 < \mu_2 \le \mu_3 \le \cdots \le \mu_{T_2}$, and each eigenvalue μ_κ, $1 \le \kappa \le T_2$ is T_1-multiple. Further,

$$||\triangle_2 u(n,m-1)||^2 = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\triangle_2 u(n,m-1)]^2 = \langle Dv, v\rangle,$$

and

$$||\triangle_2 u(n,m)||^2 = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[\triangle_2 u(n,m)]^2 = \langle Dv, v\rangle = ||\triangle_2 u(n,m-1)||^2.$$

Set $W = \{w \in E | w = \{c\}, c \in \mathbb{R}\}$ and $Y = W^\perp$. Then, $E = Y \oplus W$. Thus, for any $u \in Y$, we have

$$\begin{aligned} \lambda_2||u||^2 \le ||\triangle_1 u(n-1,m)||^2 = ||\triangle_1 u(n,m)||^2 = \langle Au, u\rangle \le \lambda_{T_1}||u||^2, \\ \mu_2||u||^2 = \mu_2||v||^2 \le ||\triangle_2 u(n,m-1)||^2 = ||\triangle_2 u(n,m)||^2 = \langle Dv, v\rangle \le \mu_{T_2}||v||^2 = \mu_{T_2}||u||^2. \end{aligned} \tag{2.3}$$

Thus, for any $w \in W$, we get

$$\begin{aligned} \|\triangle_1 w(n-1,m)\|^2 = \|\triangle_1 w(n,m)\|^2 = \langle Aw, w\rangle = 0, \\ \|\triangle_2 w(n,m-1)\|^2 = \|\triangle_2 w(n,m)\|^2 = \langle DPw, Pw\rangle = 0. \end{aligned} \tag{2.4}$$

In the following, we recall some definitions and lemmas which are useful to our main results.

Definition 2.1. *Let $I \in C^1(E,\mathbb{R})$. If any sequence $\{u_k\} \subset E$ such that $\{I(u_k)\}$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence, then I satisfies the Palais-Smale (P.S. for short) condition.*

Let B_ρ denote an open ball whose center is 0 and radius is ρ in E. Let ∂B_ρ stand for the boundary of B_ρ. The following Lemmas 2.1–2.3 are main tools to prove our results, and we can refer to [24] for detail.

Lemma 2.1. *(Mountain Pass Lemma [24]) Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ satisfy the P.S. condition. Moreover, $I(0) = 0$. Suppose*
(f_1) there exist constants ρ, $a > 0$ such that $I|_{\partial B_\rho} \geq a$;
(f_2) there exists $e \in X \backslash B_\rho$ such that $I(e) \leq 0$.
Then, I admits a critical value $c \geq a$ given by

$$c = \inf_{h\in\Gamma} \sup_{s\in[0,1]} I(h(s))$$

where

$$\Gamma = \{h \in C([0,1], X) | h(0) = 0, h(1) = e\}.$$

Lemma 2.2. *(Linking theorem [24]) Let $X = X_1 \oplus X_2$ be a real Banach space, where X_1 is a finite-dimensional subspace of X. Suppose that $I \in C^1(E,\mathbb{R})$ satisfies the P.S. condition. If*
(f_3) there exist constants ρ, $a > 0$ such that $I|_{\partial B_\rho \cap X_2} \geq a$, and
(f_4) there exist constants $e \in \partial B_1 \cap X_2$, $R_0 > \rho$ such that $I|_{\partial Q} \leq 0$, where $Q = \left(\bar{B}_{R_0} \cap X_1\right) \oplus \{re | 0 < r < R_0\}$,
then I has a critical value $c \geq a$, and

$$c = \inf_{h\in\Gamma} \max_{u\in\bar{Q}} I(h(u)),$$

where

$$\Gamma = \left\{h \in C(\bar{Q}, X) : h|_{\partial Q} = id\right\}.$$

Lemma 2.3. *(Saddle point theorem [24]) Let $X = X_1 \oplus X_2$ be a real Banach space and $X_1 \neq \{0\}$ be a finite-dimensional subspace of X. Suppose $I \in C^1(E,\mathbb{R})$ satisfies the P.S. condition. If*
(f_5) there exist constants σ and $\rho > 0$ such that $I|_{\partial B_\rho \cap X_1} \leq \sigma$, and
(f_6) there exist constants $e \in B_\rho \cap X_1$, $\omega > \sigma$ such that $I|_{e+X_2} \geq \omega$,
then I has a critical value $c \geq \omega$, and

$$c = \inf_{h\in\Gamma} \max_{u\in B_\rho\cap X_1} I(h(u)),$$

where

$$\Gamma = \left\{h \in C\left(\bar{B}_\rho \cap X_1, X\right) | h|_{\partial B_\rho \cap X_1} = id\right\}.$$

3. Main results and proofs

For convenience, we give some notations first. Write $\Omega := \mathbb{Z}(1, T_1) \times \mathbb{Z}(1, T_2)$, and

$$p_{\max} = \max_{(n,m)\in\Omega} p(n,m) > 0, \qquad p_{\min} = \min_{(n,m)\in\Omega} p(n,m) > 0,$$
$$r_{\max} = \max_{(n,m)\in\Omega} r(n,m) \geq 0, \qquad r_{\min} = \min_{(n,m)\in\Omega} r(n,m) \geq 0,$$
$$q_{\max} = \max_{(n,m)\in\Omega} q(n,m) \leq 0, \qquad q_{\min} = \min_{(n,m)\in\Omega} q(n,m) \leq 0.$$

To study (1.1), the following assumptions are needed:

(F_1) $\lim_{u\to 0} \dfrac{f((n,m),u)}{u^\eta} = 0, \qquad \forall((n,m),u) \in \Omega \times \mathbb{R}.$

(F_2) There exist constants $a_1 > 0$, $a_2 > 0$ and $\beta > \eta + 1$ such that

$$F((n,m),u) \geq a_1|u|^\beta - a_2, \qquad \forall((n,m),u) \in \Omega \times \mathbb{R}.$$

Remark 3.1. *By* (F_2)*, we have*

(F_2') $\lim_{|u|\to+\infty} \dfrac{F((n,m),u)}{u^{\eta+1}} = +\infty, \quad \forall((n,m),u) \in \Omega \times \mathbb{R}.$

Thus, (F_1) *and* (F_2') *mean that* $f((n,m),u)$ *is superlinearly increasing at both* 0 *and* ∞.

Now, we are in the position to present our main results.

Theorem 3.1. *Let* (F_1) *and* (F_2) *hold. Moreover,*

(q) *for any* $(n,m) \in \Omega$, $q(n,m) < 0$.

Then, (1.1) possesses at least two nontrivial (T_1, T_2)*-periodic solutions.*

Theorem 3.2. *Suppose* (F_1) *and* (F_2) *are satisfied. If* $\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),u) \geq 0$ *and*

(q') $q(n,m) \equiv 0$, $\forall(n,m) \in \Omega$,

then (1.1) admits at least two nontrivial (T_1, T_2)*-periodic solutions.*

Recall $C_1 = (T_1T_2)^{-\frac{\eta-1}{2(\eta+1)}}$, $\lambda_2 = 2(1 - \cos\frac{2\pi}{T_1})$ and $\mu_2 = 2(1 - \cos\frac{2\pi}{T_2})$. We have the following.

Theorem 3.3. *If* (q) *and*

(F_3)

$$\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}(-q(n,m))\left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{\eta+1}{2}}\left(\frac{1}{C_1}\right)^{\eta+1}$$
$$< \left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}} - q_{\max}\right)\left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(n,m)\right)^{\eta+1}, \qquad \forall(n,m) \in \Omega,$$

hold, then equation

$$\begin{aligned}&\triangle_1\left[p(n-1,m)\left(\triangle_1 u(n-1,m)\right)^\eta\right] + \triangle_2\left[r(n,m-1)\left(\triangle_2 u(n,m-1)\right)^\eta\right] \\ &+ q(n,m)\left(u(n,m)\right)^\eta + f(n,m) = 0, \qquad n, m \in \mathbb{Z},\end{aligned} \tag{3.1}$$

has at least a (T_1, T_2)*-periodic solution.*

Remark 3.2. *In fact, (1.2) and (1.3) are special cases of (1.1). Consider (1.1) with $r(n,m) \equiv 0$, and (1.1) can be written in the form of* (1.2)*. Meanwhile, if $q(n,m) = 0$ and $\eta = 1$, then (1.1) changes to (1.3). Moreover, our Theorems 3.1–3.3 are able to override Theorem 3.2 of [9], Theorem 3.1 of [21] and Theorem 3.3 of [9]. Consequently, (1.1) is a generalization of both (1.2) and (1.3), and our results are more universal.*

Before stating proofs of Theorems 3.1–3.3, we need to prove the compactness of I first.

Lemma 3.1. *Assume (F_2) holds. Then, I satisfies the P.S. condition on E.*

Proof. Assume that for any $\{u_k\} \subset E$ there exists a constant $M > 0$ such that

$$|I(u_k)| \leq M \qquad \text{and} \qquad I'(u_k) \to 0, \quad \text{as} \quad k \to +\infty.$$

As E is a finite-dimensional space, we only need to prove that $\{u_k\}$ is bounded. By (F_2), we have

$$\begin{aligned}
-M \leq& I(u_k)\\
=&\frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[p(n-1,m)\left(\triangle_1 u_k(n-1,m)\right)^{\eta+1} + r(n,m-1)\left(\triangle_2 u_k(n,m-1)\right)^{\eta+1}\right]\\
&-\frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)(u_k(n,m))^{\eta+1} - \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),u_k(n,m))\\
\leq&\frac{p_{\max}}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}(\triangle_1 u_k(n-1,m))^{\eta+1} + \frac{r_{\max}}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}(\triangle_2 u_k(n,m-1))^{\eta+1}\\
&-\frac{q_{\min}}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}(u_k(n,m))^{\eta+1} - a_1\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}|u_k(n,m)|^{\beta} + a_2 T_1 T_2\\
=&\frac{p_{\max}}{\eta+1}\|\triangle_1 u_k(n-1,m)\|_{\eta+1}^{\eta+1} + \frac{r_{\max}}{\eta+1}\|\triangle_2 u_k(n,m-1)\|_{\eta+1}^{\eta+1} - \frac{q_{\min}}{\eta+1}\|u_k\|_{\eta+1}^{\eta+1}\\
&- a_1\|u_k\|_{\beta}^{\beta} + a_2 T_1 T_2.
\end{aligned} \tag{3.2}$$

Moreover, due to (2.1) and (2.3), it follows that

$$\begin{aligned}
&\|\triangle_1 u_k(n-1,m)\|_{\eta+1}^{\eta+1} \leq C_2^{\eta+1}\|\triangle_1 u_k(n-1,m)\|^{\eta+1} \leq C_2^{\eta+1}\lambda_{T_1}^{\frac{\eta+1}{2}}\|u_k\|^{\eta+1},\\
&\|\triangle_2 u_k(n,m-1)\|_{\eta+1}^{\eta+1} \leq C_2^{\eta+1}\|\triangle_2 u_k(n,m-1)\|^{\eta+1} \leq C_2^{\eta+1}\mu_{T_2}^{\frac{\eta+1}{2}}\|u_k\|^{\eta+1},\\
&\|u_k\|_{\eta+1}^{\eta+1} \leq C_2^{\eta+1}\|u_k\|^{\eta+1}, \qquad \|u_k\|_{\beta}^{\beta} \geq C_3^{\beta}\|u_k\|^{\beta}.
\end{aligned} \tag{3.3}$$

Therefore, combining (3.2) with (3.3), it yields that

$$\begin{aligned}
-M \leq& \frac{p_{\max}C_2^{\eta+1}\lambda_{T_1}^{\frac{\eta+1}{2}}}{\eta+1}\|u_k\|^{\eta+1} + \frac{r_{\max}C_2^{\eta+1}\mu_{T_2}^{\frac{\eta+1}{2}}}{\eta+1}\|u_k\|^{\eta+1} - \frac{q_{\min}C_2^{\eta+1}}{\eta+1}\|u_k\|^{\eta+1}\\
&- a_1 C_3^{\beta}\|u_k\|^{\beta} + a_2 T_1 T_2.
\end{aligned}$$

That is,

$$a_1 C_3^{\beta}\|u_k\|^{\beta} - \frac{C_2^{\eta+1}\left(p_{\max}\lambda_{T_1}^{\frac{\eta+1}{2}} + r_{\max}\mu_{T_2}^{\frac{\eta+1}{2}} - q_{\min}\right)}{\eta+1}\|u_k\|^{\eta+1} \leq a_2 T_1 T_2 + M. \tag{3.4}$$

Since $\beta > \eta + 1$, (3.4) ensures that $\{u_k\} \subset E$ is a bounded sequence. Consequently, I satisfies the $P.S$. condition, and the proof is finished.

Proof of Theorem 3.1 By Lemma 3.1, I satisfies the $P.S$. condition on E. In the following, we verify conditions (f_1) and (f_2) of Lemma 2.1 to complete the proof.

In fact, from (F_1), it follows that

$$\lim_{u\to 0} \frac{F((n,m),u)}{u^{\eta+1}} = 0, \quad \forall (n,m) \in \Omega.$$

Then, there is a $\rho > 0$ such that

$$|F((n,m),u)| \le -\frac{q_{\max}}{2(\eta+1)} u^{\eta+1}, \quad \forall (n,m) \in \Omega, \quad |u| \le \rho.$$

Hence, for any $u \in E$ with $\|u\| \le \rho$, we obtain

$$I(u) \ge -\frac{q_{\max}}{\eta+1}\|u\|_{\eta+1}^{\eta+1} + \frac{q_{\max}}{2(\eta+1)}\|u\|_{\eta+1}^{\eta+1} = -\frac{q_{\max}}{2(\eta+1)}\|u\|_{\eta+1}^{\eta+1} \ge -\frac{q_{\max}C_1^{\eta+1}}{2(\eta+1)}\|u\|^{\eta+1}. \tag{3.5}$$

Take $a = -\frac{q_{\max}C_1^{\eta+1}}{2(\eta+1)}\rho^{\eta+1} > 0$, and then (3.5) ensures $I(u)|_{\partial B_\rho} \ge a > 0$. Thus, (f_1) of Lemma 2.1 is fulfilled.

Given $\omega \in E$ with $\|\omega\| = 1$ and $\alpha > 0$, we have

$$\begin{aligned}
I(\alpha\omega) =& \frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\Big[p(n-1,m)\,(\triangle_1\alpha\omega(n-1,m))^{\eta+1} + r(n,m-1)(\triangle_2\alpha\omega(n,m-1))^{\eta+1}\Big] \\
& - \frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)(\alpha\omega(n,m))^{\eta+1} - \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),\alpha\omega(n,m)) \\
\le& \frac{\alpha^{\eta+1}p_{\max}}{\eta+1}\|\triangle_1\omega(n-1,m)\|_{\eta+1}^{\eta+1} + \frac{\alpha^{\eta+1}r_{\max}}{\eta+1}\|\triangle_2\omega(n,m-1)\|_{\eta+1}^{\eta+1} - \frac{\alpha^{\eta+1}q_{\min}}{\eta+1}\|\omega\|_{\eta+1}^{\eta+1} \\
& - a_1\alpha^\beta\|\omega\|_\beta^\beta + a_2T_1T_2 \\
\le& \frac{\alpha^{\eta+1}p_{\max}C_2^{\eta+1}}{\eta+1}\|\triangle_1\omega(n-1,m)\|^{\eta+1} + \frac{\alpha^{\eta+1}r_{\max}C_2^{\eta+1}}{\eta+1}\|\triangle_2\omega(n,m+1)\|^{\eta+1} \\
& - \frac{\alpha^{\eta+1}q_{\min}C_2^{\eta+1}}{\eta+1} - a_1\alpha^\beta C_3^\beta + a_2T_1T_2 \\
\le& \frac{\alpha^{\eta+1}p_{\max}C_2^{\eta+1}\lambda_{T_1}^{\frac{\eta+1}{2}}}{\eta+1} + \frac{\alpha^{\eta+1}r_{\max}C_2^{\eta+1}\mu_{T_2}^{\frac{\eta+1}{2}}}{\eta+1} - \frac{\alpha^{\eta+1}q_{\min}C_2^{\eta+1}}{\eta+1} - a_1\alpha^\beta C_3^\beta + a_2T_1T_2 \\
\to& -\infty, \qquad\qquad \text{as} \quad \alpha \to +\infty,
\end{aligned}$$

which means that there exists $\alpha > \rho$ large enough such that $I(u_0) < 0$, where $u_0 = \alpha\omega \in E\backslash B_\rho$. Moreover, $I(0) = 0$. Thus, Lemma 2.1 guarantees that there is a critical value $c \ge a > 0$. Assume $\bar{u}$ is a critical point, namely, $I(\bar{u}) = c$ and $I'(\bar{u}) = 0$.

In the following, we look for another critical point $\tilde{u}$ for I. By (3.4), we get

$$I(u) \leq \frac{C_2^{\eta+1}\left(p_{\max}\lambda_{T_1}^{\frac{\eta+1}{2}} + r_{\max}\mu_{T_2}^{\frac{\eta+1}{2}} - q_{\min}\right)}{\eta+1} \|u_k\|^{\eta+1} - a_1 C_3^{\beta} \|u\|^{\beta} + a_2 T_1 T_2,$$

which indicates I is bounded from above. Denote the supremum of $\{I(u)\}_{u\in E}$ by $c_{\max}$, and then $\tilde{u} \in E$ and $I(\tilde{u}) = c_{\max}$. Obviously, $\tilde{u} \neq 0$. If $\bar{u} \neq \tilde{u}$, the proof is finished. Else, $c = c_{\max}$. Lemma 2.1 means that

$$c = \inf_{h\in\Gamma} \sup_{t\in[0,1]} I(h(t)),$$

where

$$\Gamma = \{h \in C([0,1], E) | h(0) = 0, h(1) = u_0\}.$$

Hence, for any $h \in \Gamma$, $c_{\max} = \max_{t\in[0,1]} I(h(t))$. In view of $I(h(t))$ being continuous with respect to t, $I(0) \leq 0$ and $I(u_0) < 0$, it follows that there exists a $t_0 \in (0,1)$ such that $I(h(t_0)) = c_{\max}$. Choose h_1, h_2 such that $\{h_1(t)|t \in (0,1)\} \cap \{h_2(t)|t \in (0,1)\} = \emptyset$, and then there exist $t_1, t_2 \in (0,1)$ such that $I(h_1(t_1)) = I(h_2(t_2)) = c_{\max}$. Thus, we obtain two different critical points $u_1 = h_1(t_1)$ and $u_2 = h_2(t_2)$. Consequently, there exist at least two nontrivial critical points which correspond to the critical value $c_{\max}$. This completes the proof.

Proof of Theorem 3.2 Let $W = \{w \in E | w = \{c\}, c \in \mathbb{R}\}$, $Y = W^{\perp}$, and then $E = W \oplus Y$. By (F_1), there exist some $\rho > 0$ and $u \in B_\rho$ such that

$$F((n,m),u) \leq \frac{p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu^{\frac{\eta+1}{2}} C_1^{\eta+1}}{2(\eta+1)C_2^{\eta+1}} |u|^{\eta+1}.$$

Then, for every $u \in (\partial B_\rho) \cap Y$, one obtains

$$\begin{aligned}
I(u) \geq & \frac{p_{\min}}{\eta+1} \|\triangle_1 u(n-1,m)\|_{\eta+1}^{\eta+1} + \frac{r_{\min}}{\eta+1} \|\triangle_2 u(n,m-1)\|_{\eta+1}^{\eta+1} - \frac{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}}\right) C_1^{\eta+1}}{2(\eta+1)C_2^{\eta+1}} \|u\|_{\eta+1}^{\eta+1} \\
\geq & \frac{p_{\min}C_1^{\eta+1}}{\eta+1} \|\triangle_1 u(n-1,m)\|^{\eta+1} + \frac{r_{\min}C_1^{\eta+1}}{\eta+1} \|\triangle_2 u(n,m-1)\|^{\eta+1} - \frac{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}}\right) C_1^{\eta+1}}{2(\eta+1)} \|u\|^{\eta+1} \\
\geq & \frac{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}}\right) C_1^{\eta+1}}{\eta+1} \|u\|^{\eta+1} - \frac{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}}\right) C_1^{\eta+1}}{2(\eta+1)} \|u\|^{\eta+1} \\
= & \frac{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}}\right) C_1^{\eta+1}}{2(\eta+1)} \|u\|^{\eta+1} \\
= & \frac{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}}\right) C_1^{\eta+1} \rho^{\eta+1}}{2(\eta+1)}.
\end{aligned} \tag{3.6}$$

Set $a = \frac{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}}\right) C_1^{\eta+1} \rho^{\eta+1}}{2(\eta+1)}$, and then (3.6) implies $I(u) \geq a$, $u \in (\partial B_\rho) \cap Y$. Thus, (f_3) of Lemma 2.2 is valid.

Let $e \in \partial B_1 \cap Y$, and for every $w \in W$ and $s \in \mathbb{R}$, set $u = se + w$. From (2.4) together with (F_2), it follows that $\triangle_1 w = \triangle_2 w = 0$, $\|e\| = 1$, and

$$\begin{aligned}
I(u) =& I(se+w) \\
=& \frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[p(n-1,m)(\triangle_1 se(n-1,m))^{\eta+1} + r(n,m-1)(\triangle_2 se(n,m-1))^{\eta+1}\right] \\
&- \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),(se+w)(n,m)). \\
\leq& \frac{s^{\eta+1}p_{\max}}{\eta+1}\|\triangle_1 e(n-1,m)\|_{\eta+1}^{\eta+1} + \frac{s^{\eta+1}r_{\max}}{\eta+1}\|\triangle_2 e(n,m-1)\|_{\eta+1}^{\eta+1} \\
&- a_1\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}|se(n,m)+w(n,m)|^{\beta} + a_2 T_1 T_2 \\
\leq& \frac{s^{\eta+1}p_{\max}C_2^{\eta+1}}{\eta+1}\|\triangle_1 e(n-1,m)\|^{\eta+1} + \frac{s^{\eta+1}r_{\max}C_2^{\eta+1}}{\eta+1}\|\triangle_2 e(n,m-1)\|^{\eta+1} \\
&- a_1 C_3^{\beta}\|se+w\|^{\beta} + a_2 T_1 T_2 \\
\leq& \frac{s^{\eta+1}C_2^{\eta+1}}{\eta+1}\left(p_{\max}\lambda_{T_1}^{\frac{\eta+1}{2}} + r_{\max}\mu_{T_2}^{\frac{\eta+1}{2}}\right) - a_1 C_3^{\beta}\|se\|^{\beta} - a_1 C_3^{\beta}\|w\|^{\beta} + a_2 T_1 T_2 \\
=& \frac{s^{\eta+1}C_2^{\eta+1}}{\eta+1}\left(p_{\max}\lambda_{T_1}^{\frac{\eta+1}{2}} + r_{\max}\mu_{T_2}^{\frac{\eta+1}{2}}\right) - a_1 C_3^{\beta}s^{\beta} - a_1 C_3^{\beta}\|w\|^{\beta} + a_2 T_1 T_2.
\end{aligned}$$

Write

$$g_1(s) = \frac{s^{\eta+1}C_2^{\eta+1}}{\eta+1}\left(p_{\max}\lambda_{T_1}^{\frac{\eta+1}{2}} + r_{\max}\mu_{T_2}^{\frac{\eta+1}{2}}\right) - a_1 C_3^{\beta}s^{\beta}, \qquad g_2(\tau) = -a_1 C_3^{\beta}\tau^{\beta} + a_2 T_1 T_2.$$

Then, both $g_1(s)$ and $g_2(\tau)$ are bounded from above. Moreover, $\beta > \eta + 1$ leads to $\lim_{s\to+\infty} g_1(s) = -\infty$ and $\lim_{\tau\to+\infty} g_2(\tau) = -\infty$. Thus, there exists a positive constant $R_0 > \rho$ such that $I(u) \leq 0$ holds for any $u \in \partial Q$ and $Q = \left(\bar{B}_{R_0} \cap W\right) \oplus \{se|0 < s < R_0\}$.

Notice that Lemma 3.1 shows I satisfies the $P.S.$ condition on E. Therefore, Lemma 2.2 ensures that I admits a critical value $c \geq a$, and

$$c = \inf_{h\in\Gamma}\max_{u\in\bar{Q}} I(h(u)), \quad \Gamma = \left\{h \in C(\bar{Q}, E) : h|_{\partial Q} = id\right\}.$$

Take $\bar{u} \in E$ to be a critical point which corresponds to c, that is, $I(\bar{u}) = c$. By (3.4), I is bounded from above. Hence, there will be a $\tilde{u} \in E$ such that

$$I(\tilde{u}) = c_{\max} = \sup_{u\in E} I(u).$$

Then, $\bar{u}$ and $\tilde{u}$ are nontrivial (T_1, T_2)-periodic solutions of (1.1). If $\bar{u} \neq \tilde{u}$, then Theorem 3.2 holds. Otherwise, $\bar{u} = \tilde{u}$, and then $c = c_{\max}$, that is,

$$\sup_{u\in E} I(u) = \inf_{h\in\Gamma}\sup_{u\in\bar{Q}} I(h(u)).$$

Choose $h = id$, and we get $\sup_{u\in\bar{Q}} I(u) = c_{\max}$. Consider $-e \in \partial B_\rho \cap Y$. With the arbitrariness of e, it follows that there is an $R_1 > \rho$ such that $I|_{\partial Q_1} \le 0$, where $Q_1 = \left(\bar{B}_{R_1} \cap W\right) \oplus \{-se|0 < s < R_1\}$. In the same way, by Lemma 2.2, I possesses a new critical value $c' \ge a > 0$, and

$$c' = \inf_{h\in\Gamma} \max_{u\in\bar{Q}_1} I(h(u)), \quad \Gamma = \left\{h \in C(\bar{Q}_1, E) : h|_{\partial Q_1} = id\right\}.$$

If $c' \ne c_{\max}$, then this proof is done. If $c' = c_{\max}$, then for any $h \in \Gamma$, we have $\max_{u\in\bar{Q}_1} I(h(u)) = c_{\max}$. Specifically, take $h = id$, and then $\max_{u\in\bar{Q}_1} I(u) = c_{\max}$. Since $I|_{\partial Q} \le 0$, $I|_{\partial Q_1} \le 0$ and $c_{\max} > 0$, the maximum value of I is given at a point inside Q and Q_1, respectively. In addition, $Q \cap Q_1 \subset W$, and for every $w \in W$, we have

$$I(w) = -\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), w(n,m)) \le 0.$$

Therefore, a point $\hat{u}$ which is different from $\tilde{u}$ must exist in E such that $I(\hat{u}) = c' = c_{\max}$. In summary, if $c < c_{\max}$, then (1.1) admits at least two nontrivial (T_1, T_2)-periodic solutions; if $c = c_{\max}$, then (1.1) admits an infinite number of nontrivial (T_1, T_2)-periodic solutions. This completes the proof.

Proof of Theorem 3.3 Similar to (2.2), the variational functional associated to (3.1) is expressed by

$$\begin{aligned}\hat{I}(u) =& \frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\Big[p(n-1,m)\left(\triangle_1 u(n-1,m)\right)^{\eta+1} + r(n,m-1)\left(\triangle_2 u(n,m-1)\right)^{\eta+1} \\ & -q(n,m)\left(u(n,m)\right)^{\eta+1}\Big] - \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(n,m)u(n,m).\end{aligned} \tag{3.7}$$

In the following, we utilize Lemma 2.3 to finish the proof. To begin with, it is to show that $\hat{I}$ meets the $P.S.$ condition on E. Suppose $\{u_k\} \subset E$, and there is a constant $\hat{M} > 0$ such that

$$|\hat{I}(u_k)| \le \hat{M}, \qquad \hat{I}'(u_k) \to 0, \quad k \to +\infty.$$

Since the dimension of E is T_1T_2, it is necessary for us to show $\{u_k\}$ is bounded in E. In view of (3.7) and the oddness of η, there holds

$$\begin{aligned}\hat{M} \ge \hat{I}(u_k) &\ge -\frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)\left(u_k(n,m)\right)^{\eta+1} - \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(i,i)u_k(n,m) \\ &\ge -\frac{q_{\max}}{\eta+1}\|u_k\|_{\eta+1}^{\eta+1} - \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{1}{2}}\left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} u_k^2(n,m)\right)^{\frac{1}{2}} \\ &\ge -\frac{q_{\max}C_1^{\eta+1}}{\eta+1}\|u_k\|^{\eta+1} - \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{1}{2}}\|u_k\|.\end{aligned} \tag{3.8}$$

Recall $\eta + 1 > 1$, and then (3.8) implies that $\{u_k\}$ is bounded in E. Therefore, $\hat{I}$ satisfies the $P.S.$ condition on E.

Next, we show (f_5) and (f_6) of Lemma 2.3 are met. For any $w = (z, \cdots, z)^T \in W$, one has

$$\hat{I}(w) = -\frac{1}{\eta+1}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)z^{\eta+1} - \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(n,m)z.$$

Take

$$z = \left(-\frac{\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(n,m)}{\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)}\right)^{\frac{1}{\eta}}, \qquad \rho = \|w\| = \sqrt{T_1T_2}\left|\frac{\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(n,m)}{\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)}\right|^{\frac{1}{\eta}}.$$

Then

$$\hat{I}(w) = \frac{\eta}{\eta+1}\frac{\left[\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(n,m)\right]^{\frac{\eta+1}{\eta}}}{\left[\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)\right]^{\frac{1}{\eta}}}.$$

Thus,

$$\hat{I}(u) = \sigma \triangleq \frac{\eta}{\eta+1}\frac{\left[\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f(n,m)\right]^{\frac{\eta+1}{\eta}}}{\left[\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} q(n,m)\right]^{\frac{1}{\eta}}}, \quad \forall u \in \partial B_\rho \cap W,$$

which means that (f_5) of Lemma 2.3 holds.

For $y \in Y$, one has

$$\begin{aligned}
\hat{I}(y) \geq& \frac{p_{\min}}{\eta+1}\|\triangle_1 y(n-1,m)\|_{\eta+1}^{\eta+1} + \frac{r_{\min}}{\eta+1}\|\triangle_2 y(n,m-1)\|_{\eta+1}^{\eta+1} - \frac{q_{\max}}{\eta+1}\|y\|_{\eta+1}^{\eta+1} \\
&- \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{1}{2}}\|y\| \\
\geq& \frac{C_1^{\eta+1}}{\eta+1}\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + r_{\min}\mu_2^{\frac{\eta+1}{2}} - q_{\max}\right)\|y\|^{\eta+1} - \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{1}{2}}\|y\| \\
\geq& -\frac{\eta}{\eta+1}\frac{\left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{\eta+1}{2\eta}}\left(\frac{1}{C_1}\right)^{\frac{\eta+1}{\eta}}}{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + \gamma_{\min}\mu_2^{\frac{\eta+1}{2}} - q_{\max}\right)^{\frac{1}{\eta}}},
\end{aligned}$$

and the last inequality is obtained by minimization with respect to $\|y\|$. Set

$$\omega_0 = -\frac{\eta}{\eta+1}\frac{\left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{\eta+1}{2\eta}}\left(\frac{1}{C_1}\right)^{\frac{\eta+1}{\eta}}}{\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + \gamma_{\min}\mu_2^{\frac{\eta+1}{2}} - q_{\max}\right)^{\frac{1}{\eta}}}.$$

Together with (F_3), this yields that

$$\hat{I}(u) \geq \omega_0 > \sigma, \quad \forall u \in Y.$$

So, (f_6) of Lemma 2.3 holds by taking $e = 0$. Thus, all conditions of Lemma 2.3 are satisfied, and (3.1) admits at least a (T_1, T_2)-periodic solution.

4. Examples

Finally, we give three examples to demonstrate the validity of our results. Let

$$\tilde{E} = \{u = \{u(n,m)\} \in S \,|\, u(n+2,m) = u(n,m) = u(n,m+2), \quad n,m \in \mathbb{Z}(1,2)\}.$$

Example 4.1. *Take $\eta = 3$. Consider Eq (1.1) with $p(n,m) > 0$, $r(n,m) \geq 0$ and $q(n,m) < 0$ for any $n,m \in \mathbb{Z}$, and*

$$f((n,m),u) = 6u^5, \qquad n,m \in \mathbb{Z}, \quad u \in \mathbb{R}.$$

Then,

$$F((n,m),u) = u^6, \qquad n,m \in \mathbb{Z}, \quad u \in \mathbb{R}.$$

Obviously, $f((n,m),u)$ satisfies all conditions of Theorem 3.1. Then, (1.1) with $f((n,m),u) = 6u^5$ admits at least two nontrivial (T_1,T_2)-periodic solutions.

Specially, let $p(n,m) = r(n,m) = 1$, $q(n,m) = -2$ and $T_1 = T_2 = 2$, and then (1.1) can be rewritten in the form of

$$\triangle_1(\triangle_1 u(n-1,m))^3 + \triangle_2(\triangle_2 u(n,m-1))^3 - 2(u(n,m))^3 + 6(u(n,m))^5 = 0. \tag{4.1}$$

Using MATLAB, we find that (4.1) has at least two nontrivial solutions $u_1 = \{u_1(n,m)\} \in \tilde{E}$ and $u_2 = \{u_2(n,m)\} \in \tilde{E}$, where

$$u_1 = (\sqrt{3}, -\sqrt{3}, \sqrt{3}, -\sqrt{3}), \qquad u_2 = (-\sqrt{3}, \sqrt{3}, -\sqrt{3}, \sqrt{3}).$$

By Remark 1.1, u_1 and u_2 are two different nontrivial $(2,2)$-periodic solutions to (4.1).

Example 4.2. *For every $n,m \in \mathbb{Z}$, consider (1.1) with $p(n,m) > 0$, $r(n,m) \geq 0$ and $q(n,m) = 0$. Set $\eta = 3$, and*

$$f((n,m),u) = 6u^5, \qquad n,m \in \mathbb{Z}, u \in \mathbb{R}.$$

Then, all conditions of Theorem 3.2 are satisfied, and (1.1) has at least two nontrivial (T_1,T_2)-periodic solutions.

Take $p(n,m) = r(n,m) = 1$ and $T_1 = T_2 = 2$, and then (1.1) becomes

$$\triangle_1(\triangle_1 u(n-1,m))^3 + \triangle_2(\triangle_2 u(n,m-1))^3 + 6(u(n,m))^5 = 0. \tag{4.2}$$

Utilizing MATLAB, (4.2) has at least two nontrivial solutions

$$u_1 = (\frac{2\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}), \qquad u_2 = (-\frac{2\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}).$$

From Remark 1.1, (4.2) has at least two different nontrivial $(2,2)$-periodic solutions u_1 and u_2.

Example 4.3. *Set $T_1 = T_2 = 2$, $\eta = 3$ and $f(n,m) = 2$. Consider (3.1) with $p(n,m) = 1$, $r(n,m) = 1$ and $q(n,m) = -2$. Clearly, $p(n,m)$, $r(n,m)$, $q(n,m)$ and $f(n,m)$ are all $(2,2)$-periodic, and (3.1) is in the following form:*

$$\triangle_1(\triangle_1 u(n-1,m))^3 + \triangle_2(\triangle_2 u(n,m-1))^3 - 2(u(n,m))^3 + 2 = 0. \tag{4.3}$$

Obviously, we only need to verify (F_3). *Simple calculation gives*

$$\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}(-q(n,m))\left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f^2(n,m)\right)^{\frac{\eta+1}{2}}\left(\frac{1}{C_1}\right)^{\eta+1} = 8\times 16^2\times 4 = 8192,$$

$$\left(p_{\min}\lambda_2^{\frac{\eta+1}{2}} + \gamma_{\min}\mu_2^{\frac{\eta+1}{2}} - q_{max}\right)\left(\sum_{n=1}^{T}\sum_{m=1}^{T_2} f(n,m)\right)^{\eta+1} = (4^2+4^2+2)\times 8^4 = 139264 > 2048.$$

Thus, Theorem 3.3 ensures that (4.3) possesses at least one nontrivial $(2,2)$*-periodic solution. By MATLAB and Remark 1.1, (4.3) admits at least a* $(2,2)$*-periodic solution*

$$u(1,1)=1,\quad u(2,1)=1,\quad u(1,2)=1,\quad u(2,2)=1.$$

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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